Asset-Return Anomalies in a Monetary Economy*

ROBERT M. TOWNSEND

The University of Chicago, Chicago, Illinois 60637

Received July 10, 1984; revised February 4, 1986

This paper provides a general equilibrium, choice theoretic, spatial model which explains the preference for holding barren money rather than interest-bearing securities or capital goods. Put somewhat differently, it examines standard asset pricing relationships in the context of a fully articulated monetary economy and delivers various asset-return anomalies. In seeking to integrate the theory of value with the theory of money, a fairly general proof of the existence of a monetary equilibrium is provided. *Journal of Economic Literature* Classification Numbers: 020, 023, 310.   ⓒ 1987 Academic Press, Inc.

1. INTRODUCTION

In an insightful and provocative article, Hicks [37] calls for an integration of the theory of value with the theory of money. By the theory of value, Hicks means the dictum that the relative value of two commodities depends on their relative marginal utilities. But for Hicks, marginal utility analysis was taken as nothing other than a general theory of choice. Thus, Hicks finds that the central monetary observation to be explained by choice theory is the preference for holding barren money rather than interest-bearing securities or capital goods.

The purpose of this paper, then, is to offer one explanation for such rate-of-return dominance and, more generally, to integrate the theory of value with the theory of money. Alternatively, the paper is motivated by the obvious questions: How are we to incorporate money into a general equilibrium model in such a way as to explain such asset return anomalies, and what are the implications more generally for asset pricing formulas.

In this endeavor, we are faced with the now familiar complaint of such diverse authors as Brunner and Meltzer [10], Cass and Shell [13], Clower [16], Hahn [27], and Wallace [71] that the standard, general

219

equilibrium, Arrow–Debreu model, in assuming unlimited market par-
ticipation and no trading frictions, does not allow a role for money. Thus,
the standard model must be altered in some way, but at the moment we do
not have many tightly specified alternatives. One class of models which
does come to mind is the class of overlapping generations models based on
the construct of Samuelson [58]. In these models money seems to flow
naturally from the assumed demographic structure, that is, from a structure
which generates missing markets, and for these models we now have well-
worked-out theorems establishing the existence of monetary equilibria and
associated optimality properties; see, for example, Belasko and Shell [4].
Unfortunately, however, as is indicated in Wallace [71], monetary
equilibria in these models are inconsistent with the existence of dominating
assets, a complaint voiced by Hahn [27] and Tobin [65], among others.

There are a variety of possible responses to this complaint. On the one
hand, one might become discouraged about the possibility of constructing
choice-theoretic models of money, preferring perhaps to start with the
anomalies rather than explaining them. On the other hand, one might take
issue with the anomalies themselves, arguing perhaps that rate-of-return
dominance is, in general, the product of exogenously imposed legal restric-
tions; this seems to be close to the position of Wallace [70]. An inter-
mediate response, however, would be to search for alternative choice-
theoretic models of money. Again, in such models, money would not enter
via exogenously imposed restrictions, say that certain goods can only be
acquired with money, but rather would emerge naturally in some way from
an underlying exchange structure.

This paper takes the intermediate route. In particular it follows Hick's
suggestion, introducing some frictions, in particular the friction of spatial
separation. That is, it is supposed that there is an absence of double-coin-
cidence of wants among households who meet with one another at distinct
spatial locations.[1] Thus, without a common medium of exchange, com-
petitive market exchange is quite limited; in effect, in the spatial model con-
sidered here, households can only consume the commodities they produce.
On the other hand, with a common medium of exchange, the commodities
produced by the household in a given period can be traded for money in
one market in a given period, money which in turn can be traded for a
market produced commodity in another market in a subsequent period.
Thus money plays a role in providing liquidity, enabling the household to
purchase commodities which it does not own. This gives it an advantage

---

[1] Spatial separation is at least implicit in the decentralized exchange models of Neihans
[53], Ostroy [54], Feldman [20], Ostroy–Starr [55], and Harris [32] and in the "costly
trips" models of Baumol [1], Tobin [63] and, more recently, Jovanovic [39]. The present
paper builds directly on Lucas' version of the Cass–Yaari [14] circle, as presented in Town-
send [66, Sect. 4].

over other assets. Indeed, the model delivers Clower's [15] dictum *endogenously* and thus is close formally to the Clower-constraint models of Grandmont and Younes [22, 23] and Lucas [46, 47, 48], among others.[2] The underlying exchange model is described in more detail in Section 2, and the resulting Clower-constraint model is the subject of Section 3.

But of course telling such exchange stories is not enough. In a choice-theoretic model of money a central task is to prove the existence of a competitive monetary equilibrium, one in which money has value. But how are we to do this? On the face of it, one might hope to follow the tradition of neoclassical growth theory, e.g., Cass [12], Koopmans [41], Brock and Mirman [9], Sargent [59], and Lucas and Stokey [49], first establishing the existence of an optimum and then, following Debreu [18], establishing that an optimum can be supported as a competitive equilibrium with suitably chosen prices. Here, though, the endogenously induced finance constraint severs the link between competitive equilibria and Pareto optima, as in Grandmont and Younes [23], Lucas [48], and Townsend [66], and we are faced with the task of establishing the existence of a competitive monetary equilibrium directly. This is a nontrivial undertaking, since, as is noted in Hahn [25], the general equilibrium theorems of Debreu [17], McKenzie [51], and others are not immediately applicable, due to discontinuity.

Existence of a monetary equilibrium for the model of this paper, one which allows for return dominance and for shocks to technology, endowments, preferences, and the money supply, is established in Section 4 of this paper, building on the insightful papers of Bewley [6] and Heller [33].[3]

---

[2] Idle balances and rate-of-return dominance are also explained in a sense in the transactions costs, general equilibrium, monetary literature; see Hahn [26, 27], Heller [33], Kurz [42, 43], and Heller–Starr [34], for example. There money is given an exogenous (less costly) advantage in facilitating exchange. Here money emerges from an exogenous specification of endowments, preferences, technology, and especially spatial frictions. Of course, market participation can be limited as in Lucas [47, 48] by the (exogenous) specification that beginning-of-period asset markets be centralized but that within-period commodity markets be decentralized, requiring the use of money. But again, the effort here is to specify (exogenously) who can trade with whom and then to let both asset and commodity trades be determined endogenously. Finally, it should be noted that Stockman [61] allows capital accumulation in a Clower constraint model and achieves rate-of-return dominance without noting the result.

[3] Existence of a temporary monetary equilibrium and of a stationary, perfect foresight monetary equilibrium for a Clower-constraint type model is established directly by Grandmont and Younes [22]. But as Hool [38] notes, their analysis does not cover the case of $k = 0$, the case which corresponds to the usual version of a Clower-constraint model, and the case which is applicable here. Hool [38] extends the analysis for temporary equilibrium only. Both Grandmont and Younes [22] and Hool [38] rule out capital accumulation and uncertainty. Lucas [46] establishes existence of a monetary equilibrium in a Clower-constraint model with essentially one (market-produced) commodity, allowing for uncertainty at the level of individual preferences (there is no aggregate uncertainty).

After all this investment, we can return to the observations that motivated the paper, the dominance of the rate of return of capital and interest-bearing securities over money, and ask whether these and other anomalies are displayed, in general, in monetary equilibria. As is established in Section 5, neither the money asset nor a privately owned capital asset seem to be priced effeciently in monetary equilibria—the returns on both generally violate stadard, intertemporal valuation relationships. Indeed, this is what delivers rate-of-return dominance. These results are apparent from the first-order conditions of the maximum problem of a "representative" consumer, and, as the product of a positive and stochastically varying Lagrange multiplier on the finance constraint, can be viewed as extensions of the results obtained earlier by Grandmont and Younes [23], Lucas [46], and Townsend [66]. Further, with some additional work, one can derive asset pricing formulas for arbitrary market securities and deliver dominance of the nominal return of one-period moncy loans over individually held "idle" money balances. More generally, one can examine the extent to which the standard asset pricing formulas of the real, general equilibrium models of Lucas [45], Breeden [7], Brock [8], and Prescott and Mehra [56],[4] for example, as taken to data in the empirical work of Grossman and Shiller [24], Hall [29], Hansen and Singleton [30, 31] and Mankiw, Rotemberg, and Summers [50], for example, are equivalent with the asset pricing formulas of the economy considered here, *an economy in which money is essential.* As it turns out, the monetary economy considered here has sufficient structure that one can obtain rather sharp, nonstandard results. Of course, the idea more generally is that the market imperfections or trading difficulties which give rise to money may well have implications for asset prices.

2. THE ECONOMY

This section describes one underlying exchange environment which generates the Clower-constraint monetary model of Section 3. Of course the entire *analysis* of the paper takes the Clower-constraint model as a starting-point; in that sense the analysis of the paper is general, and nothing hinges on the particular story given in this section. On the other hand, given the motivation for this paper, one does want to describe at least one explicit, general equilibrium, stylized environment which rationalizes the Clower-constraint set-up.

The stylized environment given here is a blend of the Lucas–Cass–Yaari model described in Sections 4 of Townsend [66] and the "turnpike" model

---

[4] See the references cited in Lucas [47] for earlier versions of the real theory.
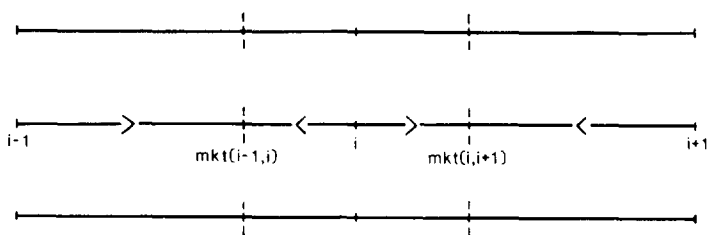
FIG. 1.   A spatial pattern of trade.

of Section 2 of Townsend [66]. The reader who finds the following description somewhat cryptic is urged to consider those sections more closely.

Briefly, imagine there is a countable set of household types and a countable set of commodities, both indexed by the positive and negative integers. Each household of type i consists of a pair of agents and is imagined to be located on some real line at integer i (see Fig. 1). There is a countable infinity of such real lines, arranged horizontally in Fig. 1, and again the integer i on each line is inhabited by a (representative) household of type i. At each data t, each member of household i is capable of moving horizontally (on the line where it is located) one-half the distance to each of the two adjacent integers $(i+1)$ and $(i-1)$. Thus, at each data t, each household i is physically capable of carrying out transactions with a household of type $(i-1)$ and a household of type $(i+1)$ in two spatially separated markets, say, $(i-1, i)$ and $(i, i+1)$. Between dates, household can move about. Each household of type i, i even, is imagined to move vertically downward to the next line, while each household of type i, i odd, stays in its fixed location. Thus, for example, debt issued by a household of type i, i even, can only be passed along to households vertically above the issuer; this construction prevents the issue of private debt. Each household of type i, whatever its location at a given date, should be taken as representative of a large (infinite) number of households in an identical situation following an identical itinerary. This rationalizes the existence of a competitive market when two representative households meet.

