Simultaneous Inference for Best Linear Predictor of the Conditional Average Treatment Effect and Other Structural Functions

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Abstract

We propose estimation and inference methods for the best linear approximation to conditional average treatment effect, conditional average partial derivative, and other structural functions where the selection among a large number of potential controls is based on machine learning tools. We approximate the structural function using a linear form of the basis functions (e.g., polynomial series) and characterize the least squares parameter as the solution to a locally robust semiparametric moment equation. We propose an estimator for the best linear approximation that features (a) pointwise Gaussian approximation, (b) simultaneous Gaussian approximation and (c) the optimal rate of simultaneous convergence. Using the data from using Yatchew and No (2001), we estimate and provide uniform confidence bands for the average price elasticity conditional on income.

Keywords: heterogeneous treatment effect, orthogonality, machine learning

JEL Codes: C14, C55

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1 Introduction and Motivation

Economists are often interested in the conditional average treatment effect (CATE) given a small set of policy-relevant covariates. However, to ensure that the treatment-control comparison reflects the causal effect (i.e., the unconfoundedness assumption holds), they must account for other controls that could impact the outcome. When the number of potential controls exceeds the sample size, data-driven control selection is difficult. Indeed, the solutions offered by current literature have either restricted the number of controls to be smaller than sample size (Abrevaya et al. (2015), Athey and Imbens (2015), Wager and Athey (2016)) or ignored rich heterogeneity (Belloni et al. (2014), Belloni et al. (2016), Chernozhukov et al. (2016a)).

The main contribution of this paper is the method for estimation and simultaneous inference about structural functions (e.g., CATE) when the number of controls is larger than sample size. We use modern machine learning techniques to select the controls. Using our method, economists can test homogeneity of treatment effect, monotonicity, and other hypotheses involving the whole function, the tasks which are beyond the scope of earlier pointwise asymptotic theory with data-driven covariate selection (Athey and Imbens (2015), Wager and Athey (2016), Oprescu et al. (2019)). Moreover, most pointwise theory can accommodate a small number of controls. In contrast, this paper incorporates a wide class of machine learning techniques, in particular regularized methods, and allows the number of controls to exceed the sample size.

This paper focuses on the conditional expectation function of a random variable, which is allowed to depend on an unknown nuisance parameter and thus may not be observed. For example, when we are interested in CATE, this variable can be taken to the difference between the outcomes in the treatment and controls groups, weighted by the propensity score of each group. Since the propensity score is an unknown function of a large number of controls, we use modern regularized techniques (e.g., lasso, ridge) to obtain its consistent estimate. Unfortunately, regularization bias of the propensity score converges slower than parametric root-$N$ rate. Therefore, plugging regularized estimate of the propensity score into the random variable produces a poor estimate of the target function.

To overcome the transmission of the first-stage bias into the second stage, we use two ideas: local robustness (a.k.a. Neyman-orthogonality) and sample splitting. A two-stage method is locally robust if the second stage is insensitive with respect to the biased estimation of the first stage (i.e., Neyman (1959), Newey (1994)). Combining local robustness and sample splitting, Chernozhukov et al. (2016a) and Chernozhukov
et al. (2016b) derive a root-$N$ consistent and asymptotically normal estimator of a fixed-dimen-
sional parameter identified by a semiparametric moment equation. To apply this idea for the estima-
tion of a structural function, we focus on the best linear approximation of this function using series expansion. We derive locally robust moment equation for the parametric component of this approximation. Then, we extend the existing asymptotic theory in two directions: (1) we allow the dimension of the parametric component to grow with sample size and (2) we approximate the target function by a Gaussian process. We apply our result to three empirically relevant cases: Conditional Average Treatment Effect, Conditional Average Partial Derivative, and Regression Function with Partially Missing Outcome.

As an empirical application, we revisit the gasoline demand analysis of Yatchew and No (2001) and estimate the price elasticity conditional on income. We find the price elasticity to be in $(-1, 0)$ for majority of income levels. The estimates based on $B$-splines are monotonically increasing in income, which is intuitive. Overall, we cannot reject the null hypothesis of constant price elasticity.

1.1 Literature Review

This paper builds on the three bodies of research within the semiparametric literature: orthogonal (debiased) machine learning, least squares series estimation, and treatment effects/missing data problems. Orthogonal machine learning (Chernozhukov et al. (2016a), Chernozhukov et al. (2016b)) concerns with the inference on a fixed-dimensional target parameter $\beta$ in presence of a high-dimensional nuisance function $\eta$ in a semiparametric moment problem. In case the moment condition is locally robust to the perturbations of $\eta$, the estimation of $\eta$ by ML methods has no first-order effect on the asymptotic distribution of the target parameter $\beta$. In particular, plugging in an estimate of $\eta$ obtained on a separate sample, results in a $\sqrt{N}$-consistent asymptotically normal estimate whose asymptotic variance is the same as if $\eta = \eta_0$ was known. This result allows one to use highly complex machine learning methods to estimate the nuisance function $\eta$, such as $l_1$ penalized methods in sparse models (Bühlmann and van der Geer (2011), Belloni et al. (2016)), $L_2$ boosting in sparse linear models (Luo and Spindler (2016)), and other methods for classes of neural nets, regression trees, and random forests.

The second building block of our paper is the literature on least squares series estimation (Newey (2007), Newey (2009), Belloni et al. (2015), Chen and Christensen (2015)), which establishes the pointwise and the uniform limit theory for least squares series estimation. We extend this theory by allowing the dependent variable of the series projection to depend upon an unknown nuisance parameter $\eta$. We show that series
properties continue to hold without any additional strong assumptions on the problem design.

The third relevant body of literature is the literature on missing data and conditional average treatment effect (Robins and Rotnitzky (1995), Hahn (1998), Graham (2011), Graham et al. (2012), Hirano et al. (2003), Abrevaya et al. (2015), Athey and Imbens (2015), Grimmer et al. (2017), Oprescu et al. (2019), etc.). In particular, we use the doubly robust scores Robins and Rotnitzky (1995) for the treatment effect and missing data problems to conjecture the locally robust condition for the target function.

2 Setup

We are interested in a conditional expectation function \( g(x) \) of a random variable \( Y \) given a variable \( X \):

\[
g(x) = E[Y|X = x],
\]

where \( Y = Y_\eta \) is allowed to depend on an identified nuisance parameter \( \eta \) and thus is unobserved. Examples of such functions include Conditional Average Treatment Effect (CATE), regression function with Partially Missing Outcome, and Conditional Average Partial Derivative (CAPD). As discussed in the introduction, the major challenge is to overcome the transmission of the biased estimation of the nuisance parameter \( \eta \) into the estimation of \( g(x) \).

To overcome this challenge, we propose a new condition - conditional local robustness - and show that it holds in empirically relevant cases. A random variable \( Y = Y_\eta \) is locally robust conditionally on \( X \) if the pathwise derivative of its conditional on \( X \) expectation with respect to the nuisance parameter \( \eta \) is equal to zero:

\[
\partial_{\eta}E[Y_{\eta_0}|X = x][\eta - \eta_0] := \partial_{\eta}E[Y_{\eta_0 + r(\eta - \eta_0)}|X = x] = 0 \quad \forall x \in \mathcal{X}.
\]

If \( Y_\eta \) is conditionally locally robust, its plug-in estimate \( Y_{\eta}^{\hat{\eta}} \) is insensitive to the biased estimation of \( \hat{\eta} \) and delivers a high-quality estimator of the target function \( g(x) \) under mild conditions. Examples below demonstrate the plausibility of this condition.

2.1 Examples

Examples below apply the proposed framework to study Conditional Average Treatment Effect, regression function with Partially Missing Outcome and Conditional Average Partial Derivative.
Example 1 (Conditional Average Treatment Effect). Let $Y_1$ and $Y_0$ be the potential outcomes, corresponding to the response of a subject with and without receiving a treatment, respectively. Let $D \in \{1, 0\}$ indicate the subject’s presence in the treatment group. The object of interest is the Conditional Average Treatment Effect

$$g(x) := \mathbb{E}[Y_1 - Y_0 | X = x].$$

Since an individual cannot be treated and non-treated at the same time, only the actual outcome $Y^o = DY_1 + (1 - D)Y_0$, but not the treatment effect $Y_1 - Y_0$, is observed.

A standard way to make progress in this problem is to assume unconfoundedness (Rosenbaum and Rubin (1983)). Suppose there exists an observable control vector $Z$ such that the treatment status $D$ is independent of the potential outcomes $Y_1, Y_0$ conditionally on $Z$, and $X \subseteq Z$.

Assumption 2.1 (Unconfoundedness). The treatment status $D$ is independent of the potential outcomes $Y_1, Y_0$ conditionally on $Z$: $\{Y_1, Y_0\} \perp D | Z$.

Define the conditional probability of treatment receipt as $s_0(Z) = \mathbb{E}[D = 1 | Z]$.

Consider a conditionally locally robust r.v. $Y$ of Robins and Rotnitzky (1995) type:

$$Y := \mu_0(1, Z) - \mu_0(0, Z) + \frac{D[Y^o - \mu_0(1, Z)]}{s_0(Z)} - \frac{(1 - D)[Y^o - \mu_0(0, Z)]}{1 - s_0(Z)}, \quad (2.1)$$

where $\mu_0(D, Z) = \mathbb{E}[Y^o | D, Z]$ is the conditional expectation function of $Y^o$ given $D, Z$. Corollary 4.1 shows that (2.1) is conditionally locally robust to the estimation error of the nuisance parameter $\eta(Z) := (s(Z), \mu(1, Z), \mu(0, Z))$.

Example 2 (Regression Function with Partially Missing Outcome). Suppose a researcher is interested in the conditional expectation given a covariate vector $X$ of a variable $Y^*$

$$g(x) := \mathbb{E}[Y^* | X = x]$$

that is partially missing. Let $D \in \{1, 0\}$ indicate whether the outcome $Y^*$ is observed, and $Y^o = DY^*$ be the observed outcome. Since the researcher does not control the presence status $D$, a standard way to make progress is to assume existence of an observable control vector $Z$ such that $Y \perp D | Z$.

Assumption 2.2 (Missingness at Random). The presence indicator $D$ is independent from the outcome $Y$ conditionally on $Z$: $Y \perp D | Z$.

Corollary 4.2 shows that the random variable $Y$, defined as
\[ Y = \mu_0(Z) + \frac{D[Y^o - \mu_0(Z)]}{s_0(Z)}, \] (2.2)

is conditionally locally robust to the estimation error of the nuisance parameter \( \eta(Z) := (s(Z), \mu(1, Z), \mu(0, Z)) \). Here, the function \( \mu_0(Z) = \mathbb{E}[Y^o|Z] = \mathbb{E}[Y|Z, D = 1] \) is the conditional expectation function of the observed outcome \( Y^o \) given \( Z \) and \( D = 1 \).

**Example 3** (Experiment with Partially Missing Outcome). Let \( Y^1 \) and \( Y^0 \) be the potential outcomes, corresponding to the response of a subject with and without receiving a treatment, respectively. Suppose a researcher is interested in the Conditional Average Treatment Effect

\[ g(x) := \mathbb{E}[Y^1 - Y^0|X = x]. \]

Conditionally on a vector of stratifying variables \( Z_D \), he randomly assigns the treatment status \( T \in \{1, 0\} \) to measure the outcome \( TY^1 + (1 - T)Y^0 \). In presence of the partially missing outcome, let \( D \in \{1, 0\} \) indicate the presence of the outcome record \( Y^o = D(TY^1 + (1 - T)Y^0) \) in the data. Since the presence indicator \( D \) may be co-determined with the covariates \( X \), estimating the treatment effect function \( g(x) \) on the observed outcomes only without accounting for the missingness may lead to an inconsistent estimate of \( g(x) \).

Since the researcher does not control presence status \( D \), a standard way to make progress is to assume Missingness at Random, namely existence of an observable control vector \( Z \) such that \( Y \perp D|Z \). Setting \( Z \) to be a full vector of observables (in particular, \( X \subseteq Z \)) makes Missingness at Random (Assumption 2.2) the least restrictive. Define the conditional probability of presence

\[ s_0(Z, T) = \mathbb{E}[D = 1|Y^o, Z, T] = \mathbb{E}[D = 1|Z, T] \]

and the treatment propensity score

\[ h_0(Z) := \mathbb{E}[T = 1|Z]. \]

A conditionally locally robust random variable \( Y \) for the CATE \( g(x) \) can be obtained as follows:

\[ Y = \mu_0(1, Z) - \mu_0(0, Z) + \frac{DT[Y^o - \mu_0(1, Z)]}{s_0(Z, T)h_0(Z)} - \frac{D(1 - T)[Y^o - \mu_0(0, Z)]}{s_0(Z, T)(1 - h_0(Z))}, \] (2.3)

where \( \mu_0(T, Z) = \mathbb{E}[Y^o|T, Z] \) is the conditional expectation function of \( Y^o \) given \( T, Z \).
Example 4 (Conditional Average Partial Derivative). Let
\[ \mu(x, w) := \mathbb{E}[Y^o | X = x, W = w] \]
be a conditional expectation function of an outcome \( Y^o \) given a set of variables \( X, W \). Suppose a researcher is interested in the conditional average derivative of \( \mu(x, w) \) with respect to \( w \) given \( X = x \), denoted by
\[ Y^* := \partial_w \mu(X, w) |_{w=W}. \]
Conditionally locally robust random variable \( Y \) of Newey and Stoker (1993) type takes the form
\[ Y := -\partial_w \log f(W|X)[Y^o - \mu(X, W)] + \partial_w \mu(X, W), \]
where \( f(W|X = x) \) is the conditional density of \( W \) conditionally on \( X = x \). The nuisance parameter \( \eta = \{\mu(X, W), f(W|X)\} \) consists of the conditional expectation function \( \mu(X, W) \) and the conditional density \( f(W|X) \). Corollary 4.3 shows that the \( Y \) is conditionally locally robust to the estimation error of the nuisance parameter \( \eta(X, W) = \{\mu(X, W), f(W|X)\} \).

