Capital Deepening and Nonbalanced Economic Growth

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We present a model of nonbalanced growth based on differences in factor proportions and capital deepening. Capital deepening increases the relative output of the more capital-intensive sector but simultaneously induces a reallocation of capital and labor away from that sector. Using a two-sector general equilibrium model, we show that nonbalanced growth is consistent with an asymptotic equilibrium with a constant interest rate and capital share in national income. For plausible parameter values, the model generates dynamics consistent with U.S. data, in particular, faster growth of employment and slower growth of output in less capital-intensive sectors, and aggregate behavior consistent with the Kaldor facts.

I. Introduction

Most models of economic growth strive to be consistent with the "Kaldor facts," that is, the relative constancy of the growth rate, the capital-output

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ratio, the share of capital income in GDP, and the real interest rate (see Kaldor 1961; Denison 1974; Homer and Sylla 1991; Barro and Sala-i-Martin 2004). Beneath this balanced picture, however, there are systematic changes in the relative importance of various sectors (see Kuznets 1957, 1973; Chenery 1960; Kongsamut, Rebelo, and Xie 2001). A recent literature develops models of economic growth and development that are consistent with such structural changes, while still remaining approximately consistent with the Kaldor facts.[1] This literature typically starts by positing nonhomothetic preferences consistent with Engel's law and thus emphasizes the demand-side reasons for nonbalanced growth; the marginal rate of substitution between different goods changes as an economy grows, directly leading to a pattern of uneven growth between sectors. An alternative thesis, first proposed by Baumol (1967), emphasizes the potential nonbalanced nature of economic growth resulting from differential productivity growth across sectors, but it has received less attention in the literature.[2]

In this paper, we present a two-sector model that highlights a natural supply-side reason for nonbalanced growth related to Baumol's (1967) thesis. Differences in factor proportions across sectors (i.e., different shares of capital) combined with capital deepening lead to nonbalanced growth because an increase in the capital-labor ratio raises output more in sectors with greater capital intensity. We illustrate this economic mechanism using an economy with a constant elasticity of substitution between two sectors and Cobb-Douglas production functions within each sector. We show that the equilibrium (and the Pareto-optimal) allocations feature nonbalanced growth at the sectoral level but are consistent with the Kaldor facts in the long run. In the empirically relevant case in which the elasticity of substitution between the two sectors is less than one, one of the sectors (typically the more capital-intensive one) grows faster than the rest of the economy, but because the relative prices move against this sector, its (price-weighted) value grows at a slower rate than

[1] See, e.g., Matsuyama (1992), Echevarria (1997), Laitner (2000), Caselli and Coleman (2001), Kongsamut et al. (2001), and Gollin, Parente, and Rogerson (2002). See also the interesting papers by Stokey (1988), Foellmi and Zweimuller (2002), Matsuyama (2002), and Buera and Kaboski (2006), which derive nonhomotheticities from the presence of a "hierarchy of needs" or "hierarchy of qualities." Finally, Hall and Jones (2007) point out that there are natural reasons for health care to be a superior good (because life expectancy multiplies utility) and show how this can account for the increase in health care spending. Matsuyama (2008) presents an excellent overview of this literature.

[2] Two exceptions are the recent independent papers by Ngai and Pissarides (2006) and Zuleta and Young (2006). Ngai and Pissarides construct a model of multisector economic growth inspired by Baumol. In their model, there are exogenous total factor productivity differences across sectors, but all sectors have identical Cobb-Douglas production functions. While both of these papers are potentially consistent with the Kuznets and Kaldor facts, they do not contain the main contribution of our paper: nonbalanced growth resulting from factor proportion differences and capital deepening.

the rest of the economy. Moreover, we show that capital and labor are continuously reallocated away from the more rapidly growing sector.[3]

Figure 1 shows that the distinctive qualitative implications of our model are consistent with the broad patterns in the U.S. data over the past 60 years. Motivated by our theory, this figure divides U.S. industries into two groups according to their capital intensity and shows that there is more rapid growth of fixed-price quantity indices (corresponding to real output) in the more capital-intensive sectors, whereas the (price-weighted) values of output and employment grow more in the less capital-intensive sectors. The opposite movements of quantity and employment (or value) between sectors with high and low capital intensities are a distinctive feature of our approach (for the theoretically and empirically relevant case of the elasticity of substitution less than one).[4]

Finally, we present a simple calibration of our model to investigate whether its quantitative as well as its qualitative predictions are broadly consistent with U.S. data. Even though the model does not feature the demand-side factors that are undoubtedly important for nonbalanced growth, it generates relative growth rates of capital-intensive sectors that are consistent with U.S. data over the past 60 years. For example, our calibration generates increases in the relative output of the more capital-intensive industries that are consistent with the changes in U.S. data between 1948 and 2004 and accounts for one-sixth to one-third of the increase in the relative employment of the less capital-intensive industries. Our calibration also shows that convergence to the asymptotic equilibrium allocation is very slow, and consistent with the Kaldor facts, along this transition path the share of capital in national income and the interest rate are approximately constant.

The rest of the paper is organized as follows. Section II presents our model of nonbalanced growth, characterizes the full dynamic equilibrium of this economy, and shows how the model generates nonbalanced sectoral growth while remaining consistent with the Kaldor facts. Section III undertakes a simple calibration of our benchmark economy to in-

[3] As we will see below, in our economy the elasticity of substitution between products will be less than one if and only if the (short-run) elasticity of substitution between labor and capital is less than one. The time-series and cross-industry evidence suggests that the short-run elasticity of substitution between labor and capital is indeed less than one. See, e.g., the surveys by Nadiri (1970) and Hamermesh (1993), which show that the great majority of the estimates are less than one. Recent works by Krusell et al. (2000) and Antras (2001) also report estimates of the elasticity that are less than one. Finally, estimates implied by the response of investment to the user cost of capital also typically imply an elasticity of substitution between capital and labor significantly less than one (see, e.g., Chirinko 1993; Chirinko, Fazzari, and Mayer 1999; Mairesse, Hall, and Mulkay 1999).

