14.461: Technological Change, Lectures 11 and 12
Innovation and Firm Dynamics

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Motivation

- Model behavior of innovating firms at the same time as aggregate innovation.
- The models so far on satisfactory in terms of their microeconomic predictions.
  - Too much "creative destruction"; in practice, most of R&D from existing firms and plants. E.g.: Bartelsman and Doms (2000) and Foster, Haltiwanger and Krizan (2000): entry and exit account for about 25% of average TFP growth, with the remaining accounted for by continuing plants.
  - Too much entry by "large firms"—entering firms are small; many are unsuccessful, but some growth rapidly (Akcigit and Kerr, 2010).
- A satisfactory microeconomic model of innovation important to understand the interactions between innovation and reallocation.
- Also for policy analysis: What are the implications of "industrial policy" on innovation in reality?
  - E.g., bailout of GM and Ford or lowering entry barriers.

Approach

- Starting point: Klette and Kortum (2004).
  - Firms have a collection of products.
  - Incumbents have an innovation advantage because they generate more product ideas (consistent with some form of Gibrat's law).
  - But no exit decision by less productive firms.
  - No reallocation.
  - No heterogeneity across different types of firms and industries.
- Here framework based on Acemoglu, Akcigit, Bloom and Kerr (2011):
  - Scarce labor, reallocated away from less to more productive firms.
  - Fixed costs of operation $\longrightarrow$ endogenous exit decisions.
  - Multi-industry, multi-product firms with entry/exit (high-tech vs. low-tech industries; important since the extent of reallocation differs between industries such as computers vs. machine tools).
  - Firms vary by managerial quality, size, maturity (age), productivity.
  - Growth comes through innovation and reallocation.

Baseline Model: Preferences

- Infinite-horizon economy in continuous time.
- Continuum 1 of individuals with preferences over the unique final good:

$$U = \int_0^\infty \exp(-\rho t) \frac{C(t)^{1-\theta} - 1}{1-\theta} dt.$$

- Supply 1 unit of labor inelastically.
- Closed economy and no investment, and thus

$$Y(t) = C(t).$$

Final Good Technology

- Unique final goods produced combining intermediate goods:

$$Y(t) = \left(\int_{\mathcal{N}(t)} y_j(t)^{\frac{\varepsilon-1}{\varepsilon}} dj \right)^{\frac{\varepsilon}{\varepsilon-1}}.$$

- $\mathcal{N}(t) \subset [0, 1]$ is the set of *active* product lines.
- The measure of $\mathcal{N}(t)$ is less than 1 due to
  1. exogenous destructive shock
  2. outdated product lines due to lack of innovations.

Intermediate Good Technology

- Each intermediate good is produced by a **monopolist** (last innovator) with a linear technology:

$$y_{j,f}(t) = q_{j,f}(t)\, l_{j,f}(t),$$

  where $q_{j,f}$ is the product-specific production technology and $l_{j,f}$ is the number of workers.
- Firms pay a fixed cost of production, ϕ in terms of labor.
- Marginal cost of producing intermediate j for firm i:

$$MC_{j,f}(t) = \frac{w(t)}{q_{j,f}(t)},$$

  where $w(t)$ is the wage rate at time t.
- Total cost

$$TC_{j,f}(y_{j,f}, t) = w(t)\left[\phi + \frac{y_{j,f}}{q_{j,f}}\right].$$

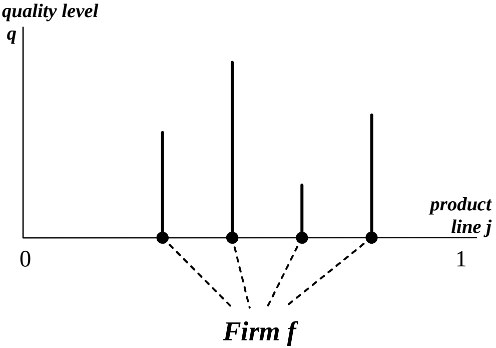
- Suppress time dependence to simplify notation.

Definition of a Firm

- A firm is defined as a collection of product qualities

$$\text{Firm } f = \mathcal{Q}_f \equiv \left\{ q_f^1, q_f^2, ..., q_f^n \right\}.$$

$n_f \equiv \left| \mathcal{Q}_f \right|$: is the number of product lines owned/operated by firm f.



Relative Quality

- Define *aggregate quality* as

$$Q(t) \equiv \left(\int_{\mathcal{N}(t)} q_j(t)^{\varepsilon-1} dj \right)^{\frac{1}{\varepsilon-1}}.$$

- In equilibrium,

$$Y(t) = Q(t) L(t),$$

  where $L(t)$ denotes employment in production.
- Define *relative quality* (of a leading-edge product) as

$$\hat{q}_j(t) \equiv \frac{q_j(t)}{Q(t)}.$$

- In equilibrium, decisions will depend on relative quality, which from the above equations determines the marginal cost of production.

R&D and Innovation

- Innovations follow a controlled Poisson Process.
- Flow rate of innovation for leader and follower given by

$$X_f = n_f^{\gamma} h_f^{1-\gamma}.$$

n_f : number of product lines.
h_f : number of researchers (here taken to be regular workers allocated to research).

- Implied costs function of achieving a flow rate of $x_f \equiv X_f / n_f$ *per product line* is

$$w(t) G(x_f) \equiv w(t) x_f^{\frac{1}{1-\gamma}}.$$

Innovation by Existing Firms

- Innovations are *undirected* across product lines.
- Let $q(j,t)$ be the latest technology in product line j at time t.
- After a successful innovation, innovation is realized in a random product line j. Then:
  1. firm f acquires product line j
  2. if technology in line j is active, i.e., $j \in \mathcal{N}(t)$, then technology in this line improves to
  $$q(j, t+\Delta t) = (1+\lambda)\, q(j,t)\,.$$
  3. if technology in line j is not active, i.e., $j \notin \mathcal{N}(t)$, then new technology is drawn from a relative quality distribution $\Lambda(\hat{q})$, i.e.,
  $$q(j, t+\Delta t) = w(t)\,\hat{q}.$$
     - spillover from the current state of technology pulling products that have fallen behind.

Entry and Exit

- A set of potential entrants can decide to enter, and will do so until the relevant free entry condition is satisfied.
  - For a flow rate of entry of x, a potential entrant needs to hire x/χ researchers.
- Exit happens in three ways:

1. *Creative destruction*. Firm f will lose each of its products at the rate $\tau > 0$ which will be determined endogenously in the economy.
2. *Exogenous destructive shock* at the rate φ.
3. *Endogenous exit due to low productivity.* When firm f's quality in a particular product line (relative to the wage rate) falls and the profit from that product falls below the fixed cost, firm f decides to stop producing that particular product.

Definition of Equilibrium

Definition A **steady-state Markov perfect equilibrum** in this economy consists of a tuple

$$\left\{y\left(\hat{q}\right), p\left(\hat{q}\right), X\left(n\right), x^{out}, r^{*}, g^{*}, \tilde{w}^{*}\right\},$$

where

- $y\left(\hat{q}\right), p\left(\hat{q}\right)$ maximizes firm's profit in a product a line where the relative productivity is $\hat{q}$,
- $X\left(n\right)$ maximizes the value of a firm with n product lines,
- x^{out} (entry) satisfies free entry condition,
- r^{*} is consistent with household maximization,
- g^{*} is consistent with the equilibrium innovation rates,
- $\tilde{w}^{*} \equiv \left(w/Y\right)^{*}$ clears the labor market.

Static Equilibrium

- Isoelastic demands imply that monopoly price for product j with quality $q_{j,f}(t)$ is

$$p_{j,f}^{*}(t)=\left(\frac{\varepsilon}{\varepsilon-1}\right)\frac{w(t)}{q_{j,f}(t)}=\frac{Q(t)}{q_{j,f}(t)}.$$

- Using the definition of relative quality and normalizing the price of the final good to one, we have

$$p_{j,f}^{*}(t)=\frac{1}{\hat{q}_{j,f}(t)} \text{ and } c_{j,f}^{*}(t)=\hat{q}_{j,f}(t)^{\varepsilon}Y(t).$$

- In equilibrium,

$$w(t)=\frac{\varepsilon-1}{\varepsilon}Q(t).$$

Static Equilibrium (continued)

- Therefore the gross equilibrium (before fixed costs) profits from a product with relative quality $\hat{q}_j$ are:

$$\begin{aligned} \pi\left(\hat{q}_{j,f}\left(t\right),t\right) &= \left(\frac{1}{\hat{q}_{j,f}\left(t\right)} - \frac{\varepsilon-1}{\varepsilon}\frac{1}{\hat{q}_{j,f}\left(t\right)}\right)c^*_{j,f}\left(t\right), \\ &= \frac{1}{\varepsilon}\hat{q}_{j,f}\left(t\right)^{\varepsilon-1}Y\left(t\right). \end{aligned}$$

- Define *normalized profits* as

$$\tilde{\pi}\left(\hat{q}_{j,f}\right) = \frac{1}{\varepsilon}\hat{q}_{j,f}^{\varepsilon-1}.$$

Dynamic Equilibrium

- Household maximization gives the growth rate of the economy, as a function of the interest rate, as

$$g(t) \equiv \frac{\dot{C}(t)}{C(t)} = \frac{\dot{Y}(t)}{Y(t)} = \frac{r(t) - \rho}{\theta}.$$

- In a *stationary equilibrium* where the growth rate and the relative quality distribution are constant, we have $g(t)$ constant and thus

$$r(t) = r^*.$$

- Let us also define *normalized value function* as

$$\tilde{V} \equiv \frac{V}{Y},$$

and *normalized wage* as

$$\tilde{w} \equiv \frac{w}{Y}.$$

Dynamic Equilibrium (continued)

- Moreover, innovation decisions also "decompose" given the above cost functions (recall that $X_f(n) = n_f x_f$).
- Then, the normalized value function of operating a set of products $\hat{\mathcal{Q}}_f$ (defined in terms of the relative qualities) can be written as follows:

Dynamic Equilibrium (continued)

$$
r^{*}\tilde{V}\left(\hat{\mathcal{Q}}_f\right) = \left[\begin{array}{c} \sum_{\hat{q}_{j,f}\in\hat{\mathcal{Q}}_f} \left\{ \begin{array}{c} \tilde{\pi}\left(\hat{q}_{jf}\right) - \tilde{w}\phi_j \\ + \frac{\partial \tilde{V}}{\partial \hat{q}_{jf}} \frac{\partial \hat{q}_{jf}}{\partial Q(t)} \frac{\partial Q(t)}{\partial t} \\ + \tau\left[\tilde{V}\left(\hat{\mathcal{Q}}_f \backslash \left\{\hat{q}_{jf}\right\}\right) - \tilde{V}\left(\hat{\mathcal{Q}}_f\right)\right] \end{array} \right\} + \\ \left|\hat{\mathcal{Q}}_f\right| \max_{x_f} \left\{ \begin{array}{c} -\tilde{w}G\left(x_f\right) \\ + x_f\left[\mathbb{E}_{\hat{q}}\tilde{V}\left(\hat{\mathcal{Q}}_f \cup (1+\lambda)\,\hat{q}_{j',f}\right) - \tilde{V}\left(\hat{\mathcal{Q}}_f\right)\right] \end{array} \right\} \\ + \varphi\left[0 - \tilde{V}\left(\hat{\mathcal{Q}}_f\right)\right] \end{array}\right.
$$

- The first line is profit from each product. The second line captures the fact that relative qualities are declining over time. The third line captures endogenous creative destruction The fourth line is the total cost of R&D. The fifth line is the benefit from R&D. The sixth line is the exogenous destructive shock.
- Here $\mathbb{E}_{\hat{q}}$ is taken over the active and inactive product distributions.

Exit Decisions

Lemma

Let firm f's decision to operate in product line j be

$$\begin{aligned} \iota_j &= 1 \text{ if operate} \\ \iota_j &= 0 \text{ otherwise.} \end{aligned}$$

Then, in equilibrium, firms's decision to operate in product line j will take the form of a cutoff rule. In particular, there exists $\hat{q}_{\min}$ such that

$$\iota_j \begin{cases} = 1 \text{ if } \hat{q}_{j,f} > \hat{q}_{\min} \\ = 0 \text{ if } \hat{q}_{j,f} < \hat{q}_{\min} \\ \in [0,1] \text{ otherwise} \end{cases}$$

Proof: The value function is monotone in $\hat{q}$ of each product line.

