Keeping the Little Guy Down: A Debt Trap for Lending with Limited Pledgeability

Ernest Liu and Benjamin N. Roth*

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Abstract

Microcredit and other forms of small-scale finance in the developing world have failed to catalyze entrepreneurship. This prompts a re-examination of the special features of informal credit markets that cause them to operate inefficiently. We present a theory that highlights two of these features. First, the borrower has limited commitment and cannot pledge the benefits of her growth. Second, borrowers and lenders bargain not only over division of surplus but also over contractual flexibility (the ease with which the borrower can invest to grow her business). These two features lead to a poverty trap for poor borrowers, while unambiguously benefitting richer borrowers. The theory features nuanced comparative statics – improving the bargaining position of rich borrowers can harm poor borrowers, as the lender tightens restrictions on them to prevent them from growing. The theory facilitates reinterpretation of a number of empirical facts about microcredit: business growth resulting from microfinance is low on average, high for businesses that are already relatively large, and microlenders have experienced low demand.

1 Introduction

Capital constraints pose a substantial obstacle to small-scale entrepreneurship in the developing world. Experimental evidence from "cash drop" studies paints a remarkably consistent picture—across a broad range of contexts including Mexico, Sri Lanka, Ghana, and India, small-scale entrepreneurs enjoy a *monthly* return to capital in the range of 5% – 10%.[1] Surprisingly

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[1]See e.g. De Mel, McKenzie, and Woodruff (2008), Fafchamps, McKenzie, and Woodruff (2014), and McKenzie and Woodruff (2008)

however, many of the entrepreneurs in these studies also had access to credit, from microfinance institutions (MFIs), moneylenders, and a variety of other informal lenders, yet they did not reach their productive capacity until an experimenter gave them cash. Moreover, many experimental evaluations of microfinance find that it has only modest impacts on entrepreneurship. This may be especially puzzling in light of the fact that the interest rates charged for microloans are well below the above estimates of marginal return to capital. Why then can't small-scale entrepreneurs use available credit to pursue their profitable investment opportunities?

We address this puzzle through a theory that posits lenders may intentionally inhibit the growth of their borrowers' income. Predatory theories of lending such as this one have deep roots in development economics (see e.g. Bhaduri (1973), and Bhaduri (1977)), and even cursory searches of popular media unearth numerous assertions that MFIs and other informal lenders are predatory.[2] However this story has fallen out of favor in the recent academic literature, especially about microfinance. In large part the departure from predatory theories can be traced to early influential work by Braverman and Srinivasan (1981) and Braverman and Stiglitz (1982), which argued that profit-maximizing lenders would always encourage efficient investments and extract the benefits through higher interest rates.

We show the above logic to be fragile in the presence of two natural features of lending relationships in the developing world. First, borrowers cannot credibly pledge the benefits that arise from investment in their businesses. The primary metaphor we will use is that richer borrowers exit the informal lending sector, either because they gain access to cheaper, more formal credit, or because they reach self sufficiency, but an inability to pledge future profits may arise for many other reasons. Second, lenders have access to *contractual restrictions* that inhibit borrowers' ability to invest their loans productively. The first feature reopens the possibility that a profit-maximizing lender would want to stymie his borrower's growth, and the second feature enables him to act on that desire.

While in our model, contractual restrictions are an abstract means for the lender to control the borrower's investment, there are many real-world examples of contractual restrictions which govern the ease with which the borrower can invest her loan to grow her business. For instance, MFIs often impose rigid repayment schedules requiring that borrowers maintain cash on hand and discouraging long-term investment. Field, Pande, Papp, and Rigol (2013) demonstrated that by relaxing this restriction, borrowers were able to invest 80% more in their business, and three years later earned 41% higher profits. Moneylenders commonly require that borrowers work on their land (tying up labor) or that borrowers must forfeit their own land for the money lender's use (tying up capital).[3] And common to both moneylenders and microfinance is the use of guarantors or joint liability. A variety of theory and empirical evidence suggest that guarantors

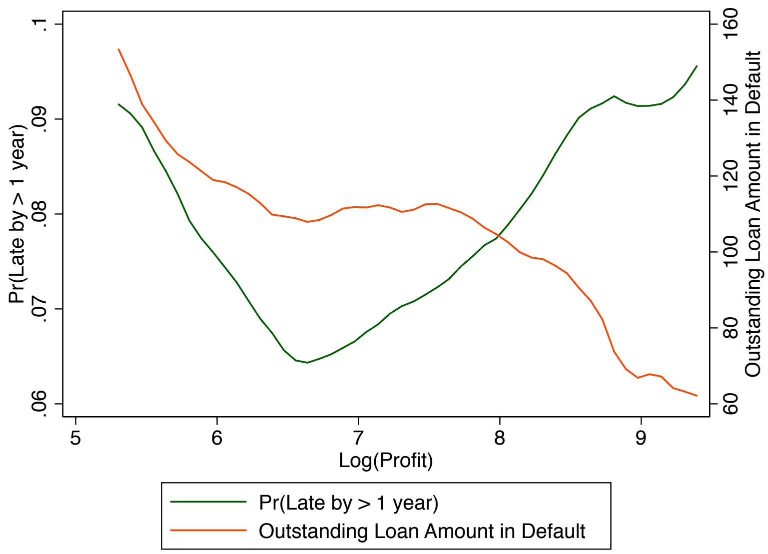
[2] See e.g. Flintoff (2010), Melik (2010), Chaudhury and Swamy (2012)

[3] See e.g. Sainath (1996)

pressure borrowers to eschew profitable but risky investments.[4]

These restrictive features are often attributed to ensuring the repayment of loans, but with the exception of a rigid repayment schedule, there is little conclusive evidence that these contractual provisions actually serve to reduce default.[5] And we argue that even in the case of repayment rigidity the story may not be clear. In fact, through a reanalysis of the Field et al. (2013) data, we find evidence that it is not the additional money taken in default that discourages MFIs from offering flexible repayment schedules, but rather that, consistent with the model's first premise, richer borrowers do not renew their loans.

Figure 1
Patterns of Default by Wealth Level



For a sample of borrowers from the Indian MFI studied in Field et al. (2013), the red line in Figure 1 plots the average amount a borrower has in default a year after loan origination as a function of business profits three years after loan origination.[6] As may be expected, there is a decreasing relationship between a borrower's profitability and the amount of money she is expected to have in default. However the blue line in Figure 1 plots the *likelihood* a borrower defaulted on any amount of money as a function of business profits three years after loan origination. In this case there is U-shaped relationship between profits and default; those with relatively low and relatively high business profits are the most likely to default. Table 1 confirms this pattern in a parametric regression.

[4]For theory see Banerjee, Besley, and Guinnane (1994), and for empirical evidence see Fischer (2013), Jack, Kremer, de Laat, and Suri (2016), and Maitra, Mitra, Mookherjee, Motta, and Visaria (2017)

[5]Gine and Karlan (2014) and Attanasio, Augsburg, De Haas, Fitzsimons, and Harmgart (2015) each provide experimental evidence that joint liability does not affect the likelihood of default. In contrast, using non-experimental variation, Carpena, Cole, Shapiro, and Zia (2013) finds that joint liability loans are more likely to be repaid.

[6]This is the publicly available measure of business profits.

Taken together these findings paint a clear picture. Borrowers with large businesses are substantially less likely to finish repaying their loans but the amount of money they (or any borrower) take in default is low. While the borrowers who do not completely repay their loans are only a subset of those who do not continue to borrow, the pattern suggests that richer borrowers renew their loans at a substantially lower rate. And to the extent that MFIs only generate profit from active borrowers, these results suggest that lenders do not benefit from the business growth of their borrowers, and therefore whatever costs are imposed on them by repayment flexibility are not offset by the much larger gains accruing to borrowers.[7]

Model Description

Motivated by these facts, we build a dynamic model of a borrower-lender relationship that captures the two special features highlighted above. Specifically we study a dynamic principal-agent model in continuous time. The principal is an MFI or informal lender, and the agent is a borrower—for concreteness suppose the borrower is a fruit vendor who operates a mobile cart. The borrower has access to two investment projects: a working capital project (e.g., buying fruits to sell throughout the week), and a fixed capital project (e.g. building a permanent stall to expand her business). The former option generates an immediate payoff, while the latter involves substantial upfront investment, requiring her to forgo immediate consumption but potentially expanding her future business capacity. Following the stochastic games literature, we model the borrower's business expansion through a discrete state space with stochastic transition rate that depends on the size of investment. For example, in the first state she operates a mobile cart, when she grows to the second state she operates a fixed stall, in the third state she owns several stalls, etc. If she reaches the final state, the game ends, with the borrower receiving a high continuation payoff (resulting from reaching the formal sector or operating at self sufficiency) and the lender receiving nothing (he has lost his customer). Crucially, the borrower cannot pledge the benefits of reaching this final state to her lender.

At each instant, the lender offers a loan to augment the borrower's budget, which specifies both an interest rate, and a *contractual restriction*. Specifically, as an abstract representation of the restrictions discussed above, we assume the lender can prevent the borrower from investing in fixed capital if she accepts the loan. We refer to the loan as *restrictive* if it specifies the borrower must invest in working capital, and we refer to the loan as *unrestrictive* if it allows her to invest freely. Accepting a loan is voluntary, so if the borrower does not find the loan's size and interest rate sufficiently attractive to offset the additional contractual restrictions, she can reject it and allocate her own, smaller budget flexibly.

[7] Indeed, while Field et al. (2013) find that borrowers afforded repayment flexibility defaulted on an extra Rs. 150 on average, they also earned an extra Rs. 450 to Rs. 900 *every week*.

We solve for the stationary Markov perfect equilibrium (MPE) of this game. In doing so, we assume that both players use strategies that condition only on the borrower's business size. By employing this solution concept we underscore both players' lack of commitment power, and preclude the usage of long-term contracts.

Summary of Results

The voluntary nature of the loan contract induces an important tradeoff. For a restrictive loan to surpass the borrower's outside option (investing her small endowment flexibly), she must be compensated along other dimensions—we focus on the interest rate. The lender's tradeoff informs both when and why we observe contractual restrictions.

Our analysis highlights an asymmetry between restrictive and unrestrictive loans. If the lender is unable to set an interest rate which leaves the borrower with exactly the level of output she would have had in the lender's absence (e.g. due to a running away constraint or asymmetric information about productivity), the borrower will retain more utility from unrestrictive loans in equilibrium. We refer to this asymmetry as the borrower's *expansion rent*. It arises because the borrower cannot commit to share the proceeds of business growth and therefore values the investment of her residual income into business growth more highly than alternative (pledgeable) investments. If this asymmetry is severe the lender only offers restrictive loans and the borrower remains in poverty. We sometimes refer to this phenomenon as a poverty trap because borrowers below a certain wealth level never reach their efficient size, and we sometimes refer to it as a debt trap, because borrowing continues in perpetuity. We show that this trap occurs if and only if the additional surplus the borrower gains from unrestrictive contracts exceeds the additional social welfare generated from business growth. Importantly, the borrower may get stuck in a debt trap even if she would have grown to her efficient size in the absence of a lender. That is, the introduction of a lender may decrease business growth relative to autarky.

We show this game admits a unique MPE, and under additional conditions the probability of a restrictive loan is single peaked in the state. The poorest and richest borrowers receive unrestrictive loans and grow their business, but those in an intermediate region receive restrictive loans and remain there forever. Relatively richer borrowers receive unrestrictive loans because they have a strong bargaining position (i.e. their outside option is attractive). The closer a borrower is to the formal sector, the more she values investing in fixed capital, so a lender finds it prohibitively costly to offer her a restrictive loan she would accept. In contrast, the farther a borrower is from the high payoff she enjoys at the formal sector, the less she values investment in fixed capital and therefore the less attractive her outside option is. So for borrowers at intermediate levels of wealth, the expansion rent can exceed the improvement in social welfare resulting from business growth, and the borrower receives restrictive loans. Finally, the poorest borrowers have a very weak bargaining position. The lender is happy to offer these borrowers unrestrictive loans, because he benefits from their improved productivity as they grow their business without fear that their bargaining position will improve too rapidly.

The model also yields nuanced comparative statics that shed light on the dynamic interlinkages of wealth accumulation. In particular, our model offers a counterpoint to the standard intuition that poverty traps are driven by impatience. We show that increasing the borrower's patience relaxes the poverty trap for rich borrowers (i.e. it increases the probability of unrestrictive loans). However, perhaps counterintuitively, this may *amplify* the poverty trap for poorer borrowers, causing them to get trapped at even lower levels of wealth. This is due to a "trickle down" effect whereby lenders anticipate that richer borrowers become more demanding, and restrict the investment of poor borrowers to ameliorate their improved bargaining position.

Similarly, improving the attractiveness of the formal sector improves the welfare of relatively richer borrowers but may harm the welfare of poorer borrowers. Notably, fixing any lender behavior, an improvement in the formal sector unambiguously increases the borrower's welfare. It is because of the lender's endogenous response that this improvement harms the borrower.

Our results help to organize a number of findings in the experimental literature on the impacts of microfinance. The low impact of microfinance on the average borrower's income can be explained by the presence of restrictive loans as an equilibrium phenomenon. We argue below that the modal microfinance borrower is likely to meet the conditions for a restrictive loan. At the same time, many experiments find that relatively wealthier borrowers do enjoy a high marginal return to microcredit, which is consistent with our model's prediction that wealthier borrowers receive unrestrictive loans.[8] Finally many experiments find that the demand for microfinance is substantially lower than once expected. In an extension of our model we show this arises as a natural prediction, as borrowers who receive restrictive loans are near indifference. While this prediction resembles those from many principal-agent models, it stands in sharp contrast to models of credit constrained borrowers, where by definition, borrowers demand more credit than they are offered.

Related Literature

Our paper contributes to the literature on credit market inefficiencies resulting from limited pledgeability. Two contributions of special relevance are Ray (1998) and Kovrijnykh and Szentes

[8] See Angelucci, Karlan, and Zinman (2015), Augsburg, De Haas, Harmgart, and Meghir (2015), Banerjee, Duflo, Glennerster, and Kinnan (2015b), and Crepon, Devoto, Duflo, and Pariente (2015). This is also consistent with abundant anecdotal evidence that MFIs relax contractual restrictions such as joint liability and rigid repayment schedules for richer borrowers, although this fact may be explained by a number of other theories.

(2007). Ray (1998) argues that a lender may deter efficient investment (via contractual restrictions) if the borrower cannot pledge to repay her loan conditional on high income realization. In contrast our model emphasizes the lender's incentive to prolong rent extraction rather than concerns about default. Kovrijnykh and Szentes (2007) highlight that a monopolist lender may induce underinvestment because of the costs of extracting non-pledgeable income. From a technical standpoint they share our focus on Markov equilibria rather than ex-ante optimal contracts. More broadly, the capital distortion brought about by the divergence between socially efficient investments and those whose output is pledgeable has a long history in corporate finance. Tirole (2010) provides a comprehensive overview of the literature.

The inability to pledge the benefits of future investments fundamentally arises from a weakness of property rights (if ownership over future income streams were explicitly defined then the borrower could commit them to the lender). In this sense, we contribute to a large literature relating the strength of property rights to wealth accumulation. For instance Chassang and Padró i Miquel (2010) study a principal-agent model in which the agent's incentive to save are diminished by the principal's inability to resist predation. In their setting it is the principal's inability to commit to future actions that diminish the agent's incentive to save, whereas in ours this effect comes from the agent's inability to commit future rents. Besley and Ghatak (2009) highlight that weak property rights may inhibit efficient trade, and Besley, Buchardi, and Ghatak (2012) study credit market inefficiencies arising from a lack of pledgeable collateral. This stands in contrast to our setting in which the inefficiency arises from an inability to pledge future income streams. Besley and Ghatak (2009) provide an overview of the ways in which weak property right hinder economic growth.

Several papers offer theories that yield comparative statics similar to ours. Notably, Petersen and Rajan (1995) and Jensen and Miller (2015) both study principal-agent models in which improving a feature of the market can make the agent worse off. In both of these models, the comparative static unambiguously harms the agent. In contrast, in our model improving the attractiveness of the formal sector only harms poor borrowers by virtue of helping richer borrowers, and the comparative static operates through a novel bargaining channel about contractual restrictions rather than through credit rationing.

Our paper also contributes to the literature on stochastic principal-agent problems with an absorbing state. These are models in which a payoff relevant state variable evolves endogenously, and once it reaches a particular value the game ends and exogenous payoffs are realized. Of particular relevance are Bergemann and Hege (1998), Bergemann and Hege (2005), and Hörner and Samuelson (2013) each of which study dynamic lending relationships. Over time the lender provides financing for the project, and is only paid once a success is achieved (and the game ends). Therefore in their models, the lender's challenge is to incentivize the borrower to invest in her project and realize a success; in ours the lender's challenge is essentially the reverse, which leads to a very different set of predictions. Finally, in contemporaneous work, Urgun (2017) studies a similar dynamic contracting model. While his application, method of analysis, proofs, and interpretation of results all differ from ours, the economic forces underlying his main results are similar in that the principal in his model has an incentive to limit the productivity of the agent.

More broadly, our paper sits in the large literature that studies how efficient investments are inhibited by frictions in credit markets, including adverse selection (e.g. Stiglitz and Weiss (1981)), moral hazard (e.g. Jensen and Meckling (1976), Banerjee and Duflo (2010)), and common agency problems (e.g. Bizer and DeMarzo (1992)). Green and Liu (2016) study common agency in a dynamic lending context, showing that the lack of commitment power to exclusive lending relationships can leave to the least investment in the most productive opportunities.

The rest of the paper proceeds as follows. In Section 2 we describe the model. Section 3 characterizes the equilibrium of our game, and Section 4 describes comparative statics. Section 5 discusses the relationship of our results with existing empirical evidence. Section 6 discusses extensions of the model where we relax some of our stylized assumptions and Section 7 concludes. The appendix contains additional extensions, a discussion institutional features of microfinance that aren't captured by our model, and all of the proofs.