Assuming \( Y \) is conditionally locally robust, we approximate the target function \( g(x) \) at a point \( x \) by a linear form \( p(x)' \beta_0 \):
\[ g(x) = p(x)' \beta_0 + r_g(x), \]
where \( p(x) \) is a \( d \)-vector of technical transformations of the covariates \( x \), \( r_g(x) \) is the misspecification error due to the linear approximation\(^1\), and \( \beta_0 \) is the Best Linear Predictor, defined by the balancing equation:
\[ \mathbb{E}p(X)[g(X) - p(X)' \beta_0] = \mathbb{E}p(X)r_g(X) = 0. \]

The two-stage estimator \( \hat{\beta} \) of Best Linear Predictor \( \beta_0 \), which we refer to as Locally Robust estimator, is constructed as follows. In the first stage we construct an estimate \( \hat{\eta} \) of the nuisance parameter \( \eta_0 \). In the second stage we construct an estimate \( \hat{Y}_i \) of r.v. \( Y_i \) as \( \hat{Y}_i := Y_i(\hat{\eta}) \) and run ordinary least squares of \( \hat{Y}_i \) on the technical regressors \( p(X_i) \). We use different samples for the estimation of \( \eta \) in the first stage and the estimation of \( \beta_0 \) in the second stage in a form of cross-fitting, described in the following definition.

\(^1\)Our analysis allows for either vanishing and non-vanishing specification error \( r_g(x) \).
Definition 2.1 (Cross-fitting). 1. For a random sample of size $N$, denote a $K$-fold random partition of the sample indices $[N] = \{1, 2, \ldots, N\}$ by $(J_k)_{k=1}^K$, where $K$ is the number of partitions and sample size of each fold is $n = N/K$. Also for each $k \in [K] = \{1, 2, \ldots, K\}$ define $J_k^c = \{1, 2, \ldots, N\} \setminus J_k$.

2. For each $k \in [K]$, construct an estimator $\hat{\eta}_k = \hat{\eta}(V_i \in J_k)^2$ of the nuisance parameter value $\eta_0$ using only the data from $J_k^c$. For any observation $i \in J_k$, define the estimate $\hat{Y}_i := Y_i(\hat{\eta}_k)$.

Definition 2.2 (Locally Robust Estimator). Given the estimates $(\hat{Y}_i)_{i=1}^N$, define Locally Robust Estimator as:

$$\hat{\beta} := \left(\frac{1}{N} \sum_{i=1}^N p(X_i)p(X_i)'\right)^{-1} \frac{1}{N} \sum_{i=1}^N p(X_i)\hat{Y}_i. \quad (2.4)$$

Under the mild conditions on $\eta$, Locally Robust delivers a high-quality estimate $p(x)\hat{\beta}$ of the pseudo-target function $p(x)'\beta_0$ with the following properties:

- W.p. $\to 1$, the mean squared error of $p(x)\hat{\beta}$ is bounded by

$$\left(\mathbb{E}_N(p(X_i)'(\hat{\beta} - \beta_0))^2\right)^{1/2} = O_P\left(\sqrt{\frac{d}{N}}\right) + z_d,$$

where $z_d$ is the effect of the misspecification error $r_g(x)$.

- The estimator $p(x)\hat{\beta}$ of the pseudo-target function $p(x)'\beta_0$ is asymptotically linear:

$$\sqrt{N} \frac{p(x)'(\hat{\beta} - \beta_0)}{\sqrt{p(x)\Omega p(x)}} = G_N(x) + o_P(1),$$

where the empirical process $G_N(x)$ converges to a tight Gaussian process with marginal distribution $N(0,1)$ uniformly over $x \in \mathcal{X}$ and the covariance matrix $\Omega$ can be consistently estimated by a sample analog $\hat{\Omega}$.

- In the case the misspecification error $r_g(x)$ is small, the pseudo-target function $p(x)'\beta_0$ can be replaced by the target function $g(x)$:

$$\sqrt{N} \frac{p(x)'\hat{\beta} - g(x)}{\sqrt{p(x)\Omega p(x)}} = G_N(x) + o_P(1).$$

\footnote{The results of this paper hold without relying on the specific choice of the first stage estimator $\hat{\eta}(Z)$. For the possible ways to estimate $\eta_0$, see the discussion.}
The results of this paper accommodate the estimation of $\hat{\eta}$ by high-dimensional/highly complex modern machine learning (ML) methods, such as random forests, neural networks, and $l_1$-shrinkage estimators, as well as earlier developed tools. The only requirement we impose on the estimation of $\hat{\eta}$ is its mean square convergence to the true nuisance parameter $\eta_0$ at a high-quality rate $o_P(N^{-1/4-\delta}/\sup_{x\in\mathcal{X}}\|p(x)\|^{1/2}), \delta > 0$. Under suitably chosen growth rate of the dimension of the basis functions $d = d(N)$, this requirement is satisfied under structured assumptions on $\eta_0$, such as approximate sparsity of $\eta_0$ with respect to some dictionary, well-approximability of $\eta_0$ by trees or by sparse neural and deep neural nets.

3 Asymptotic Theory

We shall use empirical process notation. For a generic function $f$ and a generic sample $(X_i)_{i=1}^N$, denote a sample average by

$$\mathbb{E}_N f(X_i) := \frac{1}{N} \sum_{i=1}^N f(X_i)$$

and a $\sqrt{N}$-scaled, demeaned sample average by

$$\mathbb{G}_N f(X_i) := \frac{1}{\sqrt{N}} \sum_{i=1}^N (f(X_i) - \mathbb{E} f(X_i)).$$

All asymptotic statements below are with respect to $N \to \infty$.

Assumption 3.1 (Identification). Let $Q := \mathbb{E} p(X)p(X)' = Q_d$ denote population covariance matrix of technical regressors. Assume that $\exists 0 < C_{\min} < C_{\max} < \infty$ that do not depend on $d$ s.t. $C_{\min} < \min \text{eig}(Q) < \max \text{eig}(Q) < C_{\max} \ \forall d$.

Assumption 3.1 requires that the regressors $p(X)$ are not too collinear in population, which allows identification of the best linear predictor $\beta_0$.

Assumption 3.2 (Growth Condition). We assume that the sup-norm of the technical regressors $\xi_d^2 := \sup_{x \in \mathcal{X}} \|p(x)\| = \sup_{x \in \mathcal{X}} (\sum_{j=1}^d p_j(x)^2)^{1/2}$ grows sufficiently slow:

$$\sqrt{\frac{\xi_d^2 \log N}{N}} = o(1).$$
Assumption 3.3 (Misspecification Error). There exists a sequence of finite constants \( l_d, r_d, d \to \infty \) such that the norms of the misspecification error are controlled as follows:

\[
\| r_g \|_{F,2} := \sqrt{\int r_g(x)^2 dF(x)} \lesssim r_d \quad \text{and} \quad \| r_g \|_{F,\infty} := \sup_{x \in \mathcal{X}} | r_g(x) | \lesssim l_d r_d.
\]

Assumption 3.3 introduces the rate of decay of the misspecification error. Specifically, the sequence of constants \( r_d \) bounds the mean squared misspecification error. In addition, the sequence \( l_d r_d \) bounds the worst-case misspecification error uniformly over the domain of \( X \), where \( l_d \) is the modulus of continuity of the worst-case error with respect to mean squared error.

Define the sampling error \( U \) as follows:

\[
U := Y - g(X).
\]

Assumption 3.4 (Sampling Error). The second moment of the sampling error \( U \) conditionally on \( X \) is bounded from above by \( \bar{\sigma}^2 \):

\[
\sup_{x \in \mathcal{X}} \mathbb{E}[U^2 | X = x] \lesssim_P \bar{\sigma}^2.
\]

Assumption 3.5 (Small Bias Condition). There exists a sequence \( \epsilon_N = o(1) \), such that with probability at least \( 1 - \epsilon_N \), the first stage estimate \( \hat{\eta} \), obtained by cross-fitting (Definition 2.1), belongs to a shrinking neighborhood of \( \eta_0 \), denoted by \( T_N \). Uniformly over \( T_N \), the following mean square convergence holds:

\[
B_N := \sqrt{N} \sup_{\eta \in T_N} \| \mathbb{E}p(X)[Y_\eta - Y_{\eta_0}] \| = o(1), \quad (3.1)
\]

\[
\xi_d \kappa_N := \xi_d \sup_{\eta \in T_N} (\mathbb{E}(Y_\eta - Y_{\eta_0})^2)^{1/2} = o(1). \quad (3.2)
\]

Assumption 3.5 introduces the realization set \( T_N \), where the first stage estimate \( \hat{\eta} \) belongs to with probability approaching one. It required conditional local robustness of the \( Y_\eta \) with respect to the biased estimation of \( \eta \). It also provides a joint restriction on the speed of shrinkage of the set \( T_N \) (and, hence, the quality of the first stage estimate \( \tilde{\eta} \)) and the growth speed \( \xi_d \) of the technical regressors. Section 4 shows that the Assumption 3.5 holds for Examples 1, 2, 4.

Assumption 3.6 (Tail Bounds). There exist \( m > 2 \) such that the upper bound of the
m'th moment of $|U|$ is bounded conditionally on $X$:
\[
\sup_{x \in X} \mathbb{E}[|U|^m | X = x] \lesssim 1.
\]
The norm of the outer product of the technical regressors grows sufficiently slow:
\[
\max_{1 \leq i \leq N} \|p_i p_i'\| \frac{d}{N} = o(1).
\]
Assumption 3.6 bounds the tail of the distribution of the sampling error $U$ and the regressors $p(X)$.

**Assumption 3.7** (Bound on Regression Errors). There exists a sequence $\epsilon_N = o(1)$ and a constant $q > 2$, such that with probability at least $1 - \epsilon_N$, the first stage estimate $\hat{\eta}$, obtained by cross-fitting (Definition 2.1), belongs to a shrinking neighborhood of $\eta_0$, denoted by $T_N$, constrained as follows:
\[
\sup_{\eta \in T_N} \left( \mathbb{E}|Y_\eta - Y_{\eta_0}|^q \right)^{1/q} = O(1).
\]

### 3.1 Pointwise Limit Theory

**Theorem 3.1** (Pointwise Limit Theory of LRE). Let Assumptions 3.1, 3.2, 3.3, 3.4, 3.5 hold. Then, the following statements hold:

(a) The second norm of the estimation error is bounded as:
\[
\|\hat{\beta} - \beta_0\|_2 \lesssim_P \sqrt{\frac{d}{N}} + \left[ \sqrt{d_\alpha d_{\alpha}} \wedge \xi_{d\alpha}/\sqrt{N} \right],
\]
which implies a bound on the mean squared error of the estimate $p(x)'\hat{\beta}$ of the pseudo-target function $p(x)'\beta_0$:
\[
(\mathbb{E}_N(p(X_i)'(\hat{\beta} - \beta_0))^2)^{1/2} \lesssim_P \sqrt{\frac{d}{N}} + \left[ \sqrt{d_\alpha d_{\alpha}} \wedge \xi_{d\alpha}/\sqrt{N} \right].
\]

(b) For any $\alpha \in S^{d-1} := \{\alpha \in \mathbb{R}^d : \|\alpha\| = 1\}$ the estimator $\hat{\beta}$ is approximately linear:
\[
\sqrt{N}\alpha'(\hat{\beta} - \beta) = \alpha'Q^{-1}G_Np(X_i)(U_i + r_g(X_i)) + R_{1,N}(\alpha),
\]
where the remainder term $R_{1,N}(\alpha)$ is bounded as $R_{1,N}(\alpha) \lesssim_P \sqrt{\frac{\xi_{d\alpha} \log N}{N}}(1 + \sqrt{d_\alpha d_{\alpha}}) + \xi_{d\alpha} \kappa N$. 

(c) Define the asymptotic covariance matrix of the \( p(x)'(\widehat{\beta} - \beta_0) \) as follows:

\[
\Omega = Q^{-1}E p(X)p(X)'(U + r_g(X))^2Q^{-1}.
\]

If \( R_{1,N}(\alpha) = o_p(1) \) and the Lindeberg condition holds: \( \lim_{M \to \infty} E U_1^2 I_{|U| > M} \to 0 \), then the pointwise estimator is approximately Gaussian:

\[
\lim_{N \to \infty} \sup_{t \in \mathbb{R}} \left| \mathbb{P} \left( \frac{\sqrt{N}p(x_0)'(\widehat{\beta} - \beta_0)}{\sqrt{\alpha' \Omega \alpha}} < t \right) - \Phi(t) \right| = 0. \tag{3.3}
\]

In particular, for any point \( x_0 \in \mathcal{X} \) for \( \alpha = \frac{p(x_0)}{||p(x_0)||} \), the estimator \( p(x_0)'\widehat{\beta} \) of the pseudo-target value \( p(x_0)'\beta_0 \) is asymptotically normal:

\[
\lim_{N \to \infty} \sup_{t \in \mathbb{R}} \left| \mathbb{P} \left( \frac{\sqrt{N}p(x_0)'(\widehat{\beta} - \beta_0)}{\sqrt{p(x_0)'\Omega p(x_0)}} < t \right) - \Phi(t) \right| = 0. \tag{3.4}
\]

(d) In addition, if Assumptions 3.6 and 3.7 hold, \( \Omega \) can be consistently estimated by a sample analog:

\[
\hat{\Omega} := \hat{Q}^{-1}E_N p(X_i)p(X_i)'(Y_i(\widehat{\eta}) - p(X_i)'\widehat{\beta})^2\hat{Q}^{-1}. \tag{3.5}
\]

Theorem 3.1 is our first main result. Under small bias condition, Locally Robust Estimator has the oracle rate, oracle asymptotic linearity representation and asymptotic variance \( \Omega \), where the oracle knows the true value of the first-stage nuisance parameter \( \eta_0 \).

### 3.2 Uniform Limit Theory

Let \( \alpha(x) := p(x)/||p(x)|| \) denote the normalized value of technical regressors \( p(x) \). Define their Lipshitz constant as:

\[
\xi_d^L = \sup_{x, x' \in \mathcal{X}, x \neq x'} \frac{||\alpha(x) - \alpha(x')||}{||x - x'||}.
\]

**Assumption 3.8** (Basis). Basis functions are well-behaved, namely (i) \( \xi_d^L \frac{2m}{(m-2)} \log N \leq 1 \) and \( \log \xi_d^L \lesssim \log d \).