[4] It should be noted that the differential evolutions of high and low capital intensity sectors shown in fig. 1 are distinct from the major structural changes associated with changes in the share of agriculture, manufacturing, and services. Our model does not attempt to account for these structural changes.

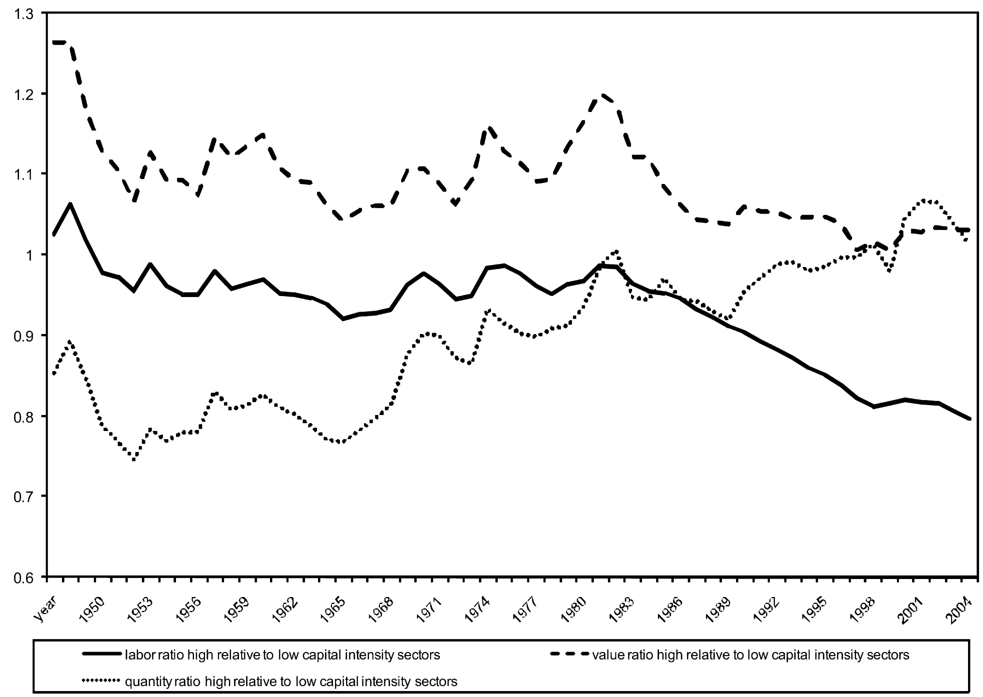

Fig. 1.—Employment, price-weighted value of output, and fixed-price quantity indices in high capital intensity sectors relative to low capital intensity sectors, 1948–2005. See Sec. III for industry classifications. Data from the National Income and Product Accounts.

vestigate whether the dynamics generated by the model are consistent with the changes in the relative output and employment of capital-intensive sectors and the Kaldor facts. Section IV presents conclusions. Appendix A contains additional theoretical results, and online Appendix B provides further details on the National Income and Product Accounts data and the sectoral classifications used in figure 1 and in Section III and additional evidence consistent with the patterns shown in figure 1.

II. A Model of Nonbalanced Growth

In this section, we present the environment, which is a two-sector model with exogenous technological change. Our working paper version (Acemoglu and Guerrieri 2006) presents results on nonbalanced growth in a more general setting as well as an extension of the model that incorporates endogenous technological change.

A. *Demographics, Preferences, and Technology*

The economy admits a representative household with the standard preferences

$$\int_0^\infty \exp\left[-(\rho - n)t\right] \frac{\tilde{c}(t)^{1-\theta} - 1}{1 - \theta}\, dt, \tag{1}$$

where $\tilde{c}(t)$ is consumption per capita at time t, ρ is the rate of time preferences, and $\theta \geq 0$ is the inverse of the intertemporal elasticity of substitution (or the coefficient of relative risk aversion). Labor is supplied inelastically and is equal to population $L(t)$ at time t, which grows at the exponential rate $n \in [0, \rho)$, so that

$$L(t) = \exp(nt)L(0). \tag{2}$$

The unique final good is produced competitively by combining the output (intermediate goods) of two sectors with an elasticity of substitution $\varepsilon \in [0, \infty)$:

$$\begin{aligned} Y(t) &= F[Y_1(t),\ Y_2(t)] \\ &= [\gamma Y_1(t)^{(\varepsilon-1)/\varepsilon} + (1-\gamma)Y_2(t)^{(\varepsilon-1)/\varepsilon}]^{\varepsilon/(\varepsilon-1)}, \end{aligned} \tag{3}$$

where $\gamma \in (0, 1)$. Both sectors use labor, L, and capital, K. Capital depreciates at the rate $\delta \geq 0$.

The aggregate resource constraint, which is equivalent to be the budget constraint of the representative household, requires consumption and investment to be less than the output of the final good:

$$\dot{K}(t) + \delta K(t) + C(t) \le Y(t), \tag{4}$$

where $C(t) \equiv \tilde{c}(t)L(t)$ is total consumption, and investment consists of new capital, $\dot{K}(t)$, and replenishment of depreciated capital, $\delta K(t)$.

The two goods Y_1 and Y_2 are produced competitively with production functions

$$Y_1(t) = M_1(t)L_1(t)^{\alpha_1}K_1(t)^{1-\alpha_1},$$
$$Y_2(t) = M_2(t)L_2(t)^{\alpha_2}K_2(t)^{1-\alpha_2}, \tag{5}$$

where K_1, L_1, K_2, and L_2 are the levels of capital and labor used in the two sectors.

If $\alpha_1 = \alpha_2$, the production function of the final good takes the standard Cobb-Douglas form. Hence, we restrict attention to the case in which $\alpha_1 \neq \alpha_2$. Without loss of generality, we make the following assumption.