Each household of type i is endowed with $w_t(i, s_t)$ units of commodity i (only) at date t, depending possibly on the state s_t but not on date t itself. Here, however, $w_t(i, s_t)$ is a constant with respect to the index i. That is, household i has $w_t(s_t)$ units of commodity i, household $i+1$ has $w_t(s_t)$ units of commodity $i+1$, and so on. (Further, this kind of symmetry across households is continued below). It is supposed that $w_t(s_t) > 0$. Here there is a finite set A of possible states s_t at each date t; see Bewley [6] for a similar formulation of uncertainty. Each household of type i also possesses a technology for transforming commodity i (only) at date t into commodity i

at date $t+1$, namely a function $f: R^1_+ \times A \to R^1_+$. That is, commodity i can be either consumed (by some household) at date t or invested by household i, say I_{t+1} units, to produce $f(I_{t+1}, s_{t+1})$ units of commodity i at date $t+1$ in state s_{t+1}. It is supposed that for every $s_t \in A$, $f(\cdot, s_t)$ is strictly concave, strictly increasing, and continuously differentiable with $f(0, s_t) = 0$, $f'(\infty, s_t) = 0$ and $f'(0, s_t) = \infty$; this is the usual, putty–putty version of the capital accumulation models cited in the introduction. As a matter of notation, let $h_t = (s_0, s_1, ..., s_t)$ denote the entire history of states through date t. Then investment by household type i at date t may depend on that history, and one writes $I_{t+1}(h_t)$.

Each household of type i cares about units of consumption c^1_t of commodity i, the so called home-produced good, and consumption c^2_t of commodity $i+1$, the so called market-produced good, both at date t, and has preferences represented by the utility function $U(c^1_t, c^2_t, s_t)$ which is strictly concave, strictly increasing, and continuously differentiable in consumptions. Further, to ensure interior solutions, it is supposed that for any $c^2_t > 0$ and any s_t, $\lim_{c^1_t \to 0} U_1[c^1_t, c^2_t, s_t] = \infty$; that for any $c^1_t > 0$ and any s_t, $\lim_{c^2_t \to 0} U_2[c^1_t, c^2_t, s_t] = \infty$; and for any nonzero, finite price vector $p = (p_1, p_2)$, the induced indirect utility function over nominal income y, namely $V(y; p_1, p_2, s_t)$, has the property that for any s_t, $\lim_{y \to 0} V_y[y; p_1, p_2, s_t] = \infty$. Note that preferences can be random at date t, a function of the state s_t. In principle, household type i's consumption of commodities i and $i+1$ can depend on the entire history h_t at date t. So let $c^1_t(h_t)$ and $c^2_t(h_t)$ denote the number of units of consumption of commodities i and $i+1$, respectively, by household i at date t under history h_t. (Again, note the imposed symmetry across households.) Thus, preferences of each household of type i over the $T+1$ periods of its lifetime are represented by the utility function

$$E_0 \sum_{t=0}^{T} \beta^t U[c^1_t(h_t), c^2_t(h_t), s_t].$$

Here the expectation E_0 is over states s_t, given s_0 and the discount rate β satisfies $0 < \beta < 1$. Also T may be either finite or infinite; it will be clear from the context below which assumption is made.

There is also fiat money in this economy, pieces of paper which represent outside indebtedness. Let M_0 denote the number of units of fiat money held by household i at date zero, a fixed initial condition, and again the same across all households. In addition, each household of type i is endowed with $z_t(h_t)$ units of fiat money at the beginning of each date t, possibly depending on the history h_t. These endowments may be viewed as random, lump-sum injections by a monetary authority or more generally, monetary instability variables. If $z_t(h_t)$ is negative, it is interpreted as a lump-sum tax.

However, if there is a tax, it is assumed that the sequence of previous transfers is such that household i can pay the tax without engaging in trade, that is, $M_0 + \sum_{\tau=0}^{t} z_\tau(h_\tau) > 0$ for each t; this avoids bankruptcy problems. (It will also be equivalent with the condition that in *equilibrium* per household money balances will always be strictly positive.) Moreover, if the horizon is finite with T as the last date, then net injections of money are necessarily taxed away in the end, in effect at date $T + 1$. This is accomplished by imposing the terminal condition $M_0 + \sum_{\tau=0}^{T} z_\tau(h_\tau) = M_{T+1}(h_T)$, where $M_{T+1}(h_T)$ is the number of units of fiat money held at the end of date T under history h_T (in effect, the tax at $T + 1$). It may be noted, following Lerner [44], that a coupling of fiat money issue with promised taxation is not unusual; see Hepburn [36] for a discussion of its use in the American colonial period, for example. This device also has a long history in the monetary economics literature, culminating with the recent efforts of Belasko and Shell [2, 3].

It should be noted now that preferences, technology, endowments, and monetary policy are all potentially random in this model. In fact, by specifying finite-state space stochastic processes on each of these, one can induce a stochastic process on states of the world s_t. Thus, the formulation is quite general and allows statistical independence as a special case. It will be supposed that the process s_t is Markov, that is, the probability that $s_{t+1} = b$ given $s_t = a$ does not depend on t. The state s_t is known by all households at the beginning of date t.

We might note now the absence of double coincidence of wants when households meet. For example, in markets $(i, i + 1)$, a household of type i has no commodity a household of type $i + 1$ wants, and this creates the possibility for monetary exchange. (Again, see Lucas' version of the Cass-Yaari [14] circle in Townsend [66, Sect. 4]). Indeed, at market $(i, i + 1)$ given history h_t, we may suppose the existence of a competitive market in which commodity $i + 1$ can be purchased with money (by a household i) at monetary price $p_t(h_t)$. Similarly, at market $(i - 1, i)$, we may suppose commodity i is sold (by a household i) for money at price $p_t(h_t)$. Again note the imposed symmetry on prices of commodities i, $i + 1$ and so on. We might also note, then, that in this set-up, money from the sale of commodity i cannot be used *contemporaneously* by household i for the purchase of commodity $i + 1$. Of course, having completed purchases and sales, agents of household i return home to consume and invest. As a matter of notation, let $M_{t+1}(h_t)$ denote the number of units of fiat money carried over to the beginning of date $t + 1$.

3. Decision Problems and the Definition
of a Monetary Equilibrium

Again limiting attention to outcomes which are symmetric across households, the problem of *each* household i is now the problem of a "representative" household, namely, taking the price sequence $p_t(h_t)$ and the tax sequence $z_t(h_t)$ parametrically,

PROBLEM 3.1.

$$\text{Max } E_0 \sum_{t=0}^{T} \beta^t U[c_t^1(h_t), c_t^2(h_t), s_t] \tag{3.1}$$

by choice of $c_t^1(h_t)$, $c_t^2(h_t)$, $I_{t+1}(h_t)$, $M_{t+1}(h_t)$ all $t \geqslant 0$, all h_t, subject to

$$p_t(h_t) c_t^1(h_t) + p_t(h_t) c_t^2(h_t) + M_{t+1}(h_t) + p_t(h_t) I_{t+1}(h_t)$$

$$\leqslant M_t(h_{t-1}) + p_t(h_t) w_t(s_t) + p_t(h_t) f[I_t(h_{t-1}), s_t] + z_t(h_t)$$

$$\text{all } t \geqslant 0, \text{ all } h_t \tag{3.2}$$

$$p_t(h_t) c_t^2(h_t) \leqslant M_t(h_{t-1}) + z_t(h_t) \qquad \text{all } t, \text{ all } h_t \tag{3.3}$$

given $M_0 + z_0(s_0) > 0$, $I_0 \geqslant 0$, $M_{T+1}(h_T) \geqslant M_0 + \sum_{\tau=0}^{T} z_\tau(h_\tau)$. Then this yields

DEFINITION 3.1. A *symmetric monetary equilibrium is a sequence of finite, positive prices* $p_t^*(h_t)$, sequences of consumptions $c_t^{1*}(h_t)$, $c_t^{2*}(h_t)$, money balances $M_{t+1}^*(h_t)$, investments $I_{t+1}^*(h_t)$ and lump-sum injections $z_t^*(h_t)$ such that the following two conditions hold.

Maximization: the sequences $c_t^{1*}(h_t)$, $c_t^{2*}(h_t)$, $M_{t+1}^*(h_t)$, $I_{t+1}^*(h_t)$ solve Problem 3.1 given the price sequence $p_t^*(h_t)$ and lump-sum injections $z_t^*(h_t)$.

Market clearing: the sequences $c_t^{1*}(h_t)$, $c_t^{2*}(h_t)$, $I_{t+1}^*(h_t)$ satisfy

$$c_t^{2*}(h_t) + c_t^{1*}(h_t) \leqslant w_t(s_t) + f[I_t^*(h_{t-1}), s_t] - I_{t+1}^*(h_t) \qquad \text{all } t, h_t. \tag{3.4}$$

Note in the market-clearing condition (3.4) that only households type i and $i-1$ consume commodity i, and only household type i is endowed with or can invest commodity i. Also, since in any market commodities are exchanged for money, commodity balance (3.4) implies equality in the demand and supply of money balances. Thus in equilibrium the representative household's decisions about $M_{t+1}(h_t)$ are coincident with aggregate money balances available, namely $M_0 + \sum_{\tau=0}^{t} z_\tau(h_\tau)$.

It may be noted also that in a symmetric monetary equilibrium there is only one maximization problem to evaluate and, in particular, market clearing is defined relative to the choice variables of the representative household. But from the point of view of the household, the economy is decentralized. For an arbitrary price vector $p_t(h_t)$; choices satisfying its budget constraint do not necessarily satisfy the market clearing condition. In particular, the amount of money it supplies in the market $(i, i+1)$ need not equal the amount it purchases in the market $(i-1, i)$. The task is to find a price vector which implies market clearing. In fact, the next section of this paper contains a proof of the existence of a symmetric monetary equilibrium for the finite horizon and infinite horizon cases.

4. Existence of a Symmetric Monetary Equilibrium

The strategy adopted here to prove the existence of a symmetric monetary equilibrium is to use the more or less standard existence arguments of Debreu [17] for the finite-horizon economy, arguments which allow for diversity among agents. This is precisely the tact taken by Heller [33] for proving existence of equilibrium in an economy with fiat money, and fortunately, as far as its use here is concerned, Heller's model allows for capital accumulation. Unfortunately, that model also exploits the properties of an exogenously specified (nontrivial) transactions cost technology, a technology which has no analog here. But it is established that a modified, Heller-style proof can be applied here.

The proof relies heavily on the taxation of fiat money in the last period, period T. In effect, this tax allows the price of money to be an arbitrary positive number in period T, and, with this, the price of money can be shown to be positive in earlier periods, no matter what the realized history.[5] So Debreu's [17] continuity and fixed-point arguments may be applied. It can be established, moreover, under specified assumptions, that money is traded in any such monetary equilibrium, that is, money does facilitate exchange. The existence of such a nontrivial monetary equilibrium for finite horizon economies is a major result of this paper. It is summarized here and proved in the Appendix.