Franchise and R&D Option Values

Lemma *The normalized value can be written as the sum of franchise values:*

$$\tilde{V}\left(\hat{\mathcal{Q}}_f\right) = \sum_{\hat{q} \in \hat{\mathcal{Q}}_f} \Upsilon\left(\hat{q}\right),$$

where the franchise value of a product of relative quality $\hat{q}$ is the solution to the differential equation (iff $\hat{q} \geq \hat{q}_{\min}$):

$$r\Upsilon\left(\hat{q}\right) - \frac{\partial \Upsilon\left(\hat{q}\right)}{\partial \hat{q}} \frac{\partial \hat{q}}{\partial Q\left(t\right)} \frac{\partial Q\left(t\right)}{\partial t} = \tilde{\pi}\left(\hat{q}\right) - \tilde{w}\phi + \Omega - \left(\tau + \varphi\right)\Upsilon\left(\hat{q}\right),$$

where Ω is the R&D option value of holding a product line,

$$\Omega \equiv \max_{x_f \geq 0} \left\{ -\tilde{w}G\left(x_f\right) + x_f \left(\mathbb{E}_{\hat{q}} \tilde{V}\left(\hat{\mathcal{Q}}_f \cup \left(1+\lambda\right)\hat{q}_{j'f} \right) - \tilde{V}\left(\hat{\mathcal{Q}}_f\right) \right) \right\},$$

and $\hat{q}_{\min} \equiv \pi^{-1}\left(\tilde{w}\phi - \Omega\right)$.

Equilibrium Value Functions and R&D

Proposition

Equilibrium normalized value functions are:

$$\Upsilon(\hat{q}) = \frac{\hat{q}_{jf}^{\varepsilon-1} Y/\varepsilon}{r+\tau+\varphi+g(\varepsilon-1)}\left[1-\left(\frac{\hat{q}_{\min}}{\hat{q}}\right)^{\frac{r+\tau+\varphi+g(\varepsilon-1)}{g}}\right] + \frac{\Omega-\tilde{w}\phi}{r+\tau+\varphi}\left[1-\left(\frac{\hat{q}_{\min}}{\hat{q}}\right)^{\frac{r+\tau+\varphi}{g}}\right],$$

and equilibrium R&D is

$$x^*(\hat{q}) = x^* = \left[\frac{(1-\gamma)\,\mathbb{E}_{\hat{q}}\Upsilon(\hat{q})}{\tilde{w}}\right]^{\frac{1-\gamma}{\gamma}}.$$

Entry

- Entry by outsiders can now be determined by the free entry condition, written in the normalized fashion:

$$\max_{x^{out}\geq 0}\left\{x^{out}\mathbb{E}_{\hat{q}}\Upsilon\left(\hat{q}\right)-\tilde{w}\chi x^{out}\right\}.$$

Labor Market Clearing

- Labor market clearing:

$$1 = \Phi(\tilde{w}) \left[\int_{\hat{q}_{\min}}^{\infty} \hat{q}^{\varepsilon-1} dF(\hat{q}) + \phi + h(\tilde{w}) \right] + \chi x^{out}(\tilde{w}),$$

where Φ is the measure of active product lines at labor share $\tilde{w}$ and F is the invariant relative quality distribution.

- The invariant relative quality distribution among active firms satisfies two functional equations, which cannot be solved explicitly (but can be proven to have Pareto tails).

Stationary Relative Quality Distribution

Proposition

Let $\hat{q}_{crit} \equiv (1+\lambda)\,\hat{q}_{\min}$. *Then:*

- *For* $\hat{q} > \hat{q}_{crit}$*:*

$$\begin{aligned}\hat{q} g f(\hat{q}) &= \hat{q}_{\min} g f(\hat{q}_{\min}) + (\tau+\varphi) F(\hat{q}) \\ &\quad - \left(\tau + x^{out}\right)\left[F\left(\frac{\hat{q}}{1+\lambda}\right) + \frac{1-\Phi}{\Phi}\hat{\Lambda}(\hat{q})\right],\end{aligned}$$

- *For* $\hat{q}_{\min} < \hat{q} \leq \hat{q}_{crit}$*:*

$$\hat{q} g f(\hat{q}) = \hat{q}_{\min} g f(\hat{q}_{\min}) + (\tau+\varphi) F(\hat{q}) - \left(\tau + x^{out}\right)\left(\frac{1-\Phi}{\Phi}\right)\hat{\Lambda}(\hat{q}),$$

where

$$\tau = x\Phi + x^{out}$$

is the endogenous creative destruction rate.

Equilibrium Growth

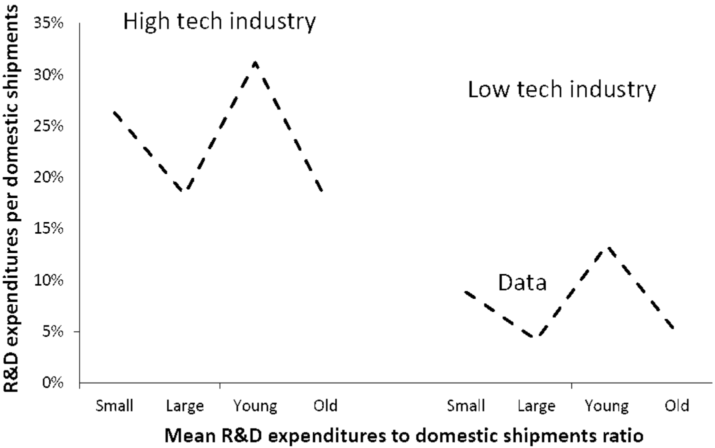
- It can be shown that (for well-behaved $\hat{\Lambda}(\cdot)$), the stationary relative quality distribution will have a Pareto tail (though the entire distribution will not generally be Pareto).
- The equilibrium growth rate can be computed as:

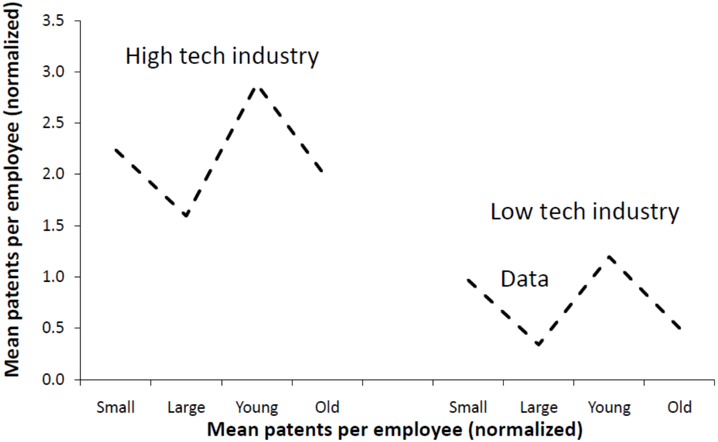
Proposition

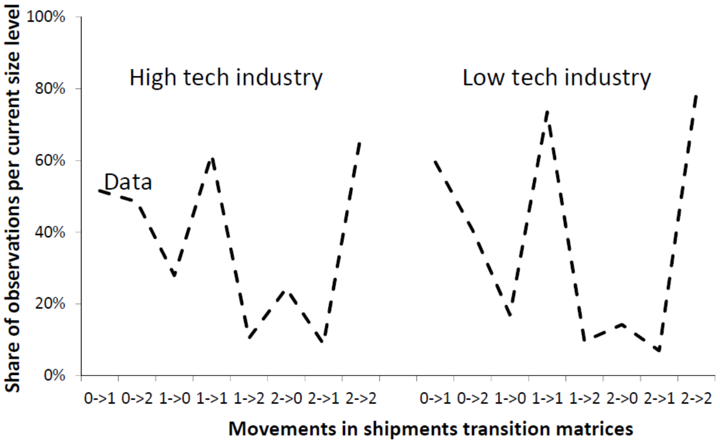
The equilibrium growth rate is

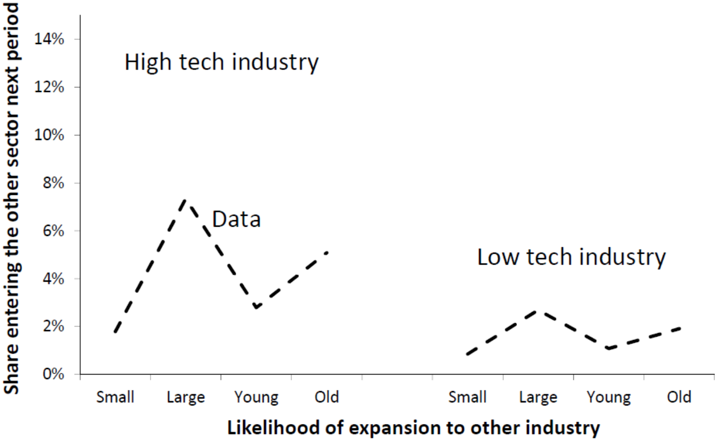
$$g = \frac{\varphi\left[\varkappa^{\varepsilon-1} - 1\right] + \tau\left[(1+\lambda)^{\varepsilon-1} - 1\right]}{\varepsilon - 1 - \left(\frac{\varepsilon-1}{\varepsilon}\right)^{\varepsilon-1}\left(\bar{\varkappa}^{\varepsilon-1} - 1\right)\hat{q}_{\min}^{\varepsilon} f\left(\hat{q}_{\min}\right)},$$

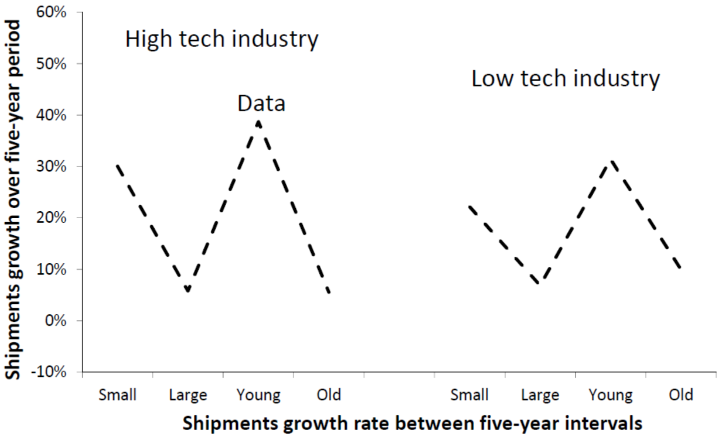
where $\varkappa \equiv \Phi\left[\int \hat{q}^{\varepsilon-1} d\Lambda\left(\hat{q}\right)\right]^{\frac{1}{\varepsilon-1}}$ *and* $\bar{\varkappa} \equiv \left[\int \hat{q}^{\varepsilon-1} d\Lambda\left(\hat{q}\right)\right]^{\frac{1}{\varepsilon-1}} / \hat{q}_{\min}$ *are the contributions of to average quality due to exogenous and endogenous exit, and* τ *is the endogenous creative destruction rate.*

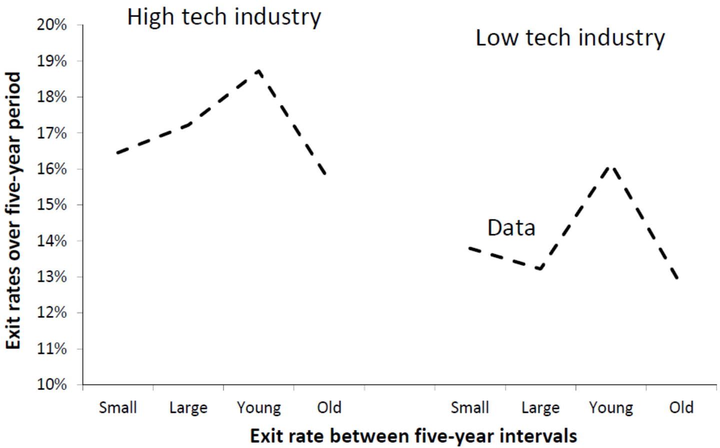

R&D expenditures per domestic shipments
35%
30%
25%
20%
15%
10%
5%
0%
High tech industry
Low tech industry
Data
Small
Large
Young
Old
Small
Large
Young
Old
Mean R&D expenditures to domestic shipments ratio


High tech industry
Low tech industry
Data
Mean patents per employee (normalized)
3.5
3.0
2.5
2.0
1.5
1.0
0.5
0.0
Small
Large
Young
Old
Small
Large
Young
Old
Mean patents per employee (normalized)


High tech industry
Low tech industry
Data
Share of observations per current size level
100%
80%
60%
40%
20%
0%
0->1 0->2 1->0 1->1 1->2 2->0 2->1 2->2
0->1 0->2 1->0 1->1 1->2 2->0 2->1 2->2
Movements in shipments transition matrices


High tech industry
Data
Low tech industry
Share entering the other sector next period
14%
12%
10%
8%
6%
4%
2%
0%
Small
Large
Young
Old
Small
Large
Young
Old
Likelihood of expansion to other industry


60%
50%
40%
30%
20%
10%
0%
-10%
Shipments growth over five-year period
High tech industry
Data
Low tech industry
Small
Large
Young
Old
Small
Large
Young
Old
Shipments growth rate between five-year intervals


High tech industry
Low tech industry
Data
20%
19%
18%
17%
16%
15%
14%
13%
12%
11%
10%
Exit rates over five-year period
Small
Large
Young
Old
Small
Large
Young
Old
Exit rate between five-year intervals

Preferences and Technology in the General Model

- Same preferences.
- Introduce two sectors with different parameters.
- Introduce managerial quality affecting the rate of innovation of each firm.
- Introduce firm age, so that young firms are potentially more innovative but also have a higher rate of failure.