ASSUMPTION 1. Sector 1 is more labor intensive (or less capital intensive), that is,

$$\Delta \equiv \alpha_1 - \alpha_2 > 0.$$

Technological progress in both sectors is exogenous and takes the form

$$\frac{\dot{M}_1(t)}{M_1(t)} = m_1 > 0, \quad \frac{\dot{M}_2(t)}{M_2(t)} = m_2 > 0. \tag{6}$$

Capital and labor market clearing require that at each date

$$K_1(t) + K_2(t) \le K(t) \tag{7}$$

and

$$L_1(t) + L_2(t) \le L(t), \tag{8}$$

where K denotes the aggregate capital stock and L is total population. The restriction that each of L_1, L_2, K_1, and K_2 has to be nonnegative is left implicit throughout.

B. *The Competitive Equilibrium and the Social Planner's Problem*

Let us denote the rental price of capital and the wage rate by R and w and the interest rate by r. Also let p_1 and p_2 be the prices of the Y_1 and

Y_2 goods. We normalize the price of the final good, P, to one at all points, so that

$$1 \equiv P(t) = [\gamma^{\varepsilon} p_1(t)^{1-\varepsilon} + (1-\gamma)^{\varepsilon} p_2(t)^{1-\varepsilon}]^{1/(1-\varepsilon)}. \tag{9}$$

A competitive equilibrium is defined in the usual way as paths for factor and intermediate goods prices $[r(t), w(t), p_1(t), p_2(t)]_{t\geq 0}$; employment and capital allocations $[L_1(t), L_2(t), K_1(t), K_2(t)]_{t\geq 0}$ such that firms maximize profits and markets clear; and consumption and savings decisions $[c(t), \dot{K}(t)]_{t\geq 0}$, which maximize the utility of the representative household.

Since markets are complete and competitive, we can appeal to the second welfare theorem and characterize the competitive equilibrium by solving the social planner's problem of maximizing the utility of the representative household.[5] This problem takes the form

$$\max_{[L_1(t), L_2(t), K_1(t), K_2(t), K(t), \tilde{c}(t)]_{t\geq 0}} \int_0^\infty \exp[-(\rho - n)t] \frac{\tilde{c}(t)^{1-\theta} - 1}{1-\theta} dt \tag{SP}$$

subject to (2), (6), (7), (8), and the resource constraint

$$\begin{aligned} &\dot{K}(t) + \delta K(t) + \tilde{c}(t)L(t) \leq Y(t) \\ &= F[M_1(t)L_1(t)^{\alpha_1}K_1(t)^{1-\alpha_1}, M_2(t)L_2(t)^{\alpha_2}K_2(t)^{1-\alpha_2}], \end{aligned} \tag{10}$$

together with the initial conditions $K(0) > 0$, $L(0) > 0$, $M_1(0) > 0$, and $M_2(0) > 0$. The objective function in this program is continuous and strictly concave, and the constraint set forms a convex-valued continuous correspondence. Thus the social planner's problem has a unique solution, and this solution corresponds to the unique competitive equilibrium.

Once this solution is characterized, the appropriate multipliers give the competitive prices. For example, given the normalization in (9), which is equivalent to the multiplier in (10) being normalized to one, the multipliers associated with (7) and (8) at time t give the rental rate, $R(t) \equiv r(t) + \delta$, and the wage rate, $w(t)$. The prices of the intermediate goods are then obtained as

$$p_1(t) = \gamma \left[\frac{Y_1(t)}{Y(t)}\right]^{-1/\varepsilon}, \quad p_2(t) = (1-\gamma)\left[\frac{Y_2(t)}{Y(t)}\right]^{-1/\varepsilon}. \tag{11}$$

C. *The Static Equilibrium*

Inspection of (SP) shows that the maximization problem can be broken into two parts. First, given the state variables $K(t)$, $L(t)$, $M_1(t)$, and

[5] See Acemoglu and Guerrieri (2006) for an explicit characterization of the equilibrium.

$M_2(t)$, the allocation of factors across sectors, $L_1(t)$, $L_2(t)$, $K_1(t)$, and $K_2(t)$, is chosen to maximize output $Y(t) = F[Y_1(t), Y_2(t)]$ so as to achieve the largest possible set of allocations that satisfy the constraint set. Second, given this choice of factor allocations at each date, the time path of $K(t)$ and $\tilde{c}(t)$ can be chosen to maximize the value of the objective function. These two parts correspond to the characterization of the static and dynamic optimal allocations, which are equivalent to the static and dynamic equilibrium allocations. We first characterize the static equilibrium and the implied competitive prices $p_1(t)$, $p_2(t)$, $r(t)$, and $w(t)$ and then turn to equilibrium dynamics.

Let us define the *maximized value* of current output given capital stock $K(t)$ at time t as

$$\Phi(K(t), t) = \max_{L_1(t), L_2(t), K_1(t), K_2(t)} F[Y_1(t), Y_2(t)] \tag{12}$$

subject to (5), (6), (7), and (8) and given $K(t) > 0$, $L(t) > 0$, $M_1(t) > 0$, and $M_2(t) > 0$. It is straightforward to see that this will involve the equalization of the marginal products of capital and labor in the two sectors, which can be written as

$$\gamma\alpha_1\left[\frac{Y(t)}{Y_1(t)}\right]^{1/\varepsilon}\frac{Y_1(t)}{L_1(t)} = (1-\gamma)\alpha_2\left[\frac{Y(t)}{Y_2(t)}\right]^{1/\varepsilon}\frac{Y_2(t)}{L_2(t)} \tag{13}$$

and

$$\gamma(1-\alpha_1)\left[\frac{Y(t)}{Y_1(t)}\right]^{1/\varepsilon}\frac{Y_1(t)}{K_1(t)} = (1-\gamma)(1-\alpha_2)\left[\frac{Y(t)}{Y_2(t)}\right]^{1/\varepsilon}\frac{Y_2(t)}{K_2(t)}. \tag{14}$$

