Completion of the Proof of Proposition 4 in "Interim Correlated Rationalizability," *Theoretical Economics* **2** (2007), 15–40.

Eddie Dekel
Northwestern University
and Tel Aviv University

Drew Fudenberg
Harvard University

Stephen Morris
Princeton University

Proposition 4 in "Interim Correlated Rationalizability" is correct as stated but the proof is incomplete. Here we provide a complete proof.[1]

Proposition 4: $R^{\mathcal{T}}_{\mathcal{F}}$ *equals* $R^{\mathcal{T}}$.

Proof. It is sufficient to prove that $R^{\mathcal{T}}$ is a best-reply set. That nothing larger can be a best-reply set is immediate. For every $a_i \in R_i^{\mathcal{T}}(t_i)$ we have that for every k there is a measurable $\sigma^k_{-i} : T_{-i} \times \Theta \to \Delta(A_{-i})$ s.t. (i) $\sigma^k_{-i}(t_{-i}, \theta)[a_{-i}] > 0 \Rightarrow a_j \in R^{\mathcal{T}}_{k,t_j}(t_j)$ and (ii) $a_i \in \arg\max_{a_i'} \sum_{\Theta \times A_{-i}} \int_{T_{-i}} g_i(a_i', a_{-i}, \theta)\, \sigma^k_{-i}(t_{-i}, \theta)[a_{-i}]\, \pi(t_i)[(\mathrm{d}t_{-i}, \theta)]$. We need to prove there exists $\sigma_{-i} : T_{-i} \times \Theta \to \Delta(A_{-i})$ s.t. (i′) $\sigma_{-i}(t_{-i}, \theta)[a_{-i}] > 0 \Rightarrow a_j \in \cap_k R^{\mathcal{T}}_{k,t_j}(t_j)$ and (ii) $a_i \in \arg\max_{a_i'} \sum_{\Theta \times A_{-i}} \int_{T_{-i}} g_i(a_i', a_{-i}, \theta)\, \sigma_{-i}(t_{-i}, \theta)[a_{-i}]\, \pi(t_i)[(\mathrm{d}t_{-i}, \theta)]$.

The proof goes as follows. In step 1 we replace σ^k_{-i} by a $\hat{\sigma}^k_{-i}$ that takes only finitely many values, at most one for each $B_{-i} \subset A_{-i}$, and that continues to satisfy (i) and (ii). Then we take limits of $\hat{\sigma}^k_{-i}$ in a manner which will satisfy (i′) and (ii).

Step 1. We mimic step IVb in the proof of Lemma 1. As in that step, for every $B_{-i} \subset A_{-i}$ let $\tau^{\mathcal{T}}_{-i,k-1}(B_{-i}) = \left\{t_{-i} \in T_{-i} : B_{-i} = R^{\mathcal{T}}_{-i,k-1}(t_{-i})\right\}$; as argued there $\tau^{\mathcal{T}}_{-i}(B_{-i}) \subset T_{-i}$ is measurable. Construct $\hat{\sigma}^k_{-i}(t_{-i}, \theta)[\cdot] \in \Delta(A_{-i})$ as follows. Map $\sigma^k_{-i}(t_{-i}, \cdot)$ into $\hat{\sigma}^k_{-i}(t_{-i}, \cdot)$ by taking all t_{-i} for whom B_{-i} is $k-1$ rationalizable, denoted $\tau^{\mathcal{T}}_{-i,k-1}(B_{-i})$, taking the

[1] We thank Saturo Takahsahi for pointing out to us that steps in the proof are missing, and Tzachi Gilboa for a helpful discussion.

conditional average of $\sigma^k_{-i}(t_{-i},\cdot)$ over those t_{-i}, and assigning that average conjecture to all t_{-i} who have that same $k-1$ rationalizable set, i.e., to $\tau_{-i,k-1}(B_{-i})$. Obviously these sets partition T_{-i}, so we can combine all those averages to get a strategy for all $t_{-i}\in T_{-i}$. (As before there is a slight issue for the case where the conditional isn't well defined because the conditioning event, $\tau^{\mathcal{T}}_{-i,k-1}(B_{-i})$, has probability zero. In that case the strategy is really irrelevant, but as we require it to be measurable and to map into the $k-1$ rationalizable set, we add that restriction by having the strategy assign probability 1 to some $k-1$ rationalizable action for all $t_{-i}\in\tau_{-i,k-1}(B_{-i})$ whenever $\pi_i(t_i)\left[\tau^{\mathcal{T}}_{-i,k-1}(B_{-i})\right]=0$. To do this, for each B_{-i} fix some $\bar{a}_{-i}(B_{-i})\in B_{-i}$.)

