ADDENDUM TO:
WHAT GOODS DO COUNTRIES TRADE?
A QUANTITATIVE EXPLORATION OF RICARDO’S IDEAS

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Abstract. This addendum provides the proofs of Lemma 1, Theorem 1, Lemma 2, and Theorem 2 and the derivation of Equation (16) in Section 4.1.2 of our main paper.

Lemma 1. Suppose that Assumptions A1-A4 hold. Then for any importer, \( j \), any pair of exporters, \( i \) and \( i' \), and any pair of goods, \( k \) and \( k' \),

\[
\ln \left( \frac{x_{ij}^k x_{ij}^{k'}}{x_{ij}^{k'} x_{ij}^k} \right) = \theta \ln \left( \frac{z_{i}^k z_{i'}^{k'}}{z_{i}^{k'} z_{i'}^k} \right) - \theta \ln \left( \frac{d_{ij}^k d_{ij}^{k'}}{d_{ij}^{k'} d_{ij}^k} \right).
\]

Proof of Lemma 1. By Assumption A4, we know that bilateral trade flows satisfy

\[
x_{ij}^k = \frac{\sum_{\omega \in \Omega_{ij}^k} [p_j^k(\omega)]^{1-\sigma_j} c_{ij}^k}{\sum_{\omega \in \Omega_j} p_j^k(\omega)^{1-\sigma_j}} \cdot \alpha_j^k w_j L_j.
\]

Since \( \Omega_{ij}^k \equiv \{ \omega \in \Omega \mid c_{ij}^k(\omega) = \min_{1 \leq i' \leq I} c_{i'j}^k(\omega) \} \), this can be rearranged as

\[
x_{ij}^k = \frac{\sum_{\omega \in \Omega} [p_j^k(\omega) \mathbb{I} \{ c_{ij}^k(\omega) = \min_{1 \leq i' \leq I} c_{i'j}^k(\omega) \}]^{1-\sigma_j}}{\sum_{\omega \in \Omega} p_j^k(\omega)^{1-\sigma_j}} \cdot \alpha_j^k w_j L_j,
\]

where the function \( \mathbb{I}\{\cdot\} \) is the standard indicator function. By Assumption A1, \( z_i^k(\omega) \) is independent and identically distributed (i.i.d.) across varieties so the same holds for \( c_{ij}^k(\omega) \).