Preferences and Technology (continued)

- Unique final goods produced from 2 sectors with a CES technology

$$Y = \left[\delta C_A^{\frac{\varepsilon-1}{\varepsilon}} + (1-\delta) C_B^{\frac{\varepsilon-1}{\varepsilon}} \right]^{\frac{\varepsilon}{\varepsilon-1}}.$$

- We normalize the price of the final good to 1.
- Each C_m, $m \in \{A, B\}$ is produced competitively with the following CES technology

$$C_m = \left(\int_{\mathcal{N}_m} y_{m,j}^{\frac{\varepsilon_m-1}{\varepsilon_m}} dj \right)^{\frac{\varepsilon_m}{\varepsilon_m-1}}.$$

- $\mathcal{N}_m(t) \subset [0,1]$ is the set of active product lines.

Intermediate Good Technology

- Each intermediate good is again produced by a monopolist with a linear technology:

$$y_{m,j,f} = q_{m,j,f} l_{m,j,f},$$

where $q_{m,j,f}$ is the firm-specific production technology and $l_{m,j,f}$ is the number of workers.

- Firms pay a fixed cost of production, ϕ_m in terms of labor.
- Marginal cost of producing intermediate j for firm i:

$$MC_{m,j,f} = \frac{w(t)}{q_{m,j,f}}.$$

where $w(t)$ is the wage rate at time t.

- Total cost

$$TC_{m,j,f}(y_{m,j,f}) = w(t)\left[\phi_m + \frac{y_{m,j,f}}{q_{m,j,f}}\right].$$

Definition of a Firm

- A firm is again defined as a technology pair and a management quality pair

$$\text{Firm } f \equiv \left\langle \left(\mathcal{Q}_{A,f}, \mathcal{Q}_{B,f} \right), \left(\theta_{A,f}, \theta_{B,f} \right) \right\rangle,$$

  where

$$\mathcal{Q}_{m,f} \equiv \left\{ q_{m,f}^{1}, q_{m,f}^{2}, ..., q_{m,f}^{n} \right\}.$$

- $n_{m,f} \equiv \left| \mathcal{Q}_{m,f} \right|$: is the number of product lines owned by firm f in industry m.

R&D and Innovation

- Innovations follow a controlled Poisson Process.
- Flow rate of innovation for leader and follower given by

$$X_{m,f} = (n_{m,f}\theta_{m,f})^{\gamma} h_{m,f}^{1-\gamma}.$$

$n_{m,f}$: number of product lines.
$\theta_{m,f}$: firm type (management quality).
$h_{m,f}$: number of researchers.

Innovation Realizations

- Innovations are *directed* across industries, *undirected* within the industry.
- Let $q_m(j,t)$ be the latest technology in industry m and product line j at time t.
- After a successful innovation, innovation is realized in a random product line j. Then:
  1. firm f acquires product line j
  2. technology in line j improves

$$q_m(j, t+\Delta t) = (1+\lambda)\, q_m(j,t).$$

- Applied to the target industry (industry m) with probability $\varrho_m \in (0,1)$, but takes place in the other industry (industry $\sim m$) with probability $(1-\varrho_m)$.

Entry

- There are potential entrants of measure 1.
- Each invests in R&D in order to innovate.
- Successful innovators enter the market.
- At the time of initial entry, each firm draws a management pair $\{\theta_A, \theta_B\}$:

$$
\begin{aligned}
\Pr\left(\{\theta_A, \theta_B\} = \left\{\theta_A^H, \theta_B^H\right\}\right) &= \alpha \\
\Pr\left(\{\theta_A, \theta_B\} = \left\{\theta_A^L, \theta_B^L\right\}\right) &= 1 - \alpha,
\end{aligned}
$$

where $\alpha \in (0, 1)$ and $\theta_m^H > \theta_m^L > 0$.

Firm Exit

- Exit happens in three ways as in the baseline model, except that now the exogenous destruction shock takes place at the rate $\varphi_f \in \left\{\varphi_y, \varphi_o\right\} > 0$ depending on whether the firm is young or old.

Young vs. Old Firms

- Over time, firms become "old" at the rate $\nu > 0$.
- When this happens, two things change:
  1. the innovativeness index of the firm declines from $\theta_{m,f}$ to $\mu\theta_{m,f}$, where $\mu < 1$, so that mature firms are less innovative,
  2. the flow rate with which the firm has to exit exogenously declines from φ_y to $\varphi_o < \varphi_y$.

Equilibrium

- Equilibrium definition and characterization similar to before (with more involved value functions and stationary distributions).
- Additional equilibrium conditions to ensure that relative prices of the two sectors are consistent.

Data Overview

- Combine two main databases:
  - Census LBD and CM manufacturing establishment production data: entry, exit, growth, etc.
  - Census firm R&D surveys: R&D expenditures by firm.
- Estimations center on combined databases from 1987 to 1997.
- Use additional materials before and after these year (e.g., to calculate firm age).
- Firms must have R&D expenditures to be included in the sample (i.e., innovative firms).

Data: LBD and Census of Manufacturing Data

- Longitudinal Business Database (LBD)
  - Annual business registry of the US from 1976 onwards.
  - Universe of establishments, so entry/exit can be modeled.
  - Establishment employee data means growth can be easily measured.
  - Describe firm age as time in registry since entry, capped at 10 years.
  - Observe complete firm age profile by 1987 starting point.
- Census of Manufacturers (CM)
  - Detailed data on inputs and outputs every five years.
  - Collected as the establishment level.
  - Shipments data from 1987-1992-1997 used for many size and transition moments.

Data: R&D Data

- NSF R&D Survey.
  - US' R&D survey, mostly biannual in timing.
  - Can be directly linked to Census operating data.
  - Firm-level survey of R&D expenditure, scientists, etc.
  - Surveys with certainty firms conducting $1m or more of R&D.
  - Sub-samples firms beneath this expenditure bar.

The SMM Methodology

- SMM chooses the model's parameters (θ) to match moments from real data to moments in the model.
- Does this using annealing to evaluate the match for different parameters $(\theta_1, \theta_2, \theta_3 ...)$ and pick the best fit.
- We have 24 parameters in the model so need at least 24 moments to match.
- In practice we target 76 moments to provide more accuracy, trying to place more weight on the more 'informative' moments.

The SMM Methodology

- Simulated Method of Moments
- $S = 16,000$ (number of simulated firms).
- $T = 10,250$ (number of simulation periods, first 10,000 burned, next 250 used to generate moments).
- SMM minimizes

$$\min_{\theta} \left[\frac{\mu(x_t) - \frac{1}{S}\sum_{s=1}^{S} \mu(x(u_t^s, \theta))}{|\mu(x_t)| + |\mu(x(u_t^s, \theta))|} \right]' \left[\frac{\mu(x_t) - \frac{1}{S}\sum_{s=1}^{S} \mu(x(u_t^s, \theta))}{|\mu(x_t)| + |\mu(x(u_t^s, \theta))|} \right]$$

- Estimate $\hat{\theta}$ is consistent as $T \rightarrow \infty$.
- **To come:** choose optimal weighting matrix W_T.

Creating Moments from the Data I

- Most data are at the firm-sector-year level, aggregating over establishments.
- As establishments have their own industry codes, we define multi-sector firms.
- Innovation intensities are defined at the firm-year level.
- We define two broad high-tech and low-tech sectors based upon industry traits.
- High-tech: communication equipment, pharma, missiles, etc.
- Low-tech: farm machinery, plastic products, fabricated metal products, auto industry, etc.
- High-tech sector accounts for 22% of firms, 22% of employment, and 20% of sales in our sample.

Creating Moments from the Data II

- 76 empirical moments are targeted.
- Both cross-sectional and dynamic moments are used.
- Age bins are created defining old firms as 10+ years in age.
- Size bins are created using sales divided at median for sector-year.
- Moments are estimated across the sector-size-age distribution.
- Some moments are estimated economy wide (e.g., epsilon, delta).
- Most moments have a scale that matches the model directly: growth rates, transition probabilities, etc.

Creating Moments from the Data III

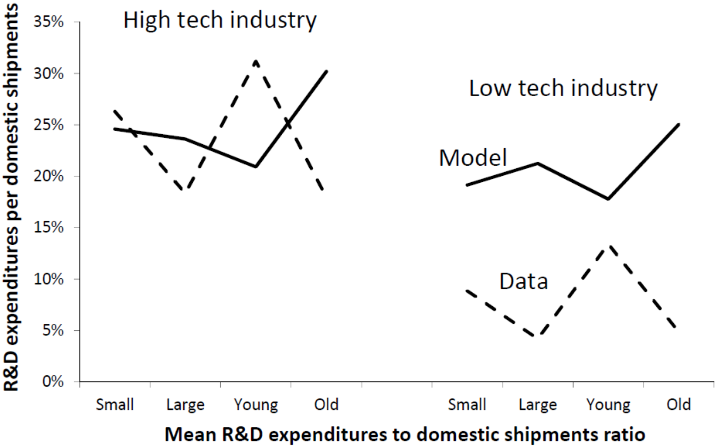
- Some of the moments are:
  - Firm entry/exit into/from the economy.
  - Firm entry/exit into/from new sector.
  - Firm age distribution.
  - Firm size distribution.
  - Transitions within the size distribution over time.
  - Firm growth by age and size.
  - R&D intensity (R&D/Sales) by age and size.

Parameter Estimates

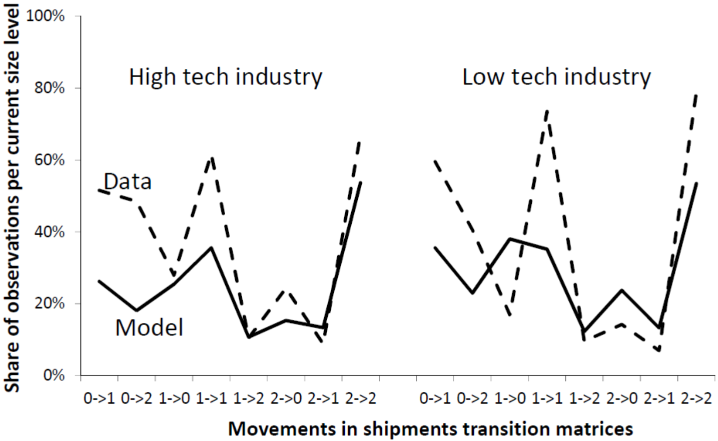
Param	Description	Value	
θ	CRRA	1.731	
ρ	discount rate	0.034	
ε	CES among A and B	1.277	
δ	share of sector A	0.436	
ε_A	CES within A	1.477	
ε_B	CES within B	1.922	
ϕ_A	fixed cost of operation in A	0.034	
ϕ_B	fixed cost of operation in B	0.101	
θ_{HH}	Management quality of high firms in high-tech	0.452	
θ_{HL}	Management quality of low firms in high-tech	0.432	
θ_{LH}	Management quality of high firms in low-tech	0.266	
θ_{LL}	Management quality of low firms in low-tech	0.201	

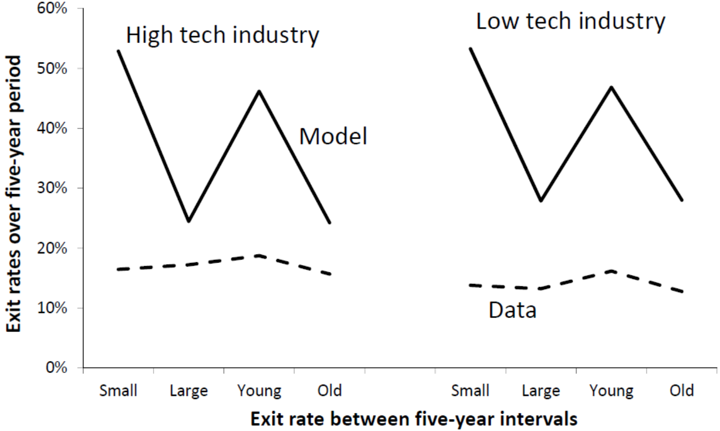
Parameter Estimates (continued)

Param	Description	Value	
λ	Innovation step size	0.138	
γ	Share of existing knowledge in the R&D production function	0.501	
ϱ_A	probability of spillover from high-tech	0.241	
ϱ_B	probability of spillover	0.039	
α	probability of drawing a high-type	0.479	
μ	reduction in the productivity	0.458	
ν	rate of becoming old	0.120	
φ_y	exogenous destruction rate for young	0.014	
φ_o	exogenous destruction rate for old	0.009	
χ_A	scaling factor of entry cost in A	0.209	
χ_B	scaling factor of entry cost in B	0.191	
Λ	Exponential distribution's shape parameter	1.183	