**Assumption 3.9** (First Stage Error Bound and Rate). There exists a sequence \( \epsilon_N = o(1) \), such that with probability at least \( 1 - \epsilon_N \), the first stage estimate \( \widehat{\eta} \), obtained by
cross-fitting (Definition 2.1), belongs to a shrinking neighborhood of \( \eta_0 \), denoted by \( T_N \), constrained as follows:

\[
\xi_d \sup_{\eta \in T_N} (\mathbb{E}(Y_\eta - Y_{\eta_0})^2)^{1/2} \sqrt{\log N} = o(1).
\]

**Theorem 3.2** (Uniform Limit Theory of LRE). Let Assumptions 3.1, 3.2, 3.3, 3.4, 3.5, 3.6, 3.8, 3.9 hold.

(a) The estimator is approximately linear uniformly over the domain \( \mathcal{X} \):

\[
|\sqrt{N} \alpha(x) ^{(\hat{\beta} - \beta_0)} - \alpha'(x) G_N p(X_i)[U_i + r_d(X_i)]| \leq R_{1,N}(\alpha(x))
\]

where \( R_{1,N}(\alpha(x)) \), summarizing the impact of unknown design and the first stage misspecification error, obeys

\[
\sup_{x \in \mathcal{X}} R_{1,N}(\alpha(x)) \lesssim_P \frac{\xi_d \log N}{N} (N^{1/m} \sqrt{\log N} + \sqrt{d_l d_r d}) + 
\xi_d k_N \sqrt{\log N} + N^{-1/2 + 1/q} \log N =: \bar{R}_{1,N}
\]

uniformly over \( x \in \mathcal{X} \). Moreover,

\[
|\sqrt{N} \alpha(x) ^{(\hat{\beta} - \beta_0)} - \alpha'(x) G_N p(X_i)[U_i + r_d(X_i)]| \leq R_{1,N}(\alpha(x)) + R_{2,N}(\alpha(x))
\]

where \( R_{2,N}(\alpha(x)) \), summarizing the impact of misspecification error, obeys

\[
R_{2,N}(\alpha(x)) \lesssim_P \sqrt{\log N} l_{d_l d_r} =: \bar{R}_{2,N}
\]

uniformly over \( x \in \mathcal{X} \).

(b) The estimator \( p(x) ^{(\hat{\beta} - \beta_0)} \) of the pseudo-target \( p(x) ^{\beta_0} \) converges uniformly over \( \mathcal{X} \):

\[
\sup_{x \in \mathcal{X}} |p(x) ^{(\hat{\beta} - \beta_0)}| \lesssim_P \frac{\xi_d}{\sqrt{N}} [\sqrt{\log N} + \bar{R}_{1,N} + \bar{R}_{2,N}].
\]

Theorem 3.2 is our second main result in the paper. Under small bias condition, Locally Robust Estimator achieves oracle asymptotic linearity representation uniformly over the domain \( \mathcal{X} \subset \mathbb{R}^r \) of the covariates of interest \( X \).

**Remark 3.1** (Optimal Uniform Rate in Holder class). Suppose the true function \( g \) belongs to the Holder smoothness class of order \( k \), denoted by \( \Sigma_k(\mathcal{X}) \). Then, the optimal
number $d$ of technical regressors that comprise a vector $p(x)$ obeys

$$d \asymp \left( \frac{\log N}{N} \right)^{r/(2k+r)}.$$ 

This choice of $d$ yields the optimal uniform rate:

$$\sup_{x \in \mathcal{X}} |\hat{g}(x) - g(x)| \lesssim_P \left( \frac{\log N}{N} \right)^{r/(2k+r)}.$$ 

Our result on strong approximation by a Gaussian process plays an important role in our second result on inference that is concerned with the weighted bootstrap. Consider a set of weights $h_1, h_2, \ldots, h_N$ that are i.i.d. draws from the standard exponential distribution and are independent of the data. For each draw of such weights, define the weighted bootstrap draw of the least squares estimator as a solution to the least squares problem weighted by $h_1, h_2, \ldots, h_N$, namely

$$\hat{\beta}^b = \arg \min_{b \in \mathbb{R}^k} \frac{1}{N} \sum_{i=1}^N h_i \left( \hat{Y}_i - p(X_i) \right)^2.$$ 

For all $x \in \mathcal{X}$, define $\hat{g}^b(x) = p(x)\hat{\beta}^b$. The following corollary establishes validity of weighted bootstrap for approximating the distribution of series process.

**Corollary 3.1 (Weighted Bootstrap Method).** (a) Let Assumption 3.6 be satisfied with $m \geq 3$. In addition, assume that $R_{1,N} = o_P(a_N^{-1})$ and $a_N^2 d^4 \xi_d^2(1+i_d^3 \xi_d^3)^2 \log^2 N/N \to 0$. Then, for some $N_d \sim N(0, J_k)$

$$\sqrt{N} \frac{\alpha(x)'(\hat{\beta} - \beta_0)}{\|\alpha(x)\Omega^{1/2}\|} = \frac{\alpha(x)'\Omega^{1/2}}{\|\alpha(x)\Omega^{1/2}\|} N_d + o_P(a_N^{-1})$$

in $l^\infty(\mathcal{X})$ so that for $e(x) = \Omega^{1/2}p(x)$,

$$\sqrt{N} \frac{p(x)'(\hat{\beta} - \beta_0)}{\|e(x)\|} = \frac{e(x)}{\|e(x)\|} N_d + o_P(a_N^{-1})$$

in $l^\infty(\mathcal{X})$.

(b) The weighted bootstrap process satisfies:

$$\sqrt{N} \alpha(x)'(\hat{\beta} - \beta) = \alpha(x)' \mathbb{E}_N [(h_i - 1)p(X_i)(U_i + r_g(X_i)) + R_{1N}^b(\alpha(x))].$$
where the remainder obeys

\[ R_{1N}^b(\alpha(x)) \lesssim_P \frac{d}{N} \sqrt{\frac{\log^3 N}{N}} \left( N^{1/m} \sqrt{\log N} + \sqrt{d} q_d \right) = o(1/\log N) =: \bar{R}_{1N}^b \]

(c) \( \sqrt{N} \frac{e(x) - \hat{g}(x) - \hat{\beta}}{\|e(x)\|} N^d + o_P(a_N^{-1}) \) in \( l^\infty(\mathcal{X}) \), and so

(d) \( \sqrt{N} \frac{\hat{g}(x) - \hat{\beta}}{\|e(x)\|} N^d + o_P(a_N^{-1}) \) in \( l^\infty(\mathcal{X}) \).

Corollary 3.1 establishes strong approximation of \( \alpha(x)'(\hat{\beta} - \beta_0) \) by a Gaussian process. Theorem 4.5 in Belloni et al. (2015) implies validity of weighted bootstrap.

4 Applications

In this section we apply the results of Section 3 for economically relevant settings, described in Examples 1, 2, 4.

4.1 Conditional Average Treatment Effect

Using the setup of Example 1, let \( Y^1 \) and \( Y^0 \) be the potential outcomes, \( D \in \{1, 0\} \) indicate the presence in the treatment group, \( Y^0 = D Y^1 + (1 - D) Y^0 \) be the actual outcome, \( s_0(Z) = \mathbb{E}[D = 1 | Z] \) be the propensity score and \( \mu_0(D, Z) = \mathbb{E}[Y^0 | D, Z] \) be the conditional expectation function. We provide sufficient low-level conditions on the regression functions \( \mu_0(1, Z), \mu_0(0, Z) \) and the propensity score \( s_0(Z) \) such that the pointwise and uniform Gaussian approximations of the target function \( g(x) \) (Theorems 3.1 and 3.2) hold.

Assumption 4.1 (Strong Overlap). A The propensity score is bounded above and below: \( \exists \bar{s}_0 > 0, 0 < \bar{s}_0 < s_0(z) < 1 - \bar{s}_0 < 1 \forall z \in Z \).

B The propensity score is bounded below: \( \exists \bar{s}_0 > 0, 0 < \bar{s}_0 < s_0(z) < 1 \forall z \in Z \).

In context of Example 1 Assumption 4.1(a) ensures that the probability of assignment to the treatment and control group is bounded away from zero. In context of Example 2 Assumption 4.1(b) ensures that the probability of observing the response \( Y^* \) is bounded away from zero.

Definition 4.1 (First Stage Rate). Given the true functions \( s_0, \mu_0 \) and sequences of shrinking neighborhoods \( S_N \) of \( s_0 \) and \( M_N \) of \( \mu_0 \), define the following rates:

\[ s_N := \sup_{s \in S_N} \left( \mathbb{E}(s(Z) - s_0(Z))^2 \right)^{1/2}, \]
\[ m_N := \sup_{\mu \in \mathcal{M}_N} (\mathbb{E}(\mu(Z) - \mu_0(Z))^2)^{1/2}, \]
where the expectation is taken with respect to \( Z \).

We will refer to \( s_N \) as the propensity score rate and \( m_N \) as the regression function rate.

**Assumption 4.2** (Assumptions on the Propensity Score and the Regression Function). Assume that there exists a sequence of numbers \( \epsilon_N = o(1) \) and sequences of neighborhoods \( S_N \) of \( s_0 \), \( M_N \) of \( \mu_0 \) with rates \( s_N, m_N \) such that the first-stage estimate \( \{\hat{s}(z), \hat{\mu}(1,z), \hat{\mu}(0,z)\} (A) \) or \( \{\hat{s}(z), \hat{\mu}(z)\} (B) \) belongs to the set \( \{S_N, M_N\} \) w.p. at least \( 1 - \epsilon_n \) and

\[ \xi_d s_N m_N \sqrt{\log N} = o(1). \]

Finally, assume that there exists \( C > 0 \) that bounds the functions in \( M_N \) uniformly over their domain \( A \): \( \sup_{\mu \in \mathcal{M}_N} \sup_{z \in \mathcal{Z}} \sup_{d \in \{1,0\}} |\mu(d,z)| < C \) or \( B \): \( \sup_{\mu \in \mathcal{M}_N} \sup_{z \in \mathcal{Z}} |\mu(z)| < C \).

Plausibility of Assumption 4.2 is discussed in the introduction of the paper. In case the propensity score and regression function can be well-approximated by a logistic (linear) high-dimensional sparse model, Assumption 4.2 holds under a low-level conditions analogous to those in Example 5.

**Corollary 4.1** (Gaussian Approximation for Conditional Average Treatment Effect). Under Assumptions 4.1(A) and 4.2(A), the robust r.v. \( Y \) given by Equation 2.1, satisfies Assumptions 3.5 and 3.9. As a result, pointwise and uniform Gaussian approximation (Theorems 3.1, 3.2) and Validity of Weighted Bootstrap (Corollary 3.1) hold with

\[ \Omega = Q^{-1} \Sigma Q^{-1}, \]

where \( \Sigma \) is:

\[ \Sigma = \mathbb{E} p(X_i)p(X_i)\left[\mu(1,Z_i) - \mu(0,Z_i) + \frac{D_i[Y_i^o - \mu(1,Z_i)]}{s_0(Z_i)} - \frac{(1 - D_i)[Y_i^o - \mu(0,Z_i)]}{1 - s_0(Z_i)} - p(X_i)' \beta_0 \right]^2. \]

### 4.2 Regression Function with Partially Missing Outcome

Using the setup of Example 2, let \( Y^* \) be a partially missing outcome variable, \( D \in \{1,0\} \) indicate the presence of \( Y^* \) in the sample, \( Y^o = DY^* \) be the observed outcome, \( s_0(Z) = \mathbb{E}[D = 1|Z] \) be the propensity score and \( \mu_0(Z) = \mathbb{E}[Y^o|D = 1, Z] \) be the conditional expectation function. We provide sufficient low-level conditions on the regression functions
\( \mu(Z), s(Z) \) such that the pointwise and uniform Gaussian approximations of the target function \( g(x) \) (Theorems 3.1 and 3.2) hold.

**Corollary 4.2** (Gaussian Approximation for Regression Function with Partially Missing Outcome). Under Assumptions 4.1(B) and Assumption 4.2(B) the robust r.v. \( Y \), given by Equation 2.2, satisfies Assumptions 3.5 and 3.9. As a result, pointwise and uniform Gaussian approximation (Theorems 3.1, 3.2) and Validity of Weighted Bootstrap (Corollary 3.1) hold with

\[
\Omega = Q^{-1}\Sigma Q^{-1},
\]

where \( \Sigma \) is:

\[
\Sigma = \mathbb{E} p(X_i)p(X_i)' \left[ \frac{D_i}{s_0(Z_i)} [Y_i^0 - \mu_0(Z_i)] + \mu_0(Z_i) - p(X_i)' \beta_0 \right]^2.
\]

Here give an example of a model and a first-stage estimator that satisfy Assumption 4.2.

**Example 5** (Partially Missing Outcome with High-Dimensional Sparse Design). Consider the setup of Example 2. Let the observable vector \((D, X, DY^*)\) consist of the covariate vector of interest \( X \) and a partially observed variable \( Y^* \), whose presence is indicated by \( D \in \{1, 0\} \). In addition, suppose there exists an observable vector \( Z \) such that Missingness at Random (Assumption 2.2) is satisfied conditionally on \( Z \). Let \( p_\mu(Z), p_s(Z) \) be high-dimensional basis functions of the vector \( Z \) that approximate the conditional expectation functions \( \mu_0(z), s_0(z) \) using the linear and logistic links, respectively:

\[
\mu_0(z) = p_\mu(Z)' \theta + r_\mu(z) \tag{4.1}
\]

\[
s_0(z) = L(p_s(Z)' \delta) + r_s(z) := \frac{\exp(p_s(Z)' \delta)}{\exp(p_s(Z)' \delta) + 1} + r_s(z) \tag{4.2}
\]

where \( \theta, \delta \) are the vectors in \( \mathbb{R}^p \) whose dimension \( p \) is allowed to be larger than the sample size \( N \), and \( r_\mu(z), r_s(z) \) are the misspecification errors of the respective link functions that vanish as described in Assumptions 4.3, 4.4. For each \( \gamma \in \{\theta, \delta\} \), denote a support set

\[
T_\gamma := \{j : \gamma_j \neq 0, j \in \{1, 2, \ldots, p\}\}
\]

and its cardinality, which we refer to as sparsity index of \( \gamma \),

\[
s_\gamma := |T| = \|\gamma\|_0 \quad \forall \quad \gamma \in \{\theta, \delta\}.
\]
We allow the cardinality of \( s_\beta, s_\theta \) to grow with \( N \). Define minimal and maximal empirical Restricted Sparse Eigenvalues (RSE) \( \phi_{\min}(m), \phi_{\max}(m) \) as

\[
\phi_{\min}(m) := \min_{1 \leq ||\nu||_0 \leq m} \frac{\nu' \mathbb{E} N Z_i Z_i' \nu}{\|\nu\|_2^2}, \quad \phi_{\max}(m) := \max_{1 \leq ||\nu||_0 \leq m} \frac{\nu' \mathbb{E} N Z_i Z_i' \nu}{\|\nu\|_2^2}.
\]

Let \( \delta_N \to 0 \) and \( \Delta_N \to 0 \) be the fixed constants approaching zero from above at a speed at most polynomial in \( N \): for example, \( \delta_N \geq \frac{1}{N^c} \) for some \( c > 0 \), \( \ell_N = \log N \), and \( c, C, \kappa', \kappa'' \) and \( \nu \in [0,1] \) are positive constants.