2 The Model

Players, Actions, and Timing: We study a dynamic game of complete information and perfectly observable actions. There are two players, a borrower (she) and a lender (he), both of whom are risk neutral. Each period lasts length dt and players discount the future at rate ρ. For analytical convenience we study the continuous time limit as dt converges to 0. The borrower's business is indexed with a state variable $w \in \{1, \ldots, n+1\}$ referred to as her business size.[9]

At the beginning of each period the borrower has an endowment E, which may be augmented by a loan from her lender. The borrower has access to two projects: a working capital project and a fixed capital project. Her working capital project, $work\,(w, i)$ produces consumption goods which she uses to repay her lender and to eat. Its two arguments are her business size w, and the amount she invests i. For simplicity, we assume a linear functional form: $work\,(w, i) = q_w i$, with $q_w > 1$.

Her fixed capital project, $fix\,(i)$ governs the rate at which her business size increases. If she invests i into the fixed capital project we assume that at the close of the period her business

[9] It is straightforward to extend the model to accommodate a countably infinite state space. See Section 6.

jumps from state w to $w+1$ according to a Poisson process with arrival rate $\frac{i}{\phi}dt$, where $\phi > 0$, and remains constant otherwise.[10]

Each period the lender makes a take it or leave it offer of a loan contract $\tilde{c} = \langle R, a\rangle \in \mathcal{C} \equiv \mathbb{R}^+ \times \{fix, work\}$, where R represents the (contractable) repayment from the borrower to the lender, and a represents the contractual restriction. If the borrower accepts the contract, the lender transfers $T_w > 0$, the borrower's endowment becomes $E + T_w$, and at the end of the period, the borrower repays R to the lender.[11] If the contract specifies $a = work$, she must invest her entire endowment in the working capital project. If instead the contract specifies $a = fix$, then she must invest $\frac{R}{q_w}$ in *work* (just enough to meet her repayment obligation) but is free to invest her residual endowment flexibly.[12] We refer to contracts which specify $a = work$ as *restrictive loans* and those that specify $a = fix$ as *unrestrictive loans.*

If the game ever reaches state $n+1$, both players cease acting. The borrower receives a continuation payoff $U \equiv \frac{u}{\rho}$ (the payoff resulting from reaching the formal sector), and the lender receives a payoff of 0 (having lost his customer).

The timing within each period is as follows:

a) The lender makes a take it or leave it offer $\tilde{c} = \langle R, a\rangle$ to the borrower.

b) The borrower chooses a decision $d \in \{\text{Accept, Reject}\}$.

   i. If she rejects the contract, she allocates her endowment E flexibly among her two projects, and the lender receives a flow payoff of 0.

   ii. If she accepts the contract the lender transfers T_w to the borrower, and she invests according to the contractual restriction a. Then the borrower repays R to the lender who consumes $R - T_w$.

c) The borrower consumes $q_w c$ where c is her investment in *work*. The borrower's business grows from state w to $w+1$ with probability $\frac{i}{\phi}dt$ and remains constant otherwise, where i is her investment in the fixed capital project.

d) If the state $w < n+1$, the period concludes and after discounting the next one begins.

The timing above can be understood through the lens of the example in our introduction. The borrower is a fruit vendor, and at state w she operates a mobile cart. At the beginning of the year

[10]This may be generalized to allow for any transition process in which the probability of transition from w to any other state scales linearly with investment. In Section 6 we allow the rate at which the borrower grows to be influenced by her business size.

[11]Note, that due to linearity of production and utility functions, we assume that T_w is fixed, and therefore do not study the lender's decision of loan size in this paper.

[12]We represent unrestrictive contracts with $a = fix$ because, as we show below, when the borrower is free to invest flexibly she will invest in fixed capital.

she has a cash endowment E (this can be understood as arising from unmodeled savings, or access to an interest free loan from a relative). If she rejects the lender's contract then she flexibly allocates her endowment between her two projects: a working capital project *work* which can be understood as purchasing fruits to sell during the week, and a fixed capital project fix which can be understood as buying raw materials to expand to a market stall from which she may have access to a broader market, improving her productivity. For every unit she invests in the working capital project, she produces $q_w > 1$ units of output. So q_w may be thought of as the markup she enjoys from selling fruits, and ϕ may be thought of as the cost of fixed investment. The more she invests in fixed capital, the more likely she is to succeed in expanding her productive capacity by moving to state $w + 1$.

If instead she accepts the contract $\langle R, a \rangle$, the lender transfers T_w working capital to the borrower and her endowment is $E + T_w$. The borrower's subsequent investment decision then depends on the contractual restriction $a \in \{fix, work\}$. Though the stylized model above does not include an detailed description of the timing of output within a period, the contractual restriction $a = work$ can be understood as the requirement of early and frequent repayments. If the lender demands that the borrower has cash on hand each day to repay a small fraction of her loan, she may not be able to initially invest in the long-term, fixed capital project which may not return output for weeks. By the time she has generated enough income through her working capital project to ensure she can repay each installment, she may no longer have enough cash on hand to meet the minimum required investment in her fixed capital project, as would be the case if she has trouble saving cash from day to day (for instance because she faces pressure from her family to share underutilized assets). In contrast, a borrower uninhibited by a restrictive repayment plan ($a = fix$) may invest freely.

Parametric Assumptions:

Arguably our most important parametric is on the range of feasible repayment rates R.

Assumption 1. *We assume that the feasible range of repayment rates satisfies* $\frac{R}{q_w} \in [0, T_w - h]$ *with* $h > 0$*.*

This guarantees that if the borrower accepts the lender's contract and sets aside $\frac{R}{q_w}$ of her endowment to invest in her working capital project for repayment, the residual endowment she can invest in either project is at least $E + h$ which necessarily exceeds the endowment E she could have invested on her own. This can be motivated in a number of ways. Most straightforwardly, the borrower might be able to hide $q_w h$ from her lender every period, and thus the repayment rate he sets is bounded above by the residual output resulting from the loan, $q_w (T_w - h)$. Alternatively one could assume that the borrower can renege on her debt in any period, in which case she must find a new lender at cost $\nu_w = q_w (T_w - h)$. Then the borrower would never repay a debt in excess of this cost.[13]

The repayment ceiling is critical to many of our results below. Because the borrower cannot commit to share the proceeds from business expansion with her lender, she values investment in fixed capital more highly than she values investment in working capital. This in turn implies that the residual endowment $E + h$ the borrower retains induces an asymmetry between the *utility* she derives from restrictive and unrestrictive contracts. When this wedge is sufficiently large, the lender will offer only restrictive contracts, trapping the borrower in poverty.

We next assume that the borrower and lender's joint production function exhibits decreasing returns to scale. In particular we assume that the flow output of *work* is increasing at a decreasing rate in the state. Let $y_w \equiv q_w(E + T_w) - T_w$.

Assumption 2. *$y_w > y_{w-1}$ for all w and $y_w - y_{w-1} \geq y_{w+1} - y_w$ for all w.*

While a number of the above s are rather stylized, they serve to isolate the central economic forces we discuss. In Section 6 we show that many of our results survive substantial generalization of the above model.

Histories and Strategies: A history $\tilde{h}_t$ is a sequence $\{\tilde{c}_{\tilde{t}}, d_{\tilde{t}}, i_{\tilde{t}}, w_{\tilde{t}}\}_{\tilde{t}<t}$ of contracts, accept/reject decisions, investment allocations and business states at all periods prior to t. We define $\tilde{H}_t$ to be the set of histories up to time t.

The lender's strategy is a sequence of (potentially mixed) contractual offers $\tilde{\mathbf{c}} = \{\tilde{c}(\tilde{h}_t)\}_{\tilde{h}_t \in \tilde{H}_t}$, where $\tilde{c}(\tilde{h}_t) \in \Delta(\mathcal{C})$ is the probability weighting of contracts he offers the borrower following history $\tilde{h}_t$. The borrower's strategy is a sequence of accept/reject decisions $\mathbf{d} = \{d(\tilde{h}_t, \tilde{c})\}_{\tilde{h}_t \in \tilde{H}_t, \tilde{c} \in \mathcal{C}}$ and investment decisions in the event of rejection $\mathbf{i} = \{i(\tilde{h}_t, \tilde{c})\}_{\tilde{h}_t \in \tilde{H}_t, \tilde{c} \in \mathcal{C}}$. Here $d(\tilde{h}_t, \tilde{c})$ denotes the probability the borrower accepts the contract $\tilde{c}$ following history $\tilde{h}_t$, and $i(\tilde{h}_t, \tilde{c})$ denotes the investment allocation the borrower undertakes following history $\tilde{h}_t$ and rejecting contract $\tilde{c}$.

Equilibrium: Our solution concept is the standard notion of *Stationary Markov Perfect Equilibrium* (henceforth *equilibrium*) which imposes that at every period agents are best responding to one another and that they only condition their strategies on payoff relevant state variables (in this case, business size). In particular, neither agent has the ability to commit to a long-term contract.

Formally, a strategy profile $(\tilde{\mathbf{c}}, \mathbf{d}, \mathbf{i})$ is an *equilibrium* if

a) $\tilde{c}(\tilde{h}_t)$ is optimal for the lender at every $\tilde{h}_t$ given the borrower's strategy $(\mathbf{d}, \mathbf{i})$.

[13] In the equilibrium of our model the borrower may not extract positive rents from the lending relationship. Thus, to take this microfoundation seriously, one can ensure that she always finds it profitable to find a new lender in the event of reneging on the first by assuming she receives an additional positive flow utility from interacting with any lender, that is unaffected by which loan she is offered. This can be motivated by an insurance benefit she receives from knowing her lender, that operates independently from the loans she receives every period.

b) $d\left(\tilde{h}_t, \tilde{c}\right)$ and $i\left(\tilde{h}_t, \tilde{c}\right)$ are optimal for the borrower at every $\tilde{h}_t$ and for every contract $\tilde{c}$ given the lender's strategy $\tilde{\mathbf{c}}$.

c) At any two histories $\tilde{h}_t$ and $\tilde{h}'_{t'}$ for which w is the same, we have $\tilde{c}\left(\tilde{h}_t\right) = \tilde{c}\left(\tilde{h}'_{t'}\right)$, $d\left(\tilde{h}_t, \tilde{c}\right) = d\left(\tilde{h}'_{t'}, \tilde{c}\right)$,and $i\left(\tilde{h}_t, \tilde{c}\right) = i\left(\tilde{h}'_{t'}, \tilde{c}\right)$.

By studying Stationary Markov Perfect Equilibria, we impose that the lender uses an impersonal strategy: any borrower with the same business size must be offered the same contract. This may be an especially plausible restriction in the context of large lenders such as microfinance institutions whose policy makers may be far removed from the recipients of their loans, rendering overly personalized contract offers infeasible.

3 Equilibrium Structure

We now describe the borrower and lender's equilibrium behavior and our main results about the structure of the equilibrium. Section 3.1 describes the borrower's autarky problem and sets forth an that guarantees the borrower will eventually reach the formal sector (state $n + 1$) in autarky. Section 3.2 describes the key incentives of the borrower and lender necessary to understand the structure of the equilibrium. Section 3.3 provides our main results: The equilibrium is unique, and the probability that the lender offers a restrictive contract is single peaked in the state. Thus, the lender's poorest and richest clients may receive unrestrictive contracts and grow faster than they would have in his absence. But borrowers with intermediate levels of wealth receive restrictive contracts every period and find themselves in a poverty trap. Notably, this poverty trap may exist even if the borrower would have reached the formal sector in autarky and even if the discounted utility from expanding to the formal sector is greater than the total surplus generated from investing the total endowment in working capital in every state. In Section 5 we argue that several well established empirical facts about microfinance can be contextualized through the lens of this equilibrium, and in Section 8.2 we discuss limitations of our model.

3.1 The Borrower's Autarky Problem

First consider the borrower's autarky problem. That is, the economic environment is as specified in Section 2, but the borrower is forced to reject the lender's contract at all times (i.e. she must choose $d\left(\tilde{h}_t, \tilde{c}\right) = 0$ for all histories $\tilde{h}_t$ and contracts $\tilde{c}$).[14]

[14] Alternatively, one can imagine that the borrower simply does not have access to a lender.

Let B_w^{aut} be the borrower's continuation value in autarky in state w. This can be decomposed into a weighted average of her flow payoff in the time interval $[t, t + dt]$, and her expected continuation utility at time $t + dt$. We have

$$B_w^{aut} = \max_i q_w (E - i)\, dt + (1 - \rho dt) \left(\frac{i}{\phi} dt B_{w+1}^{aut} + \left(1 - \frac{i}{\phi} dt \right) B_w^{aut} \right) \tag{1}$$

Fixing the optimal level of investment i in state w, rearranging, and ignoring higher order terms we have

$$B_w^{aut} = \frac{q_w (E - i)}{\rho + \frac{i}{\phi}} + \frac{\frac{i}{\phi}}{\rho + \frac{i}{\phi}} B_{w+1}^{aut} \tag{2}$$

That is, the borrower's autarky continuation value in state w is a weighted sum of her flow consumption $q_w (E - i)$ and her continuation value upon increasing business size, B_{w+1}^{aut}. Because equation 1 is linear in i (and equation 2 is monotone in i), the borrower will choose an extremal level of investment. From here on we will use the notation $\kappa \equiv \frac{E}{\phi}$, which is the maximum speed the borrower can invest in fixed capital and grow in autarky. We have the following proposition about the borrower's autarky behavior.

Proposition 1. *The borrower invests her entire income in every state iff*

$$\frac{q_w E}{\rho} \leq \alpha^{(n+1)-w} \frac{u}{\rho} \text{ for all } w,$$

where $\alpha \equiv \frac{\kappa}{\rho + \kappa}$.

The borrower's autarky problem has an attractive structure. If she chooses to invest in fixed capital in state w at every period then her continuation utility in state w is $B_w^{aut} = \alpha B_{w+1}^{aut}$. That is, she spends a fraction $(1 - \alpha)$ of her expected, discounted lifetime in the current state, and a fraction α of her expected, discounted lifetime in all future states $w + 1$ and onwards. Likewise in state w she anticipates spending a fraction α^m of her expected, discounted lifetime in state $w + m$ and onwards if she invests in fixed capital at every period until reaching state $w + m$.

This property is closely related to the Poisson arrival of jumps. Letting t denote the time of the jump and κ be the arrival intensity, v_1 be the flow utility the borrower enjoys prior to a jump and v_2 the flow utility she enjoys post jump, the borrowers utility is represented by:

$$\begin{aligned} \mathbb{E}_t \left[\int_0^t v_1 e^{-\rho s} ds + \int_t^\infty v_2 e^{-\rho s} ds \right] &= \int\limits_0^\infty \left[\int_0^t v_1 e^{-\rho s} ds + \int_t^\infty v_2 e^{-\rho s} ds \right] \kappa e^{-\kappa t} dt \\ &= (1 - \alpha) \left(\frac{v_1}{\rho} \right) + \alpha \left(\frac{v_2}{\rho} \right) \end{aligned}$$

Thus the borrower's utility from this process can be represented as the convex combination of her lifetime utility from staying in the initial state forever and her lifetime utility from staying in the post-jump state forever, where the weights on each are a function of the intensity of the arrival process. Having established that the borrower invests in fixed capital in all states in autarky if and only if $\frac{q_w E}{\rho} \leq \alpha^{(n+1)-w} \frac{u}{\rho}$ for all w, we make the following, stronger and maintain it throughout the subsequent analysis.

Assumption 3. *$\frac{q_w(E+h)}{\rho} \leq \alpha^{(n+1)-w} \frac{u}{\rho}$ for all w.*

Assumption 3 guarantees that the borrower would prefer to invest her income into fixed capital rather than invest it into working capital for any flow income stream weakly less than $q_w (E+h)$. We maintain Assumption 3 to highlight that the introduction of a lender may cause a poverty trap to emerge despite the autarkic borrower's eventual entry into the formal sector. That is, business growth among borrowers with access to credit may be *lower* than growth among their counterparts without access to credit.

3.2 Relationship Value Functions

We now outline the borrower and lender's relationship maximization problems and describe their value functions. Let B_w be the borrower's equilibrium continuation utility at the beginning of a period in state w, and let $B_w(\langle R, a\rangle)$ be her equilibrium continuation utility upon receiving the contract $\langle R, a\rangle$ in state w. Further, define

$$B_w^{Rej} \equiv \max_i q_w (E-i)\, dt + (1-\rho dt)\left(B_w + \frac{i}{\phi} dt\,(B_{w+1} - B_w)\right)$$

to be her equilibrium continuation utility upon rejecting a contract. These functions satisfy

$$B_w(\langle R, work\rangle) = \max\left\{(q_w(E+T_w) - R)\, dt + (1-\rho dt)\, B_w, B_w^{Rej}\right\}$$

and

$$B_w(\langle R, fix\rangle) = \max\left\{(1-\rho dt)\left(B_w + \frac{E + T_w - \frac{R}{q_w}}{\phi} dt\,(B_{w+1} - B_w)\right), B_w^{Rej}\right\}$$

where for both value functions above, the first expression in the brackets corresponds to the borrower's continuation utility if she accepts the contract $\langle R, a\rangle$ and the second term corresponds to her continuation utility if she rejects the contract, she is left with her smaller endowment E and chooses her own investment allocation.