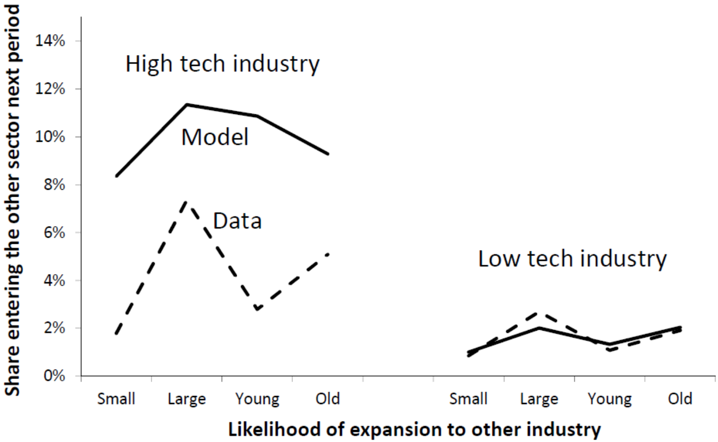


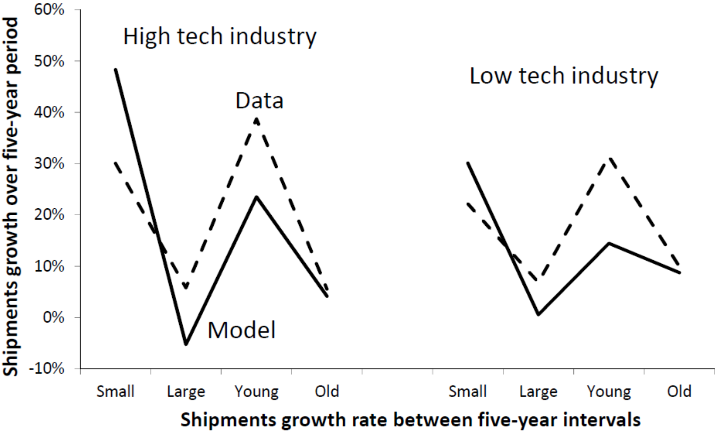
High tech industry
Low tech industry
Model
Data
R&D expenditures per domestic shipments
35%
30%
25%
20%
15%
10%
5%
0%
Small
Large
Young
Old
Small
Large
Young
Old
Mean R&D expenditures to domestic shipments ratio


Share of observations per current size level
100%
80%
60%
40%
20%
0%
High tech industry
Low tech industry
Data
Model
0->1 0->2 1->0 1->1 1->2 2->0 2->1 2->2
0->1 0->2 1->0 1->1 1->2 2->0 2->1 2->2
Movements in shipments transition matrices



60%
50%
40%
30%
20%
10%
0%
Exit rates over five-year period
High tech industry
Low tech industry
Model
Data
Small
Large
Young
Old
Small
Large
Young
Old
Exit rate between five-year intervals


Share entering the other sector next period
14%
12%
10%
8%
6%
4%
2%
0%
High tech industry
Model
Data
Low tech industry
Small
Large
Young
Old
Small
Large
Young
Old
Likelihood of expansion to other industry


High tech industry
Low tech industry
Data
Model
Shipments growth over five-year period
60%
50%
40%
30%
20%
10%
0%
-10%
Small
Large
Young
Old
Small
Large
Young
Old
Shipments growth rate between five-year intervals

Untargeted Moment

- Within-firm growth correlation

Data:	0.147
Model:	0.162

Baseline Results

- Baseline growth rate: 1.99%
- Labor Share of High-tech: 18%.
- Sales Share of High-tech: 45%.
- Share of labor employed by new entrants: 1%.

Baseline Results—Growth Decomposition

- Growth Decomposition by entrants vs incumbents

	H-T:	L-T:
Entrants	43%	56%
Incumbents	57%	44%

- by age and sector (shares in parenthesis)

	H-T:	L-T:
Young	68% (46%)	70% (49%)
Old	32% (48%)	30% (43%)

- by managerial quality (shares in parenthesis)

	H-T:	L-T:
High Quality	43% (52%)	38% (53%)
Low Quality	57% (42%)	62% (39%)

Baseline Results

- Growth decomposition by managerial quality

	H-T:	L-T:
Own Industry	97%	74%
Spillover from other Industry	3%	26%

- Reallocation of labor

	Unchanged	Reallocated within the same firm	Reallocated to another firm
H-T:	52%	29%	19%
L-T:	67%	18%	15%

- Causes of firm exit

	Creative Destruction/Competition	Exogneous Shock	Quality Dropouts
H-T:	89.9%	9.8%	0.2%
L-T:	89.6%	9.8%	0.5%

Validation Exercise

- Our ultimate objective is to apply the above methodology only to data moments generated from specific sources of variations (thus instead of SMM implementing IV-SMM).
- For now, we have focused on the simpler SMM estimation.
- We can nonetheless check that the estimated parameters are in line with credible IV estimates available in the literature.
- Here, we will compare the implications of the model to two IV exercises:
  1. The response of industries to an exogenous increase in the supply of relevant R&D scientists driven by U.S. immigration.
  2. The response of firms to changes in the cost of R&D driven by state-level R&D tax credits.

US Immigration and R&D Implications

- Immigrants comprised 24% of bachelor's level science and engineering workers in 2000, and 47% of Ph.D.s.
- Immigrants account for majority of net increase in US science and engineering workers over last couple of decades.
- Localized shocks in immigrant science and engineering workers linked to greater patenting:
  - Kerr and Lincoln (2010): city-level shocks due to annual fluctuations in H-1B program.
  - Hunt and Gauthier-Loiselle (2010): state-level shocks due to decade changes in immigration rates.

US Immigration and R&D Implications

- We examine the state-level supply shocks for science and engineering workers due to changes in H-1B visa program admissions.
- The national cap established by Congress had several dramatic shifts since 1995, going from annual levels of 65,000 admissions up to 195,000 and back down again.
- We interact the H-1B program's size with a state's initial dependency on the H-1B program (or on immigrant scientists and engineers generally).
- This leads to an instrumented elasticity of patenting to science and engineering workers of 0.694 (s.e.=0.295).
- We find a similar elasticity using changes in broader immigrant scientist and engineering labor pools across decades.
- Going forward, we will estimate firm-level elasticities to local science and engineering supply shocks due to immigration using matched patent data that identifies the ethnic composition of a firm's inventors.

Comparison with the Model-Implied Elasticity

- Recall that in the model:

$$X_f = n_f^{\gamma} h_f^{1-\gamma}.$$

- So a 1% increase in the supply of science and engineering workers (allocated fortunately across innovating firms in equilibrium) should increase R&D output (proxied here by patents) with an elasticity of $1 - \gamma$.
- Given our estimate of $\gamma \simeq 0.5$, this elasticity is estimated to be around 0.5 in our model, which is in the ballpark of 0.694.

R&D Tax Credits Background

- The US spends about $2 billion on R&D tax credits a year.
- Federal tax credits were introduced in 1981, but common to all firms.
- State tax credits introduced soon after (Minnesota was first in 1982) and apply only to R&D within that state.
- Large variation across time and states, and uncorrelated prior economic variables (Chirinko and Wilson, 2009), so provides an R&D price instrument.

Using R&D Tax Credits

- A number of studies have estimated the elasticity of R&D with respect to user cost. The estimating framework is a panel regression of the form:

$$\log(R\&D_{i,t}) = \alpha_i + \beta_t + \gamma \log(R\&D\_Cost\_of\_Capital_{i,t})$$

- Using tax changes to instrument the user cost the literature typically finds elasticities (γ) around unity, e.g:
  - Bloom, Griffith and Van Reenen (2002) find -1.088 (0.024) on a cross-country panel
  - Hall (1993) looks at firms and Baily and Lawrence (1995) and Mumuneas and Nadiri (1996) at industries, also finding about unit elasticities

Comparison with the Model-Implied Elasticity

- In the model, equilibrium R&D spending is

$$\begin{aligned} R\&D &= c_{R\&D} x^{\frac{1}{1-\gamma}} \\ &= \left(c_{R\&D}\right)^{\frac{\gamma-1}{\gamma}} \left[Y \left(1-\gamma\right) \mathbb{E}_{\hat{q}} \mathrm{Y}\left(\hat{q}\right)\right]^{\frac{1}{\gamma}}, \end{aligned}$$

  where $c_{R\&D}$ is the R&D user cost (the baseline model just the wage w). Taking logs:

$$\ln R\&D = \frac{\gamma-1}{\gamma} \ln\left(c_{R\&D}\right) + \ln\left[Y\left(1-\gamma\right)\mathbb{E}_{\hat{q}}\mathrm{Y}\left(\hat{q}\right)\right]^{\frac{1}{\gamma}}.$$

- Thus the comparable elasticity in the model is $(\gamma - 1)/\gamma \approx 1$, so again in the ballpark of the estimates in the literature.

Policy Analysis: Entry Subsidy

- Baseline

	x^{out}	x^{yl}	x^{yh}	x^{ol}	x^{oh}	Φ^{yl}	Φ^{yh}	Φ^{ol}	Φ^{oh}
H-T	0.073	0.149	0.159	0.069	0.073	0.207	0.250	0.208	0.269
L-T	0.096	0.088	0.121	0.041	0.056	0.214	0.273	0.177	0.254
	τ_A	τ_B	g						
	0.170	0.171	1.99%						

- Entry Subsidy (10%)

	x^{out}	x^{yl}	x^{yh}	x^{ol}	x^{oh}	Φ^{yl}	Φ^{yh}	Φ^{ol}	Φ^{oh}
H-T	0.082	0.144	0.153	0.066	0.071	0.217	0.259	0.204	0.256
L-T	0.105	0.084	0.113	0.038	0.052	0.222	0.278	0.175	0.243
	τ_A	τ_B	g						
	0.177	0.176	2.07%						

Policy Analysis: Exit Tax

- Baseline

	x^{out}	x^{yl}	x^{yh}	x^{ol}	x^{oh}	Φ^{yl}	Φ^{yh}	Φ^{ol}	Φ^{oh}
H-T	0.073	0.149	0.159	0.069	0.073	0.207	0.250	0.208	0.269
L-T	0.096	0.088	0.121	0.041	0.056	0.214	0.273	0.177	0.254
	τ_A	τ_B	g						
	0.170	0.171	1.99%						

- Exit Tax (exit tax of 10%)

	x^{out}	x^{yl}	x^{yh}	x^{ol}	x^{oh}	Φ^{yl}	Φ^{yh}	Φ^{ol}	Φ^{oh}
H-T	0.079	0.138	0.146	0.064	0.068	0.212	0.252	0.208	0.261
L-T	0.103	0.081	0.110	0.038	0.051	0.220	0.275	0.179	0.247
	τ_A	τ_B	g						
	0.169	0.171	1.98%						

Policy Analysis: R&D Subsidy to All existing firms

- Baseline

	x^{out}	x^{yl}	x^{yh}	x^{ol}	x^{oh}	Φ^{yl}	Φ^{yh}	Φ^{ol}	Φ^{oh}
H-T	0.073	0.149	0.159	0.069	0.073	0.207	0.250	0.208	0.269
L-T	0.096	0.088	0.121	0.041	0.056	0.214	0.273	0.177	0.254
	τ_A	τ_B	g						
	0.170	0.171	1.99%						

- Subsidizing R&D by Existing Firms by 10%

	x^{out}	x^{yl}	x^{yh}	x^{ol}	x^{oh}	Φ^{yl}	Φ^{yh}	Φ^{ol}	Φ^{oh}
H-T	0.063	0.172	0.184	0.080	0.085	0.198	0.250	0.205	0.283
L-T	0.089	0.107	0.146	0.050	0.068	0.207	0.277	0.171	0.265
	τ_A	τ_B	g						
	0.176	0.179	2.1%						

Policy Analysis: R&D Subsidy to Mature Firms

- Baseline

	x^{out}	x^{yl}	x^{yh}	x^{ol}	x^{oh}	Φ^{yl}	Φ^{yh}	Φ^{ol}	Φ^{oh}
H-T	0.073	0.149	0.159	0.069	0.073	0.207	0.250	0.208	0.269
L-T	0.096	0.088	0.121	0.041	0.056	0.214	0.273	0.177	0.254
	τ_A	τ_B	g						
	0.170	0.171	1.99%						

- Subsidizing R&D by Mature Firms by 10%

	x^{out}	x^{yl}	x^{yh}	x^{ol}	x^{oh}	Φ^{yl}	Φ^{yh}	Φ^{ol}	Φ^{oh}
H-T	0.078	0.138	0.147	0.071	0.076	0.208	0.247	0.211	0.268
L-T	0.100	0.081	0.110	0.042	0.057	0.216	0.269	0.181	0.254
	τ_A	τ_B	g						
	0.171	0.171	2.0%						