**Assumption 4.3** (Regularity Conditions for Linear Link). We assume that the following standard conditions hold. With probability \( 1 - \Delta_N \), the minimal and maximal empirical RSE are bounded from below by \( \kappa'_\mu \) and from above by \( \kappa''_\mu \):

\[
\kappa'_\mu \leq \inf_{\|\delta\|_0 \leq s_{\ell_N}, \|\delta\|_1 = 1} \|D\mu_\mu(Z)\|_{P_N, 2} \leq \sup_{\|\delta\|_0 \leq s_{\ell_N}, \|\delta\|_1 = 1} \|D\mu_\mu(Z)\|_{P_N, 2} \leq \kappa''_\mu.
\]

(b) There exist absolute constants \( B, c > 0 \): regressors \( \max_{1 \leq j \leq p} |p_{\mu, j}(Z)| \leq B \) a.s. and \( \max_{1 \leq j \leq p} c \leq \mathbb{E} p_{\mu, j}(Z)^2 \) (c) With probability \( 1 - \Delta_N \), \( \mathbb{E} N \tau^2_\mu(Z) \leq C s \log(p \vee N)/N \).

**Assumption 4.4** (Regularity Conditions for Logistic Link). We assume that the following standard conditions hold. With probability \( 1 - \Delta_N \), the minimal and maximal empirical RSE are bounded from below by \( \kappa'_s \) and from above by \( \kappa''_s \):

\[
\kappa'_s \leq \inf_{\|\delta\|_0 \leq s_{\ell_N}, \|\delta\|_1 = 1} \|p_s(Z)'\|_{P_N, 2} \leq \sup_{\|\delta\|_0 \leq s_{\ell_N}, \|\delta\|_1 = 1} \|p_s(Z)'\|_{P_N, 2} \leq \kappa''_s.
\]

(b) There exist absolute constants \( B, c > 0 \): regressors \( \max_{1 \leq j \leq p} |p_{s, j}(Z)| \leq B \) a.s. and \( \max_{1 \leq j \leq p} c \leq \mathbb{E} p_{s, j}(Z)^2 \) (c) With probability \( 1 - \Delta_N \), \( \mathbb{E} N \tau^2_s(Z) \leq C s \log(p \vee N)/N \).

Assumptions 4.3 and 4.4 are a simplification of the Assumption 6.1-6.2 in Belloni et al. (2013). The following estimators of \( \mu_0(Z) \) and \( s_0(Z) \) are available.

**Definition 4.2** (Lasso Estimator of the Regression Function). For \( \lambda = 1.1\sqrt{N} \Phi^{-1}(1 - 0.05/(N \vee p \log N)) \), define \( \hat{\theta} \) as a solution to the following optimization problem:

\[
\hat{\theta} := \arg\min_{\theta \in \mathbb{R}^p} \mathbb{E} N D_i(Y_i - Z_i' \theta)^2 + \lambda \|\theta\|_1
\]

and a first-stage estimate of \( \mu \) as

\[
\hat{\mu}(z) := z' \hat{\theta}.
\]

**Definition 4.3** (Lasso Estimator of the Propensity Score). For \( \lambda = 1.1\sqrt{N} \Phi^{-1}(1 - 0.05/(N \vee p \log N)) \) and an appropriately chosen \( \varepsilon > 0 \), define \( \hat{\delta} \) as a solution to the
following optimization problem:

\[ \hat{\delta} := \arg \min_{\delta \in \mathbb{R}^p} E_N [\log(1 + \exp(Z_i'\delta)) - D_i Z_i'\delta] + \lambda \|\delta\|_1 \]

and a first-stage estimate of \( s_0 \) as

\[ \hat{s}(z) := \max(s/2, L(z')) \].

**Lemma 4.1** (Sufficient Conditions for Assumption 3.5). Suppose Assumptions 4.3 and 4.4 hold. Define the regression rate \( m_N := \sqrt{s_\theta \log p N} \) and the propensity score rate \( s_N := \sqrt{s_\delta \log p N} \). For a sequence of numbers \( \epsilon_N = o(1) \) define the sequence of neighborhoods \( M_N \) of \( \mu_0(Z) \) and \( S_N \) of \( s_0(Z) \) as follows:

\[
M_N := \{ \mu(z) = z'\theta : \|\theta - \theta_0\|_{2,N} \leq C(-\log(\epsilon_N)) m_N \}
\]

\[
S_N := \{ s(z) = \max(s/2, L(z')) : \|\delta - \delta_0\|_{2,N} \leq C(-\log(\epsilon_N)) s_N \}
\]

Then, Assumption 4.2(B) is satisfied if the product of sparsity indices \( s_\theta s_\delta \) grows sufficiently slow:

\[
\sqrt{N} \xi_d m_N s_N \sqrt{\log N} = \xi_d \sqrt{s_\theta s_\delta \log^2 p \log N} = o(1).
\]

### 4.3 Conditional Average Partial Derivative

Using the setup of Example 4, let \( Y^0 \) be an outcome of interest, \( \mu(x,w) := \mathbb{E}[Y^0|X = x, W = w] \) be a conditional expectation of \( Y^0 \) on \( X,W \), and \( f(W|X) \) be the conditional density of \( W \) given \( X \). We provide sufficient low-level conditions on the regression functions \( f(W|X), \mu(X,W) \) such that the pointwise and uniform Gaussian approximations of the target function \( g(x) \) (Theorems 3.1 and 3.2) hold.

**Definition 4.4** (First Stage Rate). Given a true function \( f_0(W|X), \mu_0(X,W) \), let \( F_N, M_N \) be a sequence of shrinking neighborhoods of \( f_0(W|X) \) and \( \mu_0(X,W) \) constrained as follows:

\[
f_N := \sup_{f \in F_N} (\mathbb{E}(f(W|X) - f_0(W|X))^2)^{1/2}
\]

\[
m_N := \sup_{\mu \in M_N} (\mathbb{E}(\mu(X,W) - \mu_0(X,W))^2)^{1/2}
\]

where expectation is taken with respect to \( W,X \).

We will refer to \( f_N \) as the density rate and \( m_N \) as regression function rate.
**Assumption 4.5** (Assumptions on the Conditional Density and the Regression Function). Assume that there exists a sequence of numbers \( \epsilon_N = o(1) \) and a sequence of neighborhoods \( F_N, M_N \) such that the first-stage estimate \( \{\hat{f}, \hat{\mu}\} \) belongs to the set \( \{F_N, M_N\} \) w.p. at least \( 1 - \epsilon_n \). The neighborhoods \( F_N, M_N \) shrink at rates \( f_N, m_N \) such that:

\[
\xi_d f_N m_N \sqrt{\log N} = o(1).
\]

Finally, assume that there exists \( C > 0 \) that bounds the functions in \( F_N, M_N \) uniformly over their domain:

\[
\sup_{\mu \in M_N} \sup_{x,w \in X \times W} |\mu(x, w)| < C
\]

and

\[
\sup_{f \in F_N} \sup_{x,w \in X \times W} |f(W = w| X = x)| < C.
\]

**Corollary 4.3** (Gaussian Approximation for Conditional Average Partial Derivative). Let Assumption 4.5 hold. Then, the robust r.v. \( Y \), given by Equation 2.2, satisfies Assumptions 3.5 and 3.9. As a result, pointwise and uniform Gaussian approximation (Theorems 3.1, 3.2) and Validity of Weighted Bootstrap (Corollary 3.1) hold with

\[
\Omega = Q^{-1} \Sigma Q^{-1},
\]

where \( \Sigma \) is:

\[
\Sigma = \mathbb{E}p(X_i)p(X_i)' \left[ - \partial_w \log f(W_i|X_i)[Y_i^0 - \mu_0(X_i, W_i)] + \mu_0(X_i, W_i) - p(X_i)' \beta_0 \right]^2.
\]

The regression function \( \mu(X, W) \) can be estimated at \( o(N^{-1/4}) \) rate by a local linear estimator with suitably chosen bandwidth parameters. An example of a conditional density \( f(W|X) \) estimate is a kernel density estimator.

### 5 Simulation Evidence

In this section we examine the finite sample performance of the Locally Robust Estimator through the Monte Carlo experiments in context of Example 5. We compare LRE to other more naive strategies, such as Inverse Probability Weighting (IPW) and Ordinary Least Squares (OLS) on the complete data only. We show that under the small misspecification of a linear model, all three estimators have similar performance, while under larger misspecification, only Locally Robust Estimator remains valid.

Let us describe our simulation design. Using the setup of Example 5, we generate
a random sample \((D_i, X_i, Z_i, Y_i^*)\)\(_{i=1}^{N=500}\) from the following data generating process. The control vector \(Z\), \(\dim(Z) = 500\), \(Z \sim N(0, T(\rho))\), \(\rho = 0.5\) is generated from a normal distribution \(N(0, T(\rho))\), where the covariance matrix \(T(\rho)\) is the Toeplitz covariance matrix with the correlation parameter \(\rho = 0.5\). The propensity score \(s_0(z) = L(z' \delta)\), where \(L(t) := \frac{\exp t}{\exp t + 1}\) is the logistic function, and the parameter \(\delta = (1, \frac{1}{2}, \ldots, 0, \ldots, 0)\). The regression function \(\mu(z) = z' \gamma\) is a linear function of the control vector, where the parameter \(\gamma = [(1, \frac{1}{2}, \ldots, \frac{1}{(d-1)^2}), c(1, \frac{1}{2}, \ldots, \frac{1}{300})', 0, \ldots, 0]\) and \(c\) is a design constant. The outcome variable \(Y\) and the presence indicator \(D\) are generated by

\[
D \sim B(L(Z' \delta)), \\
Y^* = Z' \theta + \epsilon, \quad \epsilon \sim N(0, 1), \tag{5.1}
\]

where \(B(p)\) stands for a Bernoulli draw with probability of success \(p\). Suppose a researcher is interested in the conditional expectation of \(Y\) given the first \(d = 6\) control variables \(X = [Z_1, Z_2, ..., Z_6]\):

\[
g(x) := \mathbb{E}[Y^* | X = x].
\]

He approximates \(g(x)\) using a linear form \(p(x)' \beta\), where the vector of technical transformations

\[
p(x) := (1, x)'
\]

\(^3\) consists of the constant and a degree one polynomial of vector \(x\). Let \(Y^o = DY^*\) be the observed outcome. Having established the setup, let us describe the estimators whose finite sample performance we compare:

- Ordinary Least Squares: \(\hat{\beta}_{OLS} := (\mathbb{E}_N D_i p(X_i) p(X_i)' )^{-1} \mathbb{E}_N D_i p(X_i) Y_i^o\)
- Inverse Probability Weighting: \(\hat{\beta}_{IPW} := \left(\mathbb{E}_N \frac{D_i}{\hat{s}(Z_i)} p(X_i) p(X_i)' \right)^{-1} \mathbb{E}_N \frac{D_i}{\hat{s}(Z_i)} p(X_i) Y_i^o\)
- Locally Robust Estimator: \(\hat{\beta}_{LRE} := \left(\mathbb{E}_N p(X_i) p(X_i)' \right)^{-1} \mathbb{E}_N p(X_i) \left[ \frac{D_i}{\hat{s}(Z_i)} [Y_i^o - \hat{\mu}(Z_i)] + \hat{\mu}(Z_i) \right]\)

where the nonparametric estimates of the propensity score \(\hat{s}\) and the regression function \(\hat{\mu}\) are estimated as in Example 5 using the cross-fitting procedure in Definition 2.1.

Table 1 shows the bias, standard deviation, root mean squared error, and rejection frequency for Ordinary Least Squares, Inverse Probability Weighting, and Locally Robust

\(^3\)We omit the regressor \(p(x) = 1\) since \(\mathbb{E}Y^* = \mathbb{E}Z = 0\) by construction.
Estimator under small misspecification, which is achieved by scaling the coefficient on the omitted controls by a small constant \((c = 0.1)\). In that case, all the three estimators have small bias and good coverage property. Since the linear model is close to the true one, OLS is best linear conditionally unbiased estimator, and therefore has smaller variance than IPW and LRE.

Table 2 shows finite sample properties of IPW, LRE, and OLS under large misspecification, which is achieved by scaling the coefficient on the omitted controls by a small constant \((c = 20)\). As expected, OLS suffers from selection bias, IPW incurs the first-order bias due to the propensity score estimation error, but LRE remains valid. In the case of large misspecification, LRE achieves 8% to 100% bias reduction compared to IPW. Moreover, LRE maintains valid inference and has its rejection frequency close to the nominal under both small and large misspecification.

<table>
<thead>
<tr>
<th>(\beta_1 = 1.1)</th>
<th>OLS</th>
<th>IPW</th>
<th>LRE</th>
<th>OLS</th>
<th>IPW</th>
<th>LRE</th>
<th>OLS</th>
<th>IPW</th>
<th>LRE</th>
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<td>-0.014</td>
<td>0.049</td>
<td>0.060</td>
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<td>0.049</td>
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</tr>
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<td>St.Error</td>
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<td>0.004</td>
<td>0.101</td>
<td>0.126</td>
<td>0.124</td>
<td>0.102</td>
<td>0.126</td>
<td>0.125</td>
</tr>
<tr>
<td>RMSE</td>
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<td>0.112</td>
<td>0.116</td>
<td>0.100</td>
<td>0.116</td>
<td>0.118</td>
<td>0.100</td>
<td>0.116</td>
<td>0.118</td>
</tr>
<tr>
<td>Rej. Freq.</td>
<td>0.080</td>
<td>0.080</td>
<td>0.080</td>
<td>0.080</td>
<td>0.080</td>
<td>0.080</td>
<td>0.080</td>
<td>0.080</td>
<td>0.080</td>
</tr>
<tr>
<td>(\beta_2 = 0.5)</td>
<td>OLS</td>
<td>IPW</td>
<td>LRE</td>
<td>OLS</td>
<td>IPW</td>
<td>LRE</td>
<td>OLS</td>
<td>IPW</td>
<td>LRE</td>
</tr>
<tr>
<td>Bias</td>
<td>-0.010</td>
<td>-0.004</td>
<td>-0.009</td>
<td>0.091</td>
<td>0.120</td>
<td>0.119</td>
<td>0.092</td>
<td>0.120</td>
<td>0.119</td>
</tr>
<tr>
<td>St.Error</td>
<td>0.096</td>
<td>0.118</td>
<td>0.116</td>
<td>0.096</td>
<td>0.118</td>
<td>0.116</td>
<td>0.096</td>
<td>0.118</td>
<td>0.116</td>
</tr>
<tr>
<td>RMSE</td>
<td>0.087</td>
<td>0.109</td>
<td>0.107</td>
<td>0.087</td>
<td>0.109</td>
<td>0.107</td>
<td>0.087</td>
<td>0.109</td>
<td>0.107</td>
</tr>
<tr>
<td>Rej. Freq.</td>
<td>0.030</td>
<td>0.037</td>
<td>0.050</td>
<td>0.030</td>
<td>0.037</td>
<td>0.050</td>
<td>0.030</td>
<td>0.037</td>
<td>0.050</td>
</tr>
</tbody>
</table>

Table 1: Bias, St. Error, RMSE, Rejection Frequency of OLS, IPW, LRE. Small misspecification. Test size \(\alpha = 0.05\), design constant \(c = 0.1\), \(R^2 = 0.5\), Number of Monte Carlo Repetitions 300.