The lender's value function L_w in state w satisfies

$$L_w = \max_{\langle R,a \rangle} (R - T_w)\, dt + (1 - \rho dt) \left(L_w + \mathbb{I}_{a=I} \frac{E + T_w - \frac{R}{q_w}}{\phi} dt \left(L_{w+1} - L_w \right) \right)$$

such that

$$q_w (E + T_w) - R \geq \kappa (B_{w+1} - B_w) \quad \text{if} \quad a = work$$
$$0 \leq R \leq q_w (T_w - h)$$

Note that the lender's maximization problem and constraints assume the lender never finds it optimal to offer the borrower a contract she will reject.[15] The borrower accepts a restrictive contract $\langle R, work \rangle$ if and only if her value of consuming what the lender offers is weakly higher than that of rejecting the contract and choosing her own allocation of investment, i.e.

$$q_w (E + T_w) - R \geq \kappa (B_{w+1} - B_w) \,.$$

We refer to the above inequality as the borrower's *individual rationality constraint.*

Next, we make a technical that ensures the lender's benefit from the relationship never justifies offering an unrestrictive loan with lower than necessary repayment to speed the borrower's business growth. Were we to drop this the results below would be qualitatively unchanged.[16]

Assumption 4. $\frac{q_w T_w - h}{\rho} < \phi$ *for all* w*.*

Assumption 4 enables the following lemma.

Lemma 1. *If the lender offers an unrestrictive contract* $\langle R, fix \rangle$ *in equilibrium, it will always specify the highest possible repayment. That is,* $R = q_w (T_w - h)$*.*

Therefore, in equilibrium the lender always offers one of two contracts: an unrestrictive contract $\langle q_w (T_w - h), fix \rangle$ and a restrictive contract $\langle q_w (E + T_w) - \kappa (B_{w+1} - B_w), work \rangle$. The former contract specifies the highest possible repayment, and the latter specifies the highest *acceptable* repayment.

[15] It is straightforward to show that in any Stationary Markov perfect equilibrium, either offering the borrower a restrictive contract with the highest acceptable repayment rate or offering her an unrestrictive contract with the highest feasible repayment rate will dominate offering the borrower a contract she would reject.

[16] Without Assumption 4 there could be a set of states in which the lender and borrower use mixed strategies in equilibrium. Hoping the borrower will invest in fixed capital, the lender offers unrestrictive contracts with low repayment, and the borrower, anticipating these contracts in equilibrium, finds the present state sufficiently attractive that she no longer finds it worthwhile to invest in fixed capital. The borrower mixes at a rate that makes the lender indifferent between all repayment rates and the lender sets a repayment rate that makes the borrower indifferent between investment projects.

Expansion rents

The lender's maximization problem illuminates an important force in our model. If the lender offers the borrower a restrictive contract, he optimally offers her the most extractive repayment rate she finds acceptable, denoted by $\bar{R}_w$. This repayment rate is determined by the borrower's indifference condition between accepting the restrictive contract or investing in fixed capital at her autarkic rate. Receiving this contract at every period the borrower's continuation utility would be

$$B_w = (q_w (E + T_w) - \bar{R}_w) dt + (1 - \rho dt) B_w = (1 - \rho dt) (\kappa dt B_{w+1} + (1 - \kappa dt) B_w)$$

Rearranging and ignoring higher order terms we have

$$B_w = \frac{\kappa}{\rho + \kappa} B_{w+1} = \alpha B_{w+1}$$

That is, if the lender offers the borrower the least generous acceptable restrictive contract the borrower's continuation utility is exactly what it would be if she invested in fixed capital at her autarkic rate.

On the other hand, if the lender offers a maximally extractive unrestrictive contract in every period, the borrower's continuation value will satisfy

$$B_w = (1 - \rho dt) \left(\frac{E + h}{\phi} dt B_{w+1} + \left(1 - \frac{E + h}{\phi} dt \right) B_w \right)$$

Rearranging and ignoring higher order terms we have

$$B_w = \frac{\gamma}{\rho + \gamma} B_{w+1} = \beta B_{w+1}$$

where $\gamma \equiv \frac{E+h}{\phi}$ is the rate of expansion the borrower enjoys when she receives a maximally extractive unrestrictive contract, and $\beta \equiv \frac{\gamma}{\rho+\gamma}$ is the fraction of her discounted lifetime she expects to spend in state $w + 1$ and onwards if she invests in fixed capital at rate γ in state w. Note that if the lender offers the borrower an unrestrictive contract, the borrower's continuation utility is strictly higher than it would be in autarky, because she is allowed to invest strictly more into fixed capital than she would in autarky.

The difference between the borrower's continuation value upon receiving an unrestrictive contract and upon receiving a restrictive contract is $(\beta - \alpha) B_{w+1}$. We refer to this term as the *expansion rent* in state w. This asymmetry arises because of the ceiling on feasible repayment rates the lender may set. Recall, after transferring T_w endowment to the borrower, the lender must set a repayment weakly less than $q_w (T_w - h)$ with $h > 0$. Thus upon accepting a loan and allocating

$\frac{R}{q_w}$ to the working capital project for repayment, the borrower necessarily has a larger residual endowment to allocate to either project than she would have had on her own. Because she values investment in fixed capital more highly than she values investment in working capital, she values this extra endowment more highly when receiving unrestrictive contracts than she does when receiving restrictive contracts.

Moreover, it will be critical for our analysis that the borrower's expansion rent is increasing in her value of moving to the next business size. As we discuss below, the borrower's welfare at any business size reflects the quality of her outside option, and borrowers with attractive outside options are more demanding on the lender. Thus the size of the expansion rent will be paramount in determining when the lender offers restrictive loans to the borrower to slow her growth.

3.3 Results

Our first result about the structure of equilibrium shows that the borrower invests in fixed capital whenever she is free to do so.

Lemma 2. *The borrower invests in fixed capital following any unrestrictive contract.*

Lemma 2 follows from Lemma 1 and Assumption 3. The former ensures that if the borrower gets an unrestrictive contract in equilibrium, it comes with the highest feasible repayment and the latter ensures that the borrower prefers to grow her business rather than invest in working capital at the highest feasible repayment.

We are now in a position to state our first proposition.

Proposition 2. *An equilibrium exists and is generically unique.*

The result follows by backward induction on the state. Lemma 2 allows us to restrict attention to strategy profiles in which the lender invests in fixed capital whenever she can. In any state w the borrower's accept/reject decision is pinned down by her state w continuation value B_w and her state $w+1$ continuation value B_{w+1}. The primary subtlety arises from the fact that the borrower's welfare in state w is increasing in the probability the lender offers an unrestrictive contract in w. The more frequently the borrower anticipates unrestrictive contracts in w the *less demanding* she will be of restrictive contracts. Formally, we define $\delta(p_w) \equiv p_w\kappa + (1-p_w)\gamma$. It is straightforward to show that a borrower who expects a restrictive contract with probability p_w in state w will have a continuation utility of $B_w(p_w) = \frac{\delta(p_w)}{\rho+\delta(p_w)}B_{w+1}$ which is decreasing in p_w. The lender determines the interest rate associated with restrictive contracts, $R_w(p_w)$, to solve

$$\kappa\left(B_{w+1} - B_w(p_w)\right) = q_w\left(E + T_w\right) - R_w(p_w)$$

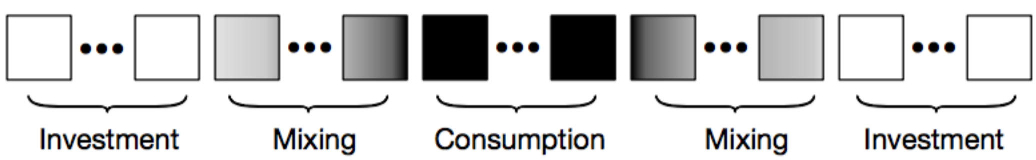
from which it is immediate that $R_w(p_w)$ is decreasing in p_w. Thus it may be that when the borrower expects a restrictive contract with certainty the lender strictly prefers to offer an unrestrictive contract, and when the borrower expects an unrestrictive contract with certainty the lender strictly prefers to offer a restrictive contract. In such a case the unique equilibrium involves a strictly interior p_w and the expansion rent is $\left(\beta_w - \frac{\kappa}{\rho+\delta(p_w)}\right) B_{w+1}$.

Equilibrium Contract Structure

Our next result regards the equilibrium organization of restrictive states and unrestrictive states under the parametric s above.

Proposition 3. *In equilibrium, the probability the lender offers a restrictive contract p_w is single peaked in w.*

This result implies that in equilibrium the states can be partitioned into three regions of consecutive states: An initial region with only unrestrictive contracts, an intermediate region in which both kinds of contracts are possible, and a final region in which only unrestrictive contracts are offered. In the intermediate region, the probability a restrictive contract is offered is increasing (potentially reaching 1) and then decreasing. This is depicted in the figure below where white states denote unrestrictive states, black states denote restrictive states and grey states denote mixing states.



Borrowers who arrive at a state in which only restrictive contracts are offered never grow beyond it. Before discussing the intuition behind Proposition 3, it is useful to understand when restrictive states arise. We have the following result.

Proposition 4. *In equilibrium, the probability the lender offers a restrictive contract $p_w = 1$ if and only if*

$$\beta\left((L_{w+1} + B_{w+1}) - \frac{q_w(E + T_w) - T_w}{\rho} - \phi\right) \leq (\beta - \alpha) B_{w+1}$$

The left hand side of the above inequality may be loosely understood as the social gain from investing in fixed capital at rate $E + h$ rather than investing everything into working capital. If the borrower invests at rate $E + h$, then she and the lender expect to spend a fraction β of their lifetime in state $w + 1$ and onwards. Once in $w + 1$ they jointly enjoy continuation values of $L_{w+1} + B_{w+1}$ but forgo the consumption they could have enjoyed in state w, $\frac{q_w(E+T_w)-T_w}{\rho}$, and the cost they incur from expansion is $\beta\phi$. In contrast, the right hand side of the inequality is the borrower's expansion rent: the additional surplus she commands from unrestrictive contracts relative to restrictive ones. Thus if this expansion rent exceeds the social gain of business expansion, the lender will offer only restrictive contracts, pinning the borrower to the current state.[17]

We are now ready to discuss the intuition behind Proposition 3. When the borrower is near the formal sector, it is extremely costly to offer her a restrictive contract. For concreteness, consider a borrower in state n. A lender who offers this borrower a restrictive contract in every period needs to compensate her with $\alpha\frac{u}{\rho}$ consumption over the life of the relationship. For u sufficiently high this is prohibitively costly. However, as the borrower becomes poorer it becomes cheaper to offer her a restrictive contract. Consider a borrower who is at state w and who expects unrestrictive contracts in all future states. A lender who offers this borrower a restrictive contract in every period needs to transfer her only $\alpha\beta^{n-w}\frac{u}{\rho}$ consumption over the lifetime of the relationship. Thus as the borrower becomes poorer it becomes exponentially cheaper to offer her the restrictive contract.

In this intermediate region the expansion rent may become important. As discussed above, when the borrower's expansion rent exceeds the social gain from business expansion, the lender offers only restrictive contracts, keeping her inefficiently small. Note that this poverty trap is created by the presence of the lender. Assumption 3 guarantees that in autarky the borrower would have grown to her efficient size.

Last, as the borrower becomes sufficiently poor her expansion rent $(\beta - \alpha) B_{w+1}$ decreases, as it is tied to her continuation value in the next state. One way to understand this is that the lender has only a weak incentive to stymie the growth of his poorest borrowers, because the rate at which these borrowers' outside option improves is very slow. Moreover, because of the decreasing returns of the output from working capital investment, the joint surplus increase from expansion becomes increasingly large as the borrower becomes poorer. Thus sufficiently poor borrowers receive unrestrictive contracts.

We close this section with a discussion of the source of this poverty trap. One crucial feature is that the minimum endowment residual of repayment the borrower enjoys when contracting with the lender, $E + h$, is strictly larger than the endowment she would have had on her own, E. We encode this fact in the following proposition.

Proposition 5. *If $h = 0$, then $p_w = 0$ for any w in which it is socially efficient to invest in fixed capital (i.e. whenever $\beta\left((L_{w+1} + B_{w+1}) - \frac{q_w(E+T_w)-T_w}{\rho} - \phi\right) > 0$).*

[17] Note that because this is not a model of transferrable utility, the left hand side of the above inequality should not literally be interpreted as a change in social welfare. Nevertheless we will sometimes abuse terminology and say that it is socially efficient to invest in fixed capital when the left hand side of the above inequality is positive.

When the lender can choose interest rates flexibly enough such that the borrower can be left with exactly the same amount of income that she would have produced alone, he offers unrestrictive contracts in any state in which the social gain from business expansion is positive. When the lender offers a restrictive contract, he gives the borrower just enough consumption to make her indifferent between accepting the contract and rejecting it and investing E in fixed capital. But if the lender instead offers the borrower a maximally extractive unrestrictive contract, the borrower remains indifferent, because the endowment she can invest into business expansion is exactly what she could have invested on her own. Since the total social surplus increases and the residual surplus accrues to the lender, he prefers unrestrictive contracts.

While the poverty trap disappears when $h = 0$ it is important to note that the unique equilibrium still features inefficiently slow business expansion relative to the social optimum. A natural question then, is what contractual flexibility is required to reach the first best level of investment in fixed capital. It is straightforward to verify that *long-term debt or equity contracts*—contracts that allow the borrower to commit a fraction of her formal sector flow payoff to the lender in exchange for favorable unrestrictive contracts—are sufficient to guarantee first best investment. However this is primarily a theoretical exercise, as the participants of informal financial markets rarely have the capacity to commit to long-term contracts.[18]

4 Comparative Statics

In this section we discuss how the equilibrium changes with respect to a number of comparative statics. Each of them emphasizes an important "trickle down" nature of our model. Namely, changes to the fundamentals of the contracting environment can have nuanced impacts on equilibrium contracts and welfare that vary depending on the borrower's business size.

4.1 Comparative Statics on the Borrower's Continuation Utility u from Entering the Formal Sector

Increasing the borrower's continuation value u from entering the formal sector shifts the entire restrictive region leftward. The poverty trap is relaxed for rich borrowers but *tightened* for poor borrowers. The intuition behind this observation relies on the fact that, when his borrower is rich enough to be in the final unrestrictive region, increasing the attractiveness of the formal sector makes restrictive contracts more expensive for the lender because it increases the borrower's

[18] Some arrangements among informal borrowers and their lenders – principally sharecropping – resemble crude equity contracts. We emphasize that it is the long-term nature of commitment, and not profit sharing, that is necessary to achieve first best levels of investment, and that even in sharecropping, long-term commitment is infeasible.

desire to expand. So the lender shifts towards unrestrictive contracts, relaxing the poverty trap for rich borrowers.

On the other hand, it is precisely this force that causes the lender to tighten the reins on poorer borrowers, increasing the likelihood he offers them restrictive contracts and trapping them at lower levels of wealth. Holding the lender's strategy fixed, increasing the attractiveness of the formal sector improves the borrower's bargaining position in all states.[19] However, the richer the borrower is, the more her bargaining position improves because of her proximity to the formal sector. Thus the lender shifts towards restrictive contracts for poorer borrowers, to prevent them from reaching higher levels of wealth where they can exercise their improved bargaining position. This is encoded in the propositions below.

Let $\bar{w} \equiv \underset{w}{\operatorname{argmax}} \{w : p_w = 1\}$, and $\underline{w} \equiv \underset{w}{\operatorname{argmin}} \{w : p_w = 1\}$.

Proposition 6. *Increasing the attractiveness of the formal sector relaxes the poverty trap for relatively rich borrowers, but tightens it for poorer borrowers.*

That is, $\frac{dp_w}{du} \leq 0$ *for* $w \geq \bar{w}$ *with strict inequality for* $0 < p_w < 1$. $\frac{dp_w}{du} \geq 0$ *for* $w < \underline{w}$ *with strict inequality for* $0 < p_w < 1$.

The intuition for the above proposition is inextricably linked to the equilibrium effects on welfare, codified in the next proposition.

Proposition 7. *Increasing u weakly decreases the lender's continuation value in all states, and strictly so for $w \leq \bar{w}$. Increasing u strictly increases the borrower's continuation utility in all states $w \geq \underline{w}$, but can decrease it in states $w < \underline{w}$.*

That is, $\frac{dL_w}{du} \leq 0$ *for all w with strict inequality for* $w \leq \bar{w}$. $\frac{dB_w}{du} > 0$ *for* $w \geq \underline{w}$. *For* $\rho > \frac{\kappa\gamma}{\kappa+\gamma}$, *if* $p_{\underline{w}-1} > 0$ *then* $\frac{dB_w}{du} < 0$ *for states* $w < \underline{w}$.

Though some of the details are cumbersome, the intuition behind these results is instructive. The comparative static for states $w > \bar{w}$ is most easily understood. Consider the largest state n at which the borrower remains in the informal sector. Increasing u makes it more expensive to offer the borrower a restrictive contract, because she finds investment in fixed capital more valuable. On the other hand, the borrower accepts any unrestrictive contract due to her expansion rent. So the lender finds unrestrictive contracts relatively more attractive and shifts towards them if he previously chose an interior solution.

The borrower's continuation utility increases for two reasons. She benefits from the increased prevalence of unrestrictive contracts, and conditional on entering the formal sector her utility increases. By , the lender at least weakly prefers to offer unrestrictive contracts and since the

[19] The borrower's bargaining position is the amount of compensation she demands in exchange for a (take it or leave it) restrictive loan offer.

utility he derives from doing so is unaffected, so is his equilibrium continuation value. This logic extends straightforwardly by backward induction to all states weakly larger than $\bar{w}$.

The story changes at or prior to $\bar{w}$. By definition of $\bar{w}$, the lender offers a restrictive contract with certainty (i.e. $p_{\bar{w}} = 1$), and therefore transfers $\alpha B_{\bar{w}+1}$ utility to the borrower over the lifetime of the relationship. Having already established that the borrower's continuation utility in state $\bar{w}+1$ increases with u, we can now see that the borrower's utility also increases in state $\bar{w}$. However, now the increase in her utility results from a direct transfer from the lender, so his continuation utility in state $\bar{w}$ decreases. A similar conclusion is reached for all states $w \in \{\underline{w}, \bar{w}\}$ by backward induction.