In addition, \( z_i^k(\omega) \) is i.i.d. across countries so \( \mathbb{I} \{ c_{ij}^k(\omega) = \min_{1 \leq i' \leq I} c_{i'j}^k(\omega) \} \) is i.i.d. across varieties as well. This implies that \( p_j^k(\omega)^{1-\sigma_j} \) and \( \mathbb{I} \{ c_{ij}^k(\omega) = \min_{1 \leq i' \leq I} c_{i'j}^k(\omega) \} \) are i.i.d. across varieties. Moreover, since \( \sigma_j^k < 1 + \theta \) we have \( E \left[ p_j^k(\omega)^{1-\sigma_j} \right] < \infty \) so we can use the strong law of large numbers for i.i.d. random variables (e.g. Theorem 22.1 in Billingsley, 1995) and the continuous mapping theorem (e.g. Theorem 18.10 (i) in Davidson, Date: April 18, 2011.)
1994) to show that
\[
x_{ij}^k = \frac{E \left[ p_j^k(\omega)^{1-\sigma_j^k} \cdot \mathbb{I} \left\{ c_{ij}^k(\omega) = \min_{1 \leq i' \leq I} c_{ij}^k(\omega) \right\} \right]}{E \left[ p_j^k(\omega)^{1-\sigma_j^k} \right]} \cdot \alpha_j^k w_j L_j.
\]
Consider \( H_{ij}^k(c_{ij}^1, \ldots, c_{ij}^k) \equiv E \left[ p_j^k(\omega)^{1-\sigma_j^k} \cdot \mathbb{I} \left\{ c_{ij}^k(\omega) = \min_{1 \leq i' \leq I} c_{ij}^k(\omega) \right\} \right] \). Assumptions A1, A3 and straightforward computations yield
\[
(A-2) \quad H_{ij}^k(c_{ij}^1, \ldots, c_{ij}^k) = \frac{(c_{ij}^k)^{-\theta}}{\Gamma(t)} \frac{1}{\sum_{i' = 1}^{I} (c_{ij}^k)^{-\theta}} (1 - \sigma_j^k)^{\theta},
\]
where \( \Gamma(\cdot) \) is the Gamma function, \( \Gamma(t) \equiv \int_0^{+\infty} v^{t-1} \exp(-v) dv \) for any \( t > 0 \). Note that
\[
E \left[ p_j^k(\omega)^{1-\sigma_j^k} \right] = \sum_{i = 1}^{I} H_{ij}^k(c_{ij}^1, \ldots, c_{ij}^k),
\]
so that by using Equation (A-2) we get
\[
(A-3) \quad E \left[ p_j^k(\omega)^{1-\sigma_j^k} \right] = \Gamma\left( \frac{\theta + 1 - \sigma_j^k}{\theta} \right) \frac{1}{\sum_{i' = 1}^{I} (c_{ij}^k)^{-\theta}} (1 - \sigma_j^k)^{\theta},
\]
and hence
\[
(A-4) \quad x_{ij}^k = \frac{(c_{ij}^k)^{-\theta}}{\sum_{i' = 1}^{I} (c_{ij}^k)^{-\theta}} \cdot \alpha_j^k w_j L_j.
\]
With iceberg trade costs, Assumption A2, we have \( c_{ij}^k = d_{ij}^k w_i / z_i^k \). Combining the previous expression with Equation (A-4) gives the result of Lemma 1.

**Theorem 1.** Suppose that Assumptions A1–A4 hold. Then for any importer, \( j \), any pair of exporters, \( i \) and \( i' \), and any pair of goods, \( k \) and \( k' \),
\[
(A-5) \quad \ln \left( \frac{\bar{x}_{ij}^k \bar{x}_{ij}^{k'}}{\bar{x}_{ij}^k \bar{x}_{ij}^{k'}} \right) = \theta \ln \left( \frac{\bar{z}_i^k \bar{z}_{i'}^{k'}}{\bar{z}_i^k \bar{z}_{i'}^{k'}} \right) - \theta \ln \left( \frac{d_{ij}^k d_{i'j}^{k'}}{d_{ij}^k d_{i'j}^{k'}} \right),
\]
where \( \bar{x}_{ij}^k \equiv x_{ij}^k / \pi_{ii}^k \).

**Proof of Theorem 1.** We make use of the following Lemma.

**Lemma 3.** Suppose that Assumption A2 holds. Then, for all countries \( i \) and goods \( k \),
\[
(A-6) \quad \Omega_i^k = \{ \omega \mid c_{ii}^k(\omega) = \min_{1 \leq i' \leq I} c_{ij}^k(\omega) \}.
\]
**Proof of Lemma 3.** We proceed by contradiction. Fix an exporter \( j \), and suppose there exists a variety \( \omega_0 \) of good \( k \) and a country \( l \neq j \) such that:

\[
\begin{align*}
& c^k_{jl}(\omega_0) = \min_{1 \leq i' \leq I} c^k_{i'l}(\omega_0); \\
& c^k_{jj}(\omega_0) \neq \min_{1 \leq i' \leq I} c^k_{i'j}(\omega_0).
\end{align*}
\]

Then, there must be an exporter \( i \neq j \) such that

\[
\begin{align*}
& d^k_{jl} \cdot w_j / z^k_j(\omega_0) \leq d^k_{ii} \cdot w_i / z^k_i(\omega_0); \\
& d^k_{ij} \cdot w_i / z^k_i(\omega_0) < d^k_{jj} \cdot w_j / z^k_j(\omega_0).
\end{align*}
\]

Since \( d^k_{jj} = 1 \), multiplying the two inequalities above gives

\[ d^k_{ij} \cdot d^k_{jl} < d^k_{ii}, \]

which contradicts Assumption A2. This completes the proof of Lemma 3.