<table>
<thead>
<tr>
<th>(\beta_1 = 1.1)</th>
<th>OLS</th>
<th>IPW</th>
<th>LRE</th>
<th>OLS</th>
<th>IPW</th>
<th>LRE</th>
<th>OLS</th>
<th>IPW</th>
<th>LRE</th>
</tr>
</thead>
<tbody>
<tr>
<td>Bias</td>
<td>-0.233</td>
<td>-0.456</td>
<td>-0.092</td>
<td>1.806</td>
<td>2.052</td>
<td>1.346</td>
<td>1.821</td>
<td>2.102</td>
<td>1.349</td>
</tr>
<tr>
<td>St.Error</td>
<td>0.025</td>
<td>0.127</td>
<td>-0.056</td>
<td>1.916</td>
<td>2.290</td>
<td>1.339</td>
<td>1.916</td>
<td>2.293</td>
<td>1.340</td>
</tr>
<tr>
<td>RMSE</td>
<td>-0.060</td>
<td>-0.067</td>
<td>-0.004</td>
<td>1.886</td>
<td>2.284</td>
<td>1.353</td>
<td>1.887</td>
<td>2.284</td>
<td>1.353</td>
</tr>
<tr>
<td>Rej. Freq.</td>
<td>0.065</td>
<td>0.061</td>
<td>-0.019</td>
<td>1.865</td>
<td>2.274</td>
<td>1.303</td>
<td>1.866</td>
<td>2.274</td>
<td>1.303</td>
</tr>
<tr>
<td>(\beta_2 = 0.5)</td>
<td>OLS</td>
<td>IPW</td>
<td>LRE</td>
<td>OLS</td>
<td>IPW</td>
<td>LRE</td>
<td>OLS</td>
<td>IPW</td>
<td>LRE</td>
</tr>
<tr>
<td>Bias</td>
<td>-0.060</td>
<td>-0.067</td>
<td>-0.004</td>
<td>1.886</td>
<td>2.284</td>
<td>1.353</td>
<td>1.887</td>
<td>2.284</td>
<td>1.353</td>
</tr>
<tr>
<td>St.Error</td>
<td>0.065</td>
<td>0.061</td>
<td>-0.019</td>
<td>1.865</td>
<td>2.274</td>
<td>1.303</td>
<td>1.866</td>
<td>2.274</td>
<td>1.303</td>
</tr>
<tr>
<td>RMSE</td>
<td>-0.060</td>
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<td>-0.004</td>
<td>1.886</td>
<td>2.284</td>
<td>1.353</td>
<td>1.887</td>
<td>2.284</td>
<td>1.353</td>
</tr>
<tr>
<td>Rej. Freq.</td>
<td>0.065</td>
<td>0.061</td>
<td>-0.019</td>
<td>1.865</td>
<td>2.274</td>
<td>1.303</td>
<td>1.866</td>
<td>2.274</td>
<td>1.303</td>
</tr>
</tbody>
</table>

Table 2: Bias, St. Error, RMSE, Rejection Frequency of OLS, IPW, LRE. Large misspecification. Test size \(\alpha = 0.1\), design constant \(c = 20\), \(R^2 = 0.2\), Number of Monte Carlo Repetitions 300.

### 6 Empirical Application

We apply our methods to study the household demand for gasoline, a question studied in Hausman and Newey (1995), Schmalensee and Stoker (1999), Yatchew and No (2001) and Blundell et al. (2012). These papers estimated the demand function and the average price elasticity for various demographic groups. The dependence of the price elasticity on the household income was highlighted in Blundell et al. (2012), who have estimated the elasticity by low, middle, and high-income groups and found its relationship with income dependencies.
to be non-monotonic. To gain more insight into this question, we estimate the average price elasticity as a function of income and provide simultaneous confidence bands for it.

The data for our analysis are the same as in Yatchew and No (2001), coming from National Private Vehicle Use Survey, conducted by Statistics Canada between October 1994 and September 1996. The data set is based on fuel purchase diaries and contains detailed information about fuel prices, fuel consumption patterns, vehicles and demographic characteristics. We employ the same selection procedure as in Yatchew and No (2001) and Belloni et al. (2011), focusing on a sample of the households with non-zero licensed drivers, vehicles, and distance driven which leaves us with 5001 observations.

The object of interest is the average predicted percentage change in the demand due to a unit percentage change in the price, holding the observed demographic characteristics fixed, conditional on income. In context of Example 4, this corresponds to the conditional average derivative

\[ g(x) = \mathbb{E}[\partial_w \mu(X, Z, W)|X = x], \]

\[ \mu(w, x, z) = \mathbb{E}[Y^o|X = x, Z = z, W = w], \]

where \( Y^o \) is the logarithm of gas consumption, \( W \) is the logarithm of price per liter, \( X \) is log income, and \( Z \) are the observed subject characteristics such as household size and composition, distance driven, and the type of fuel usage. The robust random variable \( Y \) for the target function \( g(x) \) is given by

\[ Y = -\partial_w \log f(W|X, Z)(Y^o - \mu(X, Z, W)) + \partial_w \mu(X, Z, W), \] (6.1)

where \( f(w|x, z) = f(W = w|X = x, Z = z) \) is the conditional density of the price variable \( W \) given income \( X \) and subject characteristics \( Z \). The conditional density \( f(w|x, z) \) and the conditional expectation functions \( \mu(w, x, z) \) comprise the set of the nuisance parameters to be estimated in the first stage.

The choice of the estimators in the first and the second stages is as follows. To estimate the conditional expectation function \( \mu(w, x, z) \) and its partial derivative \( \partial_w \mu(w, x, z) \), we consider a linear model that includes price, price squared, income, income squared, their interactions with 28 time, geographical, and household composition dummies. All in all, we have 91 explanatory variables. We estimate \( \mu(w, x, z) \) using Lasso with the penalty level chosen as in Belloni et al. (2014), and estimate the derivative \( \partial_w \mu(w, x, z) \) using the estimated coefficients of \( \mu(w, x, z) \). To estimate the conditional density \( f(w|x, z) \), we consider a model:

\[ W = l(X, Z) + U, \quad U \perp X, Z, \]
where \( l(x, z) = \mathbb{E}[W|X = x, Z = z] \) is the conditional expectation of price variable \( W \) given income variable \( X \) and covariates \( Z \), and \( U \) is an independent continuously distributed shock with univariate density \( \phi(\cdot) \). Under this assumption, the log density \( \partial_w \log f(W|X, Z) \) equals to

\[
\partial_w \log f(W = w|X = x, Z = z) = \frac{\phi'(w - l(x, z))}{\phi(w - l(x, z))}.
\]

We estimate \( \phi(u) : \mathcal{R} \to \mathcal{R}^+ \) by an adaptive kernel density estimator of Portnoy and Koenker (1989) with Silverman choice of bandwidth. Finally, we plug in the estimates of \( \mu(w, x, z) \), \( \partial_w \mu(w, x, z) \), \( f(w|z, x) \) into the Equation 6.1 to get an estimate of \( \hat{Y} \) and estimate \( g(x) \) by least squares series regression of \( \hat{Y} \) on \( X \). We try both polynomial basis function and B-splines to construct technical regressors.

Figures 1 and 3 report the estimate of the target function (the black line), the point-wise (the dashed blue lines) and the uniform confidence (the solid blue lines) bands for the average price elasticity conditional on income, where the significance level \( \alpha = 0.05 \). The uniform confidence bands for \( g(x) \) are chosen such that they contain the true function \( g(x) \) with probability \( 1 - \alpha = 0.95 \). The are computed as:

\[
\left[ \hat{g}(x) - \hat{e}(x)t_{1-\alpha}^*, \hat{g}(x) + \hat{e}(x)t_{1-\alpha}^* \right],
\]

where \( \hat{g}(x) \) is the estimate of the target function, \( \hat{e}(x) = \sqrt{p(x)'\hat{\Omega}p(x)} \) is the estimate of the standard error, \( \hat{\Omega} \) is the estimated asymptotic variance of \( \hat{\beta} \) (Eq. 3.5), and the \( t_{1-\alpha}^* \)-statistic is the \( (1 - \alpha) \)- empirical quantile of the \( t \)-statistic bootstrap distribution

\[
t_b := \sup_{x \in \mathcal{X}} \left| \frac{p(x)'(\hat{\beta}_b - \hat{\beta})}{\hat{e}(x)} \right|.
\]

The panels of Figure 1 correspond to different choices of the first-stage estimates of the nuisance functions \( \mu(w, x, z) \) and \( f(w|x, z) \) and dictionaries of technical regressors. The panels of Figure 3 correspond to the subsamples of large and small households and to different choices of the dictionaries.

The summary of our empirical findings based on Figure 1 and 3 is as follows. We find the elasticity to be in the range \((-1, 0)\) and significant for majority of income levels. The estimates based on B-splines (Figures 1c, 1d) are monotonically increasing in income, which is intuitive. The estimates based on polynomial functions are non-monotonic in income. For every algorithm on Figure 1 we cannot reject the null hypothesis of constant price elasticity for all income levels: for each estimation procedure, the uniform confi-
dence bands contain the constant function. Figure 3 shows the average price elasticity conditional on income for small and large households. For majority of income levels, we find large households to be more price elastic than the small ones, but the difference is not significant at any income level.

To demonstrate the relevance of demographic data $Z$ in the first stage estimation, we have shown the average predicted effect of the price change on the gasoline consumption (in logs), without accounting for the covariates in the first stage. In particular, this effect equals to $E[\partial_w \mu(X, W)|X = x]$, where $\mu(x, w) = E[Y|X = x, W = w]$ is the conditional expectation of gas consumption given income and price. This predictive effect consists two effects: the effect of price change on the consumption holding the demographic covariates fixed, which we refer to as average price elasticity, and the association of the price change with the change in the household characteristics that also affect the consumption themselves. Figure 2 shows this predictive effect, approximated by the polynomials of degree $k \in \{1, 2\}$, conditional on income. By contrast to the results in Figure 1, the slope of the polynomial of degree $k = 1$ has a negative relationship between income and price elasticity, which present evidence that the demographics $Z$ confound the relationship between income and price elasticity.

7 Proofs

7.1 Notation

We will use the following notation. Let

$$
E_{n,k} f(x_i) := \frac{1}{n} \sum_{i \in J_k} f(x_i), \quad G_{n,k} f(x_i) := \frac{1}{n} \sum_{i \in J_k} f(x_i) - E[f(x_i)|J_k^c]
$$

For an observation index $i \in J_k$ that belongs to a fold $J_k$, $k \in \{1, 2, ..., K\}$, define $Y_i(\hat{\eta}) = Y_i(\hat{\eta}_k), i \in J_k$, where $\hat{\eta}_k$ is estimated on $J_k^c$ as in Definition 2.1. Define $\hat{\eta}(Z_i) = \hat{\eta}_k(Z_i), i \in J_k$.

Let $p(x_i) := p_t$, $\hat{Q} := E_N p_t p_t'$, and $r_i := r_g(x_i) = g(x_i) - p_t' \beta_0$. For two sequences of random variables denote $a_N, b_N, n \geq 1 : a_N \preceq_P b_n$ means $a_N = O_P(b_N)$. For two sequences of numbers, denote $a_N, b_N, n \geq 1 : a_N \preceq b_n := a_N = O(b_N)$ . Let $a \wedge b = \min\{a, b\}, a \vee b = \max\{a, b\}$. The $l_2$ norm is denoted by $\| \cdot \|_2$, the $l_1$ norm is denoted by $\| \cdot \|_1$, the $l_\infty$ is denoted by $\| \cdot \|_\infty$, and the $l_{i0}$ - norm denotes the number of nonzero components of a vector.

A large household is a household with at least 4 members.
Given a vector $\delta \in \mathbb{R}^p$ and a set of indices $T \subset \{1, ..., p\}$, we denote by $\delta_T$ the vector in $\mathbb{R}^p$ in which $\delta_{Tj} = \delta_j, j \in T$ and $\delta_{Tj} = 0, j \notin T$. For a random variable $Y = Y(\eta)$, we will refer to it as a signal.

7.2 Technical Lemmas

**Theorem 7.1 (LLN for Matrices).** Let $Q_i = p_ip_i^\top$ be i.i.d symmetric non-negative $k \times k$-matrices with $d \geq e^2$. Notice that $\|Q_i\| = \|p_i\|^2 \leq \xi_d^2$. Let $Q = \frac{1}{N} \sum_{i=1}^N \mathbb{E} p_i p_i^\top$ denote average value of population covariance matrices.

$$\mathbb{E} \|\hat{Q} - Q\| \lesssim \sqrt{\frac{\xi_d^2 \log N}{N}}$$

**Proof.** Proof can be found in Rudelson (1999).

**Lemma 7.1 (Conditional Convergence Implies Unconditional).** Let $\{X_m\}_{m \geq 1}$ and $\{Y_m\}_{m \geq 1}$ be sequences of random vectors. (i) If for $\epsilon_m \to 0$, $\mathbb{P}(\|X_m\| > \epsilon_m|Y_m) \to 0$, then $\mathbb{P}(\|X_m\| > \epsilon_m) \to 0$. In particular, this occurs if $\mathbb{E}[\|X_m\|^q/\epsilon_m^q|Y_m] \to 0$ for some $q \geq 1$, by Markov inequality. (ii) Let $\{A_m\}_{m \geq 1}$ be a sequence of positive constants. If $\|X_m\| = O_P(A_m)$ conditional on $Y_m$, namely, that for any $\ell_m \to \infty$, $\mathbb{P}(\|X_m\| > \ell_m A_m|Y_m) \to 0$, then $X_m = O_P(A_m)$ unconditionally, namely, that for any $\ell_m \to \infty$, $\mathbb{P}(\|X_m\| > \ell_m A_m) \to 0$.