Finally, consider state $\underline{w}-1$, in which, by definition, the lender at least weakly prefers to offer an unrestrictive contract. Further, suppose that the preference is indeed weak, so that $p_{\underline{w}-1} \in (0,1)$ (i.e. the lender offers a restrictive contract with positive probability). First, note that since the borrower never grows beyond state $\underline{w}$, increasing u has no effect on social welfare in state $\underline{w}-1$. However, since the borrower's equilibrium continuation utility in state $\underline{w}$, $B_{\underline{w}}$ increases, so does her expansion rent in state $\underline{w}-1$. Recall that her state $\underline{w}-1$ expansion rent $\left(\beta - \frac{\kappa}{\rho+\delta\left(p_{\underline{w}-1}\right)}\right) B_{\underline{w}}$ is a fraction of her continuation utility in state $\underline{w}$. Because the borrower's share of surplus from business expansion increases but the change in social surplus accruing from business expansion does not, the lender shifts towards restrictive contracts, slowing down the borrower's growth.

Another way to understand this is the increase in the attractiveness of the formal sector trickles down and improves the borrower's bargaining position in state $\underline{w}$. Markov Perfection prevents her from committing not to exercise this improved bargaining position and, because state $\underline{w}-1$ borrowers are less affected, they become relatively more attractive and the lender shifts towards offering them restrictive contracts.

How this affects the borrower's state $\underline{w}-1$ equilibrium continuation utility $B_{\underline{w}-1}$ is in general ambiguous. That $B_{\underline{w}}$ increases is a force towards increasing $B_{\underline{w}-1}$. However, the rate at which she grows to state $\underline{w}$ slows, which is a force towards reducing $B_{\underline{w}-1}$. For sufficiently impatient borrowers the latter force dominates, as impatience amplifies the difference between slow and fast rates of expansion, and $B_{\underline{w}-1}$ decreases in the attractiveness of the formal sector. The lender is made unambiguously worse off from the increase in u, because he weakly prefers unrestrictive contracts and his state $\underline{w}$ continuation utility is decreasing in u. Thus an increase in the attractiveness of the formal sector causes a Pareto disimprovement. Because the borrower cannot commit to forgo her improved bargaining position in state $\underline{w}$, the lender traps her in state $\underline{w}-1$ to both of their detriments. And the story continues in much the same way for all states prior to $\underline{w}-1$.

That increasing u can make the borrower worse off at some business state w is especially striking in light of the following consideration. Fix any (potentially non-equilibrium) lender behavior characterized by $\{\tilde{p}_w\}$, such that in state w the lender offers a restrictive contract with probability $\tilde{p}_w$ and an unrestrictive contract with probability $1 - \tilde{p}_w$. Then increasing u unambiguously makes the borrower strictly better off in all states. The restrictive contract in state w becomes more generous when B_{w+1} improves, and the borrower's utility from receiving an unrestrictive contract in state w improves when B_{w+1} increases. This logic is codified in the following proposition.

Proposition 8. *Fixing the lender's behavior characterized by $\{\tilde{p}_w\}$ defined above, increasing u strictly improves the borrower's continuation utility in all states.*

So fixing the lender's behavior, *regardless* of what that behavior is, increasing u unambiguously improves the borrower's continuation utility. It is because of an equilibrium adjustment to the lender's behavior, namely that he shifts towards restrictive contracts, that impatient borrowers are made worse off at all business states $w < \underline{w}$.

4.2 Comparative Statics on the Borrower's Level of Patience

A standard intuition about poverty traps is that they are driven by impatience. However in this model increasing patience has a very similar effect to increasing the attractiveness of the formal sector, and hence can tighten the poverty trap and make the borrower *worse off* at some levels of wealth. Let ρ^B be the borrower's level of patience and ρ^L be the lender's level of patience (and note that decreasing ρ^B is equivalent to increasing patience).

Proposition 9. *Increasing the borrower's patience relaxes the poverty trap for relatively rich borrowers, but may tighten it for poorer borrowers.*

That is, $\frac{dp_w}{d\rho^B} \geq 0$ for $w > \bar{w}$ with strict inequality for $p_w > 0$. For $w < \underline{w}$, the sign of $\frac{dp_w}{d\rho^B}$ is ambiguous
.

For rich borrowers above the highest pure restrictive state ($w > \bar{w}$), the comparative static on ρ^B works in exactly the same way as the comparative static on u. Increasing the borrower's patience increases how much she values business expansion. This causes her to be more demanding of restrictive contracts, but leaves the lender's payoff from offering unrestrictive contracts unchanged. Thus, in all such states the lender shifts towards unrestrictive contracts, increasing the rate that these rich borrowers reach the formal sector.

For borrowers in pure restrictive states ($w \in \{\underline{w}, \ldots, \bar{w}\}$), the comparative static on the borrower's patience again works as it did for changes in the attractiveness of the formal sector. The borrower's continuation utility in state $w + 1$, B_{w+1}, increases so the amount of consumption she demands in return for a restrictive contract increases. This increases her welfare at the direct expense of the lender's.

Finally, consider state $\underline{w}-1$. Recall the borrower's expansion rent in this state is $\left(\beta - \frac{\kappa}{\rho^B+\delta\left(p_{\underline{w}-1}\right)}\right) B_{\underline{w}}$. That her utility in state $\underline{w}$, $B_{\underline{w}}$, increases is a force towards increasing her expansion rent. However, as she becomes more patient, the difference she perceives between slow and fast rates of expansion is muted. That is $\frac{d\left(\beta - \frac{\kappa}{\rho^B+\delta\left(p_{\underline{w}-1}\right)}\right)}{d\rho^B} > 0$, which is a force towards decreasing the expansion rent. Which of these two forces dominates is in general ambiguous, but we show in the appendix that these forces can resolve in favor of increasing the expansion rent. Thus, in contrast to standard models of poverty traps, increasing the borrower's patience can make this poverty trap worse.

5 Connection to Empirical Evidence

The motivating finding for this theory is that firms often fail to grow from being offered access to microfinance. This can be understood through the fact that in our model, firms who enter a state where the lender offers restrictive contracts (the black region in the figure above) never leave it, despite the fact that they would have continued to grow in autarky. That is, this is a model in which having access to a lender can *reduce* business growth.

Beyond establishing the existence of restrictive contracts in equilibrium, Proposition 4 offers guidance for when such contracts are likely. One class of vulnerable entrepreneurs are those with productive fixed capital investments (i.e. low ϕ) and low autarkic endowments (i.e. low E). The high marginal returns observed in the cash drop studies cited above suggest that many entrepreneurs may indeed have access to productive fixed capital technologies, and their inability to realize those returns on their own suggests their autarky endowment may be low. These entrepreneurs have large expansion rents, and are thus highly susceptible to a debt trap.

While the microcredit studies listed above find low marginal returns to credit on average, a number of them find considerable heterogeneity in observed returns to credit. In particular they consistently find a long right tail in returns to credit – the largest businesses in areas that randomly received access to microcredit are substantially larger than the largest businesses in areas that did not. Our model sheds light on this heterogeneity to returns as well. Firms at very low and very high business sizes grow faster in the presence of a lender than in his absence (they grow at least at rate γ rather than κ). Whereas firms at intermediate business sizes may not grow at all in the presence of a lender.

This model also offers a novel explanation for the regular finding that demand for microcredit contracts is low. Borrowers in the restrictive region are pushed exactly to their individual rationality constraint – they are indifferent between taking loans and not. While the exact indifference of these borrowers may seem an artifact of the model, the intuition that the lender can push the borrower nearer to her outside option when preventing her from investing in business expansion seems robust. Thus these borrowers may be expected to waver on their decision to accept a loan. In the appendix we discuss an extension to the model in which the lender is incompletely informed about the borrower's outside option. In equilibrium, borrowers in the restrictive region sometimes reject his offer, whereas those in the unrestrictive region never do. While this finding is intuitive, it stands in sharp contrast with standard models of credit constrained borrowers. By definition, a credit constrained borrower wants more credit than she has access to, and so these models make the opposite prediction, that borrowers will have high demand for credit.

6 Robustness and Extensions

In this section we argue that the key intuitions highlighted above survive substantial generalization of the production function and other parameters of the model, and extension to a countably infinite number of states. Then, in the appendix we explore several other extensions to the model.

First, we allow for the lender to be incompletely informed about the borrower's outside option and show that in equilibrium the borrower sometimes rejects the lender's offer of a restrictive contract, providing an explanation for the low demand of microcredit. We then discuss an extension in which we allow the borrower to flexibly allocate a fraction of her income irrespective of contractual restrictions, and show that the lender may still restrict the rate at which the borrower grows relative to autarky.

6.1 Arbitrary production functions

In this section we relax many of the s about the fixed and working capital projects. In particular, we abandon our decreasing returns Assumption 2 and so allow q_w and T_w to fluctuate arbitrarily across states. We now also index the autarky endowment E_w, the rate of fixed investment growth ϕ_w, and the repayment wedge h_w by the state and make no s about them other than that $h_w > 0$ for all w and Assumption 3 above, which guarantees that in autarky the borrower reaches the formal sector in finite time.

6.1.1 Structure of the Equilibrium

First, the proof of Proposition 2 which states that the equilibrium is unique did not rely on any of the above s, and thus goes through unchanged. In this section we discuss the structure of the unique equilibrium. A typical equilibrium is depicted below, with each circle representing a state and shaded circles representing states in which restrictive contracts are offered.

Even though in general we cannot say anything about the organization of restrictive and unrestrictive states, we argue that many of the empirical facts discussed in Section 4 can still be understood through the equilibrium above. In fact, with the exception of heterogeneity in returns to credit, our explanation of the facts in that discussion only depended on the potential for each type of contract to coexist in a single equilibrium. As such we focus our attention for the remainder of this discussion on the prediction that wealthy borrowers will receive unrestrictive contracts and thus will enjoy high returns to credit.

To do so, we first outline how to transfer the insights in the above model to one with a countably infinite number of states. Given that our results do not depend on the number of states n, or the cost of investment ϕ_w, this is a straightforward task. We define a sequence of games satisfying the above s, each with successively more states.

Let Γ^1 be an arbitrary game satisfying the s, with n business states.

For $m > 1$, let Γ^m be constructed in the following way:

- Γ^m has $2^{m-1}n$ business states, and let $q_w^m, E_w^m, T_w^m, h_w^m$, and ϕ_w^m be the corresponding parameters for game Γ^m.
- If w is even, set $q_w^m = q_{w/2}^{m-1}$, $E_w^m = E_{w/2}^{m-1}$, $T_w^m = T_{w/2}^{m-1}$, and $h_w^m = h_{w/2}^{m-1}$
- If w is odd, set
$$q_w^m \in \left[\min\left\{q_{w-1}^m, q_{w+1}^m\right\}, \max\left\{q_{w-1}^m, q_{w+1}^m\right\}\right]$$
$$E_w^m \in \left[\min\left\{E_{w-1}^m, E_{w+1}^m\right\}, \max\left\{E_{w-1}^m, E_{w+1}^m\right\}\right]$$
$$T_w^m \in \left[\min\left\{T_{w-1}^m, T_{w+1}^m\right\}, \max\left\{T_{w-1}^m, T_{w+1}^m\right\}\right]$$
and
$$h_w^m \in \left[\min\left\{h_{w-1}^m, h_{w+1}^m\right\}, \max\left\{h_{w-1}^m, h_{w+1}^m\right\}\right]$$
- If w is even, set $\phi_w^m = \frac{\phi_{w/2}^{m-1}}{2}$ and if w is odd, set $\phi_w^m = \frac{\phi_{(w-1)/2}^{m-1}}{2}$

Thus Γ^m has twice as many states at Γ^{m-1}, and even states in Γ^m correspond to states in Γ^{m-1}. The parameters in odd states take values intermediate to those in the surrounding states. Because the cost of investment in Γ^m is only half that in Γ^{m-1}, a borrower investing in fixed capital at the same rate in either game would reach the formal sector in the same expected time.

One way to understand Γ^m relative to Γ^{m-1} is that the borrower and lender appreciate more nuanced differences in the borrower's business size. Holding investment rate fixed, it takes the same amount of time to get from w to $w+2$ in Γ^{m+1} as it does to get from $\frac{w}{2}$ to $\frac{w}{2}+1$ in Γ^m, but along the way in Γ^m the borrower and lender realize an intermediate production function change. For $m' > m$, we say $\Gamma^{m'}$ is *descended from* Γ^m if there is a sequence of games $\Gamma^m, \ldots, \Gamma^{m'}$ that can be derived in this manner. We have the following result.

Proposition 10. *For any Γ^m, there is an $\bar{m}$ such that for all $m' > \bar{m}$, the equilibrium in any $\Gamma^{m'}$ descended from Γ^m features a $\tilde{w}$ such that for $w \geq \tilde{w}$ the borrower reaches the formal sector in finite time starting from state w if it is socially efficient to do so.*

The above result says that for any game with sufficiently fine discrimination between states, all sufficiently wealthy borrowers receive unrestrictive contracts in equilibrium, and thus realize high returns to credit. The intuition is simple. Because entering the formal sector is efficient, the lender is unable to profitably offer a sufficiently wealthy borrower (one who is sufficiently near to the formal sector) a restrictive contract she will accept. As the borrower and lender become arbitrarily discerning of different states, there will eventually be business states where the borrower is indeed sufficiently wealthy.

6.1.2 Comparative statics

As before, we can be fairly precise in describing how the equilibrium changes with respect to various fundamentals of the game. In this section we focus on the comparative static with respect to u.

Note that without loss of generality we can identify m disjoint, contiguous sets of states $\{\underline{w}_1, \ldots, \bar{w}_1\}, \ldots, \{\underline{w}_m, \ldots, \bar{w}_m\}$ such that $\bar{w}_m = \max\{w : p_w = 1\}$, $\underline{w}_m = \max\{w : p_w = 1, p_{w-1} < 1\}$, and in general for $k \geq 1$, $\bar{w}_k = \max\{w < \underline{w}_{k+1} : p_w = 1\}$ $\underline{w}_k = \max\{w \leq \bar{w}_k : p_w = 1, p_{w-1} < 1\}$. An arbitrary set $\{\underline{w}_k, \ldots, \bar{w}_k\}$ is a contiguous set of states where restrictive contracts are offered with probability 1, and each pure restrictive state is contained in one of these sets.

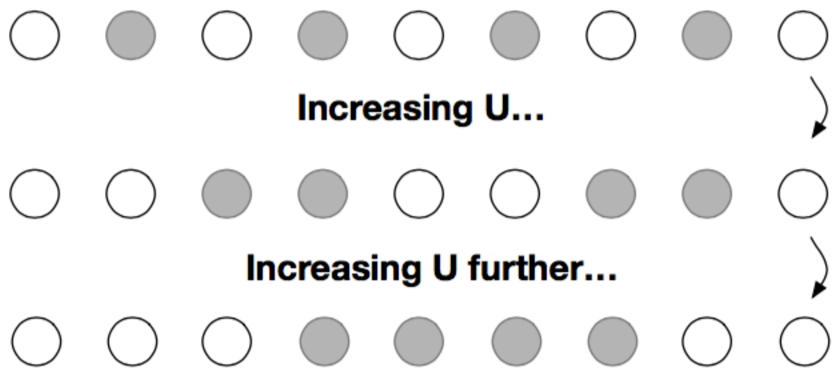
We consider an impatient borrower and establish the following result.

Proposition 11. *For impatient borrowers, the regions of contiguous restrictive states merge together as the formal sector becomes more attractive.*

That is, for $\rho > \max_w \frac{\kappa_w \gamma_w}{\kappa_w + \gamma_w}$, $\frac{dp_w}{du} < 0$ *for* $w \in \{\bar{w}_m + 1, n\}$, $\frac{dp_w}{du} > 0$ *for* $w \in \{\bar{w}_{m-1} + 1, \underline{w}_m - 1\}$,

$\frac{dp_w}{du} < 0$ *for* $w \in \{\bar{w}_{m-2}+1, \underline{w}_{m-1}-1\}$ *and so on.*

Proposition 11 states that the highest region of pure restrictive states moves leftward, the second highest region moves rightward and so on. This is depicted in the following figure.



The intuition is as follows. For $(\underline{w}_m \bar{w}_m)$, the analysis exactly follows that of Section 4.1, and hence it shifts leftward as u increases. But recall that for the impatient borrowers to the left of a restrictive state, the leftward shift *lowers* their utility. This is akin to lowering the utility of entering the formal sector, and hence for the next set of restrictive states $(\underline{w}_{m-1}\bar{w}_{m-1})$ the analysis reverses and $\bar{w}_{m-1}$ and $\underline{w}_{m-1}$ shift rightward. The rest follows by backward induction.

As u increases, the contiguous groups of restrictive states merge together. Eventually they either merge into a single contiguous group of restrictive states or disappear altogether. If they merge into one group, the comparative statics work exactly as they did in Section 4.1.

7 Concluding Remarks

Fundamentally this theory formalizes the story that MFIs and other informal lenders have so far failed to innovate new contracts that catalyze entrepreneurship because they don't profit from their customers' growth. We've shown that this simple story goes a long way towards organizing many of the established facts about microfinance. Our theory reconciles the seemingly inconsistent facts that small-scale entrepreneurs enjoy very high return to capital yet are unable to leverage microcredit and other forms of informal finance to realize those high returns. Moreover it offers an explanation for the robust findings that relatively wealthier business owners do enjoy high returns from microcredit and that on average the demand for microcredit is low.

The theory also offers nuanced predictions on comparative statics of the lending environment. Increasing the attractiveness of the formal sector improves the bargaining position of rich borrowers and hence increases their welfare and relaxes the poverty trap. However the same improvement may harm the welfare of poorer borrowers; anticipating that rich borrowers have improved bargaining position, the lender shifts towards restrictive loans for poor borrowers to prevent them from reaching higher levels of wealth and exploiting their improved position. Similarly, and counter to standard intuitions, increasing the borrower's patience (and hence her value for business expansion) can make relatively poor borrowers worse off, and tighten the poverty trap.