Since the key static decision involves the allocation of labor and capital between the two sectors, we define the shares of capital and labor allocated to the labor-intensive sector (sector 1) as

$$\kappa(t) \equiv \frac{K_1(t)}{K(t)}, \quad \lambda(t) \equiv \frac{L_1(t)}{L(t)}.$$

Clearly, we also have $1-\kappa(t) \equiv K_2(t)/K(t)$ and $1-\lambda(t) \equiv L_2(t)/L(t)$. Combining (13) and (14), we obtain

$$\kappa(t) = \left\{1 + \left(\frac{1-\alpha_2}{1-\alpha_1}\right)\left(\frac{1-\gamma}{\gamma}\right)\left[\frac{Y_1(t)}{Y_2(t)}\right]^{(1-\varepsilon)/\varepsilon}\right\}^{-1} \tag{15}$$

and

$$\lambda(t) = \left\{1 + \left(\frac{1-\alpha_1}{1-\alpha_2}\right)\left(\frac{\alpha_2}{\alpha_1}\right)\left[\frac{1-\kappa(t)}{\kappa(t)}\right]\right\}^{-1}. \tag{16}$$

Equation (16) shows that, at each time t, the share of labor in sector 1, $\lambda(t)$, is (strictly) increasing in $\kappa(t)$. We next determine how these two shares change with capital accumulation and technological change.

PROPOSITION 1. In the competitive equilibrium,

$$\frac{d\ln\kappa(t)}{d\ln K(t)} = -\frac{d\ln\kappa(t)}{d\ln L(t)} = \frac{(1-\varepsilon)\Delta[1-\kappa(t)]}{1+(1-\varepsilon)\Delta[\kappa(t)-\lambda(t)]} > 0 \Leftrightarrow \Delta(1-\varepsilon) > 0 \qquad (17)$$

and

$$\frac{d\ln\kappa(t)}{d\ln M_2(t)} = -\frac{d\ln\kappa(t)}{d\ln M_1(t)} = \frac{(1-\varepsilon)[1-\kappa(t)]}{1+(1-\varepsilon)\Delta[\kappa(t)-\lambda(t)]} > 0 \Leftrightarrow \varepsilon < 1. \qquad (18)$$

Proof. To derive these expressions, rewrite (15) as

$$\phi(\kappa, L, K, M_1, M_2) \equiv \kappa - \left[1+\left(\frac{1-\alpha_2}{1-\alpha_1}\right)\left(\frac{1-\gamma}{\gamma}\right)\left(\frac{Y_1}{Y_2}\right)^{(1-\varepsilon)/\varepsilon}\right]^{-1} = 0,$$

where, from (5),

$$\frac{Y_1}{Y_2} = \lambda^{\alpha_1}(1-\lambda)^{-\alpha_2}\kappa^{1-\alpha_1}(1-\kappa)^{-(1-\alpha_2)}\left(\frac{L}{K}\right)^{\Delta}\frac{M_1}{M_2},$$

with λ given as in (16). Applying the implicit function theorem to $\phi(\kappa, L, K, M_1, M_2)$ and using the expression for Y_1/Y_2 given here, we obtain (17) and (18). QED

Equation (17) states that the fraction of capital allocated to the labor-intensive sector increases with the stock of capital if $\varepsilon < 1$ and decreases if $\varepsilon > 1$. To obtain the intuition for this comparative static, which is useful for understanding many of the results that will follow, note that if K increased and κ remained constant, then the capital-intensive sector, sector 2, would grow by more than sector 1 because an equiproportionate increase in capital raises the output of the more capital-intensive sector by more. The prices of intermediate goods given in (11) then imply that when $\varepsilon < 1$, the relative price of the capital-intensive sector will fall more than proportionately, inducing a greater fraction of capital to be allocated to the less capital-intensive sector 1. The intuition for the converse result when $\varepsilon > 1$ is straightforward. An important implication of this proposition is that as long as $\varepsilon \neq 1$ and there is capital deepening (i.e., as long as K/L is increasing over time), growth will be nonbalanced and capital will be allocated unequally between the two sectors. This is the basis of nonbalanced growth in our model.[6] The rest of our analysis will show that nonbalanced growth in this model is asymptotically consistent with the Kaldor facts and can approximate the Kaldor facts even along transitional dynamics.

Equation (18) also implies that when the elasticity of substitution, ε, is less than one, an improvement in the technology of a sector causes the share of capital allocated to that sector to fall. The intuition is again the same: when $\varepsilon < 1$, increased production in a sector causes a more than proportional decline in its relative price, inducing a reallocation of capital away from it toward the other sector (again the converse results and intuition apply when $\varepsilon > 1$).

In view of equation (16), the results in proposition 1 also apply to λ. In particular, we have that $d\ln\lambda(t)/d\ln K(t) = -d\ln\lambda(t)/d\ln L(t) > 0$ if and only if $\Delta(1-\varepsilon) > 0$.

Next, since equilibrium factor prices, R and w, correspond to the multipliers on the constraints (7) and (8), we also obtain

$$w(t) = \gamma\alpha_1\left[\frac{Y(t)}{Y_1(t)}\right]^{1/\varepsilon}\frac{Y_1(t)}{L_1(t)} \tag{19}$$

and

$$R(t) = \Phi_K(K(t),\ t) = \gamma(1-\alpha_1)\left[\frac{Y(t)}{Y_1(t)}\right]^{1/\varepsilon}\frac{Y_1(t)}{K_1(t)}, \tag{20}$$

where $\Phi_K(K(t),\ t)$ is the derivative of the maximized output function, $\Phi(K(t),\ t)$, with respect to capital. Equilibrium factor prices take the familiar forms and are equal to the (values of) marginal products from the derived production function in (10). To obtain an intuition for the economic forces, we next analyze how changes in the state variables, L, K, M_1, and M_2, affect these factor prices. When (19) and (20) are combined, relative factor prices are obtained as

$$\frac{w(t)}{R(t)} = \frac{\alpha_1}{1-\alpha_1}\left[\frac{\kappa(t)K(t)}{\lambda(t)L(t)}\right], \tag{21}$$

and the capital share in aggregate income is

$$\sigma_K(t) \equiv \frac{R(t)K(t)}{Y(t)} = \gamma(1-\alpha_1)\left[\frac{Y(t)}{Y_1(t)}\right]^{(1-\varepsilon)/\varepsilon}\kappa(t)^{-1}. \tag{22}$$

[6] As a corollary, note that with $\varepsilon = 1$, output levels in the two sectors could grow at different rates. But there would be no reallocation of capital and labor between the two sectors, and κ and λ would remain constant.