THEOREM 4.1.   *For every horizon $T < \infty$ and for every terminal price of money $p_T(h_T) = \alpha/(1 - \alpha)$ there exists a symmetric monetary equilibrium.*

Still, one may feel uncomfortable that the end-of-horizon tax is an

---

[5] Here, and throughout the paper, positive everywhere means positive with probability one relative to the underlying (finite) event space A for states s_t with histories $h_t = (s_0, s_1, ..., s_t)$ and relative to the specified Markov transition probabilities, $\text{prob}(s_{t+1} \mid s_t)$, s_0 given.

artificial device. To counter this, two more theorems are established. First, under specified assumptions, the monetary equilibrium prices and allocations for the finite horizon economies converge to well-defined feasible limits (at least along some subsequence and, for examples, along the entire sequence) as the horizon is driven to infinity. Again this is summarized here and proved in the Appendix.

THEOREM 4.2. *There exists a subsequence of horizons T, say T_n, such that individual choices $\{c_t^1(h_t),\ c_t^2(h_t),\ I_{t+1}(h_t),\ M_{t+1}(h_t)\}^{T_n}$, $t = 0, 1,..., T_n$, and prices $\{p_t(h_t)\}^{T_n}$, $t = 0, 1,..., T_n$, converge to well-defined limits, denoted $\{c_2^1(h_t),\ c_t^2(h_t),\ I_{t+1}(h_t),\ M_{t+1}(h_t)\}^{\sim}$, $t = 0, 1,..., \infty$, and $\{p_t(h_t)\}^{\sim}$, $t = 0, 1,..., \infty$, respectively. Moreover the $^{\sim}$ choices are feasible for the individual given the $^{\sim}$ prices and satisfy market clearing as well.*

The interpretation is that the behavior of households is unaffected in the limit by a tax in the arbitrarily distant future. Thus, in principle, one might hope to use the behavior and prices of a sufficiently long-lived economy as predictions of the model.

Second, one can replace the tax in Heller's set up with utility for end-of-horizon money balances (as well as end-of-horizon capital), following Bewley [6]. But the previous arguments need little modification to establish existence of equilibrium for such T-period economies. Again, one may take a limit and, in this case, establish conditions under which the limiting behavior is maximal in the infinite horizon economy. Thus, sufficient conditions are given for the existence of a monetary equilibrium in the infinite horizon economy. Again, this is summarized here and proved in the Appendix.

THEOREM 4.3. *Consider an artificial, finite-horizon economy with no taxes on money balances at the last date, T, but with preferences modified at date T, in effect adding a date $T + 1$ term*

$$E_0\beta^{T+1}M_{T+1}(h_T) + E_0\beta^{T+1}f[I_{T+1}(h_T), s_{T+1}].$$

Then there exists a symmetric monetary equilibrium with individual choice vector $\{c_t^1(h_t),\ c_t^2(h_t),\ M_{t+1}(h_t),\ I_{t+1}(h_t)\}^T$, $t = 0, 1,..., T$, and price vector $\{p_t(h_t)\}^T$, $t = 0, 1,..., T$, for every horizon $T < \infty$. Moreover, there exists a subsequence of horizons, say T_n, such that the T_n-sequences above converge to limits $\{c_t^1(h_t),\ c_t^2(h_t),\ M_{t+1}(h_t),\ I_{t+1}(h_t)\}^{\sim}$, $t = 0, 1,..., \infty$ and $\{p_t(h_t)\}^{\sim}$, $t = 0, 1,..., \infty$, respectively, for all dates t and all histories h_t. If the $U(\cdot, s_t)$ are uniformly bounded from above by some $\bar{u} < \infty$, and below by zero, the $f(\cdot, s_t)$ are uniformly bounded from above by some $\bar{f} < \infty$, and $\lim_{T \to \infty} E_0\beta^T [\sum_{t=0}^T z_t(h_t) + M_0] = 0$ then the $^{\sim}$ choices and $^{\sim}$ prices constitute an equilibrium in the infinite horizon economy.

5. PROPERTIES OF EQUILIBRIA: THE ANOMALIES

A nice aspect of the model under study is that the first-order conditions for the solution to the representative household's maximization problem (3.1) readily deliver interesting properties of monetary equilibria—unconventional intertemporal valuation of assets, rate-of-return dominance, and nonstandard asset pricing formulas. The key is that the period-by-period liquidity constraints can bear positive Lagrange multipliers, that is, shadow prices or "liquidity premia," which must be taken into account in the standard value relationships of choice theory.[6]

To proceed, note that the choice-variables $c_t^1(h_t)$, $c_t^2(h_t)$, $M_{t+1}(h_t)$, and $I_{t+1}(h_t)$ enter into at most a finite number of terms. Thus one may use the methods of Lagrange to deliver necessary first-order conditions for an interior maximum:

$$\beta^t \text{ Prob}(h_t \mid s_0) \, U_1[c_t^1(h_t), c_t^2(h_t), s_t] - \lambda_t(h_t) \, p_t(h_t) = 0, \tag{5.1}$$

$$\beta^t \text{ Prob}(h_t \mid s_0) \, U_2[c_t^1(h_t), c_t^2(h_t), s_t] - \lambda_t(h_t) \, p_t(h_t) - \phi_t(h_t) \, p_t(h_t) = 0, \tag{5.2}$$

$$-\lambda_t(h_t) + \sum_{s_{t+1}} \phi_{t+1}(h_t, s_{t+1}) + \sum_{s_{t+1}} \lambda_{t+1}(h_t, s_{t+1}) = 0, \tag{5.3}$$

$$-p_t(h_t) \, \lambda_t(h_t) + \sum_{s_{t+1}} \lambda_{t+1}(h_t, s_{t+1}) \, p_{t+1}(h_t, s_{t+1}) \, f'[I_{t+1}(h_t), s_{t+1}] = 0, \tag{5.4}$$

where $\lambda_t(h_t)$ is the (positive) Lagrange multiplier on budget constraint (3.2), in effect the marginal utility gain of a marginal increase in income (in terms of money), and $\phi_t(h_t)$ is the (nonnegative) Lagrange multiplier on constraint (3.3), in effect the marginal gain from a marginal weakening of the liquidity constraint. We might note in particular that Eq. (5.3) is the relationship among Lagrange multipliers implied by a marginal transaction in $M_{t+1}(h_t)$, shifting income from t to $t + 1$—$M_{t+1}(h_t)$ provides income in state s_{t+1} *and weakens* the liquidity constraint in $t + 1$ for all s_{t+1}.

Conditions (5.1)–(5.4) must hold in any monetary equilibrium. That is, they must hold when evaluated at equilibrium choices and prices. First, note that in equilibrium per household money balances $M_{t+1}(h_t)$ chosen must equal per household money balances available, $M_0 + \sum_{\tau=0}^{t} z_\tau(h_t)$, a number which by assumption is strictly positive. Thus, in equilibrium the constraint that $M_{t+1}(h_t) \geqslant 0$ is nonbinding and (5.3) must hold at equality. Second, note that with strictly positive (finite) prices $p_t(h_t)$ at date t and

---

[6] An earlier draft of the paper contains numerical examples of monetary equilibria in which Lagrange multipliers are strictly positive and which display, therefore, the anomalies described below. For further details, see Townsend [67].

history h_t, $p_t(h_t) w_t(s_t) + M_t(h_{t-1}) > 0$ and there is thus a source of positive nominal income for nominal expenditure $y_t(h_t)$. Then, with $V_y[0; p_t(h_t), p_t(h_t), s_t] = \infty$, nominal expenditures can never be zero in equilibrium. Further, the Inada conditions $U_1(0, c^2, s) = \infty$ and $U_2(c^1, 0, s) = \infty$ and the strict positivity of the upper bound on $c_t^2(h_t)$, namely $[M_t(h_{t-1}) + z_t(h_t)]/p_t(h_t)$, then imply that consumptions $c_t^1(h_t)$ and $c_t^2(h_t)$ are always strictly positive as well. Finally with $f'(0, s_{t+1}) = \infty$, the constraint $I_{t+1}(h_t) \geqslant 0$ can be ignored as well, and (5.4) must hold as an equality.

Again, first-order conditions (5.1)–(5.4) are rich in their implications for asset returns, as is now argued.

5.1. Unconventional Intertemporal Valuation for the Money Asset and for the Privately-Owned Capital Good

Equations (5.1)–(5.4) and the $t+1$ counterparts of (5.1) and (5.2) can be manipulated to yield

$$U_1[c_t^1(h_t), c_t^2(h_t), s_t]$$

$$= \beta E_t U_1[c_{t+1}^1(h_t, s_{t+1}), c_{t+1}^2(h_t, s_{t+1}), s_{t+1}] \, p_t(h_t)/p_{t+1}(h_t, s_{t+1})$$

$$+ \sum_{s_{t+1}} \phi_{t+1}(h_t, s_{t+1}) \, p_t(h_t)/\beta^t \, \text{Prob}(h_t \mid s_0), \tag{5.5}$$

$$U_2[c_t^1(h_t), c_t^2(h_t), s_t]$$

$$= \beta E_t U_2[c_{t+1}^1(h_t, s_{t+1}), c_{t+1}^2(h_t, s_{t+1}), s_{t+1}] \, p_t(h_t)/p_{t+1}(h_t, s_{t+1})$$

$$+ \phi_t(h_t) \, p_t(h_t)/\beta^t \, \text{Prob}(h_t \mid s_0). \tag{5.6}$$

Equations (5.5) and (5.6) display the unusual valuation relationships for the money of this model; the money asset as a store of value generally will not seem to be "priced efficiently" relative to either the home-produced or the market-produced good; that is, a marginal increase in money holdings will not have the marginal loss in utility from deferred (current) consumption equaling the expected marginal gain in utility from increased future consumption. To see why, consider (5.5). Suppose $c_t^1(h_t)$ is marginally decreased and converted by sale to an increase in money holdings $M_{t+1}(h_t)$, money which is carried over to date $t+1$. Now the consequent increase in income might be spent directly on $c_{t+1}^1(h_t, s_{t+1})$, but if money was constraining consumption at $t+1$ and s_{t+1}, a more efficient transaction is to readjust consumption levels, marginally increasing $c_{t+1}^2(h_t, s_{t+1})$ as well. Thus, there is an "extra" gain, the added term on the right-hand side of (5.5). Similar interpretations apply in the other equation. Obviously, we must take into account the special role played by money in contemplating intertemporal substitution of consumption bundles.