**Proof.** The Lemma is a restatement of Lemma 6.1 of Chernozhukov et al. (2016a).

**Lemma 7.2 (Maximal Inequality ).** Let an i.i.d sample of size $n$ be available. Let $F$ be a function class with an envelope $F = \sup_{f \in F} |f|$ with $\|F\|_{P,q} < \infty$ for some $q \geq 2$ Let $M := \max_{i \leq n} F(W_i)$ and $\sigma^2 > 0$ be any positive constant such that

$$\sup_{f \in F} \|f\|^2_{P,2} \leq \sigma^2 \leq \|F\|^2_{P,2}$$

Suppose there exist constants $a \geq \epsilon$ and $v \geq 1$ such that

$$\log \sup_Q N(\epsilon|F|_{Q,2}, F, \cdot : \|Q,2\|) \leq v \log(a/\epsilon), 0 < \epsilon \leq 1$$

Then,

$$\mathbb{E}[\sup_{f \in F} \|G_n f\|] \leq K \left( v \sigma^2 \log \left( \frac{a\|F\|_{P,2}}{\sigma} \right) + v M \|M\|^2_{P,2} \log \left( \frac{a\|F\|_{P,2}}{\sigma} \right) \right)$$

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Lemma 7.3 (No Effect of First Stage error).
\[
\sqrt{N}\|\mathbb{E}_N p_i[Y_i(\hat{\eta}) - Y_i(\eta_0)]\| = O_P(B_N + \xi_d \kappa_N) = o(1) \tag{7.1}
\]

Proof. Define an event \(\mathcal{E}_N := \{\hat{\eta}_k \in T_N \quad \forall k \in [K]\}\), such that the nuisance parameter estimate \(\hat{\eta}_k\) belongs to the realization set \(T_N\) for each fold \(k \in [K]\). By union bound, this event holds w.h.p.
\[
P_{P_N}(\mathcal{E}_N) \geq 1 - K \epsilon_N = 1 - o(1).
\]

Conditionally on \((W_i)_{i \in J_k}\), the estimator \(\hat{\eta} = \hat{\eta}_k\) is non-stochastic. On the event \(\mathcal{E}_N\)
\[
\mathbb{E}_N p_i[Y_i(\hat{\eta}) - Y_i(\eta_0)] = \frac{1}{K} \sum_{k=1}^{K} \mathbb{E}_{n,k} p_i[Y_i(\hat{\eta}) - Y_i(\eta_0)] - \mathbb{E}_{i \in J_k} \left[ p_i[Y_i(\hat{\eta}) - Y_i(\eta_0)] \right]_{i \in J_k}
\]
\[
+ \mathbb{E}_{i \in J_k} \left[ p_i[Y_i(\hat{\eta}) - Y_i(\eta_0)] \right]_{i \in J_k} - \mathbb{E}_{i \in J_k} \left[ p_i[Y_i(\hat{\eta}) - Y_i(\eta_0)] \right]_{i \in J_k}
\]

Hence, \(\sqrt{n} I_{1,k} = O_{P_N}(\xi_d \kappa_N)\) by Lemma 7.1. To bound \(I_{2,k}\), recognize that on the event \(\mathcal{E}_N\)
\[
\mathbb{E}[\sqrt{n} I_{2,k} | \mathcal{E}_N, (W_i)_{i \in J_k}] \leq \sup_{\eta \in T_N} \mathbb{E}[p_i(Y_i(\eta) - Y_i(\eta_0))]^2 \mathbb{E}_N, (W_i)_{i \in J_k}
\]
\[
\leq \sup_{\eta \in T_N} \mathbb{E}[p_i(Y_i(\eta) - Y_i(\eta_0))]^2 \\
\leq \sup_{x \in \mathcal{X}} \|p(x)\|^2 \sup_{\eta \in T_N} \mathbb{E}(Y_i(\eta) - Y_i(\eta_0))^2 \\
\leq \xi_d^2 \kappa_N^2 \quad (Assumption 3.5)
\]

Hence, \(\sqrt{n} I_{1,k} = O_{P_N}(\xi_d \kappa_N)\) by Lemma 7.1. To bound \(I_{2,k}\), recognize that on the event \(\mathcal{E}_N\)
\[
\mathbb{E}[\sqrt{n} I_{2,k} | \mathcal{E}_N, (W_i)_{i \in J_k}] \leq \sup_{\eta \in T_N} \sqrt{n} \mathbb{E} p_i(Y_i(\eta) - Y_i(\eta_0)) (W_i)_{i \in J_k}
\]
\[
\leq \sup_{\eta \in T_N} \sqrt{n} \mathbb{E} p_i(Y_i(\eta) - Y_i(\eta_0)) \leq B_N
\]
Therefore, $\sqrt{n}I_{2,k} = O_{P_N}(B_N)$ and $\sum_{k=1}^{K} \frac{1}{K} \sqrt{N}(I_{1,k} + I_{2,k}) = O_{P_N}(\xi_d\kappa + B_N)$.

7.3 Proofs of Theorems

Proof of Theorem 3.1 (a).

\[ \|\hat{\beta} - \beta_0\| = \|\hat{Q}^{-1}E_N p_i Y_i(\hat{\eta}) - \beta_0\| \]
\[ \leq \|\hat{Q}^{-1}||E_N p_i [Y_i(\hat{\eta}) - Y_i(\eta_0)]|| + \|\hat{Q}^{-1}||E_N p_i [Y_i(\eta_0) - p'_i\beta_0]|| \]
\[ = S_1 + \|\hat{Q}^{-1}||E_N p_i [Y_i(\eta_0) - g(x_i)]|| + \|\hat{Q}^{-1}||E_N p_i [g(x_i) - p'_i\beta_0]|| \]
\[ = S_1 + ||E_N p_i u_i|| + ||\hat{Q}^{-1}||E_N p_i r_i|| \]
\[ ||E_N p_i u_i|| \lesssim_P (E[||E_n p_i u_i||^2])^{1/2} \quad \text{(Markov)} \]
\[ \lesssim (E u_i^2 p_i/N)^{1/2} \quad \text{(i.i.d data)} \]
\[ \lesssim \bar{\sigma} \sqrt{d/N} \quad \text{(7.2)} \]

\[ ||E_N p_i r_i|| \lesssim_P (E[||E_n p_i r_i||^2])^{1/2} \quad \text{(Markov)} \]
\[ \lesssim l_d r_d \sqrt{\frac{E p_i^2}{N}} = l_d r_d \sqrt{\frac{d}{N}} \quad \text{(Assumption 3.3)} \]

Alternatively,

\[ ||E_N p_i r_i|| \lesssim_P (E[||E_n p_i r_i||^2])^{1/2} \quad \text{(Markov)} \]
\[ \lesssim \xi_d \sqrt{\frac{E r_i^2}{N}} = \xi_d r_d / \sqrt{N} \quad \text{(Assumption 3.3)} \]

With high probability, $||\hat{Q}^{-1}|| \leq 2||Q^{-1}|| \leq 2/\lambda_{\min}$. Lemma 7.3 implies

\[ ||S_1|| \leq ||\hat{Q}^{-1}||E_N p_i [Y_i(\hat{\eta}) - Y_i(\eta_0)]|| \leq ||\hat{Q}^{-1}||[B_N/\sqrt{N} + \xi_d\kappa / \sqrt{N}] = o\left(\frac{1}{\sqrt{N}}\right) \]

Proof of Theorem 3.1 (b). By Definition 2.2,

\[ \hat{\beta} = \hat{Q}^{-1}E_N p_i Y_i(\hat{\eta}) \]
Decomposing

\[
\sqrt{N}[E_{NP_i}Y_i(\hat{\eta}) - \hat{Q}\beta_0] = \sqrt{N}E_{NP_i}[Y_i(\hat{\eta}) - Y_i(\eta_0)] + \sqrt{N}E_{NP_i}[Y_i(\eta_0) - \hat{p}'\beta_0]
\]
\[
= \sqrt{N}E_{NP_i}[Y_i(\hat{\eta}) - Y_i(\eta_0)] + \mathbb{G}_{NP_i}[Y_i(\eta_0) - \hat{p}'\beta_0]
\]
\[
= \sqrt{N}E_{NP_i}[Y_i(\hat{\eta}) - Y_i(\eta_0)] + \mathbb{G}_{NP_i}u_i + \mathbb{G}_{NP_i}r_i
\]

we obtain:

\[
\sqrt{N}\alpha'[\hat{\beta} - \beta_0] = \sqrt{N}\alpha'\hat{Q}^{-1}[E_{NP_i}Y_i(\hat{\eta}) - \hat{Q}\beta_0] \quad (7.3)
\]
\[
= \sqrt{N}\alpha'\hat{Q}^{-1}[E_{NP_i}[Y_i(\hat{\eta}) - Y_i(\eta_0)]]
\]
\[
+ \alpha'\hat{Q}^{-1}\mathbb{G}_{NP}[p_i[r_i + u_i]]
\]
\[
= \alpha'Q^{-1}\mathbb{G}_{NP}[p_i[r_i + u_i]]
\]
\[
+ \alpha'\hat{Q}^{-1}[\hat{Q}^{-1} - Q^{-1}]\mathbb{G}_{NP}[p_i(u_i + r_i)]
\]
\[
+ \sqrt{N}\alpha'\hat{Q}^{-1}E_{NP_i}[Y_i(\hat{\eta}) - Y_i(\eta_0)]
\]
\[
+ \sqrt{N}\alpha'\hat{Q}^{-1}E_{NP_i}[Y_i(\eta_0) - Y_i(\eta_0)]
\]
\[
= \sqrt{N}\alpha'\hat{Q}^{-1}E_{NP_i}[Y_i(\hat{\eta}) - Y_i(\eta_0)] + \alpha'\hat{Q}^{-1}[\hat{Q}^{-1} - Q^{-1}]\mathbb{G}_{NP}[p_i(u_i + r_i)]
\]

Total remainder term equals:

\[
R_{1,N}(\alpha) = I_1 + I_2 + I_3
\]

Decomposing \(I_1\) into sampling and approximation parts:

\[
I_1 = \frac{\sqrt{N}\alpha'[\hat{Q}^{-1} - Q^{-1}]\mathbb{E}_{NP_i}[u_i] + \alpha'\hat{Q}^{-1}[\hat{Q}^{-1} - Q^{-1}]\mathbb{G}_{NP_i}[r_i]}{I_{1,a}} + \frac{\alpha'\hat{Q}^{-1}[\hat{Q}^{-1} - Q^{-1}]\mathbb{G}_{NP_i}[p_i(u_i + r_i)]}{I_{1,b}}
\]

Definition of regression error \(\mathbb{E}[u_i|x_i] = 0\) and \(\mathbb{E}[u_i^2|x_i] \leq \sigma^2\) yields:

\[
\mathbb{E}[I_{1,a}(x_i)] = 0
\]
\[
\mathbb{E}[I_{1,a}^N(x_i)] \leq \alpha'[\hat{Q}^{-1} - Q^{-1}]Q[\hat{Q}^{-1} - Q^{-1}]\alpha\sigma^2
\]
\[
\leq \frac{\xi_2^2\log N}{N}\sigma^2 \quad \text{(Lemma 7.1)}
\]
Therefore, \( I_{1,a} = o_P(\sqrt{\frac{\xi_d^2 \log N}{N}}) \). Using similar argument,

\[
|I_{1,b}| \leq \sqrt{\frac{\xi_d^2 \log N}{N}} [l_d r_d \sqrt{d} \wedge \xi_d r_d]
\]

\[
|I_2| \leq_P \|Q\|^{-1} \|\sqrt{N}E_N p[\tilde{Y}_i(\tilde{\eta}) - Y_i(\eta_0)]\| \leq_P 1/\lambda_{\min}[\xi_d \kappa_N + B_N] = o_P(1)
\]

\[
|I_3| \leq \|\alpha\| \|Q^{-1} - Q^{-1}\| \|\sqrt{N}E_N p[\tilde{Y}_i(\tilde{\eta}) - Y_i(\eta_0)]\|
\]

\[
\leq_P \sqrt{\frac{\xi_d^2 \log N}{N}} \xi_d \kappa_N + B_N = o(1)
\]

Therefore, with probability approaching one,

\[
\sup_{\eta \in T} \|R_{1,N}(\alpha)\| \leq B_N \vee \xi_d \kappa_N \vee \sqrt{\frac{\xi_d^2 \log N}{N}} [1 + l_d r_d \sqrt{d} \wedge \xi_d r_d]
\]

\[
\blacksquare
\]

**Proof of Theorem 3.1 (c).** Proof of Theorem 3.1 (c) follows from Theorem 4.2 in Belloni et al. (2015).

\[
\blacksquare
\]

**Lemma 7.4** (Lemma 2, Matrix Convergence Theory Hansen (2014)). Suppose \((w_i)^N_{i=1}\) be i.i.d sequence of \(d\)-vectors such that \(\|w_i\| \leq \xi_d^2\) a.s. Suppose that for some \(p > 2\)

\[
(\mathbb{E}\|w_i\|^p)^{1/p} \leq \xi_d
\]

\[
\ell_N = \begin{cases} 
N^{1/2} \xi_d^{p/(p-2)} (\log d)^{(p-4)/(2p-4)}, & \text{if } p > 4 \\
N^{-1/2} \xi_d^{p/(p-2)} (\log d)^{(p-4)/(2p-4)}, & \text{if } 2 < p \leq 4
\end{cases}
\]

Then, w.p. \( \rightarrow 1 \),

\[
\|\mathbb{E}_N w_i w'_i - \mathbb{E} w_i w'_i\| \lesssim_P d_N
\]

**Proof.** Proof of Theorem 3.1 (d)

Step 1. We will show that \( \hat{\Sigma} := \mathbb{E}_N p_i p'_i[\hat{Y}_i - p_i \hat{\beta}]^2 \) is consistent for \( \Sigma = \mathbb{E} p_i p'_i[Y_i - p_i \beta_0]^2 \) in Step 2. Incorporating the consequence of LLN for Matrices (Lemma 7.1)

\[
\|\hat{Q} - Q\| \lesssim_P \sqrt{\frac{\xi_d^2 \log N}{N}}
\]
Therefore,

\[ \|\hat{\Omega} - \Omega\| = \|\hat{Q}^{-1}\hat{\Sigma}Q^{-1} - Q^{-1}\Sigma Q^{-1}\| \lesssim_P o(1) \]

Step 2.

\[
\|\hat{\Sigma} - \Sigma\| = \frac{E_Np_ip'_i[\hat{Y}_i - Y_i]^2}{K_1} + \frac{E_Np_ip'_i[Y_i - p'_i\beta_0]^2 - E[p_ip'_i[Y_i - p'_i\beta_0]^2]}{K_2}
+ \frac{E[p_ip'_i[p'_i\beta_0 - p'_i\hat{\beta}]^2]}{K_3}
\]

