Online Appendix for "Minimizing Justified Envy in School Choice: The Design of New Orleans' OneApp"

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S.1 Proof of Proposition 2

The example below shows that TTC-Counters, TTC-Clinch and Trade, and Equitable-TTC do not minimize justified envy in the class of Pareto efficient and strategy-proof mechanisms, when schools have multiple seats.

Consider TTC-Counters (as will become clear, the argument is similar for TTC-Clinch and Trade or Equitable-TTC). We build a Pareto efficient and strategy-proof mechanism φ that allows for strictly less justified envy than TTC-Counters. Suppose φ selects the same matching as TTC-Counters except for the following economy: there are four individuals i_1, i_2, i_3, and i_4 and three schools s_1, s_2, s_3, where $q_{s_1} = 1$ and $q_{s_2} = q_{s_3} = 3$. In this economy, φ selects a matching that is free of justified envy and Pareto efficient, which can be computed by student-proposing DA.

? characterizes priorities under which, for any individual preference, there exists a matching that is Pareto efficient and justified envy-free. Priorities must be such that there is no Ergin-cycle. A profile $\succ$ has an Ergin-cycle if there are three individuals i_1, i_2, and i_3 and two schools s_1 and s_2 such that the two conditions are satisfied:

1. *Cycle condition.* $i_1 \succ_{s_1} i_2 \succ_{s_1} i_3$ and $i_3 \succ_{s_2} i_1$,

2. *Scarcity condition.* There are (possibly empty) disjoint sets N_{s_1} and $N_{s_2} \subseteq I \backslash \{i_1, i_2, i_3\}$ s.t. $N_{s_1} \subseteq U_{s_1}(i_2)$ and $N_{s_2} \subseteq U_{s_2}(i_1)$ and $|N_{s_1}| = q_{s_1} - 1$ and $|N_{s_2}| = q_{s_2} - 1$ where $U_{s_1}(i_2)$ and $U_{s_2}(i_1)$ are the strict upper contour set of i_2 and i_1, respectively (i.e., $U_{s_1}(i_2) := \{\ell : \ell \succ_{s_1} i_2\}$ and $U_{s_2}(i_1) := \{\ell : \ell \succ_{s_2} i_1\}$).

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In the economy described above, the scarcity condition in the definition of an Ergin-cycle can never be satisfied. To see this, observe that for a school $s \in \{s_2, s_3\}$,

$$|N_s| = 3 - 1 = 2,$$

while $N_s \subseteq I \backslash \{i_1, i_2, i_3\}$ implies that

$$|N_s| \leq 1,$$

since $|I| = 4$. Therefore, sets satisfying the scarcity condition do not exist. Hence, any profile of priority relations is Ergin-acyclic.

Finally, it is enough for our purpose to build some $(P, \succ)$ where the set of blocking pairs of φ is a proper subset of the set of blocking pairs of TTC-Counters. Since φ eliminates justified envy, we only need to show that there is $(P, \succ)$ under which TTC-Counters does not eliminate justified envy.

Consider the following profile of preferences and priority relations:

P_{i_1}	P_{i_2}	P_{i_3}	P_{i_4}	$\succ_{s_1}$	$\succ_{s_2}$	$\succ_{s_3}$
s_3	s_1	s_1	s_2	i_1	i_1	i_3
	s_3			i_4	i_3	i_4
				i_2	i_2	i_2
				i_3	i_4	i_1

TTC-Counters produces:

$$\begin{pmatrix} i_1 & i_2 & i_3 & i_4 \\ s_3 & s_3 & s_1 & s_2 \end{pmatrix},$$

where (i_2, s_1) is a blocking pair.

This completes the argument for TTC-Counters. It is easy to check that TTC-Counters, TTC-Clinch and Trade, and Equitable TTC coincide to produce the same assignment for the above profile of preferences and priority relations. Hence, the same argument can be used for TTC-Clinch and Trade and Equitable TTC.

S.2 Another Justified Envy-Minimal Mechanism

When each school has a single seat, we build a mechanism different from TTC that is strategy-proof, Pareto-efficient, and justified envy-minimal. The mechanism is identical to TTC except at the following instance of priorities: we have three students i_1, i_2 and i_3 and two schools s_1 and s_2 each with a single seat. Priorities are given by

$\succ_{s_1}$	$\succ_{s_2}$
i_2	i_1
i_3	i_3
i_1	i_2

.