In addition to the theories cited in the introduction, it is worth contrasting our theory with two other classes of theories prominent in development economics. The first of these might sensibly be labeled as "blaming the borrower." These are theories that allude to the argument that many borrowers are not natural entrepreneurs, and are primarily self employed for lack of being able to find steady wage work (see e.g. Schoar (2010)). While these theories enjoy some empirical support, they are at best a partial explanation of the problem as they are inconsistent with the large impacts of cash grants, cited in our introduction.

Second are the theories that assign blame to the lender for not having worked out the right lending contract. These theories implicitly guide each of the experiments that evaluate local modifications to standard contracts (see e.g. Gine and Karlan (2014), Attanasio et al. (2015), and Carpena et al. (2013) on joint liability, Field et al. (2013) on repayment flexibility, and Feigenberg, Field, and Pande (2013) on meeting frequency). While many of these papers contribute substantially to our understanding of the ways in which microfinance operates, none have so far generated a lasting impact on the models that MFIs employ.

In contrast, ours is a theory that assumes that borrowers have the competence to grow their business, and that lenders are well aware of the constraints imposed on borrowers by the lending paradigm. Instead we focus on the rents that lenders enjoy from retaining customers, and the fact that sufficiently wealthy customers are less reliant on their informal financiers. Part of the value of this theory therefore may very well be its distance from the main lines of reasoning maintained by empirical researchers.

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8 Appendix

8.1 Additional Extensions to the Model

8.1.1 The Borrower is Privately Informed About Her Outside Option

In this section we explore an extension in which the borrower maintains some private information about her outside option. In particular we augment the model such that the borrower's autarkic endowment is privately known. If she rejects the lender's contract she receives an endowment of $E_w + \nu_t$. Let $\nu_t \overset{\text{iid}}{\sim} G$ be a random variable privately known to the borrower, and redrawn each period in an iid manner from some distribution G. Further, assume that if the borrower accepts the lender's contract, her endowment is still $E_w + T_w$. One way to understand this is that in the event that the borrower rejects the lender's contract, a relative will offer her a gift of size v_t, which she can allocate flexibly between her projects. We make the following additional on the range of G to simplify the discussion.

Assumption 5. *Let G have bounded support with minimum 0 and maximum $\bar{v}$ such that $\bar{v} < \frac{h_w}{q_w}$.*

The above guarantees that the borrower will accept any unrestrictive contract. However, if the lender offers the borrower a restrictive contract, he will now face a standard screening problem. Because he would like to extract the maximum acceptable amount of income, borrowers with unusually good outside options will reject his offer. This is encoded in the following proposition.

Proposition 12. *The borrower may reject restrictive contracts with positive probability.*

This intuitive result offers an explanation for the low takeup of microcredit contracts referenced in the introduction. Lenders who offer restrictive contracts to borrowers aim to extract all of the additional income generated by the loan, but in doing so sometimes are too demanding and therefore fail to attract the borrower. In contrast, lenders who offer unrestrictive contracts necessarily leave the borrower with excess surplus, and therefore demand for these contracts is high.

8.1.2 The Borrower Flexibly Allocates a Fraction of Her Income

In this section we explore an extension to the model in which, even when subjected to contractual restrictions, the borrower maintains flexible control over a fraction of her income. In doing so we aim to show that our main result is qualitatively robust. Rather than finding that the borrower may remain in inefficiently small forever, we now find that having access to a lender may *slow* the borrower's growth relative to her autarkic rate.

Formally, the model is as in Section 2 but after accepting a contract $\langle R, a \rangle$, the borrower is free to invest a fraction $s < 1$ of her endowment flexibly, irrespective of the contractual restriction a the lender imposes. Thus we have weakened the lender's ability to influence the borrower's project choice. We maintain all other s from Sections 2 and 3, and make the following observation.

Proposition 13. *For $s < \frac{E}{T_w+E}$, the lender may offer a restrictive contract on the equilibrium path. In such an equilibrium the borrower reaches the formal sector in finite time, but will grow more slowly than she would have in autarky.*

8.2 Microfinance Institutional Features Not Captured by the Model

The model makes a number of stylized s that are at times counter to institutional features of microfinance. In this section we address a number of these concerns and discuss how our model can be reconciled with them.

Non-profit MFIs

The debt trap in our model arises when a profit-maximizing lender prolongs the period over which he can extract rents from his borrower. Yet the low returns from microfinance have been observed across a range of microfinance institutions spanning both for-profit and non-profit business models. We believe there are a number of ways in which the forces identified in our model might similarly apply to non-profit MFIs. One such way is that there are many shared business practices among the two business models, and thus features of a business model that are adaptive for profit-maximizing MFIs may have been adopted by non-profits. Another possibility operates through the incentives of loan officers who are in charge of originating and monitoring loans. Across for-profit and non-profit MFIs many loan officers are rewarded for the number of loans they manage, and thus losing clients through graduation may not be in their self interest. Put another way, even in non-profit MFIs, the loan officers often have incentives that make them look like profit-maximizing lenders.

Infinite stream of borrowers

One crucial feature of our debt trap is that when the borrower becomes wealthy enough to leave her lender, the lender loses money. In reality there are many unserved, potential clients in the communities in which MFIs operate. Why then can't an MFI offer unrestrictive contracts and then replace borrowers who have graduated with an entirely new client? The proximate answer is that unserved clients are unserved primarily because they have no demand for loans (e.g. Banerjee, Duflo, and Hornbeck (2014)). This may be unsatisfactory as demand for microfinance would presumably increase once the lender lifted contractual restrictions and allowed the borrowers to invest more productively. But even in this case, the pool of potential borrowers who would find this appealing is likely limited. For instance, Banerjee, Breza, Duflo, and Kinnan (2015a) and Schoar (2010) argue that only a small fraction of small-scale entrepreneurs are equipped to put capital to productive use. Thus it is reasonable to assume that MFIs lose money when a borrower terminates the relationship.

Microentrepreneurs do not enjoy compounded growth

The high marginal return to capital observed in the cash drop studies cited above should not be confused with explosive business growth. Small-scale entrepreneurs often put cash to good use but seem not to reinvest it in their business for compounded growth. If these kinds of investments are unlikely to transform small-scale businesses into businesses large enough for formal loans, perhaps an MFI should be eager to support them. We argue this is not the case.

Recall that entry into the formal sector is merely a metaphor for a broader contracting friction endemic in informal markets. Any investment a borrower makes such that

a) the borrower can't credibly pledge the returns to the lender and

b) the borrower will become less reliant on the lender with positive probability

will be discouraged in the same way. We argue that both of these features are satisfied by grants that make the borrower permanently more productive but nevertheless too small to benefit from formal loans.

Competition among microlenders

Important to our model is that the borrower has no other lender to turn to when the monopolist offers her a restrictive contract. However many microfinance institutions operate in close proximity to one another. While microfinance may appear to be a highly competitive market in an ex-ante sense (i.e. for new borrowers), we argue that microfinance institutions gain private information about their borrowers over the course of the lending relationship. Indeed this is one of the primary reason cited for the increasing loan profile across borrowing cycles. The information rents that lenders enjoy over their existing clients induces imperfect competition ex-post. It is straight forward to show that the forces we identify above survive in an (ex-post) imperfectly competitive market.

8.3 Omitted Proofs

Proof of Proposition 1

In state n the borrower chooses her investment allocation i to maximize

$$B_n^{aut} = \max_{i \leq E} q_n \left(E - i\right) dt + e^{-\rho dt} \left(\left(1 - e^{-\frac{i}{\phi} dt}\right) B_{n+1}^{aut} + e^{-\frac{i}{\phi} dt} B_n^{aut}\right)$$

At the optimal level of i, the borrower's continuation utility can be rewritten as

$$\begin{aligned} B_n^{aut} &= \frac{q_n (E-i)\, dt + e^{-\rho dt}\left(\left(1-e^{-\frac{i}{\phi}dt}\right) B_{n+1}^{aut}\right)}{1-e^{-\left(\frac{i}{\phi}+\rho\right)dt}} \\ &\to \frac{q_n (E-i)}{\frac{i}{\phi}+\rho} + \frac{\frac{i}{\phi}}{\frac{i}{\phi}+\rho} B_{n+1}^{aut} \end{aligned}$$

Because the problem is stationary, the borrower's maximization problem can equivalently be written as choosing i to maximize her continuation utility above.

We now take the derivative of B_n^{aut} with respect to i.

$$\frac{dB_n^{aut}}{di} = \frac{-q_n\rho - q_n\kappa}{\left(\frac{i}{\phi}+\rho\right)^2} + \frac{\frac{\rho}{\phi}}{\left(\frac{i}{\phi}+\rho\right)^2} B_{n+1}^{aut}$$

where recall $\kappa \equiv \frac{E}{\phi}$. The denominator is positive. The numerator is positive iff

$$\begin{aligned} -q_n\rho - q_n\kappa + \frac{\rho}{\phi}B_{n+1}^{aut} &> 0 \\ &\Longleftrightarrow \\ \alpha B_{n+1}^{aut} &> \frac{q_n E}{\rho} \end{aligned}$$

where $\alpha \equiv \frac{\kappa}{\rho+\kappa}$. We conclude that if $\frac{q_n E}{\rho} < \alpha B_{n+1}^{aut}$ then the borrower's value in state n B_n^{aut} is increasing in i and otherwise it is decreasing. The result for earlier states follows similarly by backward induction. This completes our proof.

Proof of Lemma 1

First, suppose the borrower's strategy is to invest her income with probability 1. Consider an arbitrary state w. Note that $L_{w+1} \leq \max_{w'} \frac{q_{w'}T_{w'}-h}{\rho} < \phi$ where the first inequality holds because $\max_{w'} q_{w'}T_{w'} - h$ is an upper bound on the flow utility the lender can extract across any state, and the second inequality follows by Assumption 4.

If in state w the lender offers an unrestrictive contract with repayment R every period, his value in state w is

$$\left(1 - \frac{\frac{E+T_w-\frac{R}{q_w}}{\phi}}{\rho + \frac{E+T_w-\frac{R}{q_w}}{\phi}}\right)\frac{R-T_w}{\rho} + \frac{\frac{E+T_w-\frac{R}{q_w}}{\phi}}{\rho + \frac{E+T_w-\frac{R}{q_w}}{\phi}} L_{w+1}$$

Taking the first order condition with respect to R we have

$$\frac{\left(\frac{\rho}{q_w\phi}\right)}{\left(\rho+\frac{E+T_w-\frac{R}{q_w}}{\phi}\right)^2}\frac{R-T_w}{\rho}+\left(\frac{1}{\rho+\frac{E+T_w-\frac{R}{q_w}}{\phi}}\right)-\frac{\left(\frac{\rho}{q_w\phi}\right)}{\left(\rho+\frac{E+T_w-\frac{R}{q_w}}{\phi}\right)^2}L_{w+1}>$$
$$\frac{\left(\frac{\rho}{q_w\phi}\right)}{\left(\rho+\frac{E+T_w-\frac{R}{q_w}}{\phi}\right)^2}\frac{R-T_w}{\rho}+\left(\frac{1}{\rho+\frac{E+T_w-\frac{R}{q_w}}{\phi}}\right)-\frac{\left(\frac{\rho}{q_w\phi}\right)}{\left(\rho+\frac{E+T_w-\frac{R}{q_w}}{\phi}\right)^2}\phi$$

And it's straight forward to verify that this is positive. Hence, the lender's objective function is increasing in the repayment he demands, and therefore he demands the maximum feasible repayment.

Proof of Lemma 2

Suppose the borrower weakly prefers to invest in working capital when receiving an unrestrictive contract in state w. By Lemma 1, we know that unrestrictive contracts are accompanied by the maximum repayment in equilibrium. Because she weakly prefers to invest in working capital, the borrower is indifferent between restrictive and unrestrictive contracts. So if the borrower ever receives restrictive contracts in equilibrium, they will also be accompanied by maximum repayment.

Therefore her continuation utility in state w is $B_w = \frac{q_w(E+h)}{\rho}$, and by Assumption 3 this violates her preference for working capital.

Proof of Proposition 2

The existence of an equilibrium follows standard arguments (see Maskin and Tirole (2001)). In this section we prove that generically the equilibrium is unique. We do so by backward induction on the state. We first consider equilibrium behavior in state n:

The lender never offers the borrower an excessively generous restrictive contract. So in equilibrium, the lender either offers the borrower the contract $\langle q_n T_n - h, fix\rangle$, or the contract $\langle R(p), work\rangle$ for some $R(p)$ that pushes the borrower to her outside option. Now conjecture that in equilibrium the lender offers the borrower a restrictive contract with probability p.

Noting that the borrower receives the maximum of the utility from investing her outside option in fixed capital and from consuming $q_n E + h$ upon receiving a restrictive contract, we have

$$\begin{aligned} B_n(p) &= p\left(\max\left\{e^{-\rho dt}\left(\left(1-e^{-\kappa dt}\right)\frac{u}{\rho}+e^{-\kappa dt}B_n(p)\right),(q_nE+h)\,dt+e^{-\rho dt}B_n(p)\right\}\right) \\ &\quad +(1-p)\left(e^{-\rho dt}\left(\left(1-e^{-\gamma dt}\right)\frac{u}{\rho}+e^{-\gamma dt}B_n(p)\right)\right) \end{aligned}$$

$$B_n(p)=\max\left\{\frac{p\left(1-e^{-\kappa dt}\right)+(1-p)\left(1-e^{-\gamma dt}\right)}{1-pe^{-(\rho+\kappa)dt}-(1-p)\,e^{-(\rho+\gamma)dt}}e^{-\rho dt}\frac{u}{\rho},\frac{p\,(q_nE+h)\,dt+(1-p)\,e^{-\rho dt}\left(1-e^{-\gamma dt}\right)\frac{u}{\rho}}{1-pe^{-\rho dt}-(1-p)\,e^{-(\rho+\gamma)dt}}\right\}$$

where, recall that $\gamma \equiv \frac{E+h}{\phi}$. It is straightforward to verify that $\frac{dB_n(p)}{dp} < 0$. This is intuitive as a restrictive contract pushes the borrower to her individual rationality constraint (if possible), whereas an investment contract does not.

The highest possible repayment rate $R(p)$ that can be required for a restrictive contract is pinned down by the borrower's individual rationality constraint

$$\begin{aligned} (q_n(E+T_n)-R(p))\,dt+e^{-\rho dt}B_n(p) &\geq e^{-\rho dt}\left(\left(1-e^{-\kappa dt}\right)\frac{u}{\rho}+e^{-\kappa dt}B_n(p)\right) \\ &\Longrightarrow \\ q_n(E+T_n)-R(p) &= \max\left\{\frac{e^{-\rho dt}}{dt}\left(1-e^{-\kappa dt}\right)\left(\frac{u}{\rho}-B_n(p)\right),q_nE+h\right\} \end{aligned}$$

The maximal acceptable repayment rate is increasing in $B_n(p)$—this is intuitive as the higher is the borrower's continuation value in state n, the less she values investment.[20]

Now, consider the lender's decision of whether to offer an unrestrictive or restrictive contract. Fixing the borrower's expectation that the lender offers a restrictive contract with probability p, in any period in which the lender offers an unrestrictive contract his utility is

$$(q_nT_n-h-T_n)\,dt+e^{-\rho dt}\left(1-e^{-\gamma dt}\right)L_n(p)$$

If he offers a restrictive contract his utility is

$$(R(p)-T_n)\,dt+e^{-\rho dt}L_n(p)$$

So he offers a restrictive contract if and only if the following incentive compatibility constraint

[20] Note that by Assumption 3, $q_n(E+T_n)-R(1) > q_nE+h$ for sufficiently small dt, but in general $q_n(E+T_n)-R(p)$ may equal q_nE+h for some $p<1$.

holds

$$\begin{aligned}(R(p)-T_n)\,dt+e^{-\rho dt}L_n(p) \quad & \geq \quad (q_nT_n-h-T_n)\,dt+e^{-\rho dt}\left(1-e^{-\gamma dt}\right)L_n(p) \\ & \Longleftrightarrow \\ (q_nT_n-h-R(p))\,dt \quad & \leq \quad e^{-(\rho+\gamma)dt}L_n(p)\end{aligned}$$

The left hand side is the additional consumption the lender must forgo to persuade the borrower to accept a restrictive contract, and the right hand side is the discounted expected loss the lender incurs from allowing the borrower to invest in fixed capital.

Note that the lender's continuation utility $L_n(p)$ is weakly decreasing in p. This is so because the set of restrictive contracts the borrower will accept is decreasing in p, while the set of unrestrictive contracts is unchanged.

Thus the left hand side of the above inequality is weakly increasing in p, and the right hand side is weakly decreasing in p. Given the lender's incentive compatibility constraint, we argue that generically there can only be one equilibrium level of p.

If at $p=0$ (pure unrestrictive), the lender's incentive compatibility constraint for unrestrictive contracts is strictly satisfied, i.e.

$$(q_nT_n-h-R(0))\,dt>e^{-(\rho+\gamma)dt}L_n(0) \tag{3}$$

then it will be strictly satisfied for all higher levels of p, contradicting that any $p>0$ can be supported in equilibrium.

If at $p=1$ (pure restrictive) the lender's incentive compatibility constraint for restrictive contracts is strictly satisfied, i.e.

$$(q_nT_n-h-R(1))\,dt<e^{-(\rho+\gamma)dt}L_n(1) \tag{4}$$

then it will be strictly satisfied for all lower levels of p, contradicting that any $p<1$ can be supported in equilibrium.