PROPOSITION 2. In the competitive equilibrium,

$$\frac{d\ln[w(t)/R(t)]}{d\ln K(t)} = -\frac{d\ln[w(t)/R(t)]}{d\ln L(t)} = \frac{1}{1+(1-\varepsilon)\Delta[\kappa(t)-\lambda(t)]} > 0;$$

$$\begin{aligned}\frac{d\ln[w(t)/R(t)]}{d\ln M_2(t)} &= -\frac{d\ln[w(t)/R(t)]}{d\ln M_1(t)}\\ &= -\frac{(1-\varepsilon)[\kappa(t)-\lambda(t)]}{1+(1-\varepsilon)\Delta[\kappa(t)-\lambda(t)]} < 0\\ &\Leftrightarrow \Delta(1-\varepsilon) > 0;\end{aligned}$$

$$\begin{aligned}\frac{d\ln \sigma_K(t)}{d\ln K(t)} &= -\frac{(1-\varepsilon)\Delta^2[1-\kappa(t)]\kappa(t)}{[1-\alpha_1+\Delta\kappa(t)]\{1+(1-\varepsilon)\Delta[\kappa(t)-\lambda(t)]\}} < 0\\ &\Leftrightarrow \varepsilon < 1; \end{aligned} \tag{23}$$

and

$$\begin{aligned}\frac{d\ln \sigma_K(t)}{d\ln M_2(t)} &= -\frac{d\ln \sigma_K(t)}{d\ln M_1(t)}\\ &= \frac{\Delta(1-\varepsilon)[1-\kappa(t)]\kappa(t)}{[1-\alpha_1+\Delta\kappa(t)]\{1+(1-\varepsilon)\Delta[\kappa(t)-\lambda(t)]\}} < 0\\ &\Leftrightarrow \Delta(1-\varepsilon) > 0.\end{aligned} \tag{24}$$

Proof. The first two expressions follow from differentiating equation (21) and proposition 1. To prove (23) and (24), note that from (3) and (15) we have

$$\left(\frac{Y_1}{Y}\right)^{(\varepsilon-1)/\varepsilon} = \gamma^{-1}\left[1+\left(\frac{1-\alpha_1}{1-\alpha_2}\right)\left(\frac{1-\kappa}{\kappa}\right)\right]^{-1}.$$

Then, (23) and (24) follow by differentiating σ_K as given in (22) with respect to L, K, M_1, and M_2 and using the results in proposition 1. QED

The most important result in this proposition is (23), which links the impact of the capital stock on the capital share in national income to the elasticity of substitution between the two sectors, ε. Since a negative relationship between the share of capital in national income and the capital stock is equivalent to an elasticity of substitution between aggregate labor and capital that is less than one, this result also implies that, as claimed in note 3, the elasticity of substitution between capital and labor is less than one if and only if ε is less than one. Intuitively, an increase in the capital stock of the economy causes the output of the more capital-intensive sector, sector 2, to increase relative to the output in the less capital-intensive sector (despite the fact that the share of capital allocated to the less capital-intensive sector increases as shown in eq. [17]). This then increases the production of the more capital-intensive sector, and when $\varepsilon < 1$, it reduces the relative price of capital more than proportionately; consequently, the share of capital in national income declines. The converse result applies when $\varepsilon > 1$.

Moreover, when $\varepsilon < 1$, (24) implies that an increase in M_1 is "capital biased" and an increase in M_2 is "labor biased." The intuition for why an increase in the productivity of the sector that is intensive in capital is biased toward labor (and vice versa) is once again similar: when the elasticity of substitution between the two sectors, ε, is less than one, an increase in the output of a sector (this time driven by a change in technology) decreases its price more than proportionately, thus reducing the relative compensation of the factor used more intensively in that sector (see Acemoglu 2002). When $\varepsilon > 1$, we have the converse pattern, and an increase in M_1 is labor biased and an increase in M_2 is capital biased.

D. Equilibrium Dynamics

We now characterize the dynamic competitive equilibrium allocations. We again use the social planner's problem (SP) introduced above. The previous subsection characterized the static optimal (equilibrium) allocation of resources and the resulting maximized value of output $\Phi(K(t), t)$ for given values of $K(t)$, $L(t)$, $M_1(t)$, and $M_2(t)$. Given $\Phi(K(t), t)$, problem (SP) can be written as

$$\max_{[K(t),\tilde{c}(t)]_{t\geq 0}} \int_0^\infty \exp\left[-(\rho - n)t\right] \frac{\tilde{c}(t)^{1-\theta} - 1}{1-\theta}\, dt \qquad \text{(SP}')$$

subject to

$$\dot{K}(t) = \Phi(K(t),\ t) - \delta K(t) - \exp(nt) L(0)\, \tilde{c}(t) \qquad (25)$$

and the initial condition $K(0) > 0$. The constraint (25) is written as an equality since it cannot hold as a strict inequality in an optimal allocation (otherwise, consumption would be raised, yielding a higher objective value). The other initial conditions of the original problem (SP) are incorporated into the maximized value of output $\Phi(K(t), t)$ in constraint (25). This maximization problem is simpler than (SP), though it is still not equivalent to the standard problems encountered in growth models because constraint (25) is not an autonomous differential equation. To further simplify the characterization of equilibrium dynamics, we will work with transformed variables. For this purpose, we first make the following assumption.

