On the Uniqueness of Solutions for Nonlinear and Mixed Complementarity Problems

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Abstract

The aim of this paper is to establish sufficient local conditions for the uniqueness of solutions to Nonlinear Complementarity Problems (NCP) and Mixed Complementarity Problems (MCP). Our main theorems state that for NCP and MCP defined by continuously differentiable functions, the solution is unique if the Jacobian of the function is a partial P-matrix at each solution. These theorems generalize the previous uniqueness results in a number of directions, including relaxing the strict complementary slackness requirement necessary in some of these approaches. The method of proof uses and extends a recent result by Simsek-Ozdaglar-Acemoglu [14] regarding the uniqueness of generalized critical points.

1 Introduction

Let $F : \mathbb{R}^n_+ \mapsto \mathbb{R}^n$ be a continuous function, where $\mathbb{R}^n$ denotes the n-dimensional Euclidean space and $\mathbb{R}^n_+$ denotes the nonnegative orthant in $\mathbb{R}^n$. The *nonlinear complementarity problem (NCP)* is to find a vector $x \in \mathbb{R}^n_+$ that satisfies the following:

$$x \geq 0, \quad F(x) \geq 0, \tag{1}$$

$$x^T F(x) = 0. \tag{2}$$

We denote the set of solutions to (1)-(2) by NCP(F). The NCP is a powerful framework for modeling equilibrium in a diverse set of problems that arise in engineering, economics, game theory, and finance; see [5] for applications.

A problem similar to the NCP is the *mixed complementarity problem (MCP)*, which is to find a vector $x \in [a, b] \subset \mathbb{R}^n$ such that for each $i \in \{1, .., n\}$ one of the following holds:

$$x_i = a_i, \qquad F_i(x) \geq 0 \tag{3}$$

$$a_i < x_i < b_i, \qquad F_i(x) = 0. \tag{4}$$

$$x_i = b_i, \qquad F_i(x) \leq 0 \tag{5}$$

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We denote the set of solutions to (3)-(5) by $\mathrm{MCP}(F, [a, b])$. In the literature, the case when one or more of a_i, b_i are $+\infty$ or $-\infty$ is also allowed. In this work, we restrict ourselves to the case when $a, b \in \mathbb{R}^n$.

In this work, we establish sufficient conditions on F for the uniqueness of solutions to the NCP and sufficient conditions on F and $[a, b]$ for the uniqueness of solutions to the MCP. There are two lines of earlier work concerning the uniqueness of solutions to NCP. The first line, originated by Samelson-Thrall-Wesler [13] for the linear complementarity problem (i.e., complementarity problems in which the defining function F is affine) and by Cottle [3] for the NCP, assumes global P-properties of the function F (see Karamardian [8], More [11]) and, when F is continuously differentiable, global P-matrix properties or uniform boundedness of the Jacobian of F (see Cottle [3], Megiddo-Kojima [10], Facchinei-Pang [5]).[1]

The second line of work originated in Saigal and Simon's work [12], which studied the NCP for the case when F is continuously differentiable using fixed point index theory (see Guillemin-Pollack [7] and Dold [4]). Saigal and Simon established that under a set of regularity conditions, local conditions on the Jacobian of F [which only need to be satisfied at the vectors in $\mathrm{NCP}(F)$] guarantees that $\mathrm{NCP}(F)$ has an odd number of elements. Their result was used and strengthened by Kolstad-Mathiesen [9], who, under similar conditions, established the uniqueness of Cournot equilibrium, formulated as an NCP. One of the regularity conditions necessary for the approach of Saigal and Simon [12] and Kolstad-Mathiesen [9] is the strict complementarity assumption (Assumption SCS-NCP below). This assumption is not only difficult to verify without characterizing all of the solutions of the NCP, but as also recently pointed out by Gaudet-Salant [6], it may be overly restrictive.[2]

In this paper, we generalize the uniqueness result by Saigal and Simon [12] (and of Kolstad-Mathiesen [9]) by relaxing the strict complementary slackness assumption. We instead introduce local and partial P-matrix properties on the Jacobian of F and show that these are sufficient to ensure uniqueness.

Our results thus also generalize results that require global P-matrix properties. Moreover, we provide a unified study of uniqueness for both the NCP and the MCP. Our proof relies on a recent result by Simsek-Ozdaglar-Acemoglu [14] regarding the uniqueness of generalized critical points of functions defined over compact regions defined by finitely many inequality constraints. We apply and generalize this result to study uniqueness of solutions for the MCP without regularity requirements. We then relate the solutions of the NCP and the MCP to establish the uniqueness of solutions of the NCP.

The organization of the paper is as follows. In the next section, we review Saigal and Simon [12]'s approach to NCP and state our main theorems both for NCP and MCP. In section 3, we define partial P-matrices and show properties of functions whose Jacobians are partial P-matrices, which will be used for the proofs of the main theorems. These

[1] Megiddo and Kojima also studied the *globally uniquely solvable property (GUS)* of the NCP, i.e., it has a unique solution and this property will not change even if any constant term is added to the function F. They provided sufficient conditions that involve global assumptions to establish GUS property of an NCP and related these conditions to earlier conditions of Cottle [3], Karamardian [8], and More [11] that establish uniqueness of solution to the NCP.

[2] For example, Gaudet-Salant [6] show that this assumption often fail in examples of practical interest.

results might also be of independent interest. In Section 4, we present proofs of the two main theorems. Section 5 presents examples demonstrating how our theorems improve over existing uniqueness results.

Regarding notation, all vectors are viewed as column vectors, and $x^T y$ denotes the inner product of the vectors x and y. We denote the 2-norm by $\|x\| = (x^T x)^{1/2}$. x_i denotes the i^{th} component of vector x in standard coordinates. For $x, y \in \mathbb{R}^n$, $x < y$ implies $x_i < y_i$ for all $i \in \{1, .., n\}$. We let $[x, y]$ denote the closed rectangle defined by vectors x, y, i.e.

$$[x, y] = \Big\{ u \in \mathbb{R}^n \mid x_i \leq u \leq y, \ \forall \ i \in \{1, .., n\} \Big\}.$$

For a given matrix A, A^{ij} denotes its entry in i^{th} row and j^{th} column. For an $n \times n$ matrix A and $J \subset \{1, 2, .., n\}$, let $A|_J$ denote the principal sub-matrix of A that contains the entries A^{ij} where $i, j \in J$. When X is a finite set, we use $|X|$ to denote its cardinality. For a closed set M, we use the notation $U|_M$ to denote an open set containing M. For a vector $x \in \mathbb{R}^n$ and a set $J \subset \{1, 2, .., n\}$, we use the notation $x|_J$ to denote the $|J|$-dimensional vector that contains the entries x_j where $j \in J$. If f is differentiable at x, then $\nabla f(x)$ denotes the gradient of f. We say that f is continuously differentiable at $x \in A$ if f is continuously differentiable over an open set $U \subset A$ containing x.

2 Uniqueness Results for the NCP and the MCP

2.1 Uniqueness for the NCP

To avoid any difficulties related to smoothness, we assume that F is defined over an open set $\mathbb{U}^n_+ \subset \mathbb{R}^n$ containing $\mathbb{R}^n_+$. Note that this assumption does not yield any loss of generality since any function $F : \mathbb{R}^n_+ \mapsto \mathbb{R}^n$ can be extended to a function $\bar{F}$ over $\mathbb{U}^n_+$ such that the extension preserves the differentiability properties of F over $\mathbb{R}^n_+$, i.e., for any $x \in \mathbb{R}^n_+$, F is (continuously) differentiable at x (restricted to the directions interior to $\mathbb{R}^n_+$) if only if $\bar{F}$ is (continuously) differentiable at x (see Saigal-Simon [12] for an explicit construction of one such extension).