Policy Analysis: R&D Subsidy to Mature L-T Firms

- Baseline

	x^{out}	x^{yl}	x^{yh}	x^{ol}	x^{oh}	Φ^{yl}	Φ^{yh}	Φ^{ol}	Φ^{oh}
H-T	0.073	0.149	0.159	0.069	0.073	0.207	0.250	0.208	0.269
L-T	0.096	0.088	0.121	0.041	0.056	0.214	0.273	0.177	0.254
	τ_A	τ_B	g						
	0.170	0.171	1.99%						

- Subsidizing R&D by Mature Firms in Low-Tech Sector by 10%

	x^{out}	x^{yl}	x^{yh}	x^{ol}	x^{oh}	Φ^{yl}	Φ^{yh}	Φ^{ol}	Φ^{oh}
H-T	0.080	0.140	0.149	0.065	0.069	0.213	0.252	0.207	0.262
L-T	0.100	0.080	0.110	0.042	0.057	0.215	0.269	0.181	0.255
	τ_A	τ_B	g						
	0.171	0.170	2.0%						

Policy Analysis: Exit Tax on Mature L-T Firms

- Baseline

	x^{out}	x^{yl}	x^{yh}	x^{ol}	x^{oh}	Φ^{yl}	Φ^{yh}	Φ^{ol}	Φ^{oh}
H-T	0.073	0.149	0.159	0.069	0.073	0.207	0.250	0.208	0.269
L-T	0.096	0.088	0.121	0.041	0.056	0.214	0.273	0.177	0.254

τ_A	τ_B	g
0.170	0.171	1.99%

- Subsidizing Fixed Operation Cost by Mature Firms in Low-Tech Sector by 10%

	x^{out}	x^{yl}	x^{yh}	x^{ol}	x^{oh}	Φ^{yl}	Φ^{yh}	Φ^{ol}	Φ^{oh}
H-T	0.079	0.139	0.148	0.064	0.068	0.212	0.252	0.208	0.262
L-T	0.102	0.082	0.111	0.038	0.052	0.220	0.274	0.179	0.248

τ_A	τ_B	g
0.169	0.171	1.98%

Pareto Distribution

- Many quantities in economics, other social sciences and physical sciences appear to be well approximated by Pareto distribution.
- Pareto distribution or the power law has the following counter-cumulative distribution function:

$$\mathbb{G}\left(y\right) \equiv 1 - \Pr\left[\tilde{y} \leq y\right] = \Gamma y^{-\lambda},$$

  where $\lambda \geq 1$ is the shape parameter.
- When the literature refers to the Pareto or the power law distribution, this generally means that the distribution has Pareto tails, meaning that it takes this form for y large.

Zipf Distribution

- The Zipf distribution is a special case with $\lambda = 1$ (or sometimes it is used for the case where λ is approximately equal to 1).
- Empirical city size distribution and firm size distributions appear to be well approximated by this distribution, and for city size distributions, this is generally referred to as "Zipf's Law".
- This is also equivalent to a relationship of slope -1 between log rank of the city (according to city size) and log of the population.
- In other words:

$$\ln \text{Rank}_j = c - \ln y_j,$$

  where y is size. Thus

$$y_j = \frac{C}{\text{Rank}_j}.$$

- Rewriting this

$$\Pr\left(\tilde{y} > y_j\right) = \frac{C}{y_j},$$

- This may apply exactly or only in the "tails," i.e., y_j large enough.

Firm Size Distribution

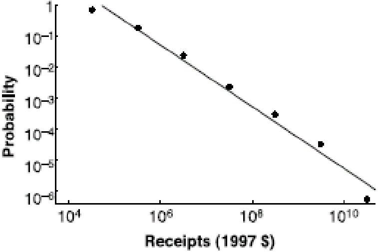
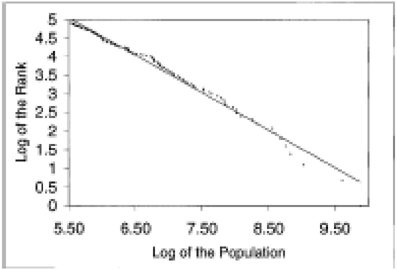


Fig. 2. Tail cumulative distribution function of U.S. firm sizes, by receipts in dollars. Data are for 1997 from the U.S. Census Bureau, tabulated in bins whose width increases in powers of 10. The solid line is the OLS regression line through the data and has slope of 0.994 (SE = 0.064; adjusted $R^2 = 0.976$).

Axtell 2001

For US Metropolitan Areas in the 1990s



Where Does the Pareto Distribution Come From?

- Many stochastic processes lead to the Pareto distribution. The most famous and most convenient is the so-called **Kesten process**, which takes the form

$$y_{t+1} = \gamma_t y_t + z_t,$$

where γ_t and z_t are independent random variables, with $\mathbb{E}\gamma_t < \infty$ and $\mathbb{E}z_t < \infty$.

- Suppose that y has a stationary distribution $\mathbb{G}$ (this is not trivial as we will see, so this assumption makes life much more straightforward).

Pareto Distribution (continued)

- $\mathbb{G}(y) \equiv 1 - \Pr[\tilde{y} \leq y] = \Gamma y^{-\lambda}$ can be written as

$$\begin{aligned} \Pr[y_{t+1} \geq y] &= \mathbb{E}\left[\mathbf{1}_{\{y_{t+1} \geq y\}}\right] \\ &= \mathbb{E}\left[\mathbf{1}_{\{\gamma_t y_t + z_t \geq y\}}\right] \\ &= \mathbb{E}\left[\mathbf{1}_{\{y_t \geq (y - z_t)/\gamma_t\}}\right], \end{aligned}$$

  where $\mathbf{1}_{\{\mathcal{P}\}}$ is the indicator function for the event $\mathcal{P}$.
- Then, by the definition of a stationary distribution $\mathbb{G}$, we have

$$\mathbb{G}(y) = \mathbb{E}_{\gamma,z}\left[\mathbb{G}\left(\frac{y-z}{\gamma}\right)\right].$$

- More generally, this is derived from the solution of **Kolmogorov's forward equation** (see below).
- The solution, which exists by assumption, will be Pareto in the tail:

$$\mathbb{G}(y) = \Gamma y^{-\lambda}$$

  for large y, and moreover, λ will be given by $\mathbb{E}_{\gamma}\gamma^{\lambda} = 1$.

Pareto Distribution (continued)

- To see this, consider the special case where $z_t = z$. Then

$$\mathbb{G}(y) = \mathbb{E}_\gamma \left[\mathbb{G}\left(\frac{y-z}{\gamma} \right) \right].$$

Suppose $\mathbb{G}(y) = \Gamma y^{-\lambda}$ for large y. Then, again for large y,

$$\begin{aligned} \Gamma y^{-\lambda} &= \mathbb{E}_\gamma \left[\Gamma (y-z)^{-\lambda} \gamma^{\lambda} \right], \text{ or} \\ \Gamma y^{-\lambda} &= \Gamma (y-z)^{-\lambda} \mathbb{E}_\gamma \gamma^{\lambda}, \end{aligned}$$

where the term on the left and the first term on the right are approximately equal for large y, giving the desired result.

- When z is random, same reasoning applies for large y.

When Does a Limiting Distribution Exist?

- Take the process
$$y_{t+1} = \gamma_t y_t,$$
with γ_t independent and mean one. This clearly does not have a limiting distribution, as the empirical distribution (as a function of time) will keep on expanding.
- Essentially for a limiting distribution to exist, the z_t term needs to make sure that there aren't too many small observations (in the exact Pareto case, y_t has to be greater than Γ.
- One way of achieving this is to have a hard or soft lower **reflecting barrier**.

When Does a Limiting Distribution Exist? (continued)

- The following is a well-known theorems from stochastic processes.

Theorem

Suppose that y follows the continuous time reflected geometric Brownian motion process

$$\frac{dy_t}{y_t} = \begin{cases} \gamma dt + \sigma dB_t & \text{if } y_t > y_{\min} \\ \max\{\gamma dt + \sigma dB_t, 0\} & \text{if } y_t \leq y_{\min}, \end{cases}$$

where $\gamma < 0$, then y_t converges in distribution to the Pareto distribution with shape parameter

$$\lambda = \frac{1}{1 - y_{\min}/\bar{y}},$$

where $\bar{y}$ is its stationary distribution average.

- This implies that if the lower barrier $y_{\min}$ is sufficiently small, the exponent is approximately 1.

Approach

- Simple framework for analysis of growth with innovations by incumbents as well as entry (creative destruction) jointly would firm size distributions.
- Need to go beyond existing models of growth.
  - Schumpeterian models thus far generate growth only by entry (powerful *Arrow's replacement effect*).
  - Models of expanding input or product variety not useful for the study of the set of questions either.

Remainder of This Lecture

- Small modification on textbook Schumpeterian growth models to incorporate innovation by incumbents as well as creative destruction.
- Two types of innovations; incremental and radical (consistent with quantitative and qualitative evidence on innovation).
- A large part of productivity growth generated by continuing establishments.
- Somewhat counterintuitive comparative statics: in contrast to the spirit of Schumpeterian models, entry barriers or taxes on entrants increase growth.
    - because it increases the value of incumbents and encourages their innovation.
- Endogenous firm dynamics (consistent with Gibrat's law, e.g., Sutton, 1997; though see Akcigit 2008; Akcigit and Kerr, 2010).
- Endogenous firm size distribution (consistent with Pareto with exponent $\simeq 1$, e.g. Axtell, 2001).

Model I

- Continuous time, extension of the baseline Schumpeterian model.
- Representative household with the standard CRRA preferences.

$$\int_0^\infty \exp(-\rho t) \frac{C(t)^{1-\theta} - 1}{1-\theta} dt.$$

- Population constant at L and labor is supplied inelastically.
- Resource constraint at time t as usual:

$$C(t) + X(t) + Z(t) \leq Y(t), \qquad (1)$$

- Production function of the unique final good:

$$Y(t) = \frac{1}{1-\beta}\left[\int_0^1 q(\nu,t)^\beta x(\nu,t \mid q)^{1-\beta} d\nu\right] L^\beta, \qquad (2)$$

where:

  - $x(\nu,t \mid q)$=quantity of the machine of type ν of quality $q(\nu,t)$.

Model (continued)

- Engine of growth: quality improvements, now driven by two types of innovations:
  1. Innovation by incumbents
  2. Creative destruction by entrants.
- "Quality ladder" for each machine type:

$$q(\nu, t) = \lambda^{n(\nu,t)} q(\nu, s) \text{ for all } \nu \text{ and } t,$$

where:

  - $\lambda > 1$ and $n(\nu, t)$ is the number of *incremental* innovations on this machine line between $s \leq t$ and t.
  - s is the date at which this particular type of technology was first invented, with quality $q(\nu, s)$ at that point.

Model (continued)

- Incumbent has fully enforced patent on machines that it has developed.
- This patent does not prevent entrants leapfrogging the incumbent's machine quality.
- At time $t = 0$, each machine line starts with some quality $q(\nu, 0) > 0$ owned by an incumbent.
- Incremental innovations only by the incumbent producer, i.e. "tinkering" innovations (consistent with case study evidence, e.g., Freemen, 1982, or Scherer, 1984):
  - If spend $z(\nu, t)\, q(\nu, t)$ of the final good for innovation on quality $q(\nu, t)$, then flow rate of innovation $\phi z(\nu, t)$ for $\phi > 0$.
  - More general version $\phi(z)$: see paper.
  - More formally: for any interval $\Delta t > 0$, the probability of one incremental innovation is $\phi z(\nu, t)\, \Delta t$ and the probability of more than one is $o(\Delta t)$ (with $o(\Delta t) / \Delta t \to 0$ as $\Delta t \to 0$).
  - Such innovation results in new machine of quality $\lambda q(\nu, t)$.

Model (continued)

- Alternatively, new firm (entrant) can innovate over existing machines in machine line ν at time t:
  - If current quality is $q(\nu, t)$, spending one unit of the final good gives flow rate of innovation $\eta(\hat{z}(\nu, t))/q(\nu, t)$.
  - $\eta(\cdot)$ is a strictly decreasing, continuously differentiable function.
  - $\hat{z}(\nu, t)$ is total of R&D by new entrants towards machine line ν at time t.
  - Innovation leads to new machine of quality $\kappa q(\nu, t)$, where $\kappa > \lambda$.
- Note:
  - Innovation by entrants more "radical" than by incumbents, supported from studies of innovation.
  - Incumbents also have access to the technology for radical innovation, but Arrow replacement effect implies they would never use it (entrants will make zero profits from it, so profits of incumbents would be negative).
  - Strictly decreasing function η, captures "external" diminishing returns (new entrants "fishing out of the same pond").