Step 1. \( K_1 \). Define an event \( \mathcal{E}_N := \{\hat{\eta}_k \in T_N \ \forall k \in [K]\} \), such that the nuisance parameter estimate \( \hat{\eta}_k \) belongs to the realization set \( T_N \) for each fold \( k \in [K] \). By union bound, this event holds w.p \( 1 - o(1) \): \( P_{P_N}(\mathcal{E}_N) \geq 1 - K\epsilon_N = 1 - o(1) \).

\[
K_1 = \frac{1}{K} \sum_{k=1}^{K} \frac{E_{n,k}p_ip'_i[\hat{Y}_i - Y_i]^2}{K_1} := \frac{1}{K} \sum_{k=1}^{K} \left[ \frac{E_{n,k}p_ip'_i[\hat{Y}_i - Y_i]^2 - E[p_ip'_i[\hat{Y}_i - Y_i]^2](W_i)_{i \in J_k^c}}{K_{1,k,1}} + \frac{E[p_ip'_i[\hat{Y}_i - Y_i]^2((W_i)_{i \in J_k^c})]}{K_{1,2,k}} \right]
\]

Conditionally on \((W_i)_{i \in J_k^c}\), the estimator \( \hat{\eta}_k \) is non-stochastic. On the event \( \mathcal{E}_N \) let us apply Lemma 7.4 for \( w_i := p_i[Y_i(\hat{\eta}) - Y_i] \). By Assumption 3.6, for \( p = q > 2 \), \( \sup_{\eta \in T_N} E|w_i|^q \leq \xi_d(E|Y_i(\eta) - Y_i(\eta_0)|^q)^{1/q} \leq \xi_d \). For \( d_N \) defined in Lemma 7.4, conditionally on \((W_i)_{i \in J_k^c}\), \( K_{1,k,1} = \|E_{n,k}w_iw'_i - E[w_iw_i(W_i)_{i \in J_k^c}]\| = O_P(d_N) = o_P(1) \). Lemma 7.1 implies that \( K_{1,k,1} = O_P(d_N) = o_P(1) \) unconditionally. The sum of \( K \) \( O_P(d_N) \) terms yields: \( \frac{1}{K} \sum_{k=1}^{K} \|K_{1,k,1}\| = O_P(d_N) \).

\[
\|K_{1,2,k}\| = \|E[p_ip'_i[\hat{Y}_i - Y_i]^2](W_i)_{i \in J_k^c}\| \leq \sup_{\alpha : \|\alpha\| = 1} \|E[(\alpha'p_i)p'_i[\hat{Y}_i - Y_i](W_i)_{i \in J_k^c}]\|^2 \\
\leq \xi_d^2\|E[p_ip'_i[\hat{Y}_i - Y_i](W_i)_{i \in J_k^c}]\|^2 \\
\leq \xi_d^2/NB_N^2 = o(1) \quad \text{(Lemma 7.3, Lemma 7.1)}
\]

Step 2. \( K_2 \). Set \( w_i := p_i[r_i + U_i] = p_i[Y_i - p'_i\beta_0] \). By Assumption 3.6, for \( p = m > 2 \), \( E|w_i|^m \leq \xi_d d r_d + \xi_d(E|U_i|^m)^{1/m} \leq \xi_d \). For \( d_N \) defined in Lemma 7.4, \( K_2 = \|E_N w_iw'_i - \)
\( \mathbb{E} \| w_i u_i \| = O_P(d_N) = o_P(1) \). To bound \( K_3 \), recognize that

\[
K_3 \lesssim_P \| \mathbb{E} N p_i p'_i \| \max_{1 \leq i \leq N} \| p'_i (\hat{\beta} - \beta_0) \|^2
\]
\[
\lesssim_P \| \mathbb{E} N p_i p'_i \| \max_{1 \leq i \leq N} (\hat{\beta} - \beta_0)' p_i p_i (\hat{\beta} - \beta_0)
\]
\[
\lesssim_P \| \mathbb{E} N p_i p'_i \| \| \hat{\beta} - \beta_0 \|^2 \max_{1 \leq i \leq n} \| p_i p'_i \|
\]
\[
\lesssim_P [\lambda_{\text{max}} + O_P(\sqrt{\frac{\epsilon_d^2 \log N}{N}})] O_P(\frac{d}{N} \max_{1 \leq i \leq n} \| p_i p'_i \|) = o_P(1) \quad \text{(Assumption 3.6)}
\]

The bounds on \( K_1 - K_3 \) conclude the proof of Theorem 3.1[d].

---

**Proof of Theorem 3.2(a).** Similar to the Equation 7.3, define

\[
I_1(x) = \alpha(x) \top [\hat{Q}^{-1} - Q^{-1}] \mathbb{G}_N [p_i (u_i + r_i)]
\]
\[
I_2(x) = \sqrt{N} \alpha(x) \top Q^{-1} \mathbb{E} N p_i [Y_i(\eta) - Y_i(\eta_0)]
\]
\[
I_3(x) = \sqrt{N} \alpha(x) \top [\hat{Q}^{-1} - Q^{-1}] \mathbb{E} N p_i [Y_i(\eta) - Y_i(\eta_0)]
\]

Decompose

\[
\sqrt{N} \alpha(x) \top (\hat{\beta} - \beta_0) = \sqrt{N} \alpha(x) \top Q^{-1} \mathbb{G}_N [p_i [r_i + u_i]] + I_1(x) + I_2(x) + I_3(x)
\]

Step 1. Bound on \( I_1(x) \) is shown in Step 1 of Lemma 4.2 of Belloni et al. (2015). We copy their proof for completeness. Let us show that

\[
\sup_{x \in \mathcal{X}} |\alpha(x) \top [\hat{Q}^{-1} - Q^{-1}] \mathbb{G}_N [p_i u_i]| \lesssim_P N^{1/m} \sqrt{\frac{\epsilon_d^2 \log^2 N}{N}}
\]

Conditional on the data, let \( T = \{t_1, t_2, \ldots, t_N\} \in \mathcal{R}^N : t_i = \alpha(x) \top (\hat{Q}^{-1} - Q^{-1}) p_i u_i, x \in \mathcal{X} \).

Define the norm \( \| \cdot \|_{N,2} = \sum_{i=1}^N t_i^2 \) on \( \mathcal{R}^N \). Let \( (\gamma_i)_{i=1}^N \) be independent Rademacher random variables \( P(\gamma_i = 1) = P(\gamma_i = -1) = \frac{1}{2} \). Symmetrization inequality (i) and Dudley (1967) (ii) imply:

\[
\mathbb{E} \sup_{x \in \mathcal{X}} |\alpha(x) \top [\hat{Q}^{-1} - Q^{-1}] \mathbb{G}_N p_i u_i| \leq 2 \mathbb{E}_\gamma \sup_{x \in \mathcal{X}} |\alpha(x) \top [\hat{Q}^{-1} - Q^{-1}] \mathbb{G}_N p_i u_i| |\gamma_i
\]
\[
\leq C \int_0^\theta (\log N(\epsilon, T, \| \cdot \|_{N,2}) \)^{1/2} d\epsilon := J
\]

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where $N(\epsilon, T, \|\cdot\|_{N,2})$ is the covering number of set $T$ and 

$$
\theta = 2 \sup_{t \in T} \|t\|_{N,2} \leq 2 \|\hat{Q}^{-1} - Q\| \|\hat{Q}\|^{1/2} \max_{1 \leq i \leq N} |u_i| 
$$

Since for any $x \neq x'$:

$$
[\alpha(x) - \alpha(x')]^\top [\hat{Q}^{-1} - Q^{-1}] p_i u_i \leq \xi_d^L \|x - x'\| \|\hat{Q}^{-1} - Q^{-1}\| \|\hat{Q}\|^{1/2} \max_{1 \leq i \leq N} |u_i| 
$$

we have for some $C > 0$:

$$
N(\epsilon, T, \|\cdot\|_{N,2}) \leq \left( \frac{\|\hat{Q}^{-1} - Q^{-1}\| \|\hat{Q}\|^{1/2} \max_{1 \leq i \leq N} |u_i|}{\epsilon} \right)^d 
$$

$$
\int_0^\theta (\log N(\epsilon, T, \|\cdot\|_{N,2}))^{1/2} d\epsilon \leq \|\hat{Q}^{-1} - Q\| \|\hat{Q}\|^{1/2} \max_{1 \leq i \leq N} |u_i| \int_0^2 \sqrt{d} \log^{1/2}(C\xi_d^L/\epsilon) d\epsilon 
$$

By Assumption 3.6, we have

$$
\mathbb{E} \max_{1 \leq i \leq N} |u_i| \leq (\mathbb{E} \max_{1 \leq i \leq N} |u_i|)^{1/m} \leq (\mathbb{E} |u_i|^m)^{1/m} \leq N^{1/m} 
$$

By Assumption 3.2, $\|\hat{Q} - Q\| \lesssim_P \sqrt{\frac{\xi_d^L \log N}{N}}$. Finally, from Assumption 3.8,

$$
\log \xi_d^L \lesssim \log d \lesssim \log N 
$$

$$
J \lesssim_P N^{1/m} \sqrt{\frac{\xi_d^L \log^2 N}{N}} 
$$

Step 2. Observe that:

$$
\sup_{x \in \mathcal{X}} |\alpha(x)^\top [\hat{Q}^{-1} - Q^{-1}] G_N[p_i r_g(x_i)]| \lesssim_P \sqrt{\frac{\xi_d^L \log N}{N}} l_d r_d \sqrt{d} 
$$

Steps 1 and 2 give the bound on $I_1(x)$. Step 3. Define an event $\mathcal{E}_N := \{\hat{\eta}_k \in T_N \ \forall k \in [K]\}$, such that the nuisance parameter estimate $\hat{\eta}_k$ belongs to the realization set $T_N$ for each fold $k \in [K]$. By union bound, this event holds w.h.p.

$$
P_{P_N}(\mathcal{E}_N) \geq 1 - K \epsilon_N = 1 - o(1).$$
Decompose $I_2(x)$ as follows:

$$I_2(x) = \alpha(x)^\top Q^{-1} \mathbb{E}_N p_i [Y_i(\tilde{\eta}) - Y_i(\eta_0)] = \frac{1}{K} \sum_{k=1}^{K} \frac{\mathbb{E}_{n,k} \alpha(x)^\top Q^{-1} p_i [Y_i(\tilde{\eta}) - Y_i(\eta_0)]}{I_{2,k}(x)}$$

$$I_3(x) = \alpha(x)^\top [\hat{Q}^{-1} - Q^{-1}] \mathbb{E}_N p_i [Y_i(\tilde{\eta}) - Y_i(\eta_0)] = \frac{1}{K} \sum_{k=1}^{K} \frac{\mathbb{E}_{n,k} \alpha(x)^\top [\hat{Q}^{-1} - Q^{-1}] p_i [Y_i(\tilde{\eta}) - Y_i(\eta_0)]}{I_{3,k}(x)}$$

It suffices to show that $\sup_{x \in X} |I_{i,k}(x)| = o_P(1)$ for all $i \in \{2, 3\}, k \in [K]$.

Define $G_{n,k}f(x_i) := \frac{1}{n} \sum_{i \in J_k} f(x_i) - \mathbb{E}[f(x_i) | J_k^c]$ Define $v_i := p_i [Y_i(\tilde{\eta}) - Y_i(\eta_0)]$

$$|I_{21,k}(x)| \leq \sup_{x \in X} \sqrt{N} \left| \alpha(x)^\top Q^{-1} G_{n,k} v_i \right|_{I_{21,k}(x)} + \sup_{x \in X} \sqrt{N} \left| \alpha(x)^\top Q^{-1} \mathbb{E} v_i | (W_i)_{i \in J_k} \right|_{I_{22,k}(x)}$$

$$|I_{22,k}(x)| \leq \|\alpha\| \|Q^{-1}\| \sup_{\eta \in T_N} \|\sqrt{N} \mathbb{E} v_i | (W_i)_{i \in J_k} \| \leq \sup_{\eta \in T_N} \|\sqrt{N} \mathbb{E} v_i \| = B_N = o(1)$$

(Assumption 3.5)

To bound $\sup_{x \in X} I_{21,k}(x)$, we apply Lemma 7.2 conditionally on $J_k^c$ so that $\tilde{\eta}_k$ can be treated as fixed. Consider a function class

$$\mathcal{F} := \{f_i = \alpha(x) Q^{-1} v_i, x \in X\}$$

with an square integrable envelope $F := \|\alpha(x)\| \|v_i\| = \|v_i\|$

$$\|F\|_{L^2} = \mathbb{E} \|v_i\|^2 \leq \|Q^{-1}\|_F^2 \sup_{\eta \in T_N} \mathbb{E} (Y_i(\eta) - Y_i(\eta_0))^2 \leq \|Q^{-1}\|_F^2 \xi_d^2 \kappa_N^2$$

Define the second moment bound $\sigma^2 := \mathbb{E} \|v_i\|^2 = O(\xi_d^2 \kappa_N^2)$ which uniformly bounds the second moment of every element in $\mathcal{F}$

$$\sup_{f \in \mathcal{F}} \mathbb{E} f^2 \leq \sigma^2 = \|v_i\|_{L^2}^2$$

To determine the bracket size, recognize that

$$|\{\alpha(x) - \alpha(x')\}^\top Q^{-1} v_i| \leq \xi_d^L \|x - x'\| \|Q^{-1}\| \|v_i\|$$
and therefore
\[
\sup_Q N(F, L^2(Q), \epsilon \| F \|_{L^2_Q}) \leq \left(\frac{\xi_d^L / \lambda_{\min}}{\epsilon} \right)^r
\]

Plugging in \(\sigma = \| F \|_{L^2_P} = \xi_d \kappa N\), \(A = \xi_d^L / \lambda_{\min}\), \(V = r\) into Lemma 7.2, we obtain:

\[
\sup_{x \in X} \left| \alpha(x)^\top Q - 1 G_{n,k} [Y_i(\tilde{\eta}) - Y_i(\eta_0)] \right| \lesssim P \sqrt{\log \xi_d^L + N^{-1/2 + 1/q} \log \xi_d^L} \quad (7.5)
\]

\[
\lesssim P \xi_d \kappa N \sqrt{\log N} + N^{-1/2 + 1/q} \log N \quad (7.6)
\]