We first review Saigal-Simon [12] and Kolstad-Mathiesen [9] results which establish the uniqueness of solutions to the NCP. Saigal-Simon [12] study the properties of the solution set of an NCP under the following assumptions.

Assumption BC (Boundary Condition): There exists a compact set $C \subset \mathbb{R}^n_+$ such that for all $x \in \mathbb{R}^n_+ - C$, there exists some $y \in C$ such that

$$\sum_{i \in \{1,..,n\}} (y_i - x_i) F_i(x) < 0.$$

Assumption ND-NCP (Non-degeneracy for the NCP): For each $x^* \in \text{NCP}(F)$, F is continuously differentiable at x^* and

$$\det(\nabla F(x^*)|_{I^{NB-NCP}(x^*)}) > 0,$$

where

$$I^{NB-NCP}(x) = \Big\{ i \in \{1,..,n\} \mid x_i > 0 \Big\}.$$

Assumption SCS-NCP (Strict Complementary Slackness for the NCP): For each $x^* \in \text{NCP}(F)$, $x^*_i = 0$ implies $F_i(x^*) > 0$, i.e.

$$I^{NB-NCP}(x^*) = I^F(x^*),$$

where

$$I^F(x^*) = \Big\{ i \in \{1,..,n\} \mid F_i(x^*) = 0 \Big\}. \tag{6}$$

The following result is implicit in Saigal-Simon [12], and is more explicitly stated by Kolstad-Mathiesen [9] (cf. Theorem 1 in [9]):

Theorem 1 Let $F : \mathbb{U}^n_+ \mapsto \mathbb{R}^n$ be a continuously differentiable function. Assume that F satisfies assumptions BC, SCS-NCP, and ND-NCP. Then, NCP(F) has a unique element.

We now introduce new assumptions that essentially relax Assumption SCS-NCP and BC to generalize Theorem 1. It can be seen that the following boundedness assumption is weaker than Assumption BC.

Assumption WBC (Weak Boundary Condition): There exists a compact set $C \subset \mathbb{R}^n_+$ such that for all $x \in \mathbb{R}^n_+ - C$, there exists some $y \in C$ and $i \in \{1,..,n\}$ such that

$$(y_i - x_i)F_i(x) < 0.$$

The following assumption is used to generalize the regularity requirements of Theorem 1, in particular, relaxing Assumption SCS-NCP. We say that $x^* \in \text{NCP}(F)$ is a *strongly non-degenerate* solution if F is continuously differentiable at x^* and

$$\det(\nabla F(x^*)|_J) > 0$$

for all J such that

$$I^{NB-NCP}(x^*) \subset J \subset I^F(x^*).$$

Assumption SND-NCP (Strong Non-degeneracy for the NCP): Each $x^* \in \text{NCP}(F)$ is strongly non-degenerate.

The following theorem, which is our main result for the uniqueness of the NCP, generalizes Theorem 1 in two directions: First, it requires a weaker boundary assumption. Second, it relaxes the strict complementary slackness assumption, Assumption SCS-NCP. It is evident that Assumptions SCS-NCP and ND-NCP together imply SND-NCP, however the converse is not true, hence our result requires weaker regularity requirements.

Theorem 2 Let $F : \mathbb{U}^n_+ \mapsto \mathbb{R}^n$ be a continuous function. Assume that F satisfies assumptions WBC and SND-NCP. Then, NCP(F) has a unique element.

Before providing a proof of this theorem, we study the related problem of uniqueness of solutions to MCP problems. MCP problems are not only important in many applications, but it is also more convenient to prove uniqueness of solutions to MCP problems before proving Theorem 2 (see Section 4.2).

2.2 Uniqueness for the MCP

Throughout this section, $a, b \in \mathbb{R}^n$ denote vectors such that $a < b$, $M = [a, b]$ denotes a closed rectangle, and $U|_M \subset \mathbb{R}^n$ denotes an open set containing M. As in the previous section, to avoid smoothness difficulties, we work with the extended function $F : U|_M \mapsto \mathbb{R}^n$ without loss of any generality. In [14], we define and study the *generalized critical points* of F over M, denoted by $\mathrm{Cr}(F, M)$, when the set M is defined by finitely many smooth inequalities. For the special case of $M = [a, b]$, it can be seen that

$$\mathrm{MCP}(F, M) = \mathrm{Cr}(F, M).$$

The following assumption is equivalent to stating that every vector $x^* \in \mathrm{MCP}(F, M) = \mathrm{Cr}(F, M)$ is complementary in the sense of [14].

Assumption SCS-MCP (Strict Complementary Slackness for the MCP): For each $x^* \in \mathrm{MCP}(F, M)$, $x^*_i = a_i$ implies $F_i(x^*) > 0$ and $x^*_i = b_i$ implies $F_i(x^*) < 0$, i.e. $I^{NB-\mathrm{MCP}}(x^*) = I^F(x^*)$, where

$$I^{NB-\mathrm{MCP}}(x) = \Big\{ i \in \{1, .., n\} \mid a_i < x_i < b_i \Big\},$$

and $I^F(x^*)$ is defined in (6).

The following assumption is equivalent to stating that every vector $x^* \in \mathrm{MCP}(F, M) = \mathrm{Cr}(F, M)$ is non-degenerate in the sense of [14].

Assumption ND-MCP (Non-degeneracy for the MCP): For each $x^* \in \mathrm{MCP}(F, M)$, F is continuously differentiable at x^* and

$$\det(\nabla F(x^*)|_{I^{NB-MCP}(x^*)}) > 0.$$

The following uniqueness result for the MCP follows directly from the main result in [14]. The subsequent corollary establishes the uniqueness of solutions to the MCP by local conditions on F at vectors in $\mathrm{MCP}(F, M)$ and is similar to Theorem 1 for the NCP case.

Theorem 3 Let $F : U|_M \mapsto \mathbb{R}^n$ be a continuous function. Assume that (F, M) satisfies Assumption SCS-MCP. Moreover, assume that for each vector $x^* \in \mathrm{MCP}(F, M)$, F is continuously differentiable at x^* and

$$\det(\nabla F(x^*)|_{I^{NB-\mathrm{MCP}}(x^*)}) \neq 0.$$

Then, MCP(F, M) has a finite (odd) number of elements and

$$\sum_{x^* \in \mathrm{MCP}(F,M)} \mathrm{sign}\Big(\det(\nabla F(x^*)|_{I^{NB-\mathrm{MCP}}(x^*)}) \Big) = 1.$$

Corollary 1 Let $F : U|_M \mapsto \mathbb{R}^n$ be a continuous function. Assume that (F, M) satisfies assumptions SCS-MCP and ND-MCP. Then, MCP(F) has a unique element.

We introduce the following assumption to further generalize the preceding corollary by relaxing the strict complementary slackness assumption. We say that $x^* \in$ MCP(F, M) is a *strongly non-degenerate* solution if F is continuously differentiable at x^* and

$$\det(\nabla F(x^*)|_J) > 0$$

for all J such that

$$I^{NB-MCP}(x^*) \subset J \subset I^F(x^*).$$

Assumption SND-MCP (Strong Non-degeneracy for the MCP): Each $x^* \in$ MCP(F) is strongly non-degenerate.