Model (continued)

- Each entrant attempting R&D is potentially small, take $\eta\left(\hat{z}\left(\nu,t\right)\right)$ as given.
- Assume that $z\eta\left(z\right)$ is strictly increasing in z: greater aggregate R&D towards a machine line increases the overall probability of discovering a superior machine.
- $\eta\left(z\right)$ satisfies Inada-type assumptions:

$$\lim_{z\to\infty}\eta\left(z\right)=0 \text{ and } \lim_{z\to 0}\eta\left(z\right)=\infty. \tag{3}$$

- Once a machine of quality $q\left(\nu,t\right)$ has been invented, any quantity can be produced at the marginal cost ψ, $\psi\equiv 1-\beta$.

Model (continued)

- Thus total amount of expenditure on the production of intermediate goods at time t:

$$X(t) = \int_0^1 \psi x(\nu, t)\, d\nu, \tag{4}$$

where $x(\nu, t)$ is the quantity of this machine used in final good production.

- Total expenditure on R&D is sum of R&D by incumbents and entrants ($z(\nu, t)$ and $\hat{z}(\nu, t)$):

$$Z(t) = \int_0^1 [z(\nu, t) + \hat{z}(\nu, t)]\, q(\nu, t)\, d\nu, \tag{5}$$

where $q(\nu, t)$ refers to the highest quality of the machine of type ν at time t (recall: higher-quality machines is proportionately more difficult).

Model (continued)

- Allocation. Time paths of $[C(t), X(t), Z(t)]_{t=0}^{\infty}$, $[z(\nu,t), \hat{z}(\nu,t)]_{\nu\in[0,1],t=0}^{\infty}$, $[p^x(\nu,t \mid q), x(\nu,t), V(\nu,t \mid q)]_{\nu\in[0,1],t=0}^{\infty}$, $[r(t), w(t)]_{t=0}^{\infty}$.
- Equilibrium. Allocation in which R&D decisions by entrants maximize their net present discounted value, pricing, quantity and R&D decisions by incumbents maximize their net present discounted value, consumers choose the path of consumption and allocation of spending across machines and R&D optimally, and the labor market clears.

Model (continued)

- Profit-maximization by the final good sector implies the demand for machines of highest-quality:

$$x(\nu, t \mid q) = p^x (\nu, t \mid q)^{-1/\beta} q(\nu, t) L \quad \text{for all } \nu \in [0, 1] \text{ and all } t, \tag{6}$$

- Unconstrained monopoly price is usual formula as a constant markup over marginal cost.
- No limit price assumption:

$$\kappa > \left(\frac{1}{1-\beta} \right)^{\frac{1-\beta}{\beta}}, \tag{7}$$

- By implication, incumbents that make further innovations can also charged the unconstrained monopoly price.

Equilibrium

- Since demand for machines in (6) is iso-elastic and $\psi = 1 - \beta$, profit-maximizing monopoly price:

$$p^x(\nu, t \mid q) = 1. \tag{8}$$

- Combining with (6):

$$x(\nu, t \mid q) = qL. \tag{9}$$

- Flow profits of a firm with monopoly rights on the machine quality q:

$$\pi(\nu, t \mid q) = \beta q L. \tag{10}$$

- Substituting (9) into (2), total output is:

$$Y(t) = \frac{1}{1-\beta} Q(t) L, \tag{11}$$

with average quality $Q(t) \equiv \int_0^1 q(\nu, t) d\nu$

Equilibrium (continued)

- Aggregate spending on machines:

$$X(t) = (1-\beta)\,Q(t)\,L. \tag{12}$$

- Labor market is competitive, wage rate:

$$w(t) = \frac{\beta}{1-\beta}Q(t)\,. \tag{13}$$

- Need to determine R&D effort levels by incumbents and entrants.
- Net present value of a monopolist with the highest quality of machine q at time t in machine line ν satisfies HJB ($V(\nu, t \mid q) = V(q)$, etc.):

$$r(t)\,V(q) - \dot{V}(q) = \max_{z(\nu,t|q)\geq o} \{\pi(q) - z(q)\,q \tag{14}$$
$$+\phi z(q)\,(V(\lambda q) - V(q)) - \eta\,(\hat{z}(q))\,\hat{z}(q) V(q)\},$$

Equilibrium (continued)

- Free entry:

$$\begin{aligned} \eta\left(\hat{z}(\nu,t\mid q)\right)V\left(\nu,t\mid\kappa q\right) &\leq q(\nu,t), \text{ and} \\ \eta\left(\hat{z}(\nu,t\mid q)\right)V\left(\nu,t\mid\kappa q\right) &= q(\nu,t) \text{ if } \hat{z}\left(\nu,t\mid q\right)>0, \end{aligned} \tag{15}$$

- Incumbent's choice of R&D effort implies similar complementary slackness condition:

$$\begin{aligned} \phi\left(V\left(\nu,t\mid\lambda q\right)-V\left(\nu,t\mid\kappa q\right)\right) &\leq q(\nu,t) \text{ and} \\ \phi\left(V\left(\nu,t\mid\lambda q\right)-V\left(\nu,t\mid\kappa q\right)\right) &= q(\nu,t) \text{ if } z\left(\nu,t\mid q\right)>0. \end{aligned} \tag{16}$$

- Consumer maximization:

$$\frac{\dot{C}\left(t\right)}{C\left(t\right)}=\frac{1}{\theta}\left(r\left(t\right)-\rho\right), \tag{17}$$

$$\lim_{t\to\infty}\left[\exp\left(-\int_0^t r\left(s\right)ds\right)\int_0^1 V\left(\nu,t\mid q\right)d\nu\right]=0 \tag{18}$$

Equilibrium and Balanced Growth Path

- Equilibrium is thus time paths of
  - $[C(t), X(t), Z(t)]_{t=0}^{\infty}$ that satisfy (1), (5), (12) and (18)
  - $[z(\nu, t), \hat{z}(\nu, t)]_{\nu \in [0,1], t=0}^{\infty}$ that satisfy (15) and (16);
  - $[p^x(\nu, t \mid q), x(\nu, t), V(\nu, t \mid q)]_{\nu \in [0,1], t=0}^{\infty}$ given by (8), (9) and (14);
  - $[w(t), r(t)]_{t=0}^{\infty}$ that satisfy (13) and (17).
- *BGP* (balanced growth path): equilibrium path in which innovation, output and consumption growth a constant rate.
- Note in BGP aggregates grow at the constant rate but there will be firm deaths and births, and the firm size distribution may also change.

Balanced Growth Path (continued)

- From Euler equation, the requirement that consumption grows at a constant rate in the BGP implies

$$r(t) = r^*$$

- In BGP, must also have $z(\nu, t \mid q) = z(q)$ and $\hat{z}(\nu, t \mid q) = \hat{z}(q)$.
- These imply in BGP $\dot{V}(\nu, t \mid q) = 0$ and $V(\nu, t \mid q) = V(q)$.
- Since profits and costs are both proportional to quality q, $z(q) = z$, $\hat{z}(q) = \hat{z}$ and $V(q) = vq$.
- Look for an "interior" BGP equilibrium (will verify below that exists and is unique).

Balanced Growth Path (continued)

- Incumbents undertake research, thus

$$\phi\left(V\left(\nu,t\mid\lambda q\right)-V\left(\nu,t\mid q\right)\right)=q(\nu,t), \tag{19}$$

- Therefore

$$V\left(q\right)=\frac{q}{\phi\left(\lambda-1\right)}. \tag{20}$$

- The free entry condition then implies $\eta\left(\hat{z}\right)V(\kappa q)=q$ and thus

$$V\left(q\right)=\frac{\beta Lq}{r^{*}+\hat{z}\eta\left(\hat{z}\right)}. \tag{21}$$

- Combining this expression with (19) and (20), we obtain

$$\frac{\phi\left(\lambda-1\right)}{\kappa\eta\left(\hat{z}\right)}=1.$$

Balanced Growth Path (continued)

- Hence the BGP R&D level by entrants $\hat{z}^*$ is defined implicitly by:

$$\hat{z}(q) = \hat{z}^* \equiv \eta^{-1}\left(\frac{\phi(\lambda-1)}{\kappa}\right) \text{ for all } q > 0. \tag{22}$$

- Combining with (21):

$$\begin{aligned} r^* &= \kappa\eta(\hat{z}^*)\beta L - \hat{z}^*\eta(\hat{z}^*) \qquad (23) \\ &= \phi(\lambda-1)\beta L - \hat{z}^*\eta(\hat{z}^*). \end{aligned}$$

- From Euler equation, growth rate of consumption and output:

$$g^* = \frac{1}{\theta}\left(\phi(\lambda-1)\beta L - \hat{z}^*\eta(\hat{z}^*) - \rho\right). \tag{24}$$

Balanced Growth Path (continued)

- (24) determines relationship between $\hat{z}^*$ and g^*. In contrast to standard Shcumpeterian models:

  Remark There is a negative relationship between $\hat{z}^*$ and g^*.

- From (24), g^* is decreasing in $\hat{z}^*\eta\left(\hat{z}^*\right)$ (which is always strictly increasing in $\hat{z}^*$).
- (24) and (22), determine BGP growth rate of the economy, but not how much of productivity growth is driven by creative destruction (entrants) and how much by incumbents.

Balanced Growth Path (continued)

- To determine this, let us write:

$$Q(t+\Delta t) = (\lambda \phi z(t) \Delta t) Q(t) + (\kappa \hat{z}(t) \eta(\hat{z}(t)) \Delta t) Q(t) + ((1 - \phi z(t) \Delta t - \hat{z}(t) \eta(\hat{z}(t)) \Delta t)) Q(t) + o(\Delta t).$$

- Subtracting $Q(t)$ from both sides, dividing by Δt and taking the limit as $\Delta t \rightarrow 0$:

$$g(t) = \frac{\dot{Q}(t)}{Q(t)} = (\lambda - 1) z(t) + (\kappa - 1) \hat{z}(t) \eta(\hat{z}(t)),$$

  which decomposes growth into the component from incumbent firms (first term) and from new entrants (second term).

- In BGP:

$$g^* = (\lambda - 1) \phi z^* + (\kappa - 1) \hat{z}^* \eta(\hat{z}^*). \tag{25}$$

Balanced Growth Path (continued)

- Can also verify that this economy does not have any transitional dynamics; if an equilibrium with growth exists, it will involve growth at g^*.
- To ensure equilibrium exists, verify R&D is profitable both for entrants and incumbents.
- The condition that r^* should be greater than ρ is sufficient for there to be positive aggregate growth.
- In addition, r^* should not be so high that transversality condition of the consumers is violated.
- Finally, need to ensure that there is also innovation by incumbents.

Balanced Growth Path (continued)

- The following condition ensures all three of these requirements:

$$\begin{aligned} & \kappa\eta\left(\hat{z}^*\right)\beta L - \left(\theta\left(\kappa-1\right)+1\right)\hat{z}^*\eta\left(\hat{z}^*\right) \\ > \quad & \rho > \left(1-\theta\right)\left(\kappa\eta\left(\hat{z}^*\right)\beta L - \hat{z}^*\eta\left(\hat{z}^*\right)\right), \end{aligned} \tag{26}$$

  with $\hat{z}^*$ given by (22).

- To obtain how much of productivity growth and innovation are driven by incumbents and how much by new entrants, from:

$$\left(\lambda-1\right)\phi z^* = \frac{1}{\theta}\left(g^* - \rho\right) - \left(\kappa-1\right)\hat{z}^*\eta\left(\hat{z}^*\right), \tag{27}$$

  with g^* given in (24) and $\hat{z}^*$ in (22).

Firm Size Dynamics

- Firm-size dynamics: size of a firm can be measured by its sales:

$$x(\nu, t \mid q) = qL \text{ for all } \nu \text{ and } t.$$

- Quality of an incumbent firm increases that the flow rate ϕz^*, with z^* given by (27), while the firm is replaced at the flow rate $\hat{z}^* \eta(\hat{z}^*)$.
- Thus, for Δt sufficiently small, the stochastic process for the size of a particular firm is:

$$x(\nu, t + \Delta t \mid q) = \begin{cases} \lambda x(\nu, t \mid q) & \text{w. p. } \phi z^* \Delta t + o(\Delta t) \\ 0 & \text{w. p. } \hat{z}^* \eta(\hat{z}^*) \Delta t + o(\Delta t) \\ x(\nu, t \mid q) & \text{w. p. } (1 - \phi z^* - \hat{z}^* \eta(\hat{z}^*)) \Delta t \\ & \quad + o(\Delta t) \end{cases} \tag{28}$$

for all ν and t.