Further manipulation of (5.1)–(5.4) yield

$$U_1[c_t^1(h_t), c_t^2(h_t), s_t]$$

$$= \beta E_t U_1[c_{t+1}^1(h_t, s_{t+1}), c_{t+1}^2(h_t, s_{t+1}), s_{t+1}]$$

$$\times f'[I_{t+1}(h_t), s_{t+1}], \tag{5.7}$$

$$U_2[c_t^1(h_t), c_t^2(h_t), s_t]$$

$$= \beta E_t U_2[c_{t+1}^1(h_t, s_{t+1}), c_{t+1}^2(h_t, s_{t+1}), s_{t+1}] f'[I_{t+1}(h_t), s_{t+1}]$$

$$+ \phi_t(h_t) \, p_t(h_t)/\beta' \, \text{Prob}(h_t \mid s_0)$$

$$- \sum_{s_{t+1}} p_{t+1}(h_t, s_{t+1}) \, \phi_{t+1}(h_t, s_{t+1}) \, f'[I_{t+1}(h_t), s_{t+1}]/\beta' \, \text{Prob}(h_t \mid s_0). \tag{5.8}$$

It is seen from (5.7) that investment levels and capital asset holdings will be efficient relative to the home-produced commodity. Equation (5.7) must hold, after all, since consumptions $c_t^1(h_t)$ and $c_t^1(h_t, s_{t+1})$ can be reallocated by the investment $I_{t+1}(h_t)$. But, capital asset holdings generally will *not* be efficient relative to the market-produced commodity. To see why, consider (5.8). Suppose $c_t^2(h_t)$ is marginally decreased, releasing income so that $I_{t+1}(h_t)$ is marginally increased, this in turn financing a marginal increase in the $c_{t+1}^2(h_t, s_{t+1})$. As before, the decrease in $c_t^2(h_t)$ is less costly than one might suppose with $\phi_t(h_t) > 0$; on the other hand, the increase in $c_{t+1}^2(h_t, s_{t+1})$ is less beneficial with $\phi_{t+1}(h_t, s_{t+1}) > 0$. Again, money as an asset has a special role to play in facilitating consumption, and this can alter conventional valuation relationships for *other* assets.[7]

---

[7] There is a more conventional investment relationship in terms of the market-produced commodity which does hold in this model:

$$E_t U_2[c_{t+1}^1(h_t, s_{t+1}), c_{t+1}^2(h_t, s_{t+1}), s_{t+1}] \, p_t(h_t)/p_{t+1}(h_t, s_{t+1})$$

$$= \beta E_t U_2[c_{t+2}^1(h_t, s_{t+1}, s_{t+2}), c_{t+2}^2(h_t, s_{t+1}, s_{t+2}), s_{t+2}]$$

$$\times p_{t+1}(h_t, s_{t+1}) f'[I_{t+1}(h_t), s_{t+1}]]/p_{t+2}(h_t, s_{t+1}, s_{t+2}). \tag{F1}$$

The rationale behind (F1) is straightforward: suppose it is given that an incrementally small amount of $c_t^1(h_t)$ is to be sacrificed at time t. This amount can be sold for money at date t, money which in turn can be sold at date $t+1$ under any state s_{t+1} for the market-produced commodity. Alternatively, this amount can be invested from dates t to $t+1$, then sold for money which in turn can purchase $c_{t+2}^2(h_t, s_{t+1}, s_{t+2})$. What strikes one about (F1) is its relationship to the usual formula

$$U_2[c_t^1(h_t), c_t^2(h_t), s_t] = \beta E_t U_2[c_{t+1}^1(h_t, s_{t+1}), c_{t+1}^2(h_t, s_{t+1}), s_{t+1}]$$

$$\times f'[I_{t+1}(h_t), s_{t+1}]. \tag{F2}$$

Relative to (F2), consumption in (F1) is updated one period (so an adjustment is made for uncertainty) and real returns are adjusted by rates of inflation.

5.2. *Rate of Return Dominance*

That the real rate of return on privately-owned capital dominates the real return on money is now apparent. From (5.5) and (5.7), for example, if any of the $\phi_{t+1}(h_t, s_{t+1})$ are positive, then

$$
\beta E_t U_1[c^1_{t+1}(h_t, s_{t+1}), c^2_{t+1}(h_t, s_{t+1}), s_{t+1}]\{f'[I_{t+1}(h_t), s_{t+1}]
$$
$$
- p_t(h_t)/p_{t+1}(h_t, s_{t+1})\} > 0. \tag{5.9}
$$

Similarly, from (5.6) and (5.8),

$$
\beta E_t U_2[c^1_{t+1}(h_t, s_{t+1}), c^2_{t+1}(h_t, s_{t+1}), s_{t+1}]\{f'[I_{t+1}(h_t), s_{t+1}]
$$
$$
- p_t(h_t)/p_{t+1}(h_t, s_{t+1})\} > 0. \tag{5.10}
$$

Thus the marginal return on capital, expressed in terms of expected marginal utility of consumption, exceeds the marginal return on money. The paradox is explained by noting that money plays a role in loosening the date $t + 1$ liquidity constraint.

To express return dominance in a more dramatic way, imagine an economy without uncertainty, that is, suppose there is no movement in the state variables s_t, so these may be suppressed from the notation. It can then be established that in a steady state the marginal return on capital is $f'(I_{t+1}) = 1/\beta > 1$, and it thus exceeds the return on money, $p_t/p_{t+1} = 1$.[8]

---

[8] To do this, consider first an artificial finite-horizon economy in which money balances and investment give direct utility in date T, as in Theorem (4.3). Now the necessary and sufficient first-order conditions for an interior maximum are the nonstochastic analogues of (5.1)–(5.4) for $t = 0,..., T$,

(G.1) $\beta^t U_1(c^1_t, c^2_t) - \lambda_t p_t = 0,$

(G.2) $\beta^t U_2(c^1_t, c^2_t) - \lambda_t p_t - \phi_t p_t = 0,$

(G.3) $-\lambda_t + \phi_{t+1} + \lambda_{t+1} = 0,$

(G.4) $-p_t \lambda_t + p_{t+1} \lambda_{t+1} f'(I_{t+1}) = 0,$

with the extra conditions for M_{T+1} and I_{T+1} at date T,

(G.5) $\beta^T - \lambda_T = 0,$

(G.6) $\beta^T - \lambda_T p_T = 0.$

Now suppose the money price of commodities were some constant, say $p_t = 1$ for all dates t. Suppose also that consumption were constant, say $(c^1_t, c^2_t) = (\bar{c}^1, \bar{c}^2)$ for all dates t. Then, if this were so, investment would be constant to satisfy (G1) and (G4), that is,

(G.7) $-\beta^t U_1(\bar{c}^1, \bar{c}^2) + \beta^{t+1} U_1(\bar{c}^1, \bar{c}^2) f'(\bar{I}) = 0.$

Thus, steady state $\bar{I}$ is determined by the condition $f'(\bar{I}) = 1/\beta$. Then, from Eq. (G.3)

(G.8) $\phi_{t+1} = \beta^t(1 - \beta) U_1(\bar{c}^1, \bar{c}^2)$

Again, the explanation seems apparent: in the steady state, money held at date t is spent entirely on consumption of the market-produced good at date $t + 1$. But capital invested at date t must be converted to money at date $t + 1$ if it is to be spent on consumption of the market-produced good. Thus there is a one-period difference in consumption dates as between money and capital holdings; hence the return to capital is discounted by the one-period discount rate β, that is, $f'(I_{t+1}) \beta = 1.$ [9]

5.3. *Nonstandard Asset Pricing Formulas for Market Securities*

Thus far we have considered two assets, the money asset, which is actively traded, and the capital asset, which is privately held though it does enter into the production process. But what about the wide variety of securities which are traded in actual market economies? What can we say about the standard asset pricing formulas which have been taken to data from actual market economies on the assumption, if only implicit, that money is inessential, just another asset? Under what conditions would such formulas continue to apply to the monetary economy under consideration here?

To proceed recall that we have taken each household of type i, whatever its location at a given date, as representative of a large (infinite) number of households in an identical situation, following an identical itinerary, in effect cohort households. (The mathematics has assumed there is only one household.) And thus, as in standard representative consumer asset pricing

---

and from (G.1), (G.2), and (G.8),

$$(G.9) \quad U_1(\bar{c}^1, \bar{c}^2) = \beta U_2(\bar{c}^1, \bar{c}^2),$$

that is, the marginal rate of substitution equals β. Condition (G.9) with steady-state market clearing condition

$$(G.10) \quad c_t^1 + c_t^2 = w + f(\bar{I}) - \bar{I}$$

determines $\bar{c}^1$ and $\bar{c}^2$. Now let the initial condition $I_0 = \bar{I}$ and let date T choice $I_{T+1} = \bar{I}$. Also let initial condition $M_0 = \bar{c}^2$ and the date T choice $M_{T+1} = \bar{c}^2$. Thus, with an inconsequential normalization of the utility function, namely, $U_1(\bar{c}^1, \bar{c}^2) = 1$, conditions (G.1)–(G.6) are all satisfied and all markets clear. Thus a steady-state monetary equilibrium for the T-period artificial economy has been constructed. Now note that the above construction was valid for arbitrary finite T. So taking a limit as $T \to \infty$, one has a sequence of monetary equilibria which, by construction, has the same steady state price and choice vectors as limits. So, by Theorem 4.3, the limit prices and choice vectors constitute an equilibrium in the infinite horizon economy.

[9] As one referee has noted, this example may be too dramatic is as much as it relies on a discount rate β strictly less than unity. Rate of return dominance is possible in the economy *with uncertainty* even if β equals unity. Additionally, it might be noted here that there *may* exist "steady state" inflationary or deflationary equilibria in the economy without uncertainty if the money supply is growing on shrinking overtime. However, a deflation greater than the rate of discount would seem to be inconsistent with equilibrium.

models, we may contemplate the potential sale and purchase of a variety of assets among cohort households which, when priced at the monetary equilibrium consumptions and Lagrange multipliers, using the obvious marginal-rate-of-substitution formulas, are never traded in equilibrium.[10]

For example, the within-cohort price of market-produced commodity at date $t+1$ under history h_t and state s_{t+1} relative to the market-produced commodity at date t under history h_t is

$$\frac{\beta \operatorname{Prob}(s_{t+1} \mid h_t) \, U_2(c_{t+1}^1(h_t, s_{t+1}), c_{t+1}^2(h_t, s_{t+1}), s_{t+1}]}{U_2[c_t^1(h_t), c_t^2(h_t), s_t]}$$

$$\equiv \frac{1}{1 + r_{t+1}^2(h_t, s_{t+1})}, \tag{5.11}$$

where $r_{t+1}^2(h_t, s_{t+1})$ is a within-cohort, contingent, real rate of interest. And, of course, with this real rate of interest,

$$U_2[c_t^1(h_t), c_t^2(h_t), s_t]$$
$$= [1 + r_{t+1}^2(h_t, s_{t+1})] \, \beta \operatorname{Prob}(s_{t+1} \mid h_t) \, U_2[c_{t+1}^1(h_t, s_{t+1}),$$
$$c_{t+1}^2(h_t, s_{t+1}), s_{t+1}], \tag{5.12}$$

and a conventional intertemporal asset pricing formula holds trivially.

Similarly, suppose that one of the (representative) households in some cohort were given the possibility of trading (with some other member of the cohort) beginning-of-money balances for end-of-period consumptions, with all other cross-island market possibilities intact. It may be verified directly from the first-order conditions of an extended maximization problem that

$$\beta^t \operatorname{Prob}(h_t \mid s_0) \, U_1[c_t^1(h_t), c_t^2(h_t), s_t] / [\lambda_t(h_t) + \phi_t(h_t)], \tag{5.13}$$

$$\beta^t \operatorname{Prob}(h_t \mid s_0) \, U_2[c_t^1(h_t), c_t^2(h_t), s_t] / [\lambda_t(h_t) + \phi_t(h_t)], \tag{5.14}$$

are the appropriate, within-cohort money prices at date t under history h_t for the home-produced and market-produced consumption goods, respectively. One may also deliver these prices in a more direct way, noting from problem (3.1) and from (3.2), (3.3) that $\lambda_t(h_t) + \phi_t(h_t)$ is the obvious "marginal-utility of money" at date t (from the point of view of date zero). Now it is apparent from (5.14) and (5.2) that the implicit, within-cohort, money price of the market-produced commodity is equal to the explicit,

---

[10] The fact that these assets are not traded in equilibrium is not a virtue of the monetary economy under consideration. But then neither is the nontrading outcome a virtue in the *real* asset pricing models of the literature.

cross-island, monetary equilibrium money price of the market-produced commodity. But from (5.13) and (5.1), that is not true of the home-produced commodity. (We shall return to this observation momentarily.)