In essence, the mechanism will rank i_3 on top of each school's ranking and run standard TTC on these modified priorities (except for some preference profiles where there is a unique efficient and stable allocation where the original priorities will still be used to run TTC). This mechanism will be denoted TTC*. Let us describe it precisely. For the instance of priorities described above and for each profile of preferences P, TTC* selects a matching as follows.

<u>Case A.</u>
If under P there is an individual who ranks all schools as unacceptable then run TTC.

<u>Case B.1.</u>
If under P both i_1 and i_2 rank s_1 first and i_1 finds s_2 unacceptable then run TTC

<u>Case B.2.</u>
If under P both i_1 and i_2 rank s_2 first and i_2 finds s_1 unacceptable then run TTC

<u>Case C.</u>
If none of the above cases apply, move i_3 to the top of each school's ranking. Run TTC on the modified priorities.

Clearly, TTC* is Pareto efficient. We prove below that it is strategy-proof.

Proposition S1. *TTC* is strategy-proof.*

Proof. Fix P falling into case A. If some student i deviates to P'_i, this cannot be profitable if we remain into case A or fall into Case B.1 or B.2 (since TTC is strategy-proof). Therefore, consider the case where we fall into Case C after i's deviation. After the deviation, all individuals rank at least one school acceptable (since we are not in Case A anymore). Since at least one individual must rank all schools unacceptable before the deviation and since we are looking at a single deviation by individual i, we conclude that P_i ranks all schools unacceptable. Hence, under TTC*, i is unmatched under P and since P_i ranks all schools unacceptable, there cannot be any profitable deviation.

Fix P falling into case B.1 (and not in case A). If some student i deviates to P'_i, this cannot be profitable if we remain into case B.1 or fall into Case A or B.2 (since TTC is strategy-proof). Therefore, consider the case where we fall into Case C after i's deviation. This must mean that before deviation both i_1 and i_2 rank s_1 first and i_1 finds s_2 unacceptable, though this is not the case anymore after deviation. Note that this must mean that i is either i_1 or i_2. Further, since P falls into Case B.1, TTC* runs standard TTC. Hence, i_2 gets matched to her top choice s_1 and so i_2 has no incentive to deviate (recall that each individual finds at least one school acceptable since we are not in Case A). Hence, let us consider $i = i_1$. The only way to reach (by a deviation of i_1) Case C is for i_1 to claim that s_2 is acceptable (while s_2 is not acceptable to i_1 under the original preferences P_i). Now, to complete the argument, we distinguish two cases. First, assume that i_3 ranks s_2 first. Then, since we fall into Case C after deviation, $\text{TTC}^*(P'_i, P_{-i})$ is given by

$$\begin{pmatrix} i_1 & i_2 & i_3 \\ i_1 & s_1 & s_2 \end{pmatrix}.$$

In particular, i_1 cannot get s_1 (the only acceptable school under P_i) so the deviation to P_i' cannot be profitable. Similarly, in the other case where i_3 ranks s_1 first, $\text{TTC}^*(P_i', P_{-i})$ is given by

$$\begin{pmatrix} i_1 & i_2 & i_3 \\ s_2 & i_2 & s_1 \end{pmatrix}.$$

Here again, i_1 fails to obtain s_1 and so the deviation is not profitable. The case under which P falls into Case B.2 can be treated in the same way.

Fix P falling into case C. If some student i deviates to P_i', this cannot be profitable if we remain into case C (since TTC is strategy-proof). Therefore, consider the case where we fall into Case A. This must mean that P_i' ranks all schools unacceptable. Since TTC* is individually rational, i's deviation cannot be profitable. So consider the case where after deviation we fall into Case B.1 (and not into Case A). This must mean that before deviation, either i_1 or i_2 does not rank s_1 first, or i_1 finds s_2 acceptable while after deviation both i_1 and i_2 rank s_1 first and i_1 finds s_2 unacceptable. This must mean that i is either i_1 or i_2. If $i = i_1$, this means that, i_2 ranks s_1 first (recall that before and after deviation, each individual has at least one acceptable school since we do not fall into Case A before and after deviation). So, in particular, after deviation, i_1 cannot get s_1 (s_1 will be allocated to i_2). Since after deviation, i_1 finds s_2 unacceptable, i_1 will not get s_2 either, and so she will remain unmatched. So the deviation cannot be profitable to i_1.