**Proof of Theorem 1 (continued).** By definition, we know that \( c^k_{ii}(\omega) = d^k_{ii} w_i / z^k_i(\omega) \). Using Lemma 3 then yields

\[
(A-7) \quad z^k_i \equiv E \left[ z^k_i(\omega) \mid \omega \in \Omega^k_i \right] = \frac{G_{ii}(c^k_{i1}, \ldots, c^k_{ii})}{\mu^k_{ii}} \cdot d^k_{ii} w_i,
\]

where we have let

\[
G_{ii}(c^k_{i1}, \ldots, c^k_{ii}) \equiv E \left[ (c^k_{ii}(\omega))^{-1} \mathbb{1} \{ c^k_{ii}(\omega) = \min_{1 \leq i' \leq I} c^k_{i'i}(\omega) \} \right],
\]

\[
\mu^k_{ii} \equiv \Pr \{ c^k_{ii}(\omega) = \min_{1 \leq i' \leq I} c^k_{i'i}(\omega) \}.
\]

The expressions for \( G_{ii}(c^k_{i1}, \ldots, c^k_{ii}) \) and \( \mu^k_{ii} \) can easily be computed from the expression for \( H^k_{ii}(c^k_{i1}, \ldots, c^k_{ii}) \) in proof of Lemma 1 when the result in Equation (A-2) is evaluated at \( \sigma^k_i = 2 \) and \( \sigma^k_i = 1 \), respectively. By Equation (A-2), we formally have

\[
G_{ii}(c^k_{i1}, \ldots, c^k_{ii}) = \Gamma \left( \frac{\theta - 1}{\theta} \right) \frac{(c^k_{ii})^{-\theta}}{\sum_{i'=1}^{I} (c^k_{i'i})^{-\theta}}^{(\theta - 1)/\theta},
\]

\[
\mu^k_{ii} = \frac{(c^k_{ii})^{-\theta}}{\sum_{i'=1}^{I} (c^k_{i'i})^{-\theta}}.
\]

Hence,

\[
(A-8) \quad z^k_i = \Gamma \left( \frac{\theta - 1}{\theta} \right) \frac{1}{\sum_{i'=1}^{I} (c^k_{i'i})^{-\theta} - 1/\theta} \cdot d^k_{ii} w_i = z^k_i \cdot \Gamma \left( \frac{\theta - 1}{\theta} \right) \left[ \frac{(c^k_{ii})^{-\theta}}{\sum_{i'=1}^{I} (c^k_{i'i})^{-\theta}} \right]^{-1/\theta}.
\]
Now, recall that we have defined \( \pi_{ii}^k \equiv x_{ii}^k / \left[ \sum_{i'=1}^I x_{i'i}^k \right] \). Using the expression for \( x_{ij}^k \) obtained in (A-4) it then follows that

\[
\pi_{ii}^k = \mu_{ii}^k = \frac{(c_{ii}^k)^{-\theta}}{\sum_{i'=1}^I (c_{i'i}^k)^{-\theta}}.
\]

Combining the two previous equations, we obtain

\[
(A-9)
\]

so combining with (A-10) and using \( e_{ij}^k = x_{ij}^k / \pi_{ii}^k \) gives

\[
(A-10)
\]

Analogously to Lemma 1, the result of Theorem 1 then follows.

\[\square\]

**Lemma 2.** Suppose that Assumptions A1-A5 hold. Adjustments in absolute productivity, \( \{Z_i\}_{i \neq i_0} \), can be computed as the solution of the system of equations

\[
\sum_{j=1}^I \sum_{k=1}^K \pi_{ij}^k \left( \frac{z_i^k / Z_i}{Z_i} \right)^{-\theta} \alpha_j^k \gamma_j = \gamma_i, \text{ for all } i \neq i_0.
\]