Firm Size Dynamics (continued)

- Thus firms have random growth, and surviving firms expand on average.
- Firms also face a probability of bankruptcy (extinction)
- Let $P(t \mid s, \nu)$ =probability that a particular incumbent firm that started production in machine line ν at time s will be bankrupt by time $t \geq s$:

$$\lim_{t \to \infty} P(t \mid s, \nu) = 1$$

so that each firm will necessarily die eventually.

Summary of Equilibrium

Proposition Consider the above-described economy starting with an initial condition $Q(0) > 0$. Suppose that (3) and (26) are satisfied. Then there exists a unique equilibrium. In this equilibrium growth is always balanced, and technology, $Q(t)$, aggregate output, $Y(t)$, and aggregate consumption, $C(t)$, grow at the rate g^* as in (24) with $\hat{z}^*$ given by (22). Equilibrium growth is driven both by innovation by incumbents and by creative destruction by entrants. Any given firm expands on average as long as it survives, but is eventually replaced by a new entrant with probability one.

The Effects of Policy on Growth

- Since Schumpeterian structure, it may be conjectured that entry barriers (or taxes on potential entrance) will have negative effects on economic growth.
- Tax τ_e on R&D expenditure by entrants and a tax τ_i on R&D expenditure by incumbents (can be negative and interpreted as subsidies).
- τ_e can also be interpreted as a more strict patent policy than in the baseline model, where the entrant did not have to pay the incumbent for partially benefiting from its accumulated knowledge.
- Focus on the case in which tax revenues are collected by the government rather than rebated back to the incumbent as patent fees.

The Effects of Policy on Growth (continued)

- Repeating the analysis above, equilibrium conditions:

$$\eta\left(\hat{z}^*\right)\right) V(\kappa q)=\left(1+\tau_e\right) q \text{ or } V(q)=\frac{q\left(1+\tau_e\right)}{\kappa \eta\left(\hat{z}^*\right)}. \tag{29}$$

- The equation that determines the optimal R&D decisions of incumbents, (19), is also modified:

$$\phi\left(V(\lambda q)-V(q)\right)=\left(1+\tau_i\right) q. \tag{30}$$

- Combining (29) with (30):

$$\phi\left(\frac{(\lambda-1)\left(1+\tau_e\right)}{\kappa \eta\left(\hat{z}^*\right)\left(1+\tau_i\right)}\right)=1.$$

- Consequently, the BGP R&D level by entrants $\hat{z}^*$, when their R&D is taxed at the rate τ_e, is given by

$$\hat{z}^* \equiv \eta^{-1}\left(\frac{\phi(\lambda-1)\left(1+\tau_e\right)}{\kappa\left(1+\tau_i\right)}\right). \tag{31}$$

The Effects of Policy on Growth (continued)

- Equation (21) still applies, so that the the BGP interest rate is

$$r^* = (1+\tau_i)^{-1} \phi (\lambda - 1) \beta L - \hat{z}^* \eta (\hat{z}^*) , \quad (32)$$

- BGP growth rate is

$$g^* = \frac{1}{\theta} \left((1+\tau_i)^{-1} \phi (\lambda - 1) - \hat{z}^* \eta (\hat{z}^*) - \rho \right) . \quad (33)$$

- From (33), g^* does not directly depend on τ_e, thus

$$\frac{dg^*}{d\tau_e} = \frac{\partial g^*}{\partial \hat{z}^*} \frac{\partial \hat{z}^*}{\partial \tau_e} > 0.$$

- Opposite of standard Schumpeterian results. Intuition?
- Moreover, as expected

$$\frac{dg^*}{d\tau_i} < 0.$$

Proposition The growth rate is decreasing in the tax rate on incumbents and increasing in the tax rate (entry barriers) on entrants.

The Effects of Policy on Growth (continued)

- Surprising result: in Schumpeterian models, making entry more difficult, either with entry barriers or by taxing R&D by entrants, has negative effects on economic growth.
- Despite the Schumpeterian nature of the current model, blocking entry and protecting the incumbents increases equilibrium growth (and welfare)
- *Intuition:*
  - Engine of growth is still quality improvements, but these improvements are undertaken both by incumbents and entrants.
  - Entry barriers, by protecting incumbents, increase their value and greater value by incumbents encourages more R&D investments and faster productivity growth.
  - Taxing entrants makes incumbents more profitable and this encourages further innovation by the incumbents.

Equilibrium Firm Size Distribution

- In equilibrium, there is entry, exit and stochastic growth of firms. $\Longrightarrow$ endogenous of firm size distribution.
- Firm growth consistent with *Gibrat's Law*
  - Gibrat's Law states that firm growth is independent of firm size.
  - Good description of actual firm size dynamics in the data (e.g., Sutton, 1997).
  - Though not always for new firms.
- What about firm size distributions?
- Axtell (2001) US firm size distribution very well approximated by the Pareto distribution with an exponent of one.
- Recall that the *Pareto distribution* is $\Pr\left[\tilde{x} \leq y\right] = 1 - \Gamma y^{-\chi}$ for $\Gamma > 0$ and $y \geq \Gamma$.

Equilibrium Firm Size Distribution (continued)

- In the current economy, the size of average firm measured by sales, $x(t)$, grows.
- To look at the firm size distribution, weneed to normalize firm sizes by the average size of firm, given by $X(t)$.
- Let the *normalized firm size* be

$$\tilde{x}(t) \equiv \frac{x(t)}{X(t)}.$$

Equilibrium Firm Size: Pareto Distribution

- Since in equilibrium $\dot{X}(t)/X(t) = g^* > 0$, law of motion for normalized size of leading firm in each industry:

$$\tilde{x}(t+\Delta t) = \begin{cases} \frac{\lambda}{1+g^*\Delta t}\tilde{x}(t) & \text{w. p. } \phi z^* \Delta t \\ \frac{\kappa}{1+g^*\Delta t}\tilde{x}(t) & \text{w. p. } \hat{z}^*\eta(\hat{z}^*)\Delta t + o(\Delta t) \\ \frac{1}{1+g^*\Delta t}\tilde{x}(t) & \text{w. p. } (1-\phi z^* - \hat{z}^*\eta(\hat{z}^*))\Delta t + o(\Delta t). \end{cases}$$

- Notice that this expression does not refer to the growth rate of a single firm, but to the leading firm in a representative industry, and in particular, when there is entry, this leads to an increase in size rather than extinction.
- It can be verified that if a stationary distribution of (normalized) firm sizes were to exist, then it would be a Pareto distribution with exponent equal to 1, i.e., $\Pr[\tilde{x} \le y] = 1 - \Gamma/y$ with $\Gamma > 0$.

Pareto Distribution of Firm Sizes

- Suppose that a stationary distribution exists.
- Consider an arbitrary time interval of $\Delta t > 0$ and write

$$\begin{aligned}
\Pr\left[\tilde{x}\left(t+\Delta t\right) \leq y\right] &= \mathbb{E}\left[\mathbf{1}_{\{\tilde{x}(t+\Delta t)\leq y\}}\right] \\
&= \mathbb{E}\left[\mathbf{1}_{\{\tilde{x}(t)\leq y/(1+g^x(t+\Delta t))\}}\right] \\
&= \mathbb{E}\left[\mathbb{E}\left[\mathbf{1}_{\{\tilde{x}(t)\leq y/(1+g^x(t+\Delta t))\}} \mid g^x\left(t+\Delta t\right)\right]\right],
\end{aligned}$$

where $\mathbf{1}_{\{\mathcal{P}\}}$ is the indicator function, so the first equation holds by definition. The second equation also holds by definition once $g^x\left(t+\Delta t\right)$ is designated as the the (stochastic) growth rate of x between t and $t+\Delta t$. Finally, the third equation follows from the Law of Iterated Expectations.

Pareto Distribution of Firm Sizes (continued)

- Next, denoting $\mathbb{G}_t(y) \equiv 1 - \Pr[\tilde{x}(t) \leq y]$:

$$\begin{aligned}\Pr[\tilde{x}(t+\Delta t) \leq y] &= 1 - \mathbb{G}_{t+\Delta t}(y) \\ &= \mathbb{E}\left[1 - \mathbb{G}_t\left(\frac{y}{1+g^x(t+\Delta t)}\right)\right].\end{aligned}$$

- Thus we obtain a simple form of Kolmogorov's forward equation:

$$\begin{aligned}\mathbb{G}_{t+\Delta t}(y) &= \mathbb{E}\left[\mathbb{G}_t\left(\frac{y}{1+g^x(t+\Delta t)}\right)\right] \\ &= \phi z^* \Delta t \mathbb{G}_t\left(\frac{(1+g^*\Delta t)\,y}{\lambda}\right) \\ &\quad + \hat{z}^* \eta(\hat{z}^*) \Delta t \mathbb{G}_t\left(\frac{(1+g^*\Delta t)\,y}{\kappa}\right) \\ &\quad + (1 - \phi z^* \Delta t - \hat{z}^* \eta(\hat{z}^*) \Delta t)\, \mathbb{G}_t((1+g^*\Delta t)\,y) + o(\Delta\end{aligned}$$

Pareto Distribution of Firm Sizes (continued)

- A stationary equilibrium will correspond to a function $\mathbb{G}(y)$ such that $\mathbb{G}_{t+\Delta t}(y) = \mathbb{G}_t(y) = \mathbb{G}(y)$ for all t and Δt and the previous equation holds.
- Let us conjecture that $\mathbb{G}(y) = \Gamma y^{-\chi}$ with $\Gamma > 0$. Then

$$\begin{aligned}
\Gamma y^{-\chi} &= \phi z^{*} \Delta t \Gamma \left(\frac{(1+g^{*}\Delta t)\, y}{\lambda} \right)^{-\chi} \\
&\quad + \hat{z}^{*} \eta\left(\hat{z}^{*}\right) \Delta t \Gamma \left(\frac{(1+g^{*}\Delta t)\, y}{\kappa} \right)^{-\chi} \\
&\quad + \left(1 - \phi z^{*} \Delta t - \hat{z}^{*} \eta\left(\hat{z}^{*}\right) \Delta t\right) \Gamma \left((1+g^{*}\Delta t)\, y \right)^{-\chi} + o(\Delta t) .
\end{aligned}$$

or

$$\begin{aligned}
& \phi z^{*} \Delta t \left(\frac{(1+g^{*}\Delta t)}{\lambda} \right)^{-\chi} + \hat{z}^{*} \eta\left(\hat{z}^{*}\right) \Delta t \left(\frac{(1+g^{*}\Delta t)}{\kappa} \right)^{-\chi} \qquad (34) \\
+ \quad & \left(1 - \phi z^{*} \Delta t - \hat{z}^{*} \eta\left(\hat{z}^{*}\right) \Delta t\right) \left(1+g^{*}\Delta t\right)^{-\chi} + o(\Delta t)\, \Gamma^{-1} y^{-\chi} = 1.
\end{aligned}$$

Pareto Distribution of Firm Sizes (continued)

- Now subtracting 1 from both sides, dividing by Δt, and taking the limit as $\Delta t \to 0$, we obtain

$$\lim_{\Delta t \to 0} \left\{ \phi z^* \left(\frac{(1 + g^* \Delta t)}{\lambda} \right)^{-\chi} + \hat{z}^* \eta\left(\hat{z}^*\right) \left(\frac{(1 + g^* \Delta t)}{\kappa} \right)^{-\chi} + \frac{\left(1 - \phi z^* \Delta t - \hat{z}^* \eta\left(\hat{z}^*\right) \Delta t\right) \left(1 + g^* \Delta t\right)^{-\chi} - 1}{\Delta t} + \frac{o\left(\Delta t\right)}{\Delta t} \Gamma^{-1} y^{-\chi} \right\}$$

- Therefore the exponent χ must satisfy

$$\phi\left(\lambda^{\chi} - 1\right) z^* + \left(\kappa^{\chi} - 1\right) \hat{z}^* \eta\left(\hat{z}^*\right) - \chi g^* = 0. \qquad (35)$$

- It can be easily verified that (35) has two solutions $\chi = 0$ and $\chi^* = 1$, since, by definition, $g^* = \phi\left(\lambda - 1\right) z^* + \left(\kappa - 1\right) \hat{z}^* \eta\left(\hat{z}^*\right)$.

Pareto Distribution of Firm Sizes (continued)

- To see that there are no other solutions, consider the derivative of this function, which is given by

$$g'(\chi) = \phi z^* \lambda^\chi \ln \lambda + \eta(\hat{z}^*) \hat{z}^* \kappa^\chi \ln \kappa - g^*.$$

Since $\ln a < a - 1$ for any $a > 1$, $g'(0) < 0$. Moreover, $g''(\chi) > 0$, so that the right-hand side of (35) is convex and as $\chi \to \infty$, it limits to infinity. Thus there is a unique nonzero solution, which as we saw above, is $\chi^* = 1$.