Now, consider the other case where deviator $i = i_2$. This means that i_1 ranks s_1 first and ranks s_2 unacceptable. This also means that before deviation, i_2 ranks s_2 first while after deviation i_2 ranks s_1 first. To complete the proof, we distinguish two cases. First, assume that i_3 ranks s_2 first. Then, at P, i_2 gets s_1 if she finds s_1 acceptable or remains unmatched. After deviation, i_2 ranks s_1 on top and i_2 must be getting s_1 after deviation, so this cannot be profitable. In the other case where i_3 ranks s_1 first, before deviation, i_2 is getting s_2, which is her top choice. So the deviation cannot improve on this. The same reasoning holds if the deviation falls into Case B.2. ∎

We fix any Pareto efficient and strategy-proof mechanism φ with less justified envy than TTC*. We claim that $\varphi = \text{TTC}^*$.

Proposition S2. *Fix any P that falls into Case A, B.1 or B.2. $\varphi(P) = TTC^*(P)$.*

Proof. Fix any P falling into Case A. Some individual must rank all schools as unacceptable. It is easy to check that, in such a case, there is a unique efficient and stable allocation that is selected by TTC (with only two students, priorities are trivially Ergin-acyclic). Hence, because

φ has less justified envy than TTC*, φ must also select the unique efficient and stable allocation, and we obtain $\varphi(P) = \text{TTC}^*(P)$.

Now, fix any P falling into Case B.1. Both i_1 and i_2 rank s_1 first, and i_1 finds s_2 unacceptable. Here again, one can check that TTC selects the unique efficient and stable allocation, and we obtain $\varphi(P) = \text{TTC}^*(P)$. A similar reasoning holds for any P falling into Case B.2. ∎

Proposition S3. *Fix any P that falls into Case C. $\varphi(P) = TTC^*(P)$.*

Proof. We assume that P falls into Case C and prove the above proposition in the four following claims.

Claim 1. Assume that $s_1 P_{i_1} s_2$ and $s_2 P_{i_2} s_1$.

$$\varphi(P) = TTC^*(P).$$

Proof. Clearly, under TTC*, i_3 is never part of any blocking pair. Hence, because φ has less justified envy than TTC*, we must have that i_3 is never part of any blocking pair under φ as well. Assume wlog that s_1 is i_3's top choice (recall that because P falls into Case C, each individual finds at least one school acceptable).

In the sequel, we claim that i_3 is assigned its top choice s_1 under matching $\varphi(P)$. If i_3 is not assigned its top choice s_1 under φ, then in order to ensure that (i_3, s_1) does not block $\varphi(P)$, we must have that i_2 is matched to s_1 under $\varphi(P)$. Now, consider two cases. First, i_3 is matched to s_2 under $\varphi(P)$. In that case, i_2 and i_3 would be better off switching their assignments, a contradiction with Pareto efficiency of φ. In the other case, i_3 must be unmatched under $\varphi(P)$. If i_1 gets matched to s_2 under $\varphi(P)$, allowing i_2 and i_1 to switch their assignments would be beneficial to both of them, again a contradiction with Pareto efficiency of φ. If i_1 is not matched to s_2 under $\varphi(P)$ then s_2 is unmatched, and by assigning it to i_2 we Pareto-improve on $\varphi(P)$, a contradiction.

Thus, we proved that $\varphi(P)(i_3) = s_1 = \text{TTC}^*(P)(i_3)$. Now, let us complete the argument and show that $\varphi(P) = \text{TTC}^*(P)$. First, consider the case where i_1 finds s_2 acceptable. TTC* yields the following matching

$$\begin{pmatrix} i_1 & i_2 & i_3 \\ s_2 & i_2 & s_1 \end{pmatrix}.$$

If under $\varphi(P)$, i_1 remains unmatched, then (i_1, s_2) would block $\varphi(P)$ while it does not block $\text{TTC}^*(P)$, a contradiction with our assumption that φ has less justified envy than TTC*. Thus, $\varphi(P)(i_1) = s_2 = \text{TTC}^*(P)(i_1)$ and so we conclude that $\varphi(P) = \text{TTC}^*(P)$. Now, consider the second case where i_1 finds s_2 unacceptable. Recall that s_2 must be acceptable to i_2, and so TTC* yields the following matching

$$\begin{pmatrix} i_1 & i_2 & i_3 \\ i_1 & s_2 & s_1 \end{pmatrix}.$$

Clearly, since we showed that i_3 gets matched to s_1 under $\varphi(P)$, i_1 remains unmatched under $\varphi(P)$. So by Pareto efficiency of φ, it must be that $\varphi(P)(i_2) = s_2 = \text{TTC}^*(P)(i_2)$. We conclude that $\varphi(P) = \text{TTC}^*(P)$. $\square$