**Proof of Lemma 2.** Throughout this proof, we use labor in country \( i_0 \) as our numeraire in the initial and counterfactual trade equilibrium: \( w_{i0} = (w_{i0})' = 1 \). By definition, we know that \( Z_i \) is chosen for any \( i \neq i_0 \) such that the value of the relative wage \( (w_i / w_{i0})' \) in the counterfactual equilibrium is the same as in the initial equilibrium \( (w_i / w_{i0}) \). Thus Assumption A5 implies

\[
(A-12)
\]

where \( (\pi_{ij}^k)' \) is the share of exports from country \( i \) in country \( j \) and industry \( k \) in the counterfactual equilibrium. Using Equation (A-4), one can easily check that

\[
(A-13)
\]
and similarly that

\[(\pi_{ij}^k)' = \frac{[(w_i)'d_{ij}^k/(z_{ij}^k)']^{-\theta}}{\sum_{i'=1}^I [(w_{i'})'d_{ij}^{k'}/(z_{ij}^{k'})']^{-\theta}}.\]

Combining Equations (A-13) and (A-14) and using the fact that the relative wages remain unchanged in the counterfactual equilibrium, we get after rearrangements

\[(\pi_{ij}^k)' = \frac{\pi_{ij}^k [z_{ij}^k/(z_{ij}^k)']^{-\theta}}{\sum_{i'=1}^I \pi_{i'j}^k [z_{ij}^{k'}/(z_{ij}^{k'})']^{-\theta}} = \frac{\pi_{ij}^k (z_{ij}^k/Z_i)^{-\theta}}{\sum_{i'=1}^I \pi_{i'j}^k (z_{ij}^{k'}/Z_{i'})^{-\theta}},\]

where the second equality uses \((z_{ij}^k) \equiv Z_i \cdot z_{ij}^k\). Equations (A-12) and (A-15) imply

\[\sum_{j=1}^K \sum_{k=1}^I \pi_{ij}^k (z_{ij}^k/Z_i)^{-\theta} \alpha_j^k \gamma_j = \gamma_i,\]

where \(\gamma_i \equiv w_i L_i / \sum_{j=1}^I w_j L_j\) is the share of country \(i\) in world income.

**Theorem 2.** Suppose that Assumptions A1-A5 hold. If we remove country \(i_0\)'s Ricardian comparative advantage, then:

1. **Counterfactual changes in bilateral trade flows,** \(x_{ij}^k\), satisfy

\[(A-16) \quad \hat{x}_{ij}^k = \frac{(z_{ij}^k/Z_i)^{-\theta}}{\sum_{i'=1}^I \pi_{i'j}^k (z_{ij}^{k'}/Z_{i'})^{-\theta}}, \text{ for all } i, j, k.\]

2. **Counterfactual changes in country \(i_0\)'s welfare,\( W_{i_0} \equiv w_{i_0}/p_{i_0}, \) satisfy

\[(A-17) \quad \hat{W}_{i_0} = \prod_{k=1}^K \left[ \sum_{i=1}^I \pi_{i_{i_0}}^k \left( \frac{z_{ij}^k}{z_{i_{i_0}}^k Z_i} \right)^{-\theta} \right]^{\alpha_{i_0}^k / \theta}.\]

**Proof of Theorem 2.** Similar to previously and throughout this proof, we use labor in country \(i_0\) as our numeraire in the initial and counterfactual trade equilibrium: \(w_{i_0} = (w_{i_0})' = 1.\)

1. **Counterfactual changes in bilateral trade flows, \(x_{ij}^k\).**

Since the relative wages are unchanged in the counterfactual equilibrium, we must have

\[\hat{x}_{ij}^k = (x_{ij}^k)' = (x_{ij}^k) / \pi_{ij}^k.\]

Combining this observation with Equation (A-15), we obtain

\[\hat{x}_{ij}^k = \frac{(z_{ij}^k/Z_i)^{-\theta}}{\sum_{i'=1}^I \pi_{i'j}^k (z_{ij}^{k'}/Z_{i'})^{-\theta}}.\]
2. **Counterfactual changes in country $i_0$’s welfare**, $W_{i_0} \equiv w_{i_0} / p_{i_0}$.