We now formalize this verbal description.

$$\hat{\sigma}^k_{-i}(t_{-i},\theta)[a_{-i}]=\begin{cases}\frac{\int_{\tau^{\mathcal{T}}_{-i,k-1}(B_{-i})}\sigma^k_{-i}(t_{-i},\theta)[a_{-i}]\pi(t_i)[(\mathrm{d}t_{-i},\theta)]}{\pi(t_i)\left[\tau^{\mathcal{T}}_{-i,k}(B_{-i})\right]} \text{ if } t_{-i}\in\tau_{-i,k-1}(B_{-i}) \text{ and } \pi(t_i)\left[\tau^{\mathcal{T}}_{-i,k-1}(B_{-i})\right]>0\\ 1 \text{ if } t_{-i}\in\tau_{-i,k-1}(B_{-i}),\ \pi(t_i)\left[\tau^{\mathcal{T}}_{-i,k-1}(B_{-i})\right]=0 \text{ and } a_{-i}=\bar{a}_{-i}(B_{-i})\\ 0 \text{ if } t_{-i}\in\tau_{-i,k-1}(B_{-i}),\ \pi(t_i)\left[\tau^{\mathcal{T}}_{-i}(B_{-i,k-1})\right]=0 \text{ and } a_{-i}\neq\bar{a}_{-i}(B_{-i})\end{cases}$$

This is measurable because it is constant on each of the finitely many measurable cells of $\{\tau_{-i,k-1}(B_{-i})\}_{B_{-i}\subset A_{-i}}$. Moreover, $\hat{\sigma}^k_{-i}(t_{-i},\theta)[a_{-i}]>0\Rightarrow a_{-i}\in R^{\mathcal{T}}_{-i,k-1}(t_{-i})$. This $\hat{\sigma}^k_{-i}$ can be used to define $\hat{\psi}_i\in\hat{\Psi}_i\left(t^*_i,R^{\mathcal{T}}_{-i,k}\right)$ by $\hat{\psi}_i[\theta,a_{-i}]=\int_{T_{-i}}\hat{\sigma}_{-i}(t_{-i},\theta)[a_{-i}]\pi_i(t_i)[(\mathrm{d}t_{-i},\theta)]$, where we are just averaging out σ^k_{-i}, so as in the proof of part IVb of Lemma 1 $\hat{\psi}_i[\theta,a_{-i}]=\psi_i[\theta,a_{-i}]$. So, for each k we have $\hat{\sigma}^k_{-i}$ that takes finitely many values, at most one for each $B_{-i}\subset A_{-i}$, that satisfies (i) and (ii).

Step 2. Now take a subsequence of $\hat{\sigma}^k_{-i}$ such that along the subsequence $\hat{\sigma}^k_{-i}\to\sigma_{-i}$ and, for each B_{-i}, $\pi(t_i)\left[\tau^{\mathcal{T}}_{-i,k-1}(B_{-i})\right]$ converge. Since the sequence determines only finitely many values such a convergent subsequence can be found.

That (i′) is satisfied is now immediate. That σ_{-i} is measurable follows because $\{t_i:\sigma_{-i}(t_{-i},\theta)=a_{-i}\}=\bigcup_{K=1}^{\infty}\bigcap_{k\geq K}\left\{t_i:\sigma^k_{-i}(t_{-i},\theta)=a_{-i}\right\}$, and since the latter is measurable so is the former. So (ii) is well defined, and convergence of the integral follows from standard results (the bounded convergence theorem). ■