If neither of the above inequalities holds even weakly then by the intermediate value theorem there will be a $\bar{p}$ at which

$$(q_nT_n-h-R(\bar{p}))\,dt=e^{-(\rho+\gamma)dt}L_n(\bar{p})$$

Note that when the borrower believes she will receive a restrictive contract with probability $\bar{p}$, the amount of consumption she demands when given a restrictive contract, $q_n(E+T_n)-R(\bar{p})$, is strictly larger than than $q_n E + h_n$ (the minimum feasible consumption the lender can leave the borrower). To see this, note that by

$$(q_n T_n - h - R(0))\, dt < e^{-(\rho+\gamma)dt} L_n(0).$$

Now, supposing that $q_n (E + T_n) - R(\bar{p}) = q_n E + h$, we'd have $L_n(\bar{p}) = L_n(0)$ (because the borrower is willing to accept all feasible contracts in both cases), which would imply that $(q_n T_n - h - R(\bar{p}))\, dt$ $e^{-(\rho+\gamma)dt} L_n(\bar{p})$ and would contradict that the lender is indifferent between restrictive and unrestrictive contracts. Therefore we know that $q_n (E + T_n) - R(\bar{p}) = q_n E + h$ and thus $\frac{dR(\bar{p})}{dp} < 0$. At $p > \bar{p}$ the lender will strictly prefer unrestrictive loans and at $p < \bar{p}$ the lender will strictly prefer restrictive loans, contradicting that any $p \neq \bar{p}$ can be supported in equilibrium.[21]

The backward induction step is similar and is thus omitted. This completes the proof that the equilibrium is generically unique.

Proof of Proposition 3

We aim to show that in equilibrium the probability the lender offers the borrower a restrictive contract in state w, p_w, is single peaked in w. In particular we show that $p_{\tilde{w}} < p_{\tilde{w}+1} \implies p_{\tilde{w}-m} \leq p_{\tilde{w}-(m-1)}$ for $m \leq k$, with strict inequality if $p_{\tilde{w}-(m-1)} > 0$:

Assume that in equilibrium the borrower invests her outside option in fixed capital in states $\tilde{w} - 1, \tilde{w}$ and $\tilde{w} + 1$. We begin by defining a function that implicitly determines the equilibrium probability $p_{\tilde{w}}$ that the lender offers the borrower a restrictive contract. To do so we first determine the borrower's value in state $\tilde{w}$ as a function of the probability p she expects a restrictive contract. This allows us to determine the maximal repayment rate $R(p)$ she is willing to accept for a restrictive contract given the probability p she expects the lender to offer a restrictive contract. Finally, $R(p)$ allows us to determine the lender's payoff from offering restrictive contracts, and by comparing this to his payoff from offering unrestrictive contracts we pin down the equilibrium probability $p_{\tilde{w}}$.

In state $\tilde{w}$, if in equilibrium the borrower receives a restrictive contract with probability $p_{\tilde{w}}$,

[21] Note that since both $R(p)$ and $L(p)$ can both be written in terms of exogenous parameters of the model, it will hold generically that neither 3 nor 4 holds with equality.

her utility is

$$
\begin{aligned}
B_{\tilde{w}}\left(p_{\tilde{w}}\right) &= e^{-\rho dt}\left(p_{\tilde{w}}\left(\left(1-e^{-\kappa dt}\right)B_{\tilde{w}+1}+e^{-\kappa dt}B_{\tilde{w}}\right)+\left(1-p_{\tilde{w}}\right)\left(\left(1-e^{-\gamma dt}\right)B_{\tilde{w}+1}+e^{-\gamma dt}B_{\tilde{w}}\right)\right)\\
B_{\tilde{w}}\left(p_{\tilde{w}}\right) &= \frac{p_{\tilde{w}}\left(e^{-\rho dt}-e^{-(\kappa+\rho)dt}\right)B_{\tilde{w}+1}+\left(1-p_{\tilde{w}}\right)\left(e^{-\rho dt}-e^{-(\gamma+\rho)dt}\right)B_{\tilde{w}+1}}{1-p_{\tilde{w}}e^{-(\rho+\kappa)dt}-\left(1-p_{\tilde{w}}\right)e^{-(\rho+\gamma)dt}}\\
&\rightarrow \frac{p_{\tilde{w}}\kappa B_{\tilde{w}+1}+\left(1-p_{\tilde{w}}\right)\gamma B_{\tilde{w}+1}}{\rho+p_{\tilde{w}}\kappa+\left(1-p_{\tilde{w}}\right)\gamma}\\
&= \frac{\delta\left(p_{\tilde{w}}\right)}{\rho+\delta\left(p_{\tilde{w}}\right)}B_{\tilde{w}+1}
\end{aligned}
$$

where $\delta\left(p_{\tilde{w}}\right)\equiv p_{\tilde{w}}\kappa+\left(1-p_{\tilde{w}}\right)\gamma$.

Further recall that in equilibrium, the maximum repayment $R\left(p\right)$ that the borrower would accept is given by

$$
\begin{aligned}
\left(q_{\tilde{w}}\left(E+T_{\tilde{w}}\right)-R(p)\right)dt+e^{-\rho dt}B_{\tilde{w}} &= e^{-\rho dt}\left(\left(1-e^{-\kappa dt}\right)B_{\tilde{w}+1}+e^{-\kappa dt}B_{\tilde{w}}\right)\\
R(p)dt &= q_{\tilde{w}}\left(E+T_{\tilde{w}}\right)dt-e^{-\rho dt}\left(\left(1-e^{-\kappa dt}\right)\left(B_{\tilde{w}+1}-B_{\tilde{w}}\right)\right)
\end{aligned}
$$

Now, given the borrower's equilibrium expectation $p_{\tilde{w}}$, we can calculate the lender's payoff from offering a maximally extractive, acceptable restrictive or unrestrictive contract. Because the problem is stationary, we can determine which contract the lender prefers by comparing the lender's lifetime utility if he were to offer only restrictive contracts or only unrestrictive contracts in state $\tilde{w}$. If the lender were to offer only restrictive contracts in state $\tilde{w}$ his utility would be

$$
\begin{aligned}
L^{R}_{\tilde{w}}\left(p_{\tilde{w}}\right) &= \left(R\left(p_{\tilde{w}}\right)-T_{\tilde{w}}\right)dt+e^{-\rho dt}L^{R}_{\tilde{w}}\left(p_{\tilde{w}}\right)\\
&= \left(q_{\tilde{w}}\left(E+T_{\tilde{w}}\right)-T_{\tilde{w}}\right)dt-e^{-\rho dt}\left(\left(1-e^{-\kappa dt}\right)\left(B_{\tilde{w}+1}-B_{\tilde{w}}\right)\right)+e^{-\rho dt}L^{R}_{\tilde{w}}\left(p_{\tilde{w}}\right)\\
&= \frac{\left(q_{\tilde{w}}\left(E+T_{\tilde{w}}\right)-T_{\tilde{w}}\right)dt-e^{-\rho dt}\left(\left(1-e^{-\kappa dt}\right)\left(\frac{\rho}{\rho+\delta\left(p_{\tilde{w}}\right)}B_{\tilde{w}+1}\right)\right)}{1-e^{-\rho dt}}\\
&\rightarrow \frac{\left(q_{\tilde{w}}\left(E+T_{\tilde{w}}\right)-T_{\tilde{w}}\right)}{\rho}-\frac{\kappa}{\rho+\delta\left(p_{\tilde{w}}\right)}B_{\tilde{w}+1}
\end{aligned}
$$

On the other hand, if the lender were to offer only unrestrictive contracts in state $\tilde{w}$ his utility would be

$$\begin{aligned} L^U_{\tilde{w}}\left(p_{\tilde{w}}\right) &= \left(q_{\tilde{w}}T_{\tilde{w}} - h - T_{\tilde{w}}\right)dt + e^{-\rho dt}\left(\left(1-e^{-\gamma dt}\right)L_{\tilde{w}+1} + e^{-\gamma dt}L^U_{\tilde{w}}\left(p_{\tilde{w}}\right)\right) \\ &= \frac{\left(q_{\tilde{w}}T_{\tilde{w}} - h - T_{\tilde{w}}\right)dt + \left(\left(e^{-\rho dt} - e^{-(\rho+\gamma)dt}\right)L_{\tilde{w}+1}\right)}{1-e^{-(\rho+\gamma)dt}} \\ &\to \frac{\left(q_{\tilde{w}}T_{\tilde{w}} - h - T_{\tilde{w}}\right) + \gamma L_{\tilde{w}+1}}{\rho+\gamma} \\ &= \left(1-\beta\right)\frac{\left(q_{\tilde{w}}T_{\tilde{w}} - h - T_{\tilde{w}}\right)}{\rho} + \beta L_{\tilde{w}+1} \end{aligned}$$

Next, consider the function $g_{\tilde{w}}\left(p\right) \equiv L^U_{\tilde{w}}\left(p\right) - L^R_{\tilde{w}}\left(p\right)$. Note that $L^U_{\tilde{w}}\left(p\right)$ is independent of p and $L^R_{\tilde{w}}\left(p\right)$ is decreasing in p, so $g_{\tilde{w}}\left(p\right)$ is increasing. If $g_{\tilde{w}}\left(1\right) < 0$, then the unique equilibrium is for the lender to offer a restrictive contract with probability 1. If $g_{\tilde{w}}\left(0\right) > 0$, then the unique equilibrium is for the lender to offer an unrestrictive contract with probability 1. Else, as shown in Proposition 2, generically there is a unique $p_{\tilde{w}} \in [0,1]$ such that $g_{\tilde{w}}\left(p_{\tilde{w}}\right) = 0$, and the unique equilibrium is for the lender to offer a restrictive contract with probability $p_{\tilde{w}}$.

We now verify that p_w is single peaked in w by considering the following five exhaustive cases:

$0 < p_{\tilde{w}} < p_{\tilde{w}+1} < 1$

$0 < p_{\tilde{w}} < p_{\tilde{w}+1} = 1$

In these cases we aim to verify that $p_{\tilde{w}-1} < p_{\tilde{w}}$

$0 = p_{\tilde{w}} < p_{\tilde{w}+1} < 1$

$0 = p_{\tilde{w}} < p_{\tilde{w}+1} = 1$

In this case we aim to verify that $p_{\tilde{w}-1} = p_{\tilde{w}} = 0$ and $g_{\tilde{w}-1}\left(0\right) > g_{\tilde{w}}\left(0\right)$

$0 = p_{\tilde{w}} = p_{\tilde{w}+1}$ and $g_{\tilde{w}}\left(0\right) > g_{\tilde{w}+1}(0)$

In this case we aim to verify that $p_{\tilde{w}-1} = p_{\tilde{w}} = 0$ and $g_{\tilde{w}-1}(0) > g_{\tilde{w}}(0)$.

We will provide the proof for the case where $0 < p_{\tilde{w}} < p_{\tilde{w}+1} < 1$ and omit the others as they are all similar.

Because $p_{\tilde{w}+1}$ is interior, we have

$$L_{\tilde{w}+1} = L^R_{\tilde{w}+1}(p_{\tilde{w}+1}) = \frac{q_{\tilde{w}+1}\left(E + T_{\tilde{w}+1}\right) - T_{\tilde{w}+1}}{\rho} - \frac{\kappa}{\rho + \delta(p_{\tilde{w}+1})}B_{\tilde{w}+2}$$

and

$$B_{\tilde{w}+1} = \frac{\delta(p_{\tilde{w}+1})}{\rho + \delta(p_{\tilde{w}+1})} B_{\tilde{w}+2}$$

Thus in state w we have

$$\begin{aligned}
g_{\tilde{w}}\left(p_{\tilde{w}}\right) &\equiv L^U_{\tilde{w}} - L^R_{\tilde{w}}(p_{\tilde{w}}) \\
&= (1-\beta)\frac{\left(q_{\tilde{w}} T_{\tilde{w}} - h - T_{\tilde{w}}\right)}{\rho} + \beta\left(\frac{q_{\tilde{w}+1}\left(E + T_{\tilde{w}+1}\right) - T_{\tilde{w}+1}}{\rho} - \frac{\kappa}{\rho + \delta(p_{\tilde{w}+1})} B_{\tilde{w}+2}\right) \\
&\quad - \left(\frac{q_{\tilde{w}}\left(E + T_{\tilde{w}}\right) - T_{\tilde{w}}}{\rho} - \frac{\kappa}{\rho + \delta(p_{\tilde{w}})} \frac{\delta(p_{\tilde{w}+1})}{\rho + \delta(p_{\tilde{w}+1})} B_{\tilde{w}+2}\right) \\
&= \beta \frac{\left(q_{\tilde{w}+1}\left(E + T_{\tilde{w}+1}\right) - T_{\tilde{w}+1}\right) - \left(q_{\tilde{w}}\left(E + T_{\tilde{w}}\right) - T_{\tilde{w}}\right)}{\rho} - (1-\beta)\frac{h + q_{\tilde{w}} E}{\rho} - \frac{\kappa}{\rho + \delta(p_{\tilde{w}+1})} B_{\tilde{w}+2}\left(\beta - \frac{\delta(p_{\tilde{w}+1})}{\rho + \delta(p_{\tilde{w}})}\right) \\
&= 0
\end{aligned}$$

Similarly we have

$$\begin{aligned}
g_{\tilde{w}-1}(p) &= L^U_{\tilde{w}-1} - L^R_{\tilde{w}-1}(p) \\
&= \beta \frac{\left(q_{\tilde{w}}\left(E + T_{\tilde{w}}\right) - T_{\tilde{w}}\right) - \left(q_{\tilde{w}-1}\left(E + T_{\tilde{w}-1}\right) - T_{\tilde{w}-1}\right)}{\rho} - (1-\beta)\frac{h + q_{\tilde{w}-1} E}{\rho} \\
&\quad - \frac{\kappa}{\rho + \delta(p_{\tilde{w}})} \frac{\delta(p_{\tilde{w}+1})}{\rho + \delta(p_{\tilde{w}+1})} B_{\tilde{w}+2}\left(\beta - \frac{\delta(p_{\tilde{w}})}{\rho + \delta(p)}\right) \\
&= 0
\end{aligned}$$

Comparing the expression for $g_{\tilde{w}-1}(p)$ to the that of $g_{\tilde{w}}(p)$ we can see that the sum of first two terms is strictly larger in the expression for $g_{\tilde{w}-1}(p)$ (by Assumption 2). That means that in order for p to set $g_{\tilde{w}-1}(p) = 0$ (if possible), we need that the third term in $g_{\tilde{w}-1}(p)$ is strictly smaller than the third term in $g_{\tilde{w}}(p)$. That is

$$\begin{aligned}
-\frac{\kappa}{\rho + \delta(p_{\tilde{w}})} \frac{\delta(p_{\tilde{w}+1})}{\rho + \delta(p_{\tilde{w}+1})} B_{\tilde{w}+2}\left(\beta - \frac{\delta(p_{\tilde{w}})}{\rho + \delta(p)}\right) &< -\frac{\kappa}{\rho + \delta(p_{\tilde{w}+1})} B_{\tilde{w}+2}\left(\beta - \frac{\delta(p_{\tilde{w}+1})}{\rho + \delta(p_{\tilde{w}})}\right) \\
&\Longleftrightarrow \\
\frac{\delta(p_{\tilde{w}+1})}{\rho + \delta(p_{\tilde{w}})}\left(\beta - \frac{\delta(p_{\tilde{w}})}{\rho + \delta(p)}\right) &> \left(\beta - \frac{\delta(p_{\tilde{w}+1})}{\rho + \delta(p_{\tilde{w}})}\right)
\end{aligned}$$

Recall that $p_{\tilde{w}+1} > p_{\tilde{w}}$ by . So $\frac{\delta(p_{\tilde{w}+1})}{\rho + \delta(p_{\tilde{w}})} < 1$. Thus

$$\begin{aligned}
\left(\beta - \frac{\delta(p_{\tilde{w}})}{\rho + \delta(p)}\right) &> \left(\beta - \frac{\delta(p_{\tilde{w}+1})}{\rho + \delta(p_{\tilde{w}})}\right) \\
&\Longrightarrow -\frac{\delta(p_{\tilde{w}})}{\rho + \delta(p)} > -\frac{\delta(p_{\tilde{w}+1})}{\rho + \delta(p_{\tilde{w}})} \\
&\Longrightarrow \frac{\delta(p_{\tilde{w}})}{\rho + \delta(p)} < \frac{\delta(p_{\tilde{w}+1})}{\rho + \delta(p_{\tilde{w}})} \\
&\Longrightarrow p < p_{\tilde{w}}
\end{aligned}$$

On the other hand, if there is no $p \geq 0$ such that $g_{\tilde{w}-1}(p) = 0$, then $g_{\tilde{w}-1}(0) > 0$, and the unique equilibrium includes $p_{\tilde{w}-1} = 0$. This completes the argument for this case. As the remaining cases are similar they are omitted.

Proof of Proposition 4

Recall from the proof of Proposition 2 that in equilibrium the lender offers a restrictive contract in state w with probability 1 if and only if $L_w^R(1) \geq L_w^U$, where $L_w^R(p) \equiv \frac{q_w(E+T_w)-T_w}{\rho} - \frac{\kappa}{\rho+\delta(p)} B_{w+1}$ and $L_w^U \equiv (1-\beta)\frac{q_w T_w - h - T_w}{\rho} + \beta L_{w+1}$. Now

$$
\begin{array}{rcl}
L_w^R(1) & \geq & L_w^U \\
 & \iff & \\
\dfrac{q_w(E+T_w)-T_w}{\rho} - \alpha B_{w+1} & > & (1-\beta)\dfrac{q_w T_w - h - T_w}{\rho} + \beta L_{w+1} \\
 & \iff & \\
-\alpha B_{w+1} & > & -(1-\beta)\left(\dfrac{q_w E + h}{\rho}\right) + \beta\left(L_{w+1} - \dfrac{q_w(E+T_w)-T_w}{\rho}\right) \\
 & \iff & \\
(\beta-\alpha) B_{w+1} & > & \beta\left(B_{w+1} + L_{w+1} - \dfrac{q_w(E+T_w)-T_w}{\rho} - \phi\right)
\end{array}
$$

which completes the proof.