In the same way, one might suppose the representative consumer could borrow and lend beginning-of-period money balances in a within-cohort market. For example, the within-cohort price of money at date $t + 1$ under history h_t and state s_{t+1} relative to money at date t would be

$$\frac{\lambda_{t+1}(h_t, s_{t+1}) + \phi_{t+1}(h_t, s_{t+1})}{\lambda_t(h_t) + \phi_t(h_t)} \equiv \frac{1}{1 + r_{t+1}(h_t, s_{t+1})}, \qquad (5.15)$$

where $r_{t+1}(h_t, s_{t+1})$ is a contingent money rate of interest. And, of course one can consider as well a within-cohort, unconditional claim on money at date $t + 1$, valid for all states s_{t+1}, as such an asset is merely a bundle of individual contingent money claims. Its price in terms of date t money is thus just

$$\frac{\sum_{s_{t+1}} [\lambda_{t+1}(h_t, s_{t+1}) + \phi_{t+1}(h_t, s_{t+1})]}{\lambda_t(h_t) + \phi_t(h_t)} \equiv \frac{1}{1 + r_t(h_t)}, \qquad (5.16)$$

where $r_t(h_t)$ is the nominal rate of interest. Moreover, by virtue of (5.3),

$$\frac{1}{1 + r_t(h_t)} = \frac{\lambda_t(h_t)}{\lambda_t(h_t) + \phi_t(h_t)}, \qquad (5.17)$$

so that the within-cohort nominal rate of interest is never negative and is positive whenever ϕ_t is positive. Thus the theory has interesting, natural implications for the nominal rate of interest.

We may also note in passing now, from (5.1) and (5.17), that the within-cohort money price of the home-produced commodity at date t is just

$$p_t(h_t)/[1 + r_t(h_t)]. \qquad (5.18)$$

Intuitively, money from the (cross-island) sale of the home-produced commodity should be discounted by the nominal rate of interest, since it takes one period to purchase the market-produced consumption good with that money. In contrast, we may recall that the within-cohort price of the market-produced commodity is equal to $p_t(h_t)$, without adjustment.

Using the within-cohort nominal rate of interest and the within-cohort money prices, it can be established that within-cohort, unconditional, one-period money loans will be priced efficiently, again in accord with standard asset pricing formulas. For again, within-cohort prices are just marginal rates of substitution, and one can easily trace out a sequence of trans-actions (of goods now for money now, money now for money later, and money later for goods later) and show that some identity is satisfied,

trivially. Of course in equilibrium no such loans are effected. Rather, we see another kind of apparent asset return dominance: money is held in "idle" balances, even though, in general, with $\phi_t(h_t) > 0$, there is a positive rate of interest. But of course money is needed for within-period transactions.

Indeed, we are now ready to consider the purchase and sale of (arbitrary) securities which are priced in terms of money at the purchase date and yield returns in terms of money at the sale date. Suppose in particular that one unit of the market-produced commodity is sold in some cohort for money at date t, that the money is used for the purchase of the security in question, and that the monetary returns from the security at date $t + 1$ and state s_{t+1} are used to purchase the market-produced consumption good at date $t + 1$ and state s_{t+1}. But since the within-cohort money price of the market-produced consumption good is equal to the explicit, cross-island monetary equilibrium price we have

$$
\begin{aligned}
U_2 &[c_t^1(h_t), c_t^2(h_t), s_t] \\
&= R_{t+1}(h_t, s_{t+1}) \, \beta \, \text{Prob}(s_{t+1} \mid h_t) \, U_2[c_{t+1}^1(h_t, s_{t+1}), \\
&\quad c_{t+1}^2(h_t, s_{t+1}), s_{t+1}],
\end{aligned} \tag{5.19}
$$

where $R_{t+1}(h_t, s_{t+1})$ is the real, gross, contingent rate of return on the asset in question, with prices at the purchase date and redemption date deflated by the nominal, *cross-island*, price level. As noted, formulas of this kind or, more specifically, the stochastic analogues which also hold here, have been taken to data. But repeating this experiment with respect to the home-produced consumption good yields a *nonstandard* formula, if monetary prices and returns are deflated in the same way, using the cross-island prices. That is, consistent with (5.18), *one should use instead an interest-rate-deflated nominal price level at the purchase and sale date.* Alternatively, one should be careful to use the within-cohort nominal price level.

In concluding this subsection we should note that the key element in pricing assets seems to be the time when money proceeds from the sale of consumption goods or resources are available for expenditures on other consumption goods, and to that extent the analysis of this paper should carry over to more elaborate models. For example if a "household produced" good such as labor is not used immediately for consumption of leisure, then it seems that, relative to the marginal utility of leisure, standard asset pricing formulas are in jeopardy.

6. CONCLUDING REMARKS

This paper tries to make some progress on two fronts. First, motivated by observations, it tries to enlarge the class of well-defined, choice-theoretic

models that we have at our disposal, developing further some of the spatial models of earlier literature. The proof of existence of a competitive monetary equilibrium can be viewed in that light. Of course one hopes that the proof is not special to the particular spatial model considered here, that at least part of it will be applicable more generally. Additional evidence that it is indeed the case is the proof of existence of a competitive equilibrium for circulating private debt in the model of Townsend and Wallace [68], a proof which follows in part the one given here, building upon Debreu, with suitable modifications.

The second front on which the paper tries to make some progress is our ability to explain observed phenomena. In this regard, the spatial model considered here appears highly stylized, and, as an explanation of money and related asset return anomalies, quite restricted. Afterall, there is only one (actively traded) *financial* asset, and the payments lag which generates the finance constraint is of a particularly simple and rigid form. On the other hand, we do not seem to have yet many formal models which capture a general yet intuitive notion of liquidity. Apparently, production technologies which limit one's ability to retrieve investments, as in Diamond and Dybvig [19], are not enough *alone*; in their model costless intermediation in claims on capital can support an optimum. Neither are costly transportation technologies enough, apparently, when considered in isolation, in an Arrow–Debreu setting. What does seem crucial is some form of limited market participation, as captured here in a formal albeit brutal way, with distinct market locations and exogenously imposed itineraries. In any event, the present model does capture some of our intuition, and one might hope that it will spur future, formal efforts.

APPENDIX: Spaces and Theorems

The economy of Section 2 is first mapped into a modified version of Heller's model. First, one wants to treat money as a separate commodity and to distinguish purchases from sales. So let $x_{0t}(h_t)$, $x_t^1(h_t)$, $x_t^2(h_t)$ denote the (nonnegative) purchases in date t under history h_t of money, home-produced goods, and market-produced goods, respectively, by the representative household. Similarly, let $y_{0t}(h_t)$ and $y_t^1(h_t)$ denote (nonnegative) sales of money and the home-produced commodity, respectively. Note that in this formulation, each household believes it can purchase additional units of the home-produced commodity but knows it cannot sell the market-produced commodity, which it does not have. Of course, in equilibrium purchases of home-produced commodity will be zero as well. Let $w_{0t}(h_t) = z_t(h_t)$ denote the "endowment" of money for each date t and let $w_t^1(s_t) = w_t(s_t)$ denote the endowment of the home-produced commodity.

(Note the "endowment" of money may be negative if there is a tax.) Let $\tilde{s}_{0t}(h_t)$ and $\tilde{s}_t^1(h_t)$ denote the (nonnegative) number of units of money and the home-produced commodity, respectively, coming out of "storage" at date t under history h_t, and $r_{0t}(h_t)$ and $r_t^1(h_t)$ denote the analogously defined (nonnegative) quantities put into "storage" at date t. Thus, at date $t = 0$, $\tilde{s}_{00}(h_0) = M_0 > 0$ and $\tilde{s}_0^1(h_0) = f[I_0, s_0]$. And thus at date T finite, $r_{0T}(h_T) \geqslant M_0 + \sum_{\tau=0}^T z_\tau(h_\tau)$. Finally, let $x_t(h_t)$ denote the relevant vector of purchases, and so on for the other variables $y_t(h_t)$, $\tilde{s}_t(h_t)$, $r_t(h_t)$, with the relevant subscripts and superscripts deleted.

Following Heller, one can define a storage technology $S_t(h_t)$ for the representative household. That is, for storage input and output vectors to be feasible it is required that

$$[(r_t(h_t), \tilde{s}_{t+1}(h_{t+1})] \in S_{t+1}(h_{t+1}), \tag{A1}$$

where $S_t(h_t)$ is a closed, convex set. Clearly the strict concavity of $f(\cdot, s_t)$ with free disposal, the linear nature of storage in fiat money with free disposal, and the fixed terminal condition on fiat money holdings are all consistent with this specification. For example, $S_{t+1}(h_{t+1}) = \{r_{0t}(h_t) \geqslant 0$, $r_t^1(h_t) \geqslant 0$, $\tilde{s}_{0,t+1}(h_{t+1}) \geqslant 0$, $\tilde{s}_{t+1}^1(h_{t+1})$: $\tilde{s}_{0,t+1}(h_{t+1}) \leqslant r_{0t}(h_t)$ and $\tilde{s}_{t+1}^1(h_{t+1}) \leqslant f[r_t^1(h_t), s_{t+1}]\}$.

Letting $c_{0t}(h_t)$, $c_t^1(h_t)$, and $c_t^2(h_t)$ denote the number of (nonnegative) units of consumption of money and commodities, respectively, by the representative household at date t under history h_t, it is clear that

$$c_{0t}(h_t) = w_{0t}(h_t) + x_{0t}(h_t) - y_{0t}(h_t) + \tilde{s}_{0t}(h_t) - r_{0t}(h_t) \geqslant 0, \tag{A.2a}$$

$$c_t^1(h_t) = w_t^1(s_t) + x_t^1(h_t) - y_t^1(h_t) + \tilde{s}_t^1(h_t) - r_t^1(h_t) \geqslant 0, \tag{A.2b}$$

$$c_t^2(h_t) = x_t^2(h_t) \geqslant 0. \tag{A.3}$$

Note also that in equilibrium, money will not be consumed, that is, $c_{0t}(h_t) \equiv 0$. Finally, let c denote the relevant vector of consumptions, that is, with components running over commodities, over dates t, and over histories h_t. Then preferences are described by the utility function $u(c) = E_0 \sum_{t=0}^T \beta^t U[c_t^1(h_t), c_t^2(h_t), s_t]$.