Claim 2. Assume that $s_1 P_{i_1} s_2$ and $s_1 P_{i_2} s_2$.

$$\varphi(P) = TTC^*(P).$$

Proof. There are two cases.

Case 1. $s_1 P_{i_3} s_2$. Because P falls into Case C, each individual ranks at least one school acceptable (since P does not fall into Case A) and i_1 finds s_2 acceptable (since P does not fall into Case B.1). Thus, TTC* yields the following matching

$$\begin{pmatrix} i_1 & i_2 & i_3 \\ s_2 & i_2 & s_1 \end{pmatrix}.$$

We first claim that under $\varphi(P)$, i_2 must remain unmatched. Indeed, if i_2 is matched under $\varphi(P)$, then consider the new preference profile where $s_2 P'_{i_2} s_1 P'_{i_2} i_2$. Note that $(P'_{i_2}, P_{-\{i_2\}})$ falls into the cases considered in Claim 1. Hence, by Claim 1, we know that $\varphi(P'_{i_2}, P_{-\{i_2\}})(i_3) = \text{TTC}^*(P'_{i_2}, P_{-\{i_2\}})(i_3) = s_1$ and $\varphi(P'_{i_2}, P_{-\{i_2\}})(i_2) = \text{TTC}^*(P'_{i_2}, P_{-\{i_2\}})(i_2) = i_2$. Thus, from profile $(P'_{i_2}, P_{-\{i_2\}})$, i_2 can misreport her preference profile as P_{i_2}. In turn, she gets matched and is strictly better-off, which contradicts the strategy-proofness of φ. Hence, under $\varphi(P)$, i_2 must be unmatched. Next, we claim that i_3 is assigned s_1 under $\varphi(P)$. Indeed, if i_3 is not assigned s_1 under $\varphi(P)$, then i_1 must be assigned s_1 since it is acceptable to her (and we already know that i_2 must be unmatched). But then (i_3, s_1) would block $\varphi(P)$ but does not block $\text{TTC}^*(P)$, which contradicts our assumption that φ has less justified envy than TTC*. To conclude, under $\varphi(P)$, i_3 gets s_1, i_2 is unmatched, and so, since s_2 is acceptable to i_1, i_1 gets matched to s_2. Thus, $\varphi(P) = \text{TTC}^*(P)$.

Case 2. $s_2 P_{i_3} s_1$. TTC* yields the following matching

$$\begin{pmatrix} i_1 & i_2 & i_3 \\ i_1 & s_1 & s_2 \end{pmatrix}.$$

We first claim that $\varphi(P)$ matches i_3 to her top choice s_2. Indeed, if i_3 is not matched to s_2 under $\varphi(P)$ then in order for (i_3, s_2) not to block $\varphi(P)$, i_1 must match to s_2. But then, in order for (i_2, s_1) not to block $\varphi(P)$, which is necessary since it does not block $\text{TTC}^*(P)$, i_2 must also match s_1. So if i_3 is not matched to s_2 under $\varphi(P)$ the only candidate for $\varphi(P)$ is

$$\begin{pmatrix} i_1 & i_2 & i_3 \\ s_2 & s_1 & i_3 \end{pmatrix}.$$

Now, let us assume that i_3 ranks s_1 acceptable under P_{i_3}. Next, consider the preference profile $(P'_{i_3}, P_{-\{i_3\}})$ where $s_1 P'_{i_3} s_2 P'_{i_3} i_3$. Note that $(P'_{i_3}, P_{-\{i_3\}})$ falls into Case 1 considered just above. Hence, we know that $\varphi(P'_{i_3}, P_{-\{i_3\}}) = \text{TTC}^*(P'_{i_3}, P_{-\{i_3\}})$ and so i_3 is matched to s_1 under $\varphi(P'_{i_3}, P_{-\{i_3\}})$. Since i_3 is unmatched under $\varphi(P)$, because we assumed that i_3 ranks s_1 acceptable under P_{i_3}, we found a profitable deviation for i_3, a contradiction with the strategy-proofness of φ. Thus, provided that P_{i_3} ranks s_1 as acceptable, we obtained $\varphi(P_{i_3}, P_{-\{i_3\}})(i_3) = \text{TTC}^*(P_{i_3}, P_{-\{i_3\}})(i_3) = s_2$.