- Finally, note that $\chi = 0$ cannot be a solution, since it would imply $\mathbb{G}(y) = \Gamma$ and thus $\mathbb{G}(y) = 0$, which would imply that all firms have normalized size equal to zero, and violate the hypothesis that a stationary firm-size distribution exists.
- It can also be verified that no other function than $\mathbb{G}(y) = \Gamma y^{-\chi}$ with $\Gamma > 0$ can satisfy this functional equation, completing the proof of the proposition.

Equilibrium Firm Size: Pareto Distribution (continued)

- Unfortunately, previous proposition stated under the hypothesis that the station redistribution exists.
- But: a stationary firm-size distribution does *not* exist.
- **Proof:**
  - A stationary distribution must take the form $\Pr[\tilde{x} \leq y] = 1 - \Gamma/y$ with $\Gamma > 0$ and Γ should be the minimum normalized firm size.
  - However, the law of motion of firm sizes shows that $\tilde{x}(t)$ can tend to zero. Therefore, Γ must be equal to 0, which implies that there does not exist a stationary firm-size distribution.
- Intuitively, given the random growth process (Gibrat's Law), the distribution of firm sizes will continuously expand.
- The "limiting distribution" will involve all firms being arbitrarily small relative to the average $X(t)$ and a vanishingly small fraction of firms becomes arbitrarily large.

Equilibrium Firm Size: Economy with Imitation

- Can we have a stationary firm size distribution?
- Let us introduce a third type of technology of innovation, "*imitation*".
- A new firm can enter in sector $\nu \in [0,1]$ with a technology of $q^e(\nu,t) = \omega Q(t)$, where $\omega \geq 0$ and $Q(t)$ is average quality of machines.
- The cost of this type of innovation is $\omega Q(t) / (\phi(\lambda - 1))$ (not necessary, but simplifies life a lot).
- This implies that if a firm could enter into a particular sector, become the monopolist and obtain the BGP value (20), it would be happy to enter (or more precisely, it would be indifferent between entering and not entering, and we suppose that in this case it does enter).

Equilibrium Firm Size (continued)

- This structure implies that they will enter into any sector with quality $q \leq \epsilon Q(t)$, where $\epsilon = \epsilon(\omega)$ is determined endogeneously.
- Assume $\mu_e > \underline{\mu}$ so that an imitator does not enter too early, i.e. $\epsilon \leq \omega (1-\beta)^{(1-\beta)/\beta}$ but once entered, it can charge the monopolistic price without potential threat from the incumbent. Moreover $\mu_e < \overline{\mu}$, so that there is positive entry by imitation.

Roadmap for Derivation of Size Distribution

1. Characterize optimal policies in the economy with imitation and show that they are approximately linear.
2. Derive the Kolmogorov forward equation for the law of motion overlooked a firm size and there are stationary distribution.
3. Derive the growth rate.
4. Conjecture an exact Pareto distribution for the tail.
5. Solve out for the Pareto tail and show that it is approximately 1.

Step 1: Value Functions

- We look for an equilibrium in which

$$V(\nu, t|q) = Q(t)\,\widehat{V}\left(\frac{q(\nu,t)}{Q(t)}\right)$$

- We choose as state variable the relative productivity $\widetilde{q} \equiv q/Q$, which gives the HJB equation

$$\begin{aligned}(r-g)\,\widehat{V}_g(\widetilde{q}) - g\widehat{V}_g'(\widetilde{q}) &= \beta L\widetilde{q} \\ &+ \left\{\max_{z(\nu,t)\geq 0}\left\{\phi(z(\nu,t))\left(\widehat{V}_g(\lambda\widetilde{q}) - \widehat{V}_g(\widetilde{q})\right) - z(\nu,t)\,\widetilde{q}\right\}\right. \\ &-\widehat{z}(\nu,t)\,\eta(\widehat{z}(\nu,t))\,\widehat{V}_g(\widetilde{q}),\end{aligned} \tag{36}$$

- Additional source of capital gains/losses from changes in aggregate quality, and the threat of being replaced by an imitator.
- When ω and ϵ small, value approximately linear as before.

Step 1: Free Entry Conditions

- Free-entry condition for **Innovative Entrants**:

$$\eta\left(\widehat{z}\left(\nu, t\right)\right) \widehat{V}\left(\kappa \widetilde{q}\left(v, t\right)\right) = \widetilde{q}\left(v, t\right).$$

- Free-entry condition for **Imitators**:

$$\widehat{V}\left(\omega\right) = \mu_e \omega.$$

ϵ is such that $\widehat{V}\left(\epsilon\right) = 0$: entrants enter whenever $q\left(\nu, t\right) \leq \epsilon Q\left(t\right)$.

- Thus boundary condition for value function.

Step 1: Optimal Strategy of Incumbents

- From the value function $\widehat{V}(\widetilde{q})$ we obtain the optimal strategy of incumbents

$$z(\widetilde{q}) = \arg\max_{z} \phi(z) \left(\widehat{V}(\lambda \widetilde{q}) - \widehat{V}(\widetilde{q}) \right) - z\widetilde{q}$$

and of the entrants

$$\widehat{z}(\widetilde{q}) = \eta^{-1} \left(\widetilde{q} / \widehat{V}(\kappa \widetilde{q}) \right).$$

Step 2: Stationary Distribution

- Functional differential equation of the stationary distribution with cdf $F(.)$ (with intuition similar to the constant growth case)
- If $y > \omega$

$$0 = F'(y)\, yg - \int_{\frac{y}{\lambda}}^{y} \phi(z(\tilde{q}))\, dF(\tilde{q}) - \int_{\frac{y}{\kappa}}^{y} \hat{z}(\tilde{q})\, \eta(\hat{z}(\tilde{q}))\, dF(\tilde{q})\,. \tag{37}$$

- If $y < \omega$, then

$$0 = F'(y)\, yg - F'(\epsilon)\, \epsilon g - \int_{\frac{y}{\lambda}}^{y} \phi(z(\tilde{q}))\, dF(\tilde{q}) - \int_{\frac{y}{\kappa}}^{y} \hat{z}(\tilde{q})\, \eta(\hat{z}(\tilde{q}))\, dF \tag{38}$$

and

$$F(y) = 0 \text{ for } y \leq \epsilon.$$

Step 3: Growth Rate

- From the investment of incumbents and entry decision entrants and imitators, as well as stationary distribution we obtain

$$T[g] = \frac{(\lambda - 1)\,\mathbb{E}_{F_g}\left[\phi\left(z\left(\widetilde{q}\right)\right)\widetilde{q}\right] + (\kappa - 1)\,\mathbb{E}_{F_g}\left[\widehat{z}_g\left(\widetilde{q}\right)\eta\left(\widehat{z}_g\left(\widetilde{q}\right)\right)\widetilde{q}\right]}{1 - \epsilon F_g'\left(\epsilon\right)\left(\omega - \epsilon\right)}.$$

- Equilibrium growth rate such that

$$g = T[g].$$

- $T[g^*] = +\infty$ and $T[\overline{g}] < \overline{g}$. Therefore there exists $g(\omega) \in (g^*, \overline{g})$ such that $T[g] = g$.

Step 4: Pareto Distribution in the Tail

- For y large, we have
  $0 = F'(y)\,yg - \int_{\frac{y}{\lambda}}^{y} \phi\left(z\left(\widetilde{q}\right)\right) dF\left(\widetilde{q}\right) - \int_{\frac{y}{\kappa}}^{y} \widehat{z}\left(\widetilde{q}\right) \eta\left(\widehat{z}\left(\widetilde{q}\right)\right) dF\left(\widetilde{q}\right)$,
  which implies (again for y large)

$$0 = F'(y)\,yg - \mathbb{E}_{F_g}[\phi]\left[F(y) - F\left(\frac{y}{\lambda}\right)\right] - \mathbb{E}_{F_g}\left[\widehat{z}\eta\left(\widehat{z}\right)\right]\left[F(y) - F\left(\frac{y}{\kappa}\right.\right.$$

- Conjecture: $F(y) = 1 - \Gamma y^{-\chi}$ for large y.
- Then:

$$\mathbb{E}_{F_g}[\phi]\left[\lambda^{\chi} - 1\right] - \mathbb{E}_{F_g}\left[\widehat{z}\eta\left(\widehat{z}\right)\right]\left[\kappa^{\chi} - 1\right] - \chi g = 0$$

- Solution χ gives the tail Pareto shape coefficient.

Step 5: Pareto Shape Parameter Approximately 1

- When $\epsilon \to 0$,

$$T[g] = \frac{(\lambda - 1)\,\mathbb{E}_{F_g}\left[\phi\left(z\left(\widetilde{q}\right)\right)\widetilde{q}\right] + (\kappa - 1)\,\mathbb{E}_{F_g}\left[\widehat{z}_g\left(\widetilde{q}\right)\eta\left(\widehat{z}_g\left(\widetilde{q}\right)\right)\widetilde{q}\right]}{1 - \epsilon F_g'\left(\epsilon\right)\left(\omega - \epsilon\right)}$$

$$\to \mathbb{E}_{F_g}\left[\phi\right]\left[\lambda - 1\right] - \mathbb{E}_{F_g}\left[\widehat{z}\eta\left(\widehat{z}\right)\right]\left[\kappa - 1\right]$$

- Thus

$$\chi \to 1.$$

Formal Derivation of Stationary Distribution

- The stationary distribution has an approximate Pareto tail with the Pareto exponent $\chi = \chi(\omega) > 1$ such that: $\forall \xi > 0$ there exist $\overline{A}$,$\underline{A}$ and y_0 such that

$$f(y) < 2\overline{A}y^{-(\chi-1-\xi)}, \forall y \geq y_0$$

  and

$$f(y) > \frac{1}{2}\underline{A}y^{-(\chi-1+\xi)}, \forall y \geq y_0.$$

  In other words, $f(y) = y^{-\chi-1}\varphi(y)$, where $\varphi(y)$ is a slow-varying function. Moreover

$$\lim_{\omega \longrightarrow 0} \chi(\omega) = 1.$$

- Proof: Value function is approximately linear for large firms. The R&D investment and entry are approximately constant. Therefore we obtain a constant growth. Combined with lowest barriers $\Longrightarrow$Approximate Pareto distribution.

Simulation result

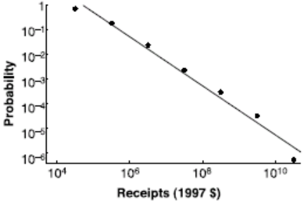
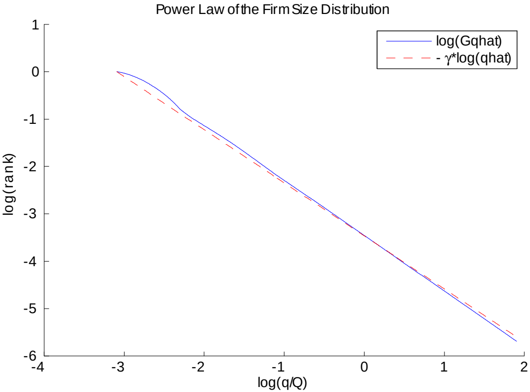


Fig. 2. Tail cumulative distribution function of U.S. firm sizes, by receipts in dollars. Data are for 1997 from the U.S. Census Bureau, tabulated in bins whose width increases in powers of 10. The solid line is the OLS regression line through the data and has slope of 0.994 (SE = 0.064; adjusted R^2 = 0.976).

Axtell 2001



Stationary Distribution of Firm Size

Existence of the Epsilon Economy

Theorem (Existence of the Epsilon Economy)

Suppose that

$$\max_{z>0} \varepsilon_\eta (z) \leq 1 - \frac{1}{\kappa^\theta}.$$

There exists an interval $\left(\underline{\mu}, \overline{\mu}\right)$ *and* $\Delta > 0$, $\overline{\omega} > 0$ *such that given* $\mu_e \in \left(\underline{\mu}, \overline{\mu}\right)$ *and for each* $\omega \in (0, \overline{\omega})$, *there is a BGP with the following properties :*

1. *An imitator pays* $\mu_e \omega Q(t)$ *to buy a product quality* $\omega Q(t)$ *and to enter into a sector* ν *if* $q(\nu, t) \leq \epsilon(\omega) Q(t)$, *where*
$0 < \epsilon(\omega) \leq \omega (1-\beta)^{\frac{1-\beta}{\beta}}$

2. *The incumbents solve the net present value maximization problem but they take into account the behavior of the imitators as well as the innovative entrants.*

Existence of the Epsilon Economy (continued)

Theorem (Existence of the Epsilon Economy)

Continued

3. *The equilibrium growth rate of the aggregate product quality is* $g(\omega) \in (g^*, g^* + \Delta)$ *which satisfies*

$$\lim_{\omega \longrightarrow 0} g(\omega) = g^*.$$

4. *The stationary distribution is approximately Pareto with the tail* $\chi(\omega)$

$$\lim_{\omega \longrightarrow 0} \chi(\omega) = 1.$$

Existence of the Epsilon Economy (Sketch of Proof)