Now let $p_t'(h_t)$ denote the price of any commodity in terms of some abstract unit of account at date t under history h_t (recall we are searching for a symmetric monetary equilibrium). Also let $p_{0t}(h_t)$ denote the corresponding price of money. Prices are normalized so that $p_{0t}(h_t) + p_t'(h_t) = 1$ with $p_{0t}(h_t) \geqslant 0$, $p_t'(h_t) \geqslant 0$. Call this simplex $P_t(h_t)$. Here then $p_t(h_t) \in P_t(h_t)$ is a two-dimensional vector, unlike previous notation. However, once an equilibrium is discovered with prices in the simplex, one can take the ratio $p_t'(h_t)/p_{0t}(h_t)$ as the price of commodities $p_t(h_t)$ as before. In what follows, the price of money at date T will be fixed at some α,

$0 < \alpha < 1$ for all histories h_T, that is, $p_{0T}(h_T) = \alpha$, $p'_T(h_T) = 1 - \alpha$, all h_T. Let $P_T(h_T)$ denote the set consisting of this unique vector. Then let P^{α} be the cross-product space of the $P_t(h_t)$ and $P_T(h_T)$ with generic price system p.

Following Heller, the fact that all trade involves money is written as

$$p_{0t}(h_t)\, x_{0t}(h_t) \leqslant p'_t(h_t)\, y^1_t(h_t), \tag{A.4}$$

$$p_{0t}(h_t)\, y_{0t}(h_t) \geqslant p'_t(h_t)\, x^1_t(h_t) + p'_t(h_t)\, x^2_t(h_t). \tag{A.5}$$

In words, constraint (A.4) says that money purchases cannot exceed in value commodity sales, and constraint (A.5) says that purchases of commodities cannot exceed in value money sales. Note that constraints (A.4) and (A.5) imply the usual budget constraint

$$p_{0t}(h_t)\, x_{0t}(h_t) + p'_t(h_t)\, x^1_t(h_t) + p'_t(h_t)\, x^2_t(h_t)$$
$$\leqslant p_{0t}(h_t)\, y_{0t}(h_t) + p'_t(h_t)\, y^1_t(h_t). \tag{A.6}$$

Finally one has the liquidity constraint not included in Heller,

$$x^2_t(h_t)\, p'_t(h_t) \leqslant p_{0t}(h_t)\, \tilde{s}_{0t}(h_t) + p_{0t}(h_t)\, w_{0t}(h_t). \tag{A.7}$$

Now define the vector of choice variables $b_t(h_t) = [x_t(h_t),\ y_t(h_t),\ r_t(h_t),\ \tilde{s}_t(h_t)]$ and let b be the relevant vector of the $b_t(h_t)$'s. Then define $B(p)$ as the set of b's which satisfy (A.1)–(A.7) under the price system p. Thus, the representative household maximizes $u(c)$ over $B(p)$. Let the set of maximizing b's be denoted $\gamma(p)$. With this notation one may now repeat the definition of a symmetric monetary equilibrium,

DEFINITION A.1. A symmetric monetary equilibrium is a strictly positive price vector p^* and choice vector b^* such that the following two conditions hold:

Maximization: vector $b^* \in \gamma(p^*)$.

Market clearing: $x^*_{0t}(h_t) = y^*_{0t}(h_t)$ all t, all h_t, for money

$x^{1*}_t(h_t) + x^{2*}_t(h_t) - y^{1*}_t(h_t) = 0$ all t, all h_t, for commodities.

Finally one obtains

THEOREM A.1. *For every horizon $T < \infty$, and for every terminal price of money α, $0 < \alpha < 1$, there exists a symmetric monetary equilibrium.*

Proof of Theorem A.1. First, as in Heller [33], it can be established that if the $p_{0t}(h_t)$, $p'_t(h_t)$ are all strictly positive, then $B(p)$ has an interior point. The addition of constraint (A.7) does not alter the essential part of

the argument: for the representative household one can find a sequence of the $b_t(h_t)$ with all components strictly positive, with all the components of c strictly positive, with constraints (A.4)–(A.7) all at strict inequality, and with storage input–output vectors all in the interior of the $S_t(h_t)$. This sequence is the interior point. Second, as in Heller, it can be established that the budget set $B(p)$ is a closed, convex set for all $p \in P^\alpha$ and is a continuous correspondence for all $p \in P^\alpha$ satisfying $p_{0t}(h_t) > 0$, $p'_t(h_t) > 0$, for all dates t and all histories h_t.

Now following Debreu [17], consider a cube K, the same dimension as $B(p)$, which contains the set of *attainable* choice vectors for representative household in its *interior*. This is possible since for any date t and any history h_t resources are finite and aggregate money holdings are finite; thus purchases, sales, and storage input–output vectors are bounded. Take $\bar{B}(p)$ to be the intersection of $B(p)$ with K.

Let $\bar{\gamma}(p)$ denote the set of utility maximizing elements in $\bar{B}(p)$. Clearly $\bar{\gamma}(p)$ is nonempty, bounded, convex, and by a theorem of Berge [5] and the second result above, is upper semicontinuous and hence closed for the $p_{0t}(h_t) > 0$, $p'_t(h_t) > 0$. Now, following Heller again (closely), consider the correspondence $\hat{\gamma}(p)$ obtained from $\bar{\gamma}(p)$ by taking the union of $\bar{\gamma}(p)$ and the set of limit points of the graph of $\bar{\gamma}(p)$. Then $\hat{\gamma}(p)$ is upper semicontinuous for all $p \in P^\alpha$ since its graph is closed and $\hat{\gamma}(p) = \bar{\gamma}(p)$ for $p_{0t}(h_t) > 0$, $p'_t(h_t) > 0$ since $\bar{\gamma}(p)$ is upper semicontinuous there. Finally, let $\delta(p)$ be the convex hull of $\hat{\gamma}(p)$.

Now let

$$\mu_t[b_t(h_t)] = \{\, p_t(h_t) \in P_t(h_t) \mid p_{0t}(h_t)[x_{0t}(h_t) - y_{0t}(h_t)] + p'_t(h_t)$$
$$\times [x^1_t(h_t) + x^2_t(h_t) - y^1_t(h_t)] \text{ is maximal in } P_t(h_t),\ t < T\},$$

$$\mu_T[b_T(h_T)] = p^\alpha_T, \text{ a fixed vector.}$$

Let $\mu(b)$ denote the associated cross product of the μ_t's. Now consider the correspondence $\phi(b, p) = \delta(p) \times \mu(b)$ defined on $K \times P^\alpha$. All the properties of Kakutani's fixed point theorem are satisfied, so there exists some p^*, b^* such that $p^* \in \mu(b^*)$ and $b^* \in \delta(p^*)$.

Now the budget inequality (A.6) at prices p^* yields the first inequality in (A.8),

$$0 \geqslant p^*_{0t}(h_t)[x^*_{0t}(h_t) - y^*_{0t}(h_t)] + p^{*\prime}_t(h_t)[x^{1*}_t(h_t) + x^{2*}_t(h_t) - y^{1*}_t(h_t)]$$
$$\geqslant p_{0t}(h_t)[x^*_{0t}(h_t) - y^*_{0t}(h_t)] + p'_t(h_t)[x^1_t(h_t) + x^2_t(h_t) - y^{1*}_t(h_t)], \qquad t < T.$$
$$\text{(A.8)}$$

The second inequality in (A.8) holds *for all* $p_t(h_t) \in P_t(h_t)$, *including the endpoints*, from the definition μ_t. Thus each of the terms in brackets in

(A.8) is nonpositive; there is (positive) excess demand in neither money nor commodities. So suppose some $p_{0t}^*(h_t)$ were zero for $t < T$. With $p_{0T}^*(h_t) \equiv \alpha > 0$ there would be excess demand for money, a contradiction. Similarly, suppose $p_t^{*\prime}(h_t) = 0$. Then there would be excess demand for commodities, a contradiction. Thus prices are all strictly positive. Finally, the budget constraints (A.6) will hold at equality yielding an equality in the first line of (A.8) with strictly positive prices. Thus excess demand cannot be negative and markets clear for $t < T$. For date T, consider (A.2a) and $t = T$. Then, with zero consumption of money at date T (since b^* is maximizing), and with the specified end-of-horizon terminal condition on money balances, there is zero excess demand for money at date T, for all h_T. Again, the budgets at equality and the imposed positive price of commodities at T yield zero excess demand for commodities at T. Thus the existence of an equilibrium for the bounded economy has been established.

Now it is claimed that b^* must also be maximizing in the unbounded economy, that is, that $b^* \in \gamma(p^*)$. From market clearing and the choice of K, $b^* \in \bar{\gamma}(p^*)$ is in the *interior* of $\bar{B}(p^*)$. Suppose there existed some $\hat{b} \in \gamma(p^*)$ which did better than b^* under the utility function $u(c)$. Then $b(\lambda) = \lambda \hat{b} + (1 - \lambda) b^*$ is utility improving also by the convexity of preferences and is in $\bar{B}(p^*)$ for sufficiently small λ. This contradicts $b^* \in \bar{\gamma}(p^*)$, and this completes the proof.

Remark. It is reassuring that there exist *nontrivial* monetary equilibria, that is, with active trade in money. It is fairly easy to construct examples. For suppose the utility function is separable, that is, of the form $U(c_t^1, c_t^2, s_t) = V(c_t^1, s_t) + W(c_t^2, s_t)$ with $W'(0, s_t) = \infty$. Then some of the market-produced commodity will be purchased with money at *any* positive price, and of course prices *are* positive by construction in a monetary equilibrium.

The monetary equilibria of Theorem (A.1) do depend on the terminal price of money, which is fixed at an arbitrary constant α, $0 < \alpha < 1$, and on the length of the horizon T. To remove the latter indeterminacy, at least, a limit result is now established. Let p^T and b^T denote the price system and choice vector of a monetary equilibrium for the T-period economy, $T < \infty$ with $p_{0T}(h_t) \equiv \alpha$. Clearly one can generate a sequence of monetary equilibria as the horizon T goes to infinity. It is somewhat reassuring that there exists a subsequence of these monetary equilibria which converges to some limit and that the limiting behavior is feasible under the limiting price system. This is stated formally in

THEOREM A.2.   *There exists a subsequence of horizons T, say T_n, such that the $b_t^{T_n}(h_t)$ and the $p_t^{T_n}(h_t)$ converge to some limits, say $\bar{b}(h_t)$ and $\tilde{p}_t(h_t)$*

for all dates t and histories h_t as $n \to \infty$. Moreover, $\bar{b}$ satisfies (A.1)–(A.7) *under $\bar{p}$ and the market clearing condition in Definition* A.1.

Proof of Theorem A.2. First, note that the $p_t^T(h_t) \in P_t(h_t)$ for every T, and thus $p_t^T(h_t)$ is uniformly bounded with respect to T. Similarly, as in the proof of Theorem A.1, one may take $b_t^T(h_t)$ to be in some bounded set for every T. Now there are countably many of these components $p_t^T(h_t)$, $b_t^T(h_t)$ as $T \to \infty$. Following Bewley's [6] Cantor diagonal argument (7.4, p. 194), one can find a subsequence of T such that all these components converge. Inequality constraints (A.2)–(A.7), market-clearing condition (ii), and storage feasibility (A.1) (where the $S_{t+1}(h_{t+1})$ are closed) are all satisfied at every T (sufficiently large) and thus passage to the limit is immediate. This completes the proof.