Let us now assume that i_3 ranks s_1 unacceptable under P_{i_3}. Consider a deviation of i_3 to P'_{i_3} satisfying $s_2 P'_{i_3} s_1 P'_{i_3} i_3$, i.e., where s_1 is ranked as acceptable. We just saw that, in such a case, $\varphi(P'_{i_3}, P_{-\{i_3\}})(i_3) = \text{TTC}^*(P'_{i_3}, P_{-\{i_3\}})(i_3) = s_2$ and so i_3 gets matched to s_2 under $\varphi(P'_{i_3}, P_{-\{i_3\}})$. Here again, we find a profitable deviation for i_3, which contradicts the strategy-proofness of φ.

We conclude that $\varphi(P)$ matches i_3 to her top choice s_2. Now, $\text{TTC}^*(P)$ matches i_2 with s_1, and, in order not have the blocking pair (i_2, s_1) under $\varphi(P)$, i_2 and s_1 must also be matched together under $\varphi(P)$. We conclude that $\varphi(P) = \text{TTC}^*(P)$. □

Claim 3. Assume that $s_2 P_{i_1} s_1$ and $s_2 P_{i_2} s_1$.

$$\varphi(P) = TTC^*(P).$$

Proof. The proof is similar to that of Claim 2. □

Claim 4. Assume that $s_2 P_{i_1} s_1$ and $s_1 P_{i_2} s_2$.

$$\varphi(P) = TTC^*(P).$$

Proof. Without loss of generality, assume that $s_2 P_{i_3} s_1$ (the same argument applies when $s_1 P_{i_3} s_2$). TTC^* yields the following matching

$$\begin{pmatrix} i_1 & i_2 & i_3 \\ i_1 & s_1 & s_2 \end{pmatrix}.$$

If, under $\varphi(P)$, i_3 is not matched to her top choice s_2, then in order for (i_3, s_2) not to block $\varphi(P)$ (which is necessary, since it does not block $\text{TTC}^*(P)$), i_1 must be matched to s_2 under $\varphi(P)$. But then for (i_2, s_1) not to block $\varphi(P)$ (which is necessary since it does not block $\text{TTC}^*(P)$), i_2 must be matched to s_1. Thus, if i_3 is not matched to her top choice s_2, the only candidate for $\varphi(P)$ is

$$\begin{pmatrix} i_1 & i_2 & i_3 \\ s_2 & s_1 & i_3 \end{pmatrix}.$$

Now, consider P'_{i_1} such that $s_1 P'_{i_1} s_2 P'_{i_1} i_1$. Since i_1 ranks s_2 acceptable under P'_{i_1}, $(P'_{i_1}, P_{-\{i_1\}})$ falls in to the profile of preferences considered in Claim 2. Hence, $\varphi(P'_{i_1}, P_{-\{i_1\}})$ and $\text{TTC}^*(P'_{i_1}, P_{-\{i_1\}})$ both yield the same matching given by

$$\begin{pmatrix} i_1 & i_2 & i_3 \\ i_1 & s_1 & s_2 \end{pmatrix}.$$

Now, if the true preference profile is $(P'_{i_1}, P_{-\{i_1\}})$ and i_1 misreports to P_{i_1}, then i_1 gets matched to s_2 under $\varphi(P_{i_1}, P_{-\{i_1\}})$. Hence, the misreport P_{i_1} is profitable to i_1, which contradicts the strategy-proofness of φ.

We conclude that under $\varphi(P)$, i_3 must be matched to her top choice s_2. But now, if i_2 is not matched to s_1 under $\varphi(P)$ then (i_2, s_1) blocks $\varphi(P)$ but does not block $\text{TTC}^*(P)$, which is a contradiction. Hence, i_2 must be matched to s_1, and we conclude that $\varphi(P) = \text{TTC}^*(P)$. □

These four claims together establish the proposition. ∎