Lemma 7.1 implies that the bounds on \(I_{21,k}(x)\) and \(I_{22,k}(x)\) are unconditional. Therefore,

\[
\sup_{x \in X} |I_{21}(x)| \lesssim P \frac{1}{K} \sum_{k=1}^K \left( \sup_{x \in X} |I_{21,k}(x)| + \sup_{x \in X} |I_{22,k}(x)| \right)
\]

\[
\lesssim P \xi_d \kappa N \sqrt{\log N} + N^{-1/2 + 1/q} \log N + B_N
\]

Step 4. We wish to bound \(I_3(x)\).

\[
|I_{3,k}(x)| \leq \sup_{x \in X} |\alpha(x)^\top (Q^{-1} - Q^{-1}) G_{n,k} v_i| + \sup_{x \in X} \sqrt{N} |\alpha(x)^\top (Q^{-1} - Q^{-1}) E[v_i|J_k]| 
\]

\[
\leq \|Q^{-1} - Q^{-1}\| \|G_{n,k} v_i\| + \sup_{x \in X} \|\alpha(x)\| \sqrt{\frac{\xi_d^2 \log N}{N} B_N} 
\]

\[
\leq \sqrt{\frac{\xi_d^2 \log N}{N} (\xi_d \kappa N \sqrt{\log N} + N^{-1/2 + 1/q} \log N)} + O_P(\sqrt{\frac{\xi_d^2 \log N}{N} B_N})
\]

Theorem 3.2(b) is established in Belloni et al. (2015). Proof of Corollary 3.1 follows from Theorem 3.2 (a) and Theorem 4.4 of Belloni et al. (2015).

\[\square\]

**Proof of Corollary 4.1.** Let us show that \(Y_\eta\) given by Equation 2.1 is an efficient signal that satisfies Assumption 3.5.
\[
Y_\eta - Y_{\eta_0} = \left[ \mu(1, Z) - \mu_0(1, Z) \right] \left[ 1 - \frac{D}{s_0(Z)} \right] s_1 - \left[ \mu(0, Z) - \mu_0(0, Z) \right] \left[ 1 - \frac{1 - D}{1 - s_0(Z)} \right] s'_1 \\
+ [s_0(Z) - s(Z)][Y^1 - \mu_0(1, Z)] \frac{D}{s(Z)s_0(Z)} s_2 - [Y^0 - \mu_0(0, Z)] \left( 1 - \frac{1 - D}{1 - s(Z)(1 - s_0(Z))} \right) s'_2 \\
+ [s_0(Z) - s(Z)][\mu(1, Z) - \mu_0(1, Z)] \frac{D}{s(Z)s_0(Z)} s_3 - [\mu(1, Z) - \mu_0(0, Z)] \left( 1 - \frac{1 - D}{1 - s(Z)(1 - s_0(Z))} \right) s'_3
\]

Let us see that Assumption 3.5 (1) is satisfied with \( B_N := \xi_d m_N s_N \). The terms \( S_i, S'_i, i \in \{1, 2\} \) are mean zero conditionally on \( Z \). Assumption 2.1 implies that any technical regressor \( p(X) \) is uncorrelated with \( S_1 + S'_1 + S_2 + S'_2 \).

\[
E\left( [S_1 + S'_1 + S_2 + S'_2]|Z \right) = 0 \Rightarrow E[p(X)|[S_1 + S'_1 + S_2 + S'_2]] = 0
\]

\[
||E[p(X)|S_3 + S'_3]|Z\|^2 = \sum_{j=1}^d (E[p_j(X)|S_3 + S'_3])^2 \leq \sum_{j=1}^d (E[p_j^2(X)E(S_3 + S'_3)]^2 \leq \xi_d^2 (m_N)^2 (s_N)^2
\]

Assumption 3.5 (2) is satisfied with \( \kappa_N := [m_N^1 \lor m_N \lor s_N] \).

\[
E[S_1^2] = E[\mu(1, Z) - \mu_0(1, Z)]^2 E\left[ 1 - \frac{D}{s_0(Z)} \right]^2 |Z = z] \preceq (m_N)^2
\]

\[
E(S'_1)^2 = E[\mu(0, Z) - \mu_0(0, Z)]^2 E\left[ 1 - \frac{1 - D}{1 - s_0(Z)} \right]^2 |Z = z] \leq (m_N)^2
\]

\[
E_s^2 = E[s(Z) - s_0(Z)]^2 E\left[ [Y^1 - \mu_0(1, Z)]^2 |Z = z] \frac{1}{s_0(Z)s_0(Z)} \right] \leq s_N^2 \frac{1}{s_0^2}
\]

\[
E_s^2 = E[s(Z) - s_0(Z)]^2 E\left[ [Y^0 - \mu_0(0, Z)]^2 |Z = z] \frac{1}{1 - s_0(Z)(1 - s(Z))} \right] \leq s_N^2 \frac{1}{s_0}
\]

To sum up,

\[
\xi_d (E(Y_\eta - Y_{\eta_0})^2)^{1/2} \leq_P \xi_d (m_N \lor s_N)
\]

**Proof of Corollary 4.2.** Let us show that \( Y_\eta \) given by Equation 2.2 is an efficient signal that satisfies Assumption 3.5
Let us see that Assumption 3.5 (1) is satisfied with \( B_N := \xi_d m_N s_N \). The terms \( S_1, S_2 \) are mean zero conditionally on \( Z \). Assumptions 2.2 implies that any technical regressor \( p(X) \) is uncorrelated with \( S_1 + S_2 \).

\[
\|\mathbb{E}p(X)S_3\| \lesssim \xi_d m_N s_N
\]

\[
\mathbb{E}S_1^2 = \mathbb{E}[\mu(Z) - \mu_0(Z)]^2 \mathbb{E}[(1 - \frac{D}{s_0(Z)})^2 | Z = z] \lesssim m_N^2
\]

\[
\mathbb{E}S_2^2 = \mathbb{E}[s(Z) - s_0(Z)]^2 \mathbb{E}[(Y - \mu_0(Z))^2 | Z = z] \frac{1}{s_0(Z)s^2(Z)} \lesssim s_N^2 \frac{1}{s_0^3}
\]

\[
\mathbb{E}S_3^2 = \mathbb{E}[s_0(Z) - s(Z)]^2 [\mu(Z) - \mu_0(Z)]^2 \frac{1}{s_0^2 s_0(Z)} \lesssim \min(m_N^2, s_N^2) \frac{1}{s_0^3}
\]

To sum up,

\[
\xi_d (\mathbb{E}(Y - Y_{\theta})^2)^{1/2} \lesssim_p \xi_d [m_N \lor s_N]
\]

**Proof of Corollary 4.3.** Let us show that \( Y_\theta \) given by Equation 2.2 is an efficient signal that satisfies Assumption 3.5

\[
Y_\theta - Y_{\theta_0} = \underbrace{-\partial_w \log f_0(W | X)[\mu(X, W) - \mu_0(X, W)] + \partial_w [\mu(X, W) - \mu_0(X, W)]}_{S_1}
\]

\[
+ \underbrace{[\partial_w \log f_0(W | X) - \partial_w \log f(W | X)][Y - \mu_0(X, W)]}_{S_2}
\]

\[
+ \underbrace{[\partial_w \log f_0(W | X) - \partial_w \log f(W | X)][\mu(X, W) - \mu_0(X, W)]}_{S_3}
\]

Let us see that Assumption 3.5 (1) is satisfied with \( B_N := \xi_d f_N m_N \). The terms \( S_1, S_2 \) are mean zero conditionally on \( Z \). Since \( X \subset Z \), any technical regressor \( p(X) \) is uncorrelated with \( S_1 + S_2 \)

\[
\|\mathbb{E}p(X)S_3\| = \|\mathbb{E}p(X)S_3\| \lesssim \xi_d f_N m_N
\]
\[ E S_1^2 \leq E(-\partial_w \log f_0(W|X)[\mu(X,W) - \mu_0(X,W)])^2 + E(\partial_w[\mu(X,W) - \mu_0(X,W)])^2 \preceq m_N^2 \]
\[ E S_2^2 \leq (E[|Y_\eta - \mu_0(X,W)|^2|X,W]) f_N^2 \preceq f_N^2 \]
\[ E S_3^2 \leq (E[|\mu(X,W) - \mu_0(X,W)|^2]) f_N^2 \preceq \min[f_N^2 \lor m_N^2] \]
\[ \xi_d (E(Y_\eta - Y) \eta - \mu_0(X,W))^2 \] \[ \preceq \xi_d [f_N \lor m_N] \]

7.4 Proof of Example 5

We establish the argument in the following steps.

Proof. Define \( \tilde{Y} = DY, \tilde{p}_\mu(Z) = Dp_Z(Z), \tilde{r}_\mu(Z) = Dr_\mu(Z), \) and \( \tilde{\epsilon} = D[Y - \mu(Z)]. \) Step 1. Let us show that the original coefficient \( \theta, \) defined in Equation (4.1), satisfies

\[ \tilde{Y} = \tilde{D}' \theta + \tilde{r}_\mu(Z) + \tilde{\epsilon}, \quad E[\tilde{\epsilon}|\tilde{D}] = 0 \]

Indeed,

\[ E[\tilde{Y}|D, Z] = E[DY|D, Z] = D E[Y|Z, D] = D[\mu_0(Z)] = Dp_\mu(Z)' + Dr(Z) \]
\[ = \tilde{p}_\mu(Z)' + \tilde{r}_\mu(Z) \]

Therefore,

\[ E[\tilde{\epsilon}|\tilde{D}] = E[[\tilde{Y} - \tilde{p}_\mu(Z)' + \tilde{r}_\mu(Z)]|\tilde{D}] \]
\[ = E[E[|\tilde{Y} - \tilde{p}_\mu(Z)' + \tilde{r}_\mu(Z)||D, Z]|\tilde{D}] \]
\[ = 0 \]

Step 2. Recognize that Assumption 4.3 implies that an analog of Assumption 4.3 holds for \( \tilde{p}_\mu(Z), \tilde{Y} \). Assumption 4.3 (a) directly assumes bounded Restricted Sparse Eigenvalues for Observed Regressors \( \tilde{p}_\mu(Z) = p_\mu(Z)D \). Assumption 4.3 (b) is satisfied with \( E[\tilde{p}_\mu(Z)]^2 = EDp_\mu^2(Z) \geq \xi d E[p_\mu^2(Z)] = c' \) where \( c' := c_s \) is the new lower bound on the moments of \( E[\tilde{p}_\mu(Z)] \) for observed regressors. Assumption 4.3 (c) is satisfied with the \( E[r_\mu(Z)]^2 \) \( \preceq \xi d [f_N \lor m_N] \).

\[ \blacksquare \]
References


(a) Step 2: $l(x,z)$ is estimated by Lasso. Step 3: polynomials of degree 3.

(b) Step 2: $l(x,z)$ is estimated by random forest. Step 3: polynomials of degree 3.

(c) Step 2: $l(x,z)$ is estimated by Lasso. Step 3: $B$-splines of order 2 with 1 knot.

(d) Step 2: $l(x,z)$ is estimated by random forest. $B$-splines of order 2 with 1 knot.

Figure 1: 95% confidence bands for the best linear approximation of the average price elasticity conditional on income with accounting for the demographic controls in the first stage. The black line is the estimated function, the dashed (solid) blue lines are the pointwise (uniform) confidence bands. The estimation algorithm has three steps: (1) first-stage estimation of the conditional expectation function $\mu(w|x,z)$, (2) second-stage estimation of the conditional density $f(w|x,z)$, and (3) third-stage estimation of the target function $g(x)$ by least squares series. Step 1 is performed using Lasso with standardized covariates and the penalty choice $\lambda = 2.2\sqrt{n}\hat{\sigma}\Phi^{-1}(1 - \gamma/2p)$, where $\gamma = 0.1/\log n$ and $\hat{\sigma}$ is the estimate of the residual variance. Step 2 is performed by estimating the regression function of $l(x,z) = E[W|X = x, Z = z]$ and estimating the density $f(w - l(x,z))$ of the residual $w - l(x,z)$ by adaptive kernel density estimator of Portnoy and Koenker (1989) with the Silverman choice of bandwidth. The regression function $l(x,z)$ is estimated lasso (1a, 1c) and random forest (1b, 1d). Step 3 is performed using $B$-splines of order 2 with the number of knots equal to one (1c, 1d) and polynomial functions of order 3. (1a, 1b). $B = 200$ weighted bootstrap repetitions.
Figure 2: 95% confidence bands for the best linear approximation of the average price elasticity conditional on income without accounting for the demographic controls in the first stage. The black line is the estimated function, the dashed blue lines and the solid blue lines are the pointwise and the uniform confidence bands. The estimation algorithm has three steps: (1) first-stage estimation of the conditional expectation function \( \mu(w, x) = \mathbb{E}[Y | W = w, X = x] \), (2) second-stage estimation of the conditional density \( f(W = w | X = x) \), and (3) third-stage estimation of the target function \( g(x) \) by least squares series. Step 1 is performed using least squares series regression using polynomial functions \( \{1, x, \ldots, x^q\} \), \( q = 3 \) whose power \( q \) is chosen by cross-validation out of \( \{1, 2, 3\} \). Step 2 is performed by kernel density estimator with the Silverman choice of bandwidth. Step 3 is performed using polynomial functions \( \{1, x, \ldots, x^q\} \) and is shown for \( q = 1 \) and \( q = 2 \). \( B = 200 \) weighted bootstrap repetitions.
Figure 3: 95% confidence bands for the best linear approximation of the average price elasticity conditional on income with accounting for the demographic controls in the first stage by household size. The black line is the estimated function, the dashed (solid) blue lines are the pointwise (uniform) confidence bands. The estimation algorithm has three steps: (1) first-stage estimation of the conditional expectation function \( \mu(w, x, z) \), (2) second-stage estimation of the conditional density \( f(w|x, z) \), and (3) third-stage estimation of the target function \( g(x) \) by least squares series. Step 1 is performed using Lasso with standardized covariates and the penalty choice \( \lambda = 2.2 \sqrt{n\hat{\sigma}}\Phi^{-1}(1 - \gamma/2p) \), where \( \gamma = 0.1/\log n \) and \( \hat{\sigma} \) is the estimate of the residual variance. Step 2 is performed by estimating the regression function of \( l(x, z) = \mathbb{E}[W|X = x, Z = z] \) and estimating the density \( f(w - l(x, z)) \) of the residual \( w - l(x, z) \) by adaptive kernel density estimator of Portnoy and Koenker (1989) with the Silverman choice of bandwidth. The regression function \( l(x, z) \) is estimated lasso. Step 3 is performed using B-splines of order 2 with the number of knots equal to one (3c, 3d) and using non-orthogonal polynomial functions of degree 3 (3a, 3b). \( B = 200 \) weighted bootstrap repetitions.