Theorem A.2 does not establish that the limiting prices and allocations constitute a monetary equilibrium in the infinite horizon economy. To obtain sufficient conditions for this, the finite-horizon economies are altered somewhat, by changing preferences in period T. In addition, it is supposed that the $U(\cdot, s_t)$ are uniformly bounded from above and from below, the production functions $f(\cdot, s_t)$ are uniformly bounded from above, and the growth rate of monetary injections is bounded in mean by the discount rate. More formally, consider

THEOREM A.3. (*Existence of a Monetary Equilibrium for the Infinite Horizon Economy*). *Consider an artificial, finite-horizon economy with no taxes on money balances at the last date T, but with preferences modified at T, adding the term $E_0 \beta^{T+1} \tilde{s}_{0,T+1}(h_{T+1}) + E_0 \beta^{T+1} \tilde{s}_{T+1}^1(h_{T+1})$. There exists a symmetric monetary equilibrium (p^T, b^T) in the artificial economy for every horizon $T < \infty$. Moreover, there exists a subsequence of horizons T_n such that the $b_t^{T_n}(h_t)$, $p_t^{T_n}(h_t)$ converge to some limits $\tilde{b}_t(h_t)$, $\tilde{p}_t(h_t)$ for all dates t and histories h_t. If the $U(\cdot, s_t)$ are uniformly bounded from above by some $\bar{u} < \infty$ and below by zero, the $f(\cdot, s_t)$ are uniformly bounded from above by some $\bar{f} < \infty$, and $\lim_{T \to \infty} E_0 \beta^T [\sum_{t=0}^T z_t(h_t) + M_0] = 0$ then $(\tilde{p}, \tilde{b})$ constitutes a monetary equilibrium in the infinite horizon economy.*

Proof of Theorem A.3. First, follow the proof of Theorem A.1 to establish the existence of a symmetric monetary equilibrium in the artificial economy, but with prices at date T, $p_T(h_T)$ unrestricted, except for the requirement $p_T(h_T) \in P_T(h_T)$, the unit simplex. Define the mappings μ_t as before, now with a similar construction of μ_T. At the fixed point, there can be excess demand for neither money nor commodities, as before. Note that $p_{0T}^*(h_T) \neq 0$ now given the *direct* utility for money balances at date T. This implies in turn that the maximizing demands satisfy market clearing.

Now as in Theorem A.2, the sequence of equilibrium prices and choice

vectors converges to some feasible limit $\tilde{p}_t(h_t)$, $\tilde{b}_t(h_t)$ for some subsequence of horizons T_n going to infinity. And as before these limits are in $B(\tilde{p})$ and satisfy market clearing condition (ii) in the infinite horizon economy. It will be argued that $\tilde{p}_t(h_t) \gg 0$ and that $\tilde{b}$ is maximizing in the infinite horizon economy. This will complete the proof.

It is first established that $\tilde{p}_{0t}(h_t) > 0$ and $\tilde{p}'_t(h_t) > 0$ for all dates t and all histories h_t. For, suppose $p'_t(h_t) = 0$ for some date t and some history h_t. Then, for sufficiently large T_n, by its definition, $p_t'^{T_n}(h_t)$ would be arbitrarily close to zero. But then to suppose $x_t'^{T_n}(h_t) \equiv 0$ is to contradict maximization for sufficiently large T_n, and otherwise there is a contradiction to equilibrium (market-clearing in commodities). Similarly, suppose $\tilde{p}_{0t}(h_t) = 0$ for some date t under some history h_t. But then for sufficiently large T_n, $p_{0t}^{T_n}(h_t)$ would be arbitrarily close to zero. It follows that the $p_{0\tau}^{T_n}(h_t, s_{t+1}, ..., s_\tau)$ must also have limits of zero for all dates $\tau > t$ and all histories $h_t, s_{t+1}, ..., s_\tau$. For suppose $\tilde{p}_{0\tau}(h_t, s_{t+1}, ..., s_\tau) > 0$ for at least one date τ under at least one history $h_t, s_{t+1}, ..., s_\tau$. Then, to suppose $\tilde{s}_{0t}^{T_n}(h_t) \leqslant [M_0 + \sum_{u=0}^{t} z_u(h_u)]$ is to contradict maximization for sufficiently large T_n, and otherwise there is a contradiction to equilibrium, market-clearing in money balances. That the $p_{0\tau}^{T_n}(h_t, s_{t+1}, ..., s_\tau)$ all have limits of zero implies in turn that the consumptions $c_\tau^{2T_n}(h_t, s_{t+1}, ..., s_\tau)$ all have limits of zero, by constraints (3.3). Finally, it can be established directly from the first-order conditions for a maximum (see (5.1)·(5.4)) that the $p_t'^{T_n}(h_t)$ can be defined recursively by

$$p_t^T(h_t) = \beta^t \, \text{Prob}(h_t \mid s_0) \, U_1[c_t^{1T}(h_t), c_t^{2T}(h_t), s_t] / \lambda_t(h_t),$$

$$\lambda_t^T(h_t) = \sum_{s_{t+1}} \frac{\beta^{t+1} \, \text{Prob}(h_t, s_{t+1} \mid s_0) \, U_2[c_{t+1}^{1T}(h_t, s_{t+1}), c_{t+1}^{2T}(h_t, s_{t+1}), s_{t+1}]}{p_{t+1}^T(h_t, s_{t+1})},$$

for all $T = T_n$, where the superscript T indexes the length of the horizon T and here, for the moment $p_t(h_t) = p'_t(h_t)/p_{0t}(h_t)$. This, with the above-derived consumption limits and $U_2[c_t^1, 0, s_t] = \infty$, implies $\lim_{T_n} p_t'^{T_n}(h_t) = 0$, a contradiction to $\tilde{p}_{0t}(h_t) = 0$.

Now let V denote the value of the infinite horizon objective function of the representative household under $\tilde{b}$ with prices $\tilde{p}$. Also, let F denote the supremum of the objective function under all feasible policies, that is, all $b \in B(\tilde{p})$. Clearly, $V \leqslant F$. We want to argue that $V \geqslant F$ as well.[11] So, let G^{T_n} denote the maximal value of the objective function by choice of $b \in B(\tilde{p})$ but with utility functions $U(\cdot, s_t)$ fixed at their upper bounds for all states s_t at all dates $t > T_n$. Clearly, $G^{T_n} \geqslant F$, so $\lim_{n \to \infty} G^{T_n} \geqslant F$ (taking subsequence of n if necessary). Now let H^{T_n} denote the value of the objective

function of the representative household in a monetary equilibrium for the artificial economy with finite horizon T_n, under prices $p_t^{T_n}(h_t)$. Suppose it can be established that $\lim_{n \to \infty} H^{T_n} = V$ and $\lim_{n \to \infty} |G^{T_n} - H^{T_n}| = 0$. Then $V \geqslant F$ and we are done.

That $\lim_{n \to \infty} |H^{T_n} - V| = 0$ is easy to establish. First, pick some $\varepsilon > 0$. Then choose n sufficiently large that $\sum_{t=T_n+1}^{\infty} \beta^t \bar{u} < \varepsilon/3$, $E_0 \beta^{T_n+1} \sum_{t=0}^{T_n} [z_t(h_t) + M_0] < \varepsilon/6$, $\beta^{T_n+1} \bar{f} < \varepsilon/6$. Thus the utility value of the tail of the objective function for the programming problem determining V is arbitrarily small and the utility values of beginning-of-period $T_n + 1$ money balances and commodity storage, respectively, in the programming problem determining H^{T_n}, are arbitrarily small. Then by continuity of the $U(\cdot, s_t)$ and convergence of the $b_t^{T_n}(h_t)$ to $\tilde{b}_t(h_t)$, n can be chosen, still larger if necessary, such that

$$\left| E_0 \sum_{t=0}^{T_n} \beta^t U[c_t^{T_n}(h_t), s_t] - E_0 \sum_{t=0}^{T_n} \beta^t U[\tilde{c}_t(h_t), s_t] \right| < \varepsilon/3.$$

Thus $|H^{T_n} - V| < \varepsilon$ for sufficiently large n.

That $\lim_{n \to \infty} |H^{T_n} - G^{T_n}| = 0$ is established as follows. As before, for $\varepsilon > 0$, choose n sufficiently large that $\sum_{t=T_n+1}^{\infty} \beta^t \bar{u} < \varepsilon/3$, $E_0 \beta^{T_n+1} (\sum_{t=0}^{T_n} z_t(h_t) + M_0) < \varepsilon/6$, and $\beta^{T_n+1} \bar{f} < \varepsilon/6$. Then it only remains to show that the utility values of the objective function over the first T_n periods, under the programs defining H^{T_n} and G^{T_n}, can be made arbitrarily close. First, note the prices $\tilde{p}_t(h_t)$ and $p_t^{T_n}(h_t)$ are all strictly positive. Now pick the solution $b \in B(\tilde{p})$ which yields the value G^{T_n}. Since $\lim_{n \to \infty} p^{T_n}(h_t) = \tilde{p}_t(h_t)$, b can be modified slightly to make it a feasible choice for the problem defining H^{T_n}, namely the T_n-period problem of the representative household in a monetary equilibrium under $p_t^{T_n}(h_t)$. In fact, for sufficiently large n, no more than $\varepsilon/3$ utility units need be lost in this modification in the first T_n periods. So $H^{T_n} - E_0 \beta^{T_n+1} \tilde{s}_{0,T_n+1}^*(h_{T_n+1}) - E_0 \beta^{T_n+1} \tilde{s}_{T_n+1}^{1*}(h_{T_n+1})$ is at most $\varepsilon/3$ units from below of $G^{T_n} - \sum_{t=T_n+1}^{\infty} \beta^t \bar{u}$. Similarly, $G^{T_n} - \sum_{t=T_n+1}^{\infty} \beta^t \bar{u}$ is no more than $\varepsilon/3$ units from below of $H^{T_n} - E_0 \beta^{T_n+1} \tilde{s}_{0,T_n+1}^*(h_{T_n+1}) - E_0 \beta^{T_n+1} \tilde{s}_{T_n+1}^{1*}(h_{T_n+1})$. So $|H^{T_n} - G^{T_n}| \leqslant |H^{T_n} - E_0 \beta^{T_n+1} \tilde{s}_{0,T_n+1}^*(h_{T_n+1}) - E_0 \beta^{T_n+1} \tilde{s}_{T_n+1}^{1*}(h_{T_n+1}) - G^{T_n} + \sum_{t=T_n+1}^{\infty} \beta^t \bar{u}| + |E_0 \beta^{T_n+1} \tilde{s}_{0,T_n+1}^*(h_{T_n+1})| + |E_0 \beta^{T_n+1} \tilde{s}_{T_n+1}^{1*}(h_{T_n+1})| + |\sum_{t=T_n+1}^{\infty} \beta^t \bar{u}| < \varepsilon$ for sufficiently large n.

REFERENCES

1. W. J. BAUMOL, The transactions demand for cash: an inventory theoretic approach, *Quart. J. Econ.* **66** (1952), 545–56.
2. Y. BALASKO, AND K. SHELL, "Lump-Sum Taxation: The Static Economy," Caress Working Paper, No. 83-08RR, Univ. of Pennsylvania, revised October 1983.