Proof of Proposition 5

Suppose $h = 0$ and consider the lender's behavior in state n. Fixing a probability p_n that the borrower anticipates a restrictive contract in equilibrium, (and recalling that we can consider the lender's continuation utility in state n from a fixed action due to the stationarity of the problem), the lender's utility from offering a restrictive contract is

$$
\begin{aligned}
L_n^R(p_n) &= \frac{q_n(E+T_n)-T_n}{\rho} - \frac{\kappa}{\rho+\delta(p_n)}\frac{u}{\rho} \\
&= \frac{q_n(E+T_n)-T_n}{\rho} - \alpha\frac{u}{\rho}
\end{aligned}
$$

where the equality follows from the fact that $h = 0$.

On the other hand, his utility from offering an unrestrictive contract is

$$\begin{aligned} L_n^U(p_n) &= \frac{q_n T_n - h - T_n}{\rho}(1-\beta) \\ &= \frac{q_n T_n - h - T_n}{\rho}(1-\alpha) \end{aligned}$$

The lender prefers offering an unrestrictive contract over a restrictive one if and only if

$$\begin{aligned} \frac{q_n T_n - h - T_n}{\rho}(1-\alpha) &\geq \frac{q_n(E+T_n) - T_n}{\rho} - \alpha\frac{u}{\rho} \\ (1-\alpha)\frac{q_n T_n - h - T_n}{\rho} + \alpha\frac{u}{\rho} &\geq \frac{q_n(E+T_n) - T_n}{\rho} \end{aligned} \tag{5}$$

The left hand side of inequality 5 is the sum of the borrower and lender's welfares if the borrower invests in fixed capital at the slowest possible rate in the relationship, and the right hand side is the sum of their welfares from if the borrower invests in working capital. Thus if it is socially efficient to invest in fixed capital the lender strictly prefers unrestrictive contracts, irrespective of the borrower's expectation p_n, and thus in equilibrium in state n the lender chooses unrestrictive contracts with probability 1.

Moving backwards, the proof proceeds similarly.

Proof of Propositions 6 and 7

Lemma 3. *For any state $w > \bar{w}$, $\frac{dp_w}{du} \leq 0$ with strict inequality if $p_w > 0$.*

Proof. By definition, in states $w > \bar{w}$, $p_w < 1$. Thus in equilibrium the lender at least weakly prefers offering the borrower an unrestrictive contract. We can thus write the lender's continuation utility in each such state as the utility he derives from offering an unrestrictive contract at every period (fixing the borrower's expectation at p_w).[22] That is

$$L_w = L_w^U \equiv (1-\beta)\frac{q_w T_w - h - T_w}{\rho} + \beta L_{w+1}$$

On the other hand, if the lender were to offer a minimally generous restrictive contract at every

[22]Note that in full generality he may offer the borrower an unrestrictive contract with positive transfer in state w. If so his continuation utility is $L_w^U = \left(1 - \frac{\frac{q_w(E+T_w)-R}{\phi}}{\frac{q_w(E+T_w)-R}{\phi}+\rho}\right)\frac{R-T_w}{\rho} + \left(\frac{\frac{q_w(E+T_w)-R}{\phi}}{\frac{q_w(E+T_w)-R}{\phi}+\rho}\right)L_{w+1}$, but otherwise the argument goes through unchanged.

period (again, fixing the borrower's expectation at p_w) he would receive a continuation utility of

$$L_w = L_w^R(p_w) \equiv \frac{q_w(E+T_w)-T_w}{\rho} - \frac{\kappa}{\rho+\delta(p_w)}B_{w+1}$$

In state n $L_n^U = (1-\beta)\frac{q_nT_n-h-T_n}{\rho}$ which is not a function of u. On the other hand $L_n^R(p_n) = \frac{q_n(E+T_n)-T_n}{\rho} - \frac{\kappa}{\rho+\delta(p_n)}\frac{u}{\rho}$ is decreasing in u. Hence if in state n $L_n^U > L_n^R(0)$, then $p_n = 0$ and $\frac{dp_n}{du} = 0$. The lender's continuation utility is $L_n = (1-\beta)\frac{q_nT_n-h-T_n}{\rho}$ and $\frac{dL_n}{du} = 0$. The borrower's utility is $B_n = \beta\frac{u}{\rho}$ so $\frac{dB_n}{du} > 0$.

If $L_n^I = L_n^C(p_n)$ for some $p_n \in [0,1]$ then p_n is the solution to

$$g(p_n) \equiv (1-\beta)\frac{q_nT_n-h-T_n}{\rho} - \left(\frac{q_n(E+T_n)-T_n}{\rho} - \frac{\kappa}{\rho+\delta(p_n)}\frac{u}{\rho}\right) = 0$$

Since δ is a decreasing function, it is clear that $\frac{dp_n}{du} < 0$. But we still have $L_n = (1-\beta)\frac{q_nT_n-h-T_n}{\rho}$ so that $\frac{dL_n}{du} = 0$. $B_n = \frac{\delta(p_n)}{\rho+\delta(p_n)}\frac{u}{\rho}$, so

$$\frac{dB_n}{du} = \frac{\left(\frac{d\delta(p_n)}{dp_n}\frac{dp_n}{du}\right)\rho}{(\rho+\delta(p_n))^2}\frac{u}{\rho} + \frac{\delta(p_n)}{\rho+\delta(p_n)} > 0$$

.

Proceeding backward to any state $w > \bar{w}$, suppose $\frac{dB_{w+1}}{du} > 0$, $\frac{dL_{w+1}}{du} = 0$. Then the proof proceeds exactly as above. This completes the proof of the lemma. □

We next consider the comparative static in states $w \in \{\underline{w}, \ldots, \bar{w}\}$.

Lemma 4. *For $w \in \{\underline{w}, \ldots, \bar{w}\}$, generically $\frac{dp_w}{du} = 0$, $\frac{dB_{\bar{w}+1}}{du} > 0$ and $\frac{dL_{\bar{w}}}{du} < 0$.*

Proof. By definition $p_w = 1$ for $w \in \{\underline{w}, \ldots, \bar{w}\}$. Generically this preference will be strict and thus $\frac{dp_{\bar{w}}}{du} = 0$.

Recall that in Lemma 3 we established $\frac{dB_{\bar{w}+1}}{du} > 0$. We also know that $L_{\bar{w}} = L_{\bar{w}}^R(1) \equiv \frac{q_{\bar{w}}(E+T_{\bar{w}})-T_{\bar{w}}}{\rho} - \alpha B_{\bar{w}+1}$. Hence generically $\frac{dL_{\bar{w}}}{du} < 0$. Further, $B_{\bar{w}} = \alpha B_{\bar{w}+1}$ so $\frac{dB_{\bar{w}}}{du} = \alpha\frac{dB_{\bar{w}+1}}{du} > 0$.

For the remainder of the states $w \in \{\underline{w}, \ldots, \bar{w}\}$, the result follows from straightforward induction. □

We now consider the comparative statics for $w < \underline{w}$ in the following three lemmas.

Lemma 5. *Suppose $p_{\underline{w}-1} = 0$. Then generically $\frac{dp_w}{du} = 0$, $\frac{dL_w}{du} < 0$, and $\frac{dB_w}{du} > 0$ for all $w < \underline{w}$.*

Proof. If $p_{\underline{\mathrm{w}}-1} = 0$ and $L^{U}_{\underline{\mathrm{w}}-1} > L^{R}_{\underline{\mathrm{w}}-1}(0)$, then the lender's preference for unrestrictive contracts is strict so $\frac{dp_{\underline{\mathrm{w}}-1}}{du} = 0$. Further, Lemma 4 established that $\frac{dL_{\underline{\mathrm{w}}}}{du} < 0$ and $\frac{dB_{\underline{\mathrm{w}}}}{du} > 0$. Therefore, because $L_{\underline{\mathrm{w}}-1} = (1-\beta)\frac{q_{\underline{\mathrm{w}}-1}T_{\underline{\mathrm{w}}-1}-h}{\rho} + \beta L_{\underline{\mathrm{w}}}$, we know $\frac{dL_{\underline{\mathrm{w}}-1}}{du} < 0$. And $B_{\underline{\mathrm{w}}-1} = \beta B_{\underline{\mathrm{w}}}$ so $\frac{dB_{\underline{\mathrm{w}}-1}}{du} > 0$. Moving backwards proceeds by straightforward induction. □

The remainder of the proof deals with the case for which $p_{\underline{\mathrm{w}}-1} > 0$. We split the analysis into two cases based on the players' level of patience.

Lemma 6. *Suppose $p_{\underline{w}-1} > 0$ and $\rho > \frac{\kappa\gamma}{\kappa+\gamma}$. Then $\frac{dp_w}{du} > 0$, $\frac{dL_w}{du} < 0$, and $\frac{dB_w}{du} < 0$ for all $w < \underline{w}$.*

Proof. Consider first state $\underline{\mathrm{w}} - 1$. We know $p_{\underline{\mathrm{w}}-1}$ is the solution to $g\left(p_{\underline{\mathrm{w}}-1}\right) = 0$. So

$$\beta\frac{\left(q_{\underline{\mathrm{w}}}\left(E+T_{\underline{\mathrm{w}}}\right)-T_{\underline{\mathrm{w}}}\right)-\left(q_{\underline{\mathrm{w}}-1}\left(E+T_{\underline{\mathrm{w}}-1}\right)-T_{\underline{\mathrm{w}}-1}\right)}{\rho}-(1-\beta)\frac{h+q_{\underline{\mathrm{w}}-1}E}{\rho}-\left(\beta-\frac{\delta\left(p_{\underline{\mathrm{w}}}\right)}{\rho+\delta(p_{\underline{\mathrm{w}}-1})}\right)\frac{\kappa}{\rho+\delta(p_{\underline{\mathrm{w}}})}B_{\underline{\mathrm{w}}+1}=0$$

$$\Longleftrightarrow$$

$$\beta\frac{\left(q_{\underline{\mathrm{w}}}\left(E+T_{\underline{\mathrm{w}}}\right)-T_{\underline{\mathrm{w}}}\right)-\left(q_{\underline{\mathrm{w}}-1}\left(E+T_{\underline{\mathrm{w}}-1}\right)-T_{\underline{\mathrm{w}}-1}\right)}{\rho}-(1-\beta)\frac{h+q_{\underline{\mathrm{w}}-1}E}{\rho}=\left(\beta-\frac{\delta\left(p_{\underline{\mathrm{w}}}\right)}{\rho+\delta(p_{\underline{\mathrm{w}}-1})}\right)\frac{\kappa}{\rho+\delta(p_{\underline{\mathrm{w}}})}B_{\underline{\mathrm{w}}+1}$$

Note that the left hand side of the above equation is constant in u.[23] Thus

[23] The right hand side of the above equation can be simplified by noting that $p_{\underline{\mathrm{w}}} = 1$, but we leave it in this more general form to economize on notation in the backward induction step.

$$\begin{aligned}
\frac{d\left(\beta\frac{\kappa}{\rho+\delta\left(p_{\underline{\mathbf{W}}}\right)}-\frac{\kappa}{\rho+\delta\left(p_{\underline{\mathbf{W}}-1}\right)}\frac{\delta\left(p_{\underline{\mathbf{W}}}\right)}{\rho+\delta\left(p_{\underline{\mathbf{W}}}\right)}\right)B_{\underline{\mathbf{W}}+1}}{du} &= 0 \qquad (6)\\
&\Longrightarrow\\
-\beta\frac{\kappa\frac{d\delta\left(p_{\underline{\mathbf{W}}}\right)}{dp_{\underline{\mathbf{W}}}}\frac{dp_{\underline{\mathbf{W}}}}{du}}{\left(\rho+\delta\left(p_{\underline{\mathbf{W}}}\right)\right)^2}B_{\underline{\mathbf{W}}+1}+\frac{\kappa\frac{d\delta\left(p_{\underline{\mathbf{W}}-1}\right)}{dp_{\underline{\mathbf{W}}-1}}\frac{dp_{\underline{\mathbf{W}}-1}}{du}}{\left(\rho+\delta\left(p_{\underline{\mathbf{W}}-1}\right)\right)^2}\frac{\delta\left(p_{\underline{\mathbf{W}}}\right)}{\rho+\delta\left(p_{\underline{\mathbf{W}}}\right)}B_{\underline{\mathbf{W}}+1}&\\
-\frac{\kappa}{\rho+\delta\left(p_{\underline{\mathbf{W}}-1}\right)}\frac{\left(\frac{d\delta\left(p_{\underline{\mathbf{W}}}\right)}{dp_{\underline{\mathbf{W}}}}\frac{dp_{\underline{\mathbf{W}}}}{du}\right)\left(\rho+\delta\left(p_{\underline{\mathbf{W}}}\right)\right)-\delta\left(p_{\underline{\mathbf{W}}}\right)\frac{d\delta\left(p_{\underline{\mathbf{W}}}\right)}{dp_{\underline{\mathbf{W}}}}\frac{dp_{\underline{\mathbf{W}}}}{du}}{\left(\rho+\delta\left(p_{\underline{\mathbf{W}}}\right)\right)^2}B_{\underline{\mathbf{W}}+1}&\\
+\left(\beta\frac{\kappa}{\rho+\delta\left(p_{\underline{\mathbf{W}}}\right)}-\frac{\kappa}{\rho+\delta\left(p_{\underline{\mathbf{W}}-1}\right)}\frac{\delta\left(p_{\underline{\mathbf{W}}}\right)}{\rho+\delta\left(p_{\underline{\mathbf{W}}}\right)}\right)\frac{dB_{\underline{\mathbf{W}}+1}}{du} &= 0\\
&\Longrightarrow\\
\frac{\kappa\frac{d\delta\left(p_{\underline{\mathbf{W}}-1}\right)}{dp_{\underline{\mathbf{W}}-1}}\frac{dp_{\underline{\mathbf{W}}-1}}{du}}{\left(\rho+\delta\left(p_{\underline{\mathbf{W}}-1}\right)\right)} &< 0\\
&\Longrightarrow\\
\frac{dp_{\underline{\mathbf{W}}-1}}{du} &> 0
\end{aligned}$$

Where the second implication follows by removing positive terms from the right hand side and noting that $p_{\underline{\mathbf{W}}} > p_{\underline{\mathbf{W}}-1}$ which implies that $\frac{\delta\left(p_{\underline{\mathbf{W}}}\right)}{\rho+\delta\left(p_{\underline{\mathbf{W}}-1}\right)} < \frac{\delta\left(p_{\underline{\mathbf{W}}-1}\right)}{\rho+\delta\left(p_{\underline{\mathbf{W}}-1}\right)} \leq \beta$.

Next, note that

$$L_{\underline{\mathbf{W}}-1} = L^U_{\underline{\mathbf{W}}-1} = (1-\beta)\frac{q_{\underline{\mathbf{W}}-1}T_{\underline{\mathbf{W}}-1}-h}{\rho}+\beta L_{\underline{\mathbf{W}}} \qquad (7)$$

so by Lemma 4 we know that $\frac{dL_{\underline{\mathbf{W}}-1}}{du} < 0$.[24]

To find the sign of $\frac{dB_{\underline{\mathbf{W}}-1}}{du}$ recall that $\frac{\rho+\delta\left(p_{\underline{\mathbf{W}}-1}\right)}{\delta\left(p_{\underline{\mathbf{W}}-1}\right)}B_{\underline{\mathbf{W}}-1} = B_{\underline{\mathbf{W}}} = \frac{\delta\left(p_{\underline{\mathbf{W}}}\right)}{\rho+\delta\left(p_{\underline{\mathbf{W}}}\right)}B_{\underline{\mathbf{W}}+1}$. Hence we know

[24]Note that in full generality the lender may offer the borrower an investment loan with positive transfer in state $\underline{\mathbf{w}}-1$. If so his continuation utility is $L^U_{\underline{\mathbf{W}}-1} = \left(1-\frac{\frac{q_{\underline{\mathbf{W}}-1}\left(E+T_{\underline{\mathbf{W}}-1}\right)-R}{\phi}}{\frac{q_{\underline{\mathbf{W}}-1}\left(E+T_{\underline{\mathbf{W}}-1}\right)-R}{\phi}+\rho}\right)\frac{R-T_{\underline{\mathbf{W}}-1}}{\rho}+\left(\frac{\frac{q_{\underline{\mathbf{W}}-1}\left(E+T_{\underline{\mathbf{W}}-1}\right)-R}{\phi}}{\frac{q_{\underline{\mathbf{W}}-1}\left(E+T_{\underline{\mathbf{W}}-1}\right)-R}{\phi}+\rho}\right)L_{\underline{\mathbf{W}}}$, and the interest rate becomes weakly higher but otherwise the argument to follow goes through unchanged.

that

$$
\begin{aligned}
\frac{d\left(\beta\frac{\kappa}{\rho+\delta\left(p_{\underline{W}}\right)}-\frac{\kappa}{\rho+\delta\left(p_{\underline{W}-1}\right)}\frac{\delta\left(p_{\underline{W}}\right)}{\rho+\delta\left(p_{\underline{W}}\right)}\right)B_{\underline{W}+1}}{du} &= 0\\
\frac{d\left(\beta\frac{\kappa}{\delta\left(p_{\underline{W}}\right)}-\frac{\beta\frac{\kappa}{\delta\left(p_{\underline{W}}\right)}\rho-\kappa}{\delta\left(p_{\underline{W}-1}\right)}\right)B_{\underline{W}-1}}{du} &=\\
\frac{d\left(\beta+\frac{\beta\rho-\kappa}{\delta\left(p_{\underline{W}-1}\right)}\right)B_{\underline{W}-1}}{du} &=\\
\frac{-\left(\beta\rho-\kappa\right)\frac{d\delta\left(p_{\underline{W}-1}\right)}{dp_{\underline{W}-1}}\frac{dp_{\underline{W}-1}}{dU}}{\left(\delta\left(p_{\underline{W}-1}\right)\right)^{2}}B_{\underline{W}-1}+\frac{dB_{\underline{W}-1}}{du}\left(\beta+\frac{\beta\rho-\kappa}{\delta\left(p_{\underline{W}-1}\right)}\right)
\end{aligned}
\tag{8}
$$

where the second equality follows from noting that $p_{\underline{W}}=1$. Reducing we have

$$
\frac{dB_{\underline{W}-1}}{du}=NEG\left(\beta\rho-\kappa\right) \tag{9}
$$

where NEG is a negative constant. The one subtle algebraic reduction in this final step is that $\left(\beta+\frac{\beta\rho-\kappa}{\delta\left(p_{\underline{W}-1}\right)}\right)=\frac{\rho+\delta\left(p_{\underline{W}-1}\right)}{\delta\left(p_{\underline{W}-1}\right)}\left[\beta-\frac{\kappa}{\rho+\delta\left(p_{\underline{W}-1}\right)}\right]>0$.