ASSUMPTION 2. Either (i) $m_1/\alpha_1 < m_2/\alpha_2$ and $\varepsilon < 1$, or (ii) $m_1/\alpha_1 > m_2/\alpha_2$ and $\varepsilon > 1$.

This assumption ensures that the *asymptotically dominant sector* will be the labor-intensive sector, sector 1. The asymptotically dominant sector is the sector that determines the long-run growth rate of the economy. Observe that this condition compares not the exogenous rates of technological progress, m_1 and m_2, but m_1/α_1 and m_2/α_2, which we refer to as the *augmented* rates of technological progress. The reason is that the two sectors differ in terms of their capital intensities, and technological change will be augmented by the differential rates of capital accumulation in the two sectors. For example, with equal rates of Hicks-neutral technological progress in the two sectors, the adjustment of the capital stock to technological change implies that the labor-intensive sector 1 will have a lower augmented rate of technological progress than sector 2.

Notice that when $\varepsilon < 1$, the sector with the lower rate of augmented technological progress will be the asymptotically dominant sector. The reason is that when $\varepsilon < 1$, the outputs of the two sectors are highly complementary and the slower-growing sector will determine the asymptotic growth rate of the economy. When $\varepsilon > 1$, the converse happens and the more rapidly growing sector determines the asymptotic growth rate of the economy and is the asymptotically dominant sector. In this light, assumption 2 implies that sector 1 is the asymptotically dominant sector. Appendix A shows that parallel results apply when the converse of assumption 2 holds and sector 2 is asymptotically dominant. The empirically relevant case is part i of assumption 2; as already argued, $\varepsilon < 1$ provides a good approximation to the data, and $\alpha_1 > \alpha_2$ from assumption 1. Therefore, as long as m_1 and m_2 are close to each other, the economy will be in case i of assumption 2.

Let us next introduce the following normalized variables,

$$c(t) \equiv \frac{\tilde{c}(t)}{M_1(t)^{1/\alpha_1}}, \quad \chi(t) \equiv \frac{K(t)}{L(t)M_1(t)^{1/\alpha_1}}, \tag{26}$$

which represent consumption and capital per capita normalized by the augmented technology of the asymptotically dominant sector, which, in view of assumption 2, is sector 1 (and thus the corresponding augmented technology is $M_1(t)^{1/\alpha_1}$). The next proposition shows that the solution to (SP)—and thus the dynamic equilibrium—can be expressed in terms of three autonomous differential equations in c, χ, and κ.

PROPOSITION 3. Suppose that assumptions 1 and 2 hold. Then a competitive equilibrium satisfies the following three differential equations:

$$\frac{\dot{c}(t)}{c(t)} = \frac{1}{\theta}[(1-\alpha_1)\gamma\eta(t)^{1/\varepsilon}\lambda(t)^{\alpha_1}\kappa(t)^{-\alpha_1}\chi(t)^{-\alpha_1} - \delta - \rho] - \frac{m_1}{\alpha_1},$$

$$\frac{\dot{\chi}(t)}{\chi(t)} = \lambda(t)^{\alpha_1}\kappa(t)^{1-\alpha_1}\chi(t)^{-\alpha_1}\eta(t) - \chi(t)^{-1}c(t) - \delta - n - \frac{m_1}{\alpha_1},$$

$$\frac{\dot{\kappa}(t)}{\kappa(t)} = \frac{[1-\kappa(t)]\{\Delta[\dot{\chi}(t)/\chi(t)] + m_2 - (\alpha_2/\alpha_1)m_1\}}{(1-\varepsilon)^{-1} + \Delta[\kappa(t) - \lambda(t)]}, \tag{27}$$

where

$$\eta(t) \equiv \gamma^{\varepsilon/(\varepsilon-1)}\left\{1 + \left(\frac{1-\alpha_1}{1-\alpha_2}\right)\left[\frac{1-\kappa(t)}{\kappa(t)}\right]\right\}^{\varepsilon/(\varepsilon-1)}, \tag{28}$$

with initial conditions $\chi(0)$ and $\kappa(0)$, and also satisfies the transversality condition

$$\lim_{t\to\infty} \exp\left\{-\left[\rho - \frac{(1-\theta)m_1}{\alpha_1} - n\right]t\right\}\chi(t) = 0. \tag{29}$$

Moreover, any allocation that satisfies (27)–(29) is a competitive equilibrium.

Proof. See Appendix A.

The first equation in (27) is the standard Euler equation, written in terms of the normalized variables. The first term in brackets, $(1-\alpha_1)\gamma\eta(t)^{1/\varepsilon}\lambda(t)^{\alpha_1}\kappa(t)^{-\alpha_1}\chi(t)^{-\alpha_1}$, is the marginal product of capital. The second equation in (27) is the law of motion of the normalized capital stock, $\chi(t)$. The third equation, in turn, specifies the evolution of the share of capital between the two sectors. We also impose the following parameter condition, which ensures that the transversality condition (29) holds.

Assumption 3. $\rho - n \geq (1-\theta)(m_1/\alpha_1)$.