3. Y. BALASKO AND K. SHELL, "Lump-Sum Taxes and Transfers: The Overlapping Generations Model with Money," Caress Working Paper, No. 83-06R, Univ. of Pennsylvania, resived October 1983.
4. Y. BALASKO AND K. SHELL, The overlapping generation model II: The case of pure exchange with money, *J. Econ. Theory* **24** (1981), 112–142.
5. C. BERGE, "Topological Spaces," Macmillan, New York, 1963.
6. T. BEWLEY, The optimum quantity of money, *in* "Models of Monetary Economies" (J. H. Kareken and N. Wallace, Eds.), pp. 169–210, Federal Reserve Bank of Minneapolis, 1980.
7. D. T. BREEDEN, An intertemporal asset pricing model with stochastic consumption and investment opportunities, *J. Finan. Econ.* **7** (1979), 265–96.
8. W. A. BROCK, Asset Prices in a Production Economy, *in* "The Economics of Uncertainty" (J. J. McCall, Ed.), Univ. of Chicago Press, Chicago, 1980.
9. W. A. BROCK AND L. MIRMAN, Optimal economic growth and uncertainty: The discounted case, *J. Econ. Theory* **4** (1972), 479–513.
10. K. BRUNNER AND A. H. MELTZER, The uses of money: Money in the theory of an exchange economy, *Amer. Econ. Rev.* **61** (1971), 784–805.
11. J. BRYANT, Are money and capital 'separable'?, manuscript, Rice Univ. Houston, 1982.
12. D. CASS, Optimal growth in aggregate model of capital accumulation, *Rev. Econ. Stud.* **32** (1965), 233–240.
13. D. CASS AND K. SHELL, Do sunspots matter? Caress Working Paper, No. 81-09R, Univ. of Pennsylvania; *J. Polit. Econ.* (1983).
14. D. CASS AND M. YAARI, A re-examination of the pure consumption loans model, *J. Polit. Econ.* **74** (1966), 353–67.
15. R. W. CLOWER, A reconsideration of the microfoundations of monetary theory, *Western Econ. J.* **6** (1967), 1–8.
16. R. W. CLOWER, Theoretical foundations of monetary policy, *in* "Monetary Theory and Monetary Policy in the 1970's" (G. Clayton, J. C. Gilbert and R. Sedgwick, Eds.), Oxford Univ. Press, London, 1971.
17. G. DEBREU, "The Theory of Value," Wiley, New York, 1959.
18. G. DEBREU, Valuation equilibrium and pareto optimum, *Proc. Natl. Acad. Sci. U.S.A.* **40** (1954), 500–592.
19. D. DIAMOND AND P. DYBVIG, Bank runs, deposit insurance, and liquidity, *J. Polit. Econ.* **91** (1983), 401–419.
20. A. M. FELDMAN, Bilateral trading processes, pairwise optimality, and pareto optimality, *Rev. Econ. Stud.* **40** (1973), 463–474.
21. M. FRIEDMAN, "The Optimum Quantity of Money and Other Essays," Aldine, Chicago, 1969.
22. J. M. GRANDMONT AND Y. YOUNES, On the role of money and the existence of a monetary equilibrium, *Rev. Econ. Stud.* **39** (1972), 355–372.
23. J. M. GRANDMONT AND Y. YOUNES, On the efficiency of a monetary equilibrium, *Rev. Econ. Stud.* **40** (1973), 149–166.
24. S. J. GROSSMAN AND R. J. SHILLER, The determinants of the variability of stock market prices, *Amer. Econ. Rev.* (Papers and Proceedings) **71** (1981), 222–227.
25. F. H. HAHN, On some problems of proving existence of an equilibrium in a monetary economy, *in* "The Theory of Interest Rates" (F. H. Hahn and F. R. P. Brechling, Eds.), pp. 126–135, Macmillan, London, 1965.
26. F. H. HAHN, Equilibrium with transactions costs, *Econometrica* **39** (1971), 417–440.
27. F. H. HAHN, On the foundations of monetary theory, *in* "Essays in Modern Economics" (M. Parkin and A. R. Nobay, Eds.), pp.230–242, Harper & Row, New York, 1973.
28. F. H. HAHN, On transaction costs, inessential sequence economies, and money, *Rev. Econ. Stud.* **40** (1973), 449–462.

29. R. E. HALL, Intertemporal substitution in consumption, mimeograph, Hoover Institution and the Department of Economics, Stanford University, 1981.

30. L. P. HANSEN AND K. J. SINGLETON, Generalized instrumental variables estimation of nonlinear rational expectations models, *Econometrica* **50** (1982), 1269–1286.

31. L. P. HANSEN AND K. J. SINGLETON, Stochastic consumption, risk aversion, and the temporal behavior of stock market returns, *J. Polit. Econ.* **11** (1983), 249–265.

32. M. HARRIS, Expectations and money in a dynamic exchange economy, *Econometrica* **47** (1979), 1403–1419.

33. W. P. HELLER, The holding of money balances in general equilibrium, *J. Econ. Theory* **7** (1974), 93–108.

34. W. P. HELLER AND R. M. STARR, Equilibium with nonconvex transactions costs: Monetary and nonmonetary economies, *Rev. Econ. Stud.* **43** (1976), 195–216.

35. E. HELPMAN, Optimal spending and money holdings in the presence of liquidity constraints, *Econometrica* **49** (1981), 1559–1570.

36. A. B. HEPBURN, "A History of Currency in the United States," Macmillan, New York, 1924.

37. J. R. HICKS, A Suggestion for simplifying the theory of money, *Econometrica* (1935), 1–19, reprinted *in* "Readings in Monetary Theory," selected by the American Association, R. D. Irwin, 1951.

38. B. HOOL, Money, Expectations, and the existence of a temporary equilibrium, *Rev. Econ. Stud.* **43** (1976), 439–446.

39. B. JOVANOVIC, Inflation and welfare in the steady state, *J. Polit. Econ.* **90** (1982), 561–577.

40. M. KOHN, In defense of the finance constraint, *Econ. Inquiry* **19** (1981), 177–195.

41. T. KOOPMANS, On the concept of optimal economic growth, *Pontif. Acad. Sci.* **28** (1965), 225–300.

42. M. KURZ, Equilibrium in a finite sequence of markets with transactions costs, *Econometrica* **42** (1974), 1–20.

43. M. KURZ, Equilibrium with transaction costs and money in a single market exchange economy, *J. Econ. Theory* **7** (1974), 418–452.

44. A. P. LERNER, Money as a creature of the state, procedings of the *Amer. Econ. Assoc.* **37** (1947), 312–317.

45. R. E. LUCAS, JR., Asset prices in an exchange economy, *Econometrica* **46** (1978), 1426–1446.

46. R. E. LUCAS, JR., Equilibrium in a pure currency economy *in* "Models of Monetary Economics" (J. H. Kareken and N. Wallace, Eds.), Federal Reserve Bank of Minneapolis, 1980.

47. R. E. LUCAS, JR., Interest rates and currency prices in a two-country world, *J. Monet. Econ.* **10** (1982), 335–359.

48. R. E. LUCAS, JR., Money in a theory of finance, manuscript presented at the Carnegie–Rochester Conference on Public Policy, Bar-Harbour, November 1983.

49. R. E. LUCAS, JR. AND N. L. STOKEY, Optimal fiscal and monetary policy in an economy without capital, *J. Monet. Econ.* **12** (1984), 55–93.

50. N. MANKIW, J. ROTEMBERG, AND L. H. SUMMERS, "Intertemporal Substitution in Macro-Economics," NBER, April 1982.

51. L. W. McKENZIE, On the existence of general economic equilibrium for a competitive market, *Econometrica* **27** (1959), 54–71.

52. R. MEHRA AND E. C. PRESCOTT, A test of the intertemporal asset pricing model, manuscript, February 1982.

53. J. NIEHANS, Money in a static theory of optimal payment arrangements, *J. Money, Credit, and Banking* **1** (1969), 706–726.

54. J. M. OSTROY, The informational efficiency of monetary exchange, *Amer. Econ. Rev.* **63** (1973), 597–610.

55. J. M. OSTROY AND R. M. STARR, Money and the decentralization of exchange, *Econometrica* **42** (1974), 1093–1113.
56. E. C. PRESCOTT, AND R. MEHRA, Recursive competitive equilibrium: The case of homogeneous households, *Econometrica* **48** (1980), 1365–1380.
57. D. H. ROBERTSON, Saving and hoarding, *in* "Essays in Monetary Theory," Reprint London, 1940.
58. P. A. SAMUELSON, An exact consumption-loan model of interest with or without the social contrivance of money, *J. Polit. Econ.* **66** (1958), 467–482.
59. T. J. SARGENT, Tobin's q' and the rate of investment in general equilibrium, *in* "On the State of Macroeconomics" (K. Brunner and A. Meltzer, Eds.), North-Holland, Amsterdam, 1979.
60. R. J. SHILLER, Consumption, asset markets and macroeconomic fluctuations, presented at the Carnegie–Rochester Conference on Public Policy, Pittsburgh, November 1981.
61. A. C. STOCKMAN, Anticipated inflation and the capital stock in a cash-in-advance economy, *J. Monet. Econ.* **8** (1981), 387–394.
62. J. TOBIN, A dynamic aggregative model, *J. Polit. Econ.* **63** (1955), 103–115.
63. J. TOBIN, The interest-elasticity of the transactions demand for cash, *Rev. Econ. Stat.* **38** (1956), 241–247.
64. J. TOBIN, Money and economic growth, *Econometrica* **33** (1965), 671–684.
65. J. TOBIN, Discussion *in* "Models of Monetary Economies" (J. H. Kareken and N. Wallace, Eds.), pp. 83–90, Federal Reserve Bank of Minneapolis, 1980.
66. R. M. TOWNSEND, Models of money with spatially separated agents, *in* "Models of Monetary Economies" (J. H. Kareken and N. Wallace, Eds.), pp. 265–303, Federal Bank of Minneapolis, 1980.
67. R. M. TOWNSEND, Asset prices in a monetary economy: Explaining rate of return dominance and other apparent anomalies, working paper, 1982.
68. R. M. TOWNSEND AND N. WALLACE, "Circulating Private Debt: An Example with a Coordination Problem," Federal Reserve Bank of Minneapolis Staff Report, No. 83, September 1983.
69. S. C. TSIANG, Keynes's finance demand for liquidity, Robertson's loanable funds theory, and Friedman's monetarism, *Quart. J. Econ.* **14** (1980), 467–492.
70. N. WALLACE, A legal restriction theory of the demand for "money" and the role of monetary policy, *Fed. Res. Bank Minneapolis Quart. Rev.* **7** (1983), 1–7.
71. N. WALLACE, The overlapping generations model of fiat money, *in* "Models of Monetary Economies" (J. H. Kareken and N. Wallace, Eds.), pp. 49–78, Federal Reserve Bank of Minneapolis, 1980.
72. K. WICKSELL, "Lectures on Political Economy," Macmillan Co., New York, 1935.