The following theorem is our main result for the uniqueness of solutions to the MCP.

Theorem 4 Let $F : U|_M \mapsto \mathbb{R}^n$ be a continuous function. Assume that (F, M) satisfies Assumption SND-MCP. Then, MCP(F) has a unique element.

We prove Theorem 4 in Section 4. For the proof, we need some preliminary results regarding properties of square matrices, which we present in the next section.

3 Partial P-Matrices and Properties

We first define the P-matrix property for a square matrix.

Definition 1 An $n \times n$ matrix A is called a *P-matrix* if the determinant of each of its principal sub-matrices is positive, i.e. if

$$\det(A|_J) > 0, \qquad \forall\ J \subset \{1, 2, .., n\}.$$

P-matrices play an important role in establishing global univalance of continuous maps[3] (see the celebrated Gale-Nikaido Theorem [1]). The P-matrix property is weaker than positive definiteness when the matrix is not necessarily symmetric. Every positive definite matrix is also a P-matrix, yet the converse statement is only true if the matrix is

[3]Given $X, Y \subset \mathbb{R}^n$, a map $\phi : X \mapsto Y$ is called *globally univalent* if ϕ is a homeomorphism between X and Y, i.e., ϕ is a one-to-one map of X to Y such that the two maps ϕ and its inverse ϕ^{-1} are both continuous.

symmetric (see [1], [2]). This makes the P-matrix property a useful notion for analyzing vector-valued functions whose Jacobian is not necessarily symmetric.

P-matrix properties of the Jacobian of F are also relevant in studying the uniqueness of solutions to the NCP in (1)-(2). The appropriate generalization of non-degeneracy when the complementary slackness assumption is relaxed requires assumptions regarding certain principal minors of the matrix $\nabla F(x^*)|_{I^F(x^*)}$. The P-matrix property constrains all of the principal minors of a matrix and would be too strong for our purposes. We, therefore, introduce and study the weaker notion of a partial P-matrix.

Definition 2 Given an index set $I \subset \{1, 2, .., n\}$ and an $n \times n$ matrix A, we say that A is a *partial P-matrix with respect to* I if

$$\det(A|_J) > 0, \qquad \forall\ J \text{ with } I \subset J \subset \{1, 2, .., n\}.$$

Clearly, every P-matrix is a partial P-matrix with respect to any subset $I \subset \{1, .., n\}$. Note also that the strong non-degeneracy assumptions of the previous section are closely related to the partial P-matrix properties of $\nabla F(x)$. It can be seen that F satisfies Assumption SND-NCP [resp. (F, M) satisfies SND-MCP] if and only if $\nabla F(x^*)|_{I^F(x^*)}$ is a partial P-matrix with respect to $I^{NB-NCP}(x^*)$ [resp. $I^{NB-MCP}(x^*)$] for all $x^* \in \text{NCP}(F)$ [resp. $\text{MCP}(F, M)$].

We note the following result which provides a sufficient condition for a matrix to be a P-matrix and which therefore could be useful in establishing Assumptions SND-NCP and SND-MCP in applications (for the proof, see [15]).

Lemma 1 Let A be an $n \times n$ *positive row diagonally dominant* matrix, i.e. assume that for all $i \in \{1, .., n\}$,

$$A^{ii} - \sum_{j \neq i} |A^{ij}| > 0.$$

Then, A is a P-matrix.

In the remainder of this section, we study properties of partial P-matrices, which we need for the proofs of our main results, and which could also be of independent interest. The following lemma is a generalization of the similar property for P-matrices (see Theorem 7.8.2 in [1]).

Lemma 2 Let $I \subset \{1, 2, .., n\}$ and let A be an $n \times n$ partial P-matrix with respect to I. Let D be an $n \times n$ diagonal matrix with diagonal entries d_i such that

$$d_i = 0,\ \forall\ i \in I \ \text{ and } \ d_i \geq 0,\ \forall\ i \notin I.$$

Then, $A + D$ is also a partial P-matrix with respect to I.

Proof. Without loss of generality, assume that $I = \{k + 1, ..., n\}$. Let D^i denote the diagonal matrix with diagonal entries $(0, .., 0, d_i, 0, ..0)$. We first claim that $A + D^1$ is a partial P-matrix with respect to I. Let $J = \{j_1, .., j_m\}$ be an index set such that

$I \subset J \subset \{1, 2, .., n\}$. If $1 \notin J$, then $\det((A + D^1)|_J) = \det(A_J) > 0$. Else if $1 \in J$ (assume $j_1 = 1$), then we have

$$\begin{aligned}
\det((A + D^1)|_J) = \quad & \det \begin{pmatrix} A^{11} & A^{1j_2} & .. & A^{1j_m} \\ A^{j_2 1} & A^{j_2 j_2} & .. & A^{j_2 j_m} \\ .. & .. & .. & .. \\ A^{j_m 1} & A^{j_m j_2} & .. & A^{j_m j_m} \end{pmatrix} \\
& + d_1 \det \begin{pmatrix} A^{j_2 j_2} & .. & A^{j_2 j_m} \\ .. & .. & .. \\ A^{j_m j_2} & .. & A^{j_m j_m} \end{pmatrix} \\
= \quad & \det(A|_J) + d_1 \det(A|_{J-\{1\}})
\end{aligned}$$

Since $1 \notin I$ and $I \subset J$, we have $I \subset J - \{1\}$. Then, since A is a partial P-matrix with respect to I, both of the determinants above are positive. Since $d_1 \geq 0$, we have $\det((A + D^1)|_J) > 0$, showing that $A + D^1$ is a partial P-matrix with respect to I. Now, repeating the same argument recursively, we have that

$$A + D^1 + D^2 + .. + D^k = A + D$$

is a P-matrix with respect to I. **Q.E.D.**

Lemma 3 Let $I \subset \{1, 2, .., n\}$ and let A be an $n \times n$ partial P-matrix with respect to I. Then, there exists $\mu_1, \mu_2 > 0$ such that, for any $v \neq 0$ such that

$$|(Av)_i| \leq \mu_1 \|v\|, \ \forall \ i \in I,$$

there exists $j \notin I$ such that

$$v_j (Av)_j > \mu_2 \|v\|^2.$$

Proof. Consider the function $f : \mathbb{R}^n \mapsto \mathbb{R}$ given by

$$f(v) = \max_{i \in I} |(Av)_i|$$

and the function $g : \mathbb{R}^n \mapsto \mathbb{R}$ given by

$$g(v) = \max_{i \notin I} v_i (Av)_i.$$

To prove this result, we show that there exists some scalar $\mu > 0$ such that the following optimization problem either is infeasible or has a positive optimal value:

$$\begin{aligned}
& \min g(v) && (7) \\
& \text{subject to } \|v\| = 1, \ f(v) \leq \mu.
\end{aligned}$$

For this purpose, we consider

$$\begin{aligned} &\min g(v) \\ &\text{subject to } \|v\| = 1,\ \ f(v) \le 0. \end{aligned} \tag{8}$$

If (8) is infeasible, i.e., for all v with $\|v\| = 1$, we have $f(v) > 0$, then, by the continuity of f, we have $\min_{\|v\|=1} f(v) > 0$, showing that there exists some $\mu > 0$ such that (7) is infeasible and the result follows. Thus, we assume that problem (8) is feasible.