By definition, we know that

$$\hat{p}_{i_0}^k = (p_{i_0}^k)' / p_{i_0}^k = \left[ \frac{\sum_{\omega \in \Omega} \left[ p_{i_0}^k (\omega) \right]' \left( 1 - \sigma_{i_0}^k \right)}{\sum_{\omega \in \Omega} p_{i_0}^k (\omega) \left( 1 - \sigma_{i_0}^k \right)} \right]^{1/(1 - \sigma_{i_0}^k)}.$$  

By invoking the strong law of large numbers for i.i.d. random variables and the continuous mapping theorem as we did in Theorem 1, then using Equation (A-3), we can rearrange the previous expression as

$$(A-18)\quad \hat{p}_{i_0}^k = \left[ \frac{\sum_{i=1}^I \left[ (w_i)' d_{i_0}^k / (z_i)^{\theta} \right]}{\sum_{i=1}^I (w_i d_{i_0}^k / z_i)^{-\theta}} \right]^{-1/\theta}.$$  

Combining Equations (A-13) and (A-18) and using the fact that the relative wages remain unchanged in the counterfactual equilibrium, we get after rearrangements

$$\hat{p}_{i_0}^k = \left[ \sum_{i=1}^I \pi_i^k \left( \frac{z_i^k}{z_i^0 Z_i^0} \right) \right]^{-1/\theta}.$$  

By definition of $p_{i_0} \equiv \prod_{k=1}^K \left( p_{i_0}^k \right)^{\alpha_{i_0}^k}$, we therefore have

$$\hat{p}_{i_0} = \prod_{k=1}^K \left[ \sum_{i=1}^I \pi_i^k \left( \frac{z_i^k}{z_i^0 Z_i^0} \right) \right]^{-\alpha_{i_0}^k / \theta},$$  

which immediately implies Equation (A-17). \hfill \Box

We conclude this online appendix by showing that for any pair of goods, $k$ and $k'$, and any pair of countries, $i$ and $i'$, Assumptions A1-A3 imply

$$\frac{z_i^k z_i'^k}{z_i^k z_i'^k} = E \left[ p_i^k (\omega) \left| \Omega_i^k \right. \right] E \left[ p_i^{k'} (\omega) \left| \Omega_i^{k'} \right. \right],$$

as stated in Equation (16) of Section 4.1.2 in our main paper. $E \left[ p_i^k (\omega) \left| \Omega_i^k \right. \right]$ can be readily computed from the expression for $H_{i_0}^k (c_{i_1}^k, \ldots, c_{i_I}^k)$ in the proof of Lemma 1 when the result in Equation (A-2) is evaluated at $\sigma_i^k = 0$ and $\sigma_i^k = 1$, respectively. Specifically, we have

$$E \left[ p_i^k (\omega) \left| \Omega_i^k \right. \right] = \Gamma \left( \frac{\theta + 1}{\theta} \right) \left[ \frac{1}{\sum_{i'=1}^I (c_i^{k,i'})^{-\theta}} \right]^{1/\theta}.$$
By Equation (A-8), we also know that
\[ z_i^k = \Gamma \left( \frac{\theta - 1}{\theta} \right) \frac{1}{\left[ \sum_{i'=1}^{J} (c_{ii'}^k)^{-\theta} \right]^{-1/\theta}} \cdot w_i, \]
where we have used the fact that \( d_{ii}^k = 1 \). Combining the two previous expressions for any pair of goods, \( k \) and \( k' \), we obtain
\[ \frac{z_i^k}{z_i^{k'}} = \frac{E \left[ p_i^{k'}(\omega) \right]}{E \left[ p_i^k(\omega) \right]} \left[ \frac{\Omega_i^{k'}}{\Omega_i^k} \right], \]
from which the desired result follows.

References