Since we have assumed $\rho>\frac{\kappa\gamma}{\kappa+\gamma}$, which is equivalent to $\rho\beta>\kappa$ we have $\frac{dB_{\underline{W}-1}}{du}<0$.

Moving backward to state $\underline{w}-2$, suppose $p_{\underline{W}-2}>0$ (or $p_{\underline{W}-2}=0$, but $L^{I}_{\underline{W}-2}-L^{C}_{\underline{W}-2}\left(0\right)=0$). Then $p_{\underline{W}-2}$ is the solution to $g_{\underline{W}-2}\left(p_{\underline{W}-2}\right)=0$. That is

$$
\left(1-\beta\right)\frac{q_{\underline{W}-2}T_{\underline{W}-2}-h}{\rho}+\beta L_{\underline{W}-1}-\left(\frac{q_{\underline{W}-2}\left(E+T_{\underline{W}-2}\right)}{\rho}-\frac{\kappa}{\rho+\delta\left(p_{\underline{W}-2}\right)}B_{\underline{W}-1}\right)=0
$$

Differentiating both sides with respect to u we see

$$
\beta\frac{dL_{\underline{W}-1}}{du}+\left(\frac{-\kappa\frac{d\delta\left(p_{\underline{W}-2}\right)}{dp_{\underline{W}-2}}\frac{dp_{\underline{W}-2}}{du}}{\left(\rho+\delta\left(p_{\underline{W}-2}\right)\right)^{2}}B_{\underline{W}-1}+\frac{\kappa}{\rho+\delta\left(p_{\underline{W}-2}\right)}\frac{dB_{\underline{W}-1}}{du}\right)=0 \tag{10}
$$

We know that $\frac{dL_{\underline{W}-1}}{du}<0$ and $\frac{dB_{\underline{W}-1}}{du}<0$. Hence $\frac{dp_{\underline{W}-2}}{du}>0$. Further

$$
L_{\underline{W}-2}=\left(1-\beta\right)\frac{q_{\underline{W}-2}T_{\underline{W}-2}-h-T_{\underline{W}-2}}{\rho}+\beta L_{\underline{W}-1}
$$

so $\frac{dL_{\underline{W}-2}}{du} < 0$. And $B_{\underline{W}-2} = \frac{\delta\left(p_{\underline{W}-2}\right)}{\rho+\delta\left(p_{\underline{W}-2}\right)} B_{\underline{W}-1}$ so

$$\frac{dB_{\underline{W}-2}}{du} = \frac{\frac{d\delta\left(p_{\underline{W}-2}\right)}{dp_{\underline{W}-2}} \frac{dp_{\underline{W}-2}}{du} \rho}{\left(\rho + \delta\left(p_{\underline{W}-2}\right)\right)^2} B_{\underline{W}-1} + \frac{\delta\left(p_{\underline{W}-2}\right)}{\rho + \delta\left(p_{\underline{W}-2}\right)} \frac{dB_{\underline{W}-1}}{du} < 0 \tag{11}$$

If instead we had $p_{\underline{W}-2} = 0$ and $L^U_{\underline{W}-2} > L^R_{\underline{W}-2}(0)$, then $\frac{dp_{\underline{W}-2}}{du} = 0$. $L_{\underline{W}-2} = (1-\beta) \frac{q_{\underline{W}-2} T_{\underline{W}-2} - h}{\rho} + \beta L_{\underline{W}-1}$ so $\frac{dL_{\underline{W}-2}}{du} < 0$. $B_{\underline{W}-2} = \beta B_{\underline{W}-1}$ so $\frac{dB_{\underline{W}-2}}{du} < 0$.

Because we used only that $\frac{dB_{\underline{W}-1}}{du} < 0$ and $\frac{dL_{\underline{W}-1}}{du} < 0$, moving backwards from state $\underline{w} - 2$ to state 0 is straightforward induction. □

We now complete the proof of Propositions 6 and 7 by considering a patient borrower.

Lemma 7. *Suppose $p_{\underline{w}-1} > 0$ and $\rho < \frac{\kappa\gamma}{\kappa+\gamma}$. Then $\frac{dp_w}{du} > 0$ and $\frac{dL_w}{du} < 0$ for all $w < \underline{w}$.*

Proof. In state $\underline{w} - 1$ everything follows as it did in Lemma 6 except that $\frac{dB_{\underline{W}-1}}{du}$, determined by equation (9) is positive. In state $\underline{w} - 2$, the considerations are similar. $\frac{dp_{\underline{W}-2}}{du} > 0$ is determined by equation (6) (reducing all indices by 1) and $\frac{dL_{\underline{W}-2}}{du} < 0$ is determined by (7) (reducing all indices by 1). However the sign of $\frac{dB_{\underline{W}-2}}{du}$, determined by (8) is now ambiguous.

Moving back to an arbitrary state $w < \underline{w}$ such that $\frac{dB_{w+1}}{du} > 0$, the considerations will be exactly the same as for $\underline{w} - 2$. In any state $w < \underline{w}$ for which $\frac{dB_{w+1}}{du} < 0$, $\frac{dp_w}{du} > 0$ is determined by equation (10), $\frac{dL_w}{du} < 0$ is determined by (7), and $\frac{dB_w}{du} < 0$ is determined by (11). This completes the proof. □

Together Lemmas 3 through 7 complete the proof of Propositions 6 and 7.

Proof of Proposition 8

Fixing the lender's behavior, the borrower's continuation utility in state n is

$$B_n(p_n) = \frac{p_n\left(1 - e^{-\kappa dt}\right) + (1 - p_n)\left(1 - e^{-\gamma dt}\right)}{1 - p_n e^{-(\rho+\kappa)dt} - (1 - p_n) e^{-(\rho+\gamma)dt}} e^{-\rho dt} \frac{u}{\rho}$$

which increases linearly in u. Moving backward, suppose B_{w+1} is increasing in u. Then, noting that

$$B_w(p_w) = \frac{p_w\left(1 - e^{-\kappa dt}\right) + (1 - p_w)\left(1 - e^{-\gamma dt}\right)}{1 - p_w e^{-(\rho+\kappa)dt} - (1 - p_w) e^{-(\rho+\gamma)dt}} e^{-\rho dt} B_{w+1}$$

increases linearly in B_{w+1} completes the proof.

Proof of Proposition 9

The proof for states $w \geq \underline{w}$ proceeds exactly as in Proposition 6 and is thus omitted. In this section we provide an example in which $\frac{dp_{\underline{w}-1}}{d\rho^B} < 0$ so that making the borrower more patient can strengthen the poverty trap.

We prove this result with a three state model $w \in \{1, 2, 3\}$ where the game ends if the borrower ever reaches state 3. We take

$E = .15$, $q_1 = q_2 \equiv q = 2$, $\phi = \frac{1}{2}$, $h = 100$, $T_1 = 600$, $T_2 = 1000$, $\frac{u}{\rho^B} = 2000$ and $\rho^B = \rho^L = 1$. It is easily verified that Assumption 3 hold in states 1 and 2. That is,

$$\alpha^2 \frac{u}{\rho^B} = \left(\frac{.3}{1.3}\right)^2 2000 > \frac{qE + h}{\rho^B} = 100.3$$

Now we verify that in state 2 the lender offers the borrower a restrictive contract with probability 1. If the borrower expects a restrictive contract with probability 1 then the lender gets the following continuation utility if he offers the borrower a restrictive contract in state 2.

$$L_2^R(1) = \frac{q(E + T_2) - T_2}{\rho^L} - \alpha \frac{u}{\rho^B} = 1000.3 - \frac{.3}{1.3} 2000 \approx 539$$

If instead the lender offers the borrower an unrestrictive contract at every period in state 2, his continuation utility is

$$L_2^U = \frac{qT_2 - h - T_2}{\rho^L}(1 - \beta) \approx 9$$

Because the lender finds it *least* appealing to offer a restrictive contract when the borrower expects restrictive contracts with probability 1, we conclude that in the unique equilibrium the lender offers the borrower a restrictive contract with probability 1.

We next verify that in equilibrium, the lender mixes between restrictive and unrestrictive contracts in state 1.

First, consider the lender's continuation utility in state 1 from offering the borrower a restrictive contract with probability 1 *when she expects an restrictive contract with probability p.*

$$\begin{aligned} L_1^R(p) &= \frac{q(E+T_1)-T_1}{\rho^L} - \max\left\{\frac{qE+h}{\rho^L}, \frac{\kappa}{\rho^B+\delta(p)}B_2\right\} \\ &\approx 600.3 - \max\left\{100.3, \left(\frac{.3}{1+.3p+100(1-p)}\right)\left(\frac{.3}{1.3}\right)2000\right\} \end{aligned}$$

Note that the repayment rate the lender must set is the larger of $qT-h$ and what the lender must set so that the borrower achieves the utility she would have received from investing E in fixed capital.

If instead the lender were to offer an unrestrictive contract with probability 1, her state 1 continuation utility would be

$$L_1^I = \frac{qT-h-T_1}{\rho^L}(1-\beta)+\beta L_2 \approx \frac{500}{101}+\frac{100}{101}539$$

It is easily verified that $L_1^R(0) > L_1^U > L_1^R(1)$ and hence the unique equilibrium in state 1 involves the lender mixing between restrictive and unrestrictive contracts. The probability p_1 that the lender offers a restrictive contract is determined by the following equation.

$$\begin{aligned} \frac{qT_1-h-T_1}{\rho^L}(1-\beta)+\beta L_2 &= \frac{q(E+T_1)-T_1}{\rho^L} - \frac{\kappa}{\rho^B+\delta(p_1)}B_2 \\ &\Longrightarrow \\ \frac{500}{101}+\frac{100}{101}\left(1000.3-\frac{.3}{1.3}2000\right) &\approx 600.3 - \frac{.3}{1+.3p_1+100(1-p_1)}\left(\frac{.3}{1.3}2000\right) \\ &\Longrightarrow \\ p_1 &\approx .99 \end{aligned}$$

Now, recall the investment rent in state 1 is

$$\left(\beta - \frac{\kappa}{\rho^B+\delta(p_1)}\right)B_2 \approx \left(\frac{100}{101} - \frac{.3}{1+.3p_1+(1-p_1)100}\right)B_2$$

We have

$$\frac{d\left(\beta-\frac{\kappa}{\rho^B+\delta(p)}\right)B_2}{d\rho^B} = \frac{d\left(\beta-\frac{\kappa}{\rho^B+\delta(p)}\right)}{d\rho^B}B_2 + \left(\beta-\frac{\kappa}{\rho^B+\delta(p)}\right)\frac{d}{d\rho^B}B_2$$

Now,

$$\begin{aligned}
\frac{d}{d\rho^B}B_2 &= \frac{d\frac{.3}{\rho^B+.3}\frac{u}{\rho^B}}{d\rho^B} \\
&\approx -\frac{.3}{(\rho^B+.3)^2}\frac{u}{\rho^B}-\frac{.3}{\rho^B+.3}\frac{u}{(\rho^B)^2} \\
&\approx -\frac{.3}{(1.3)^2}2000-\frac{.3}{1.3}2000 \\
&\approx -816.56
\end{aligned}$$

And,

$$\begin{aligned}
\frac{d}{d\rho^B}\left(\frac{100}{\rho^B+100}-\frac{.3}{\rho^B+.3p+(1-p)\,100}\right) &\approx \left(-\frac{100}{(\rho^B+100)^2}+\frac{.3}{(\rho^B+.3p+(1-p)\,100)^2}\right) \\
&\approx .05
\end{aligned}$$

So,

$$\frac{d}{d\rho^B}\left[\left(\beta-\frac{\kappa}{\rho^B+\delta}\right)B_2\right]\approx -675.87<0$$

Therefore making the borrower more patient increases the investment rent and reduces p_1.

Proof of Proposition 10

Fix a game Γ with n states, and a cost of fixed investment $\{\phi_w\}$. Then for game Γ^m with $m>0$, a borrower investing in fixed capital rate i in state $2^m n$ will derive a state $2^m n$ continuation value of

$$\frac{\frac{i}{\phi_{2^m n}}}{\rho+\frac{i}{\phi_{2^m n}}}\frac{u}{\rho}$$

which converges to $\frac{u}{\rho}$ as m becomes large. If the borrower's equilibrium expectation is that the lender will offer the restrictive contract with probability 1, then the lender's continuation utility in state $2^m n$ from doing so is

$$L^R_{2^m n}(1)=\frac{q_{2^m n}\left(E_{2^m n}+T_{2^m n}\right)-T_{2^m n}}{\rho}-\frac{\kappa_{2^m n}}{\rho+\kappa_{2^m n}}\frac{u}{\rho}$$

which for sufficiently large m will be negative when it is socially efficient to invest.

On the other hand the lender's continuation utility in state n if he offers an unrestrictive contract in every period is

$$L^U_{2^m n} = \frac{q_{2^m n} T_{2^m n} - h - T_{2^m n}}{\rho} \left(1 - \frac{\gamma_{2^m n}}{\rho + \gamma_{2^m n}}\right)$$

which is positive for all $m > 0$. Thus for sufficiently high m, the lender will offer an unrestrictive contract with positive probability in state $2^m n$, completing the proof.

Proof of Proposition 11

For states $w > \bar{w}_{m-1}$, the proof closely follows that of Proposition 6. Specifically, for states $w > \bar{w}_m$, the proof follows that of Lemma 3. For states $w \in \{\underline{w}_m, \ldots, \bar{w}_m\}$ the proof follows that of Lemma 4. And for states $w \in \{\bar{w}_{m-1} + 1, \ldots, \underline{w}_m - 1\}$ the proof follows that of Lemma 6.

For $w \in \{\bar{w}_{m-2} + 1, \bar{w}_{m-1}\}$ the logic of Proposition 6 is reversed, as $\frac{dB_{\bar{w}_{m-1}+1}}{du} < 0$. That is, in the state directly beyond the pure restrictive state $\bar{w}_{m-1}$, the borrower's continuation utility is decreasing in u. In contrast, in the state directly beyond the pure restrictive state $\bar{w}_m$, the borrower's continuation utility is *increasing* in u. So, the comparative static for $w \in \{\bar{w}_{m-2} + 1, \bar{w}_{m-1}\}$ comes directly from reversing the signs in Lemmas 4 and 6.

The proof proceeds similarly for all consumption regions $\{\underline{w}_{\tilde{m}} \ldots, \bar{w}_{\tilde{m}}\}$ for $\tilde{m} \leq m - 2$.

Proof of Proposition 12

Lemma 8. *The borrower may reject a restrictive contract on the equilibrium path.*

Proof. To prove this lemma we need only find an example in which the borrower rejects a restrictive contract with positive probability. To do so we modify the example from the proof of Proposition 9. Specifically consider the one state example in which we take $E = .15$, $q = 2$, $\phi = \frac{1}{2}$, $h = 100$, $T = 1000$, $\frac{u}{\rho^B} = 2000$ and $\rho^B = \rho^L = 1$. We define the distribution G such that $\nu = 0$ with probability $1 - \varepsilon$, and $\nu = 45$ with probability ε.

We verified in the proof of Proposition 9 that this example satisfies Assumption 3 and that in equilibrium the lender offers the borrower a restrictive contract with probability 1. Clearly for sufficiently small ε, the lender would prefer to offer the least generous restrictive contract that borrowers of type $\nu = 0$ would accept. The loss the lender suffers from being rejected with probability ε is vanishing. In contrast, if the lender offers a contract that both types of borrowers would accept, he incurs a first order loss in order to compensate the high type borrower for the $\nu = 45$ additional forgone investment. □

Proof of Proposition 13

Define $\beta_s \equiv \frac{\gamma - \frac{s}{\phi}}{\rho + \gamma - \frac{s}{\phi}}$. Now suppose in state w the borrower anticipates a restrictive contract with probability 1. It is straightforward to show that the borrower's expansion rent when she can invest a fraction of her endowment s flexibly no matter the contractual restriction is $(\beta_s - \alpha) B_{w+1}$. Further as in Proposition 4, in equilibrium the borrower receives a restrictive contract with certainty in state w if and only if

$$(\beta_s - \alpha) B_{w+1} \geq \beta \left(B_{w+1} + L_{w+1} - \frac{q_w (E + T_w) - T_w}{\rho} - \phi \right).$$

For $s < \frac{E}{T_w + E}$, the borrower will grow more slowly in equilibrium than in autarky in any state in which the above condition is satisfied.

Table 1
Default is U-Shaped in Income in Field et. al. Data

	(1) Doesn't Complete Repayment	(2) Doesn't Complete Repayment
Log Profits	-0.153	-0.167*
	(0.0955)	(0.0945)
Log Proits Sq.	0.0108	0.0118*
	(0.00684)	(0.00672)
N	660	660
Controls	No	Yes

Notes: Robust standard errors in parentheses. All columns are borrower level OLS regressions. The outcome variable is whether the borrower has completed repayment a year after disbursal. Controls are those included in Field et. al. (2013) and include borrower education, household size, religion, literacy, marital status, age, household shocks, business ownership at baseline, financial control, home ownership, and whether the houshold has a drain. Log Profits are measured three years after loan disbursal—this is the publicly available measure.

* $p < 0.10$, ** $p < 0.05$, *** $p < 0.01$