We next show that if problem (8) has a positive optimal value then so does problem (7). We denote the optimal value of problem (7) by $p(\mu)$ and show that $p(\mu)$ is right continuous at $\mu = 0$. Consider a scalar sequence $\{\mu^k\}$ with $\mu^k \downarrow 0$. Since (8) is feasible, problem (7) with $\mu = \mu^k$ is feasible for all k, which by the continuity of $g(v)$ and the compactness of the constraint set, implies the existence of some v^k with $\|v^k\| = 1$ and

$$g(v^k) = p(\mu^k) \le p(0), \qquad f(v^k) \le \mu^k.$$

Since the sequence $\{v^k\}$ is bounded, it has a limit point $\bar{v}$ with $\|\bar{v}\| = 1$. Taking the limit as $k \to \infty$ along the relevant subsequence in the preceding relations and using the continuity of g and f, we obtain

$$g(\bar{v}) = \lim_{k\to\infty} p(\mu^k) \le p(0), \qquad f(\bar{v}) \le 0.$$

This implies that $\bar{v}$ is feasible for problem (8), and therefore $g(\bar{v}) = p(0)$, establishing the right continuity of $p(\mu)$ at $\mu = 0$ by the preceding relation, and showing the claim.

We, finally, show that (8) has a positive optimal value. Assume the contrary, i.e. for every vector v that is feasible for (8),

$$g(v) = v_i(Av)_i \le 0, \qquad \forall\ i \notin I. \tag{9}$$

Consider some v feasible for (8) and let $J = \{i \mid v_i \ne 0\}$. By Eq. (9), for all $i \in J - I$, there exists $\alpha_i \ge 0$ such that $(Av)_i = -\alpha_i v_i$. Moreover, by feasibility of v, for all $i \in I$, $(Av)_i = 0$. Let $\alpha_i = 0$ for $i \in I$, and consider the $n \times n$ diagonal matrix D with diagonal entries α_i. Then, we have

$$\Big((A + D)v\Big)\Big|_{I\cup J} = 0.$$

Since $v_i = 0$ for $i \notin I \cup J$, this implies that

$$(A + D)|_{(I\cup J)}\ v|_{(I\cup J)} = 0. \tag{10}$$

By Lemma 2, $A + D$ is a partial P-matrix with respect to I, which implies, in particular, that $\det\Big((A + D)|_{(I\cup J)}\Big) > 0$. By the preceding relation, this implies that $v = 0$, contradicting the feasibility of v for (8), and showing the desired result. **Q.E.D.**

We next present two propositions related to local univalence properties of a differentiable mapping.

Proposition 1 Let $I \subset \{1, 2, .., n\}$, $U \subset \mathbb{R}^n$ be an open set, $F : U \mapsto \mathbb{R}^n$ be a function and $x \in U$ be a vector. Assume that F is differentiable at x and $\nabla F(x)$ is a partial P-matrix with respect to I. Then, there exists an open set U_x containing x such that the set

$$E = \Big\{ y \in U_x - \{x\} \ \Big| \quad \begin{array}{l} F_i(y) = F_i(x), \ \forall\, i \in I, \\ (y_i - x_i)\Big(F_i(y) - F_i(x)\Big) \le 0, \ \forall\, i \notin I \Big\} \end{array}$$

is empty.

Note that some special cases of this result are known. Assume that F is continuously differentiable at x. When $I = \{1, 2, .., n\}$, $\nabla F(x)$ is a partial P-matrix with respect to I if and only if $\det(\nabla F(x)) > 0$. Then, from the Inverse Function Theorem, F is locally invertible, which implies, in particular that there exists an open set U_x containing x such that for all $y \in U_x$, $F(y) \neq F(x)$, implying the result of the proposition for this case. When $I = \emptyset$, then $\nabla F(x)$ is a partial P-matrix with respect to I if and only if it is a P-matrix. Then, by the continuity of the determinants of the sub-matrices of ∇F, there exists an open rectangle U_x containing x such that F is continuously differentiable over U_x and $\nabla F(y)$ is a P-matrix for all $y \in U_x$. Then, F is a P-function on U_x implying the result of the proposition for this case (cf. Proposition 3.5.9 in [5]).

Proof of Proposition 1 By using a translation argument, we can assume, without loss of generality, that $F(x) = 0$. Since F is differentiable at x, there exists an open set V_x containing x such that, for all $y \in V_x$,

$$F(y) = \nabla F(x)^T (y - x) + \epsilon(y)\|y - x\| \tag{11}$$

where $\epsilon : V_x \mapsto \mathbb{R}^n$ is a continuous error function such that $\epsilon(x) = 0$. By Lemma 3, there exists $\mu_1, \mu_2 > 0$ such that for all $v \neq 0$ such that

$$\Big|(\nabla F(x)v)_i\Big| \le \mu_1 \|v\|, \ \forall\, i \in I$$

there exists $j \notin I$ such that

$$v_j(\nabla F(x)v)_j > \mu_2 \|v\|^2.$$

Let $U_x \subset V_x$ be an open set containing x such that $|\epsilon_i(y)| < \min(\mu_1, \mu_2)$, for all i and for all $y \in U_x$. Assume, to get a contradiction, that there exists $y \in U_x$, $y \neq x$ such that

$$F_i(y) = 0 \text{ for all } i \in I,$$

$$(y_i - x_i)F_i(y) \le 0 \text{ for all } i \notin I. \tag{12}$$

Since $F_i(y) = 0$ for all $i \in I$, by Eq. (11), we have for all $i \in I$

$$\Big|\Big(\nabla F(x)(y - x)\Big)_i\Big| = |\epsilon_i(y)|\|y - x\| < \mu_1 \|y - x\|.$$

Therefore, by the choice of μ_1 and μ_2, there exists $j \notin I$ such that

$$(y_j - x_j)\Big(\nabla F(x)(y-x)\Big)_j > \mu_2 \|y-x\|^2. \tag{13}$$

Multiplying Eq. (11) with $(y_j - x_j)$ from the left and considering the j^{th} component equation, we have

$$\begin{aligned} (y_j - x_j)F_j(y) &= (y_j - x_j)\Big(\nabla F(x)(y-x)\Big)_j + (y_j - x_j)\epsilon_j(y)\|y-x\| \\ &> \mu_2\|y-x\|^2 - |(y_j - x_j)|\mu_2\|y-x\| \end{aligned}$$

Since $|y_j - x_j| \leq \|y-x\|$, we have

$$(y_j - x_j)F_j(y) > 0,$$

for some $j \notin I$, contradicting Eq. (12). **Q.E.D.**

The following proposition strengthens Proposition 1.

Proposition 2 Let $I_1 \subset I_2 \subset \{1, 2, .., n\}$, $U \subset \mathbb{R}^n$ be an open set, $F : U \mapsto \mathbb{R}^n$ be a function, and $x \in U$ be a vector. Assume that F is differentiable at x and $\nabla F(x)|_{I_2}$ is a partial P-matrix with respect to I_1. Then, there exists an open set U_x containing x such that the set

$$E = \Big\{ y \in U_x - \{x\} \;\Big|\; \begin{array}{l} y_i = x_i, \ \forall\, i \notin I_2, \\ F_i(y) = F_i(x), \ \forall\, i \in I_1, \\ (y-x)_i(F_i(y) - F_i(x)) \leq 0, \ \forall\, i \in I_2 - I_1 \end{array} \Big\} \tag{14}$$

is empty.