Our next task is to use proposition 3 to provide a tighter characterization of the dynamic equilibrium allocation. For this purpose, let us first define a *constant growth path* (CGP) as a dynamic competitive equilibrium that features constant aggregate consumption growth.[7] The next theorem will show that there exists a unique CGP that is a solution to the social planner's problem (SP) and will provide closed-form solutions for the growth rates of different aggregates in this equilibrium. The notable feature of the CGP will be that despite the constant growth rate

[7] Kongsamut et al. (2001) refer to this as a "generalized balanced growth path."

of aggregate consumption, growth will be nonbalanced because output, capital, and employment in the two sectors will grow at different rates. Let us define

$$\frac{\dot{L}_s(t)}{L_s(t)} \equiv n_s(t), \quad \frac{\dot{K}_s(t)}{K_s(t)} \equiv z_s(t), \quad \frac{\dot{Y}_s(t)}{Y_s(t)} \equiv g_s(t) \quad \text{for } s = 1, 2,$$

$$\frac{\dot{K}(t)}{K(t)} \equiv z(t), \quad \frac{\dot{Y}(t)}{Y(t)} \equiv g(t),$$

so that n_s and z_s denote the growth rates of labor and capital stock, m_s denotes the growth rate of technology, and g_s denotes the growth rate of output in sector s. Moreover, whenever they exist, we denote the corresponding asymptotic growth rates by asterisks, so that $n_s^* = \lim_{t\to\infty} n_s(t)$, $z_s^* = \lim_{t\to\infty} z_s(t)$, and $g_s^* = \lim_{t\to\infty} g_s(t)$. Similarly, let us denote the asymptotic capital and labor allocation decisions by asterisks, that is,

$$\kappa^* = \lim_{t\to\infty} \kappa(t), \quad \lambda^* = \lim_{t\to\infty} \lambda(t).$$

Then we have the following characterization of the unique CGP.

THEOREM 1. Suppose that assumptions 1–3 hold. Then there exists a unique CGP in which consumption per capita grows at the rate $g_c^* = m_1/\alpha_1$, $\kappa^* = 1$,

$$\chi^* = \left[\frac{(\theta m_1/\alpha_1) + \rho + \delta}{\gamma^{\varepsilon/(\varepsilon-1)}(1-\alpha_1)}\right]^{-1/\alpha_1}, \tag{30}$$

and

$$c^* = \gamma^{\varepsilon/(\varepsilon-1)}(\chi^*)^{1-\alpha_1} - \chi^*\left(\delta + n + \frac{m_1}{\alpha_1}\right). \tag{31}$$

Moreover, the growth rates of output, capital, and employment in the two sectors are

$$g^* = g_1^* = z^* = z_1^* = n + \frac{m_1}{\alpha_1}, \tag{32}$$

$$z_2^* = g^* - (1-\varepsilon)\omega, \tag{33}$$

$$g_2^* = g^* + \varepsilon\omega, \tag{34}$$

and

$$n_1^* = n, \quad n_2^* = n - (1 - \varepsilon)\omega, \tag{35}$$

where

$$\omega \equiv m_2 - \frac{\alpha_2}{\alpha_1} m_1.$$

Proof. From proposition 3, a competitive equilibrium satisfies (27)–(29). A CGP requires that $\lim_{t\to\infty} \dot{c}(t)/c(t)$ is constant; thus from the first equation of (27), $\eta(t)^{1/\varepsilon}\lambda(t)^{\alpha_1}\kappa(t)^{-\alpha_1}\chi(t)^{-\alpha_1}$ must be constant as $t \to \infty$. Consequently, as $t \to \infty$,

$$\frac{1}{\varepsilon}\frac{\dot{\eta}(t)}{\eta(t)} + \alpha_1\frac{\dot{\lambda}(t)}{\lambda(t)} - \alpha_1\frac{\dot{\kappa}(t)}{\kappa(t)} - \alpha_1\frac{\dot{\chi}(t)}{\chi(t)} = 0. \tag{36}$$

Differentiating (16) and (28) with respect to time and using the third equation of the system (27), we obtain expressions for $\dot{\eta}(t)/\eta(t)$, $\dot{\lambda}(t)/\lambda(t)$, and $\dot{\chi}(t)/\chi(t)$ in terms of $\dot{\kappa}(t)/\kappa(t)$. Substituting these into (36) and rearranging, we can express (36) as an autonomous first-order differential equation in $\kappa(t)$ as

$$\frac{\dot{\kappa}(t)}{\kappa(t)} = G(\kappa(t))\alpha_1\alpha_2(1-\varepsilon)\left(\frac{m_2}{\alpha_2} - \frac{m_1}{\alpha_1}\right), \tag{37}$$

where

$$G(\kappa(t)) \equiv \frac{[\Delta\kappa(t) + (1-\alpha_1)][1-\kappa(t)]}{\Delta\kappa(t) + \alpha_2(1-\alpha_1)}.$$

Clearly, $G(0) = 1/\alpha_2$, $G(1) = 0$, and $G'(\kappa) < 0$ for any κ. These observations together with assumption 2 imply that (37) has a unique solution with $\kappa(t) \to \kappa^* = 1$. Next, $\dot{\kappa}(t)/\kappa(t) = 0$ combined with (16) and (28) implies that $\dot{\eta}(t)/\eta(t) = \dot{\lambda}(t)/\lambda(t) = 0$. Moreover, given $\kappa(t) \to \kappa^* = 1$, (16) implies $\lambda(t) \to \lambda^* = 1$ and (28) implies $\eta(t) \to \eta^* = \gamma^{\varepsilon/(\varepsilon-1)}$. Equation (36) then implies that $\lim_{t\to\infty}\chi(t) = \chi^* \in (0, \infty)$ must also exist; thus $\dot{\chi}(t)/\chi(t) \to 0$. Now setting the first two equations in (27) equal to zero and using the fact that $\kappa^* = \lambda^* = 1$ and $\eta^* = \gamma^{\varepsilon/(\varepsilon-1)}$, we obtain (30) and (31). By construction, there are no other allocations with constant $\dot{c}(t)/c(t)$.

To derive equations (32)–(35), combine (26) with the result that $\dot{\chi}(t)/\chi(t) \to 0$, which implies $z^* = n + (m_1/\alpha_1)$. Moreover, $\kappa^* = \lambda^* = 1$ together with the market-clearing conditions (7) and (8)—holding as equalities—give $z_1^* = z^*$ and $n_1^* = n$. Finally, differentiating (3), (5), (13), and (14) and using the preceding results, we obtain (32)–(35) as unique solutions.