Proof. Without loss of generality, assume that $I_2 = \{1, 2, .., m\}$ for some $m \leq n$. Given some set $A \subset \mathbb{R}^n$, we use the notation $A|_{I_2}$ to denote a subset of $\mathbb{R}^m$ given by

$$A|_{I_2} = \Big\{ u \in \mathbb{R}^m \ |(u_1, .., u_m, x_{m+1}, .., x_n) \in A \Big\}.$$

Consider the function $G : U|_{I_2} \mapsto \mathbb{R}^m$ defined by

$$G_i(u) = F_i(u_1, .., u_m, x_{m+1}, .., x_n), \qquad \forall\, i \in I_2, \ \forall\, u \in U|_{I_2}. \tag{15}$$

Then, G is differentiable at $(x_1, .., x_m) \in U|_{I_2}$. Moreover, we have

$$G_k(x_1, .., x_m) = F_k(x), \qquad \forall\, k \in I_2,$$

and

$$\nabla G(x) = \nabla F(x)|_{I_2},$$

which implies that $\nabla G(x)$ is a partial P-matrix with respect to I_1. Then, by Proposition 1, there exists an open set $U'_x \subset U|_{I_2} \subset \mathbb{R}^m$ containing $(x_1, .., x_m)$ such that the set

$$E' = \Big\{(u_1, .., u_m) \in U'_x \quad - \quad \{(x_1, .., x_m)\} \;\Big|\; G_i(u_1, .., u_m) = G_i(x_1, .., x_m),\ \forall\, i \in I_1,$$
$$(u_i - x_i)\Big(G_i(u_1, .., u_m) - G_i(x_1, .., x_m)\Big) \leq 0,\ \forall\, i \in I_2 - I_1\Big\}$$

is empty. We claim that the open subset of U given by

$$U_x = \{u \in U \mid (u_1, .., u_m) \in U'_x\}$$

satisfies the claim of the proposition. Assume the contrary, that E is not empty. Let $y \in E$. By the definition of U_x, we have $(y_1, .., y_m) \in U'_x$. Further, since $y \neq x$ and $y_i = x_i$ for all $i \notin I_2$, it follows that $(y_1, .., y_m) \neq (x_1, .., x_m)$. Then, by Eq. (15) $y \in E$ implies that $(y_1, .., y_m) \in E'$, contradicting the fact that E' is empty. Thus, we conclude that E is empty as desired. **Q.E.D.**

4 Proofs of the Uniqueness Results for the MCP and the NCP

In this section, we provide proofs of our two main theorems Theorems 2 and 4. We start in reverse order with Theorem 4, and then use this theorem to provide a simpler proof of Theorem 2.

4.1 Proof of Theorem 4

As in Section 2.2, throughout this section, we let $a, b \in \mathbb{R}^n$ denote vectors such that $a < b$, $M = [a, b]$ denote a closed rectangle, and $U|_M \subset \mathbb{R}^n$ be an open set containing M. We introduce the notion of an irregular pair to prove Theorem 4.

Definition 3 Let $F : U|_M \mapsto \mathbb{R}^n$ be a function. We say that (i, x^*) is an *irregular pair* if

$$x^* \in \text{MCP}(F, M), \qquad i \in I^F(x^*) - I^{NB-MCP}(x^*),$$

i.e. the inequality corresponding to i in either Eq. (3) or Eq. (5) is not strict. We denote the set of irregular pairs of F over M by

$$\mathcal{A}(F, M) = \Big\{(i, x^*) \mid x^* \in \text{MCP}(F, M),\ i \in I^F(x^*) - I^{NB-MCP}(x^*)\Big\}.$$

We note that (F, M) satisfies Assumption SCS-MCP if and only if $\mathcal{A}(F, M) = \emptyset$. The following lemma, which shows that we can recursively remove the irregular pairs.

Lemma 4 Let $F : U|_M \mapsto \mathbb{R}^n$ be a continuous function. Assume that $x^* \in \text{MCP}(F, M)$ is a strongly non-degenerate solution. Then,
(i) There exists an open set U_{x^*} containing x^* such that $U_{x^*} \cap \text{MCP}(F, M) = \{x^*\}$, i.e. x^* is an isolated solution to the MCP.

(ii) Assume that $(k, x^*) \in \mathcal{A}(F, M)$. Then, there exists a function $\tilde{F} : U|_M \mapsto \mathbb{R}^n$ such that

(a) $\text{MCP}(\tilde{F}, M) = \text{MCP}(F, M)$.
(b) $\mathcal{A}(\tilde{F}, M) = \mathcal{A}(F, M) - \{(k, x^*)\}$.
(c) $x^* \in \text{MCP}(\tilde{F}, M)$ is a strongly non-degenerate solution.

Proof. For notational convenience, for each $x \in M$ we let

$$I^{NB}(x) = I^{NB-\text{MCP}}(x) = \Big\{ i \in \{1, .., n\} \mid a_i < x_i < b_i \Big\}$$

denote the set of non-binding indices. Since x^* is a strongly non-degenerate MCP solution for F, $\nabla F(x^*)|_{I^F(x^*)}$ is a partial P-matrix with respect to $I^{NB}(x^*)$. Then by Proposition 2, there exists an open set $V_{x^*} \subset \mathbb{R}^n$ containing x^* such that the set

$$E = \Big\{ y \in V_{x^*} - \{x^*\} \;\Big|\; \begin{array}{l} y_i = x_i^*, \ \forall\, i \notin I^F(x^*), \\ F_i(y) = F_i(x^*), \ \forall\, i \in I^{NB}(x^*), \\ (y_i - x_i^*)\Big(F_i(y) - F_i(x^*)\Big) \le 0, \ \forall\, i \in I^F(x^*) - I^{NB}(x^*) \Big\} \end{array}$$

is empty. Let $U_{x^*} \subset V_{x^*}$ be an open set which is sufficiently small such that $F_i(x^*) > 0$, [resp. $F_i(x^*) < 0$] [resp. $a_j < x_j^* < b_j$] implies $F_i(u) > 0$ [resp. $F_i(u) < 0$] [resp. $a_j < u_j < b_j$] for all $u \in U_{x^*}$. We will show that U_{x^*} satisfies part (i) of the lemma.

(i) Assume, to get a contradiction, that there exists some $y \in U_{x^*} \cap \text{MCP}(F, M)$ such that $y \ne x^*$. Let $J = \{i \mid y_i \ne x_i^*\}$ and consider $i \in J$. We first claim that $i \in I^{NB}(y)$. If $i \in I^{NB}(x^*)$, by choice of U_{x^*}, $i \in I^{NB}(y)$. Else if $x_i^* = a_i$ or $x_i^* = b_i$, since $y_i \ne x_i^*$ and $y \in M$, we have $a_i < y_i < b_i$, i.e. $i \in I^{NB}(y)$, showing the claim. Since $y \in \text{MCP}(F, M)$, we further have $i \in I^F(y)$. Then, by choice of U_{x^*}, we also have $i \in I^F(x^*)$. Thus, we have shown

$$J \subset I^F(y) \text{ and } J \subset I^F(x^*). \tag{16}$$

We next claim that $y \in E$. We have $y_i = x_i^*$ for all $i \notin I^F(x^*)$. Since $I^{NB}(x^*) \subset I^F(x^*)$ and $I^{NB}(y) \subset I^F(y)$ [in view of the fact that $y \in \text{MCP}(F, M)$], we also have $F_i(x^*) = F_i(y)$ for all $i \in I^{NB}(x^*)$. Let $i \in I^F(x^*) - I^{NB}(x^*)$. If $i \notin J$, then $x_i^* = y_i$ and

$$(x_i^* - y_i)\Big(F_i(x^*) - F_i(y)\Big) = 0 \le 0.$$

If $i \in J$, then, by Eq. (16), $F_i(x^*) = F_i(y) = 0$, thus

$$(x_i^* - y_i)\Big(F_i(x^*) - F_i(y)\Big) = 0 \le 0,$$

hence $y \in E$, contradicting the fact that E is empty. Thus, $U_{x^*} \cap \text{Cr}(F, M) = \{x^*\}$ as desired.

(ii) Since $k \notin I^{NB}(x^*)$, we have either $x_k^* = a_k$ or $x_k^* = b_k$. Assume $x_k^* = a_k$. Let $w : \mathbb{R}^n \mapsto \mathbb{R}$ be a continuously differentiable weight function such that

$$\begin{cases} w(x^*) = 1, \\ w(u) \ge 0, \ \text{if } u \in U_{x^*} \\ w(u) = 0, \ \text{if } u \notin U_{x^*}. \end{cases}$$

Let $\tilde{F} : U \mapsto \mathbb{R}^n$ be given by

$$\begin{cases} \tilde{F}_k(u) = F_k(u) + w(u), \text{ for all } u \in U, \\ \tilde{F}_i(u) = F_i(u), \text{ for all } i \neq k \text{ and } u \in U. \end{cases} \tag{17}$$

We will show that the function $\tilde{F}$ satisfies the claims of the lemma. We have,

$$\tilde{F}_k(x^*) > 0, \text{ and } \tilde{F}_i(x^*) = F_i(x^*) \text{ for } i \neq k. \tag{18}$$

Since $x^* \in \text{MCP}(F, M)$, $\tilde{F}(x^*)$ satisfies (3)-(5) and thus $x^* \in \text{MCP}(\tilde{F}, M)$. We have

$$I^{\tilde{F}}(x^*) = I^F(x^*) - \{k\}. \tag{19}$$

By Eq. (19) and the definition in (17), we have

$$\nabla \tilde{F}(x^*)|_{I^{\tilde{F}}(x^*)} = \nabla F(x^*)|_{I^{\tilde{F}}(x^*)}. \tag{20}$$

Since x^* is a strongly non-degenerate solution for F, $\nabla F(x^*)|_{I^F(x^*)}$ is a partial P-matrix with respect to $I^{NB}(x^*)$. Then, by Eqs. (19) and (20), $\nabla \tilde{F}(x^*)|_{I^{\tilde{F}}(x^*)}$ is also a partial P-matrix with respect to $I^{NB}(x^*)$. This shows that $x^* \in \text{MCP}(\tilde{F}, M)$ is a strongly non-degenerate solution and hence that $\tilde{F}$ satisfies the claim in (ii)-(c).

We next claim that $\text{MCP}(\tilde{F}, M) \cap U_{x^*} = \{x^*\}$. Assume that there exists an MCP solution y for $\tilde{F}$ in U_{x^*} such that $y \neq x^*$. Let $J = \{i \mid y_i \neq x_i^*\}$. As shown in the proof of part (i),

$$J \subset I^{\tilde{F}}(x^*) \subset I^F(x^*) \text{ and } J \subset I^{\tilde{F}}(y). \tag{21}$$

We claim that $y \in E$. We have $y_i = x_i^*$ for all $i \notin I^F(x^*)$ and $F_i(x^*) = F_i(y)$ for all $i \in I^{NB}(x^*)$. Let $i \in I^F(x^*) - I^{NB}(x^*)$. If $i \notin J$, then $x_i^* = y_i$ and

$$(x_i^* - y_i)\Big(F_i(x^*) - F_i(y)\Big) = 0 \leq 0.$$

If $i \in J$, then, by Eq. (16), $F_i(x^*) = \tilde{F}_i(y) = 0$. If $i \neq k$, then since $F_i(y) = \tilde{F}_i(y) = 0$, we have

$$(x_i^* - y_i)\Big(F_i(x^*) - F_i(y)\Big) = 0 \leq 0.$$

Else if $i = k$, then $y_i > x_i^*$ since $x_i^* = x_k^* = a_k$. We have

$$0 = (x_i^* - y_i)\tilde{F}_i(y) = (x_i^* - y_i)F_i(y) + (x_i^* - y_i)w(y)$$

which, since $w(y) \geq 0$, implies

$$(x_i^* - y_i)\Big(F_i(x^*) - F_i(y)\Big) \leq 0. \tag{22}$$

Hence, Eq. (22) holds for all $i \in I^F(x^*) - I^{NB}(x^*)$, showing that $y \in E$. This contradicts the fact that E is empty and completes the proof of part (ii)-(a) of the lemma.

Since $\tilde{F}_k(x^*) = w(x^*) > 0$, $(k, x^*) \notin \mathcal{A}(F, M)$. Let $(i, x^*) \in \mathcal{A}(F, M)$ for some $i \neq k$. Then, since $\tilde{F}_i = F_i$, we have $(i, x^*) \in \mathcal{A}(\tilde{F}, M)$. Let $(i, y) \in \mathcal{A}(F, M)$ for some $y \in \mathrm{Cr}(F, M)$ such that $y \neq x^*$. Then by part (i) of this Lemma, $y \notin U_{x^*}$ and thus $F(y) = \tilde{F}(y)$. This implies, $(i, y) \in \mathcal{A}(\tilde{F}, M)$. We conclude that $\mathcal{A}(\tilde{F}, M) = \mathcal{A}(F, M) - \{k, x^*\}$, completing the proof of part (ii)-(b) of the lemma.

The proof for the case when $k \in I^{\max}(x)$ can be analogously given with the continuously differentiable weight function chosen such that

$$\begin{cases} w(x^*) = -1, \\ w(u) \leq 0, \text{ if } u \in U_{x^*} \\ w(u) = 0, \text{ if } u \notin U_{x^*}. \end{cases}$$

Q.E.D.

We are now ready to prove Theorem 4.

Proof of Theorem 4. We first claim that $\mathrm{MCP}(F, M)$ is a compact set. Let

$$A_i^1 = \{x \in M \mid F_i(x) = 0\}$$
$$A_i^2 = \{x \in M \mid F_i(x) \geq 0 \text{ and } x_i = a_i\}$$

and

$$A_i^3 = \{x \in M \mid F_i(x) \leq 0 \text{ and } x_i = b_i\}.$$

Since each of A_i^1, A_i^2, A_i^3 is compact, so is $A_i = A_i^1 \cup A_i^2 \cup A_i^3$. By equations (3)-(5), we have

$$\mathrm{MCP}(F, M) = \bigcap_{i \in \{1,2,..,n\}} A_i.$$

Then, being the intersection of compact sets, $\mathrm{MCP}(F, M)$ is compact. We next claim that $\mathrm{MCP}(F, M)$ has a finite number of elements. By part (i) of Lemma 4, for each $x^* \in \mathrm{MCP}(F, M)$, there exists an open set U_{x^*} containing x^* such that $U_{x^*} \cap \mathrm{Cr}(F, M) = \{x^*\}$. Then, $\{U_{x^*} \mid x^* \in \mathrm{MCP}(F, M)\}$ is an open covering of the compact set $\mathrm{MCP}(F, M)$, which implies that it has a finite sub-covering. This further implies that $\mathrm{MCP}(F, M)$ has a finite number of elements.