To complete the proof that this allocation is the unique CGP, we need to establish that it satisfies the transversality condition (29). This follows immediately from the fact that $\lim_{t\to\infty}\chi(t) = \chi^*$ exists and is finite combined with assumption 3, which implies that $\lim_{t\to\infty}\exp\{-[\rho - (1 - \theta)(m_1/\alpha_1) - n]t\} = 0$. QED

There are a number of important implications of this theorem. First, growth is nonbalanced, in the sense that the two sectors grow at different asymptotic rates (i.e., at different rates even as $t \to \infty$). The intuition for this result is more general than the specific parameterization of the model and is driven by the juxtaposition of factor proportion differences between sectors and capital deepening. In particular, suppose that there is capital deepening (which here is due to technological progress). Now, if both capital and labor were allocated to the two sectors in constant proportions, the more capital-intensive sector, sector 2, would grow faster than sector 1. The faster growth in sector 2 would reduce the price of sector 2, leading to a reallocation of capital and labor toward sector 1. However, this reallocation cannot entirely offset the greater increase in the output of sector 2, since, if it did, the change in prices that stimulated the reallocation would not take place. Therefore, growth must be nonbalanced. In particular, if $\varepsilon < 1$, capital and labor will be reallocated away from the more rapidly growing sector toward the more slowly growing sector. In this case, the more slowly growing sector, sector 1, becomes the asymptotically dominant sector and determines the growth rate of aggregate output as shown in equation (32). Note that sector 1 is the one growing more slowly because assumption 2, together with $\varepsilon < 1$, implies $m_1/\alpha_1 < m_2/\alpha_2$. Hence, $Y_1/Y_2 \to 0$. Appendix A shows that similar results apply when the converse of assumption 2 holds.

Second, the theorem shows that in the CGP the shares of capital and labor allocated to sector 1 tend to one (i.e., $\kappa^* = \lambda^* = 1$). Nevertheless, at all points in time both sectors produce positive amounts and both sectors grow at rates greater than the rate of population growth (so this limit point is never reached). Moreover, in the more interesting case in which $\varepsilon < 1$, equation (34) implies $g_2^* > g^* = g_1^*$, so that the sector that is shedding capital and labor (sector 2) is growing faster than the rest of the economy, even asymptotically. Therefore, the rate at which capital and labor are allocated away from this sector is determined in equilibrium to be exactly such that this sector still grows faster than the rest of the economy. This is the sense in which nonbalanced growth is not a trivial outcome in this economy (with one of the sectors shutting down), but results from the positive and differential growth of the two sectors.

Finally, it can be verified that the share of capital in national income and the interest rate are constant in the CGP. For example, under assumption 2, we have $\sigma_K^* = 1 - \alpha_1$. This implies that the asymptotic capital share in national income will reflect the capital share of the dominant sector. Also, again under assumption 2, the asymptotic interest rate is

$$r^* = (1 - \alpha_1)\gamma^{\varepsilon/(\varepsilon-1)}(\chi^*)^{-\alpha_1} - \delta.$$

These results are the basis of the claim in the introduction that this economy features both nonbalanced growth at the sectoral level and aggregate growth consistent with the Kaldor facts. In particular, the CGP matches both the Kaldor facts and generates unequal growth between the two sectors. However, at the CGP one of the sectors has already become (vanishingly) small relative to the other. Therefore, this theorem does not answer the question of whether we can have a situation in which both sectors have nontrivial employment levels and the capital share in national income and the interest rate are approximately constant. This question is investigated quantitatively in the next section. Before doing so, we establish the stability of the CGP.

E. *Dynamics and Stability*

The previous subsection demonstrated that there exists a unique CGP with nonbalanced sectoral growth; that is, there is aggregate output growth at a constant rate together with differential sectoral growth and reallocation of factors of production across sectors. We now investigate whether the competitive equilibrium will approach the CGP. We focus on allocations in the neighborhood of the CGP, thus investigating only local (saddle-path) stability. Because there are two predetermined (state) variables, χ and κ, with initial values $\chi(0)$ and $\kappa(0)$, this type of stability requires the linearized system in the neighborhood of the asymptotic path to have a (unique) two-dimensional manifold of solutions converging to c^*, χ^*, and κ^*. The next theorem states that this is the case.

THEOREM 2. Suppose that assumptions 1–3 hold. Then the competitive equilibrium, given by (27), is locally (saddle-path) stable, in the sense that in the neighborhood of c^*, χ^*, and κ^*, there is a unique two-dimensional manifold of solutions that converge to c^*, χ^*, and κ^*.

Proof. Let us rewrite the system (27) in a more compact form as

$$\dot{x} = f(x), \tag{38}$$

where $x \equiv (c \quad \chi \quad \kappa)'$ is the transpose of the row vector $(c \quad \chi \quad \kappa)$. To investigate the dynamics of the system (38) in the neighborhood of the steady state, consider the linear system

$$\dot{z} = J(x^*)z,$$

where $z \equiv x - x^*$ and x^* is such that $f(x^*) = 0$, and $J(x^*)$ is the Jacobian of $f(x)$ evaluated at x^*. Differentiation and some algebra enable us to write this Jacobian matrix as

$$J(x^*) = \begin{pmatrix} 0 & a_{c\chi} & a_{c\kappa} \\ -1 & a_{\chi\chi} & a_{\chi\kappa} \\ 0 & 0 & a_{\kappa\kappa} \end{pmatrix}.$$

Hence, the determinant of the Jacobian is $\det J(x^*) = -a_{\kappa\kappa}a_{c\chi}$, where

$$a_{\kappa\kappa} = -(1-\varepsilon)\left(m_2 - \frac{\alpha_2}{\alpha_1}m_1\right),$$

$$a_{c\chi} = -\gamma^{\varepsilon/(\varepsilon-1)}(\chi^*)^{-\alpha_1-1}\frac{\alpha_1(1-\alpha_1)}{\theta}.$$

The above expressions show that both $a_