We finally claim that there exists a function $G : U|_M \mapsto \mathbb{R}^n$ such that $\mathrm{MCP}(G, M) = \mathrm{MCP}(F, M)$ and (G, M) satisfies assumptions SCS-MCP and ND-MCP. Let $F^0 = F$ and, for any $j \geq 0$ such that $\mathcal{A}(F^j, M) \neq \emptyset$, define

$$F^{j+1} = \tilde{F^j}$$

where $\tilde{F^j}$ is the modified function which satisfies the claim of part (ii) of Lemma 4 for the function F^j and an arbitrary $(k, x) \in \mathcal{A}(F^j, M)$. By part (ii) of Lemma 4,

$$|\mathcal{A}(F^{j+1}, M)| = |\mathcal{A}(F^j, M)| - 1.$$

Since $\mathrm{MCP}(F, M)$ has finitely many elements, $\mathcal{A}(F, M)$ has finitely many elements, which implies that there exists an integer $m \geq 0$ such that $\mathcal{A}(F^m, M) = \emptyset$. We let, $G = F^m$. Since $\mathcal{A}(G, M) = \emptyset$, (G, M) satisfies Assumption SCS-MCP. Also, by part (iii) of Lemma 4, every $x^* \in \mathrm{MCP}(G, M)$ is strongly non-degenerate, which means that (G, M) satisfies Assumption ND-MCP. Then, G satisfies the claim and Corollary 1 applies to G, showing that $\mathrm{MCP}(G, M)$ has a unique element. Since $\mathrm{MCP}(F, M) = \mathrm{MCP}(G, M)$, we conclude that $\mathrm{MCP}(F, M)$ also has a unique element, as desired. **Q.E.D.**

4.2 Proof of Theorem 2

We first prove that when F satisfies Assumption WBC, $\mathrm{NCP}(F) = \mathrm{MCP}(F, M)$ for an appropriately chosen $M = [a, b]$.

Lemma 5 Let $F : \mathbb{U}_+^n \mapsto \mathbb{R}^n$ be a function which satisfies Assumption WBC. For each i, let $b_i > 0$ be sufficiently large such that $y_i < b_i$ for all y in the compact set C of Assumption WBC. Let $M = [0, b]$. Then, we have
(i) $\mathrm{NCP}(F) = \mathrm{MCP}(F, M)$.
(ii) $I^{NB-\mathrm{NCP}}(x^*) = I^{NB-\mathrm{MCP}}(x^*)$, for all $x^* \in \mathrm{NCP}(F) = \mathrm{MCP}(F, M)$.
(iii) F satisfies Assumption SCS-NCP (resp. ND-NCP), (resp. SND-NCP) if and only if (F, M) satisfies Assumption SCS-MCP (resp. ND-MCP), (resp. SND-MCP).

Proof. (i) By the choice of b_i, we have

$$C \subset M. \tag{23}$$

Let $x^* \in \mathrm{NCP}(F)$. We first claim that $x^* \in C$. Assume, to get a contradiction, that $x^* \notin C$. Then, by Assumption WBC, there exists $y \in C$ and $i \in \{1, .., n\}$ such that

$$(y_i - x_i^*)F_i(x^*) < 0. \tag{24}$$

If $x_i^* > 0$, then since $x^* \in \mathrm{NCP}(F)$, we have $F_i(x^*) = 0$, contradicting Eq. (24). Else if $x_i^* = 0$, then $y_i \geq x_i^*$ and $F_i(x^*) \geq 0$ implies

$$(y_i - x_i^*)F_i(x^*) \geq 0,$$

thus Eq. (24) yields a contradiction, showing that $x^* \in C$, and by (23) that $x^* \in M$. Since x^* also satisfies Eqs. (3)-(5), we have $x^* \in \mathrm{MCP}(F, M)$.

Conversely, let $x^* \in \mathrm{MCP}(F, M)$. We claim that $x^* \in C$. Assume, to get a contradiction, that $x^* \notin C$. By Assumption WBC, there exists $y \in C$ and $i \in \{1, .., n\}$ such that Eq. (24) holds. If $0 < x_i^* < b_i$, then we have $F_i(x^*) = 0$, contradicting Eq. (24). Else if $x_i^* = 0$, then $y_i \geq x_i$ and $F_i(x^*) \geq 0$ implies

$$(y_i - x_i^*)F_i(x^*) \geq 0,$$

which contradicts Eq. (24). Finally, if $x_i^* = b_i$, we have $y_i < b_i$ by choice of b_i and $F_i(x^*) \leq 0$ since $x^* \in \mathrm{Cr}(F, M)$, implying that

$$(y_i - x_i^*)F_i(x^*) \geq 0,$$

once again contradicting Eq. (24) and showing that $x^* \in C$. Then, by choice of b_i, we have $x_i^* < b_i$ for all i, which by Eqs. (4) and (3) implies that $x^* \in \mathrm{NCP}(F)$. We have shown,

$$\mathrm{NCP}(F) \subset \mathrm{MCP}(F, M) \subset C \text{ and } \mathrm{MCP}(F, M) \subset \mathrm{NCP}(F) \subset C,$$

which implies that $\mathrm{NCP}(F) = \mathrm{MCP}(F, M)$ as desired.

(ii) Since $\mathrm{NCP}(F) \subset C$, for all $x^* \in \mathrm{NCP}(F)$ and i, we have $x_i^* < b_i$. Then,

$$I^{NB-\mathrm{NCP}}(x^*) = \{i \mid 0 < x_i^*\}$$

is equal to

$$I^{NB-\mathrm{MCP}}(x^*) = \{i \mid 0 < x_i^* < b_i\}$$

as desired.

(iii) Follows in view of part (ii) and the definitions of the assumptions. **Q.E.D.**

Now, Lemma 5 enables us to use Theorem 4 to provide a simple proof of Theorem 2.

Proof of Theorem 2. Let $M = [0, b]$ be appropriately chosen such that Lemma 5 holds. Since F satisfies Assumption SND-NCP, by part (iii) of Lemma 5, (F, M) satisfies Assumption SND-MCP. Then, by Theorem 4, $\mathrm{MCP}(F, M)$ has a unique element. Finally, by part (i) of Claim 5, $\mathrm{NCP}(F) = \mathrm{MCP}(F, M)$ has a unique element, as desired. **Q.E.D.**

5 Examples

In this section, we demonstrate how our two main theorems improve over earlier results.

5.1 Example For Theorem 2

The following example illustrates the improvement of Theorem 2 over Theorem 1.

Example 1 Let $d_1, .., d_n \geq 0$ be scalars and consider the function $F : \mathbb{U}_+^n \mapsto \mathbb{R}^n$ be given by,

$$F_i(p) = -\frac{d_i}{e^{p_i}} + \sum_{j \neq i} \frac{d_j}{p^{-j}} + 1 \tag{25}$$

where

$$p^{-j} = 1 + \sum_{k \neq j} e^{p_k}$$

for all $i \in \{1, .., n\}$ and $p \in \mathbb{U}_+^n$. We now use Theorem 2 to show that $\mathrm{NCP}(F)$ has a unique element. First, we claim that F satisfies Assumption WBC. By the definition in (25), there exists $p^{\max} \in \mathbb{R}$ sufficiently large such that for every vector $p \in \mathbb{R}_+^n$ and i such that $p_i > p^{\max}$,

$$F_i(p) > 0. \tag{26}$$

Let C be the rectangular region defined by

$$C = \Big[0, (p^{\max}, .., p^{\max})\Big].$$

For any $p \in \mathbb{R}_+^n - C$, there exists $i \in \{1, .., n\}$ such that $p_i > p^{\max}$, hence using Eq. (26), for $y = 0 \in C$, we have

$$(y_i - p_i) F_i(p) < 0.$$

Then, F satisfies Assumption WBC with the compact set C, showing our first claim. We next claim that F satisfies Assumption SND-NCP. Let $p \in \text{NCP}(F)$. It can be seen that

$$\nabla F(p)^{ii} = \frac{d_i}{e^{p_i}} - e^{p_i} \sum_{j \neq i} \frac{d_j}{(p^{-j})^2}$$

and for $k \neq i$

$$\nabla F(p)^{ik} = e^{p_k} \sum_{j \neq i,k} \frac{d_j}{(p^{-j})^2}.$$

We claim that $\nabla F(p)|_{I^F(p)}$ is positive row diagonally dominant. For $i \in I^F(p)$, using $F_i(p) = 0$, we have

$$\begin{aligned}
\nabla F(p)^{ii} - \sum_{k \neq i} \nabla F(p)^{ik} &= 1 + \sum_{j \neq i} \frac{d_j}{p^{-j}} - e^{p_i} \sum_{j \neq i} \frac{d_j}{(p^{-j})^2} - \sum_{k \neq i} e^{p_k} \sum_{j \neq i,k} \frac{d_j}{(p^{-j})^2} \\
&= 1 + \sum_{j \neq i} \frac{d_j}{p^{-j}} \left(1 - \frac{e^{p_i}}{p^{-j}} - \sum_{k \neq i,j} \frac{e^{p_k}}{p^{-j}}\right) \\
&= 1 + \sum_{j \neq i} \frac{d_j}{p^{-j}} \left(1 - \frac{\sum_{k \neq j} e^{p_k}}{p^{-j}}\right) \geq 1,
\end{aligned}$$

where the inequality follows by the definition of p^{-j}. Since $\nabla F(p)^{ik} \geq 0$ for $i \neq k$, this shows, in particular, that

$$\nabla F(p)^{ii} - \sum_{k \neq i, k \in I^F(p)} \left|\nabla F(p)^{ik}\right| > 0,$$

i.e. $\nabla F(p)|_{I^F(p)}$ is positive row diagonally dominant. Then, by Lemma 1, $\nabla F(p)|_{I^F(p)}$ is a P-matrix, showing, in particular, that it is a partial P-matrix with respect to $I^{NB-NCP}(p)$. Then, F satisfies Assumption SND-NCP as desired. Since F satisfies assumptions WBC and SND-NCP, we conclude that $\text{NCP}(F)$ has a unique element.

Note that Theorem 1 could not have been used to assert uniqueness in this problem since whether F satisfies Assumption SCS-NCP cannot be verified. In fact, when $d_1 = 1, d_2 = .. = d_n = 0$, it is evident that the unique solution to the NCP is $p^* = (0, .., 0)$ and F does not satisfy Assumption SCS-NCP since $F_1(p^*) = 0$. Theorem 2 by relaxing the strict complementary slackness assumption, which is difficult to establish for a given problem, can be applied to this class of problems.

It is also noteworthy that earlier uniqueness results that require $\nabla F(p)$ to be a P-matrix for all $p \in \mathbb{R}^n_+$ cannot be used to assert uniqueness in this problem. Our proof shows that $\nabla F(p)|_{I^F(p)}$ is a P-matrix for $p \in \text{NCP}(F)$, yet it can be seen that $\nabla F(p)$ is not necessarily a global P-matrix [i.e., it is not a P-matrix when $p \notin \text{NCP}(F)$].

5.2 Example For Theorem 4

The following example demonstrates Theorem 4 and its improvement over Corollary 1.

Example 2 Let $0 < a < b$ be vectors in $\mathbb{R}^n$, $d_1, .., d_n > 0$ be scalars, and $M = [a, b]$ be a rectangular region in $\mathbb{R}^n$. Consider the function $f : \mathbb{R}^n \mapsto \mathbb{R}$ given by

$$f(p) = \sum_{i \in \{1,..,n\}} -d_i \log\left(\frac{p_i}{p^{-j}}\right)$$

where

$$p^{-j} = 1 + \sum_{k \neq j} p_k.$$

Let $F : \mathbb{R}^n \mapsto \mathbb{R}^n$ be given by $F = \nabla f$, i.e. for $i \in \{1, .., n\}$ [4],

$$F_i(p) = \frac{-d_i}{p_i} + \sum_{j \neq i} \frac{d_j}{p^{-j}}.$$

We will use Theorem 4 to show that MCP(F, M) has a unique element. We claim that (F, M) satisfies Assumption SND-MCP. Let $p \in$ MCP(F, M). It can be seen that

$$\nabla F(p)^{ii} = \frac{d_i}{p_i^2} - \sum_{j \neq i} \frac{d_j}{(p^{-j})^2} \tag{27}$$

and for $k \neq i$,

$$\nabla F(p)^{ik} = \sum_{j \neq i,k} -\frac{d_j}{(p^{-j})^2}. \tag{28}$$

We claim that $\nabla F(p)|_{I^F(p)}$ is a P-matrix. Let

$$C = P \nabla F(p) P$$

where P is the $n \times n$ diagonal matrix with entries p_i in the diagonal, and note that $\nabla F(p)|_{I^F(p)}$ is a P-matrix if and only if $C|_{I^F(p)}$ is a P-matrix. We claim that $C|_{I^F(p)}$ is positive row diagonally dominant. For $i \in I^F(p)$, using $F_i(p) = 0$, we have

$$\begin{aligned}
C^{ii} + \sum_{k \neq i} C^{ik} &= \sum_{j \neq i} \frac{d_j p_i}{p^{-j}} - \sum_{j \neq i} \frac{d_j p_i^2}{(p^{-j})^2} - \sum_{k \neq i} \sum_{j \neq i,k} \frac{d_j p_i p_k}{(p^{-j})^2} \\
&= \sum_{j \neq i} d_j \frac{p_i}{p^{-j}} - \sum_{j \neq i} d_j p_i \frac{\sum_{k \neq j} p_k}{(p^{-j})^2} \\
&> \sum_{j \neq i} d_j \frac{p_i}{p^{-j}} - \sum_{j \neq i} d_j \frac{p_i}{p^{-j}} = 0,
\end{aligned}$$

where the inequality follows from the definition of p^{-j}. Since $\nabla F^{ik} < 0$ for all $k \neq i$, this shows, in particular, that

$$C^{ii} - \sum_{k \neq i, k \in I^F(p)} \left| C^{ik} \right| > 0,$$

[4] We picked F to be the gradient vector of a relatively simple scalar valued function to show that such examples are likely to appear in simple applications.

hence $C|_{I^F(p)}$ is positive row diagonally dominant. Then by Lemma 1, $C|_{I^F(p)}$ is a P-matrix and hence $\nabla F(p)|_{I^F(p)}$ is a P-matrix. Then, p is a strongly non-degenerate solution and (F, M) satisfies Assumption SND-MCP, showing the claim. By Theorem 4, MCP(F, M) has a unique element as desired.

We note that Theorem 3 could not have been used to show uniqueness in this example, since there is no obvious way to prove (or in fact disprove) whether F satisfies Assumption SCS-MCP. Theorem 4 enables us to establish uniqueness without taking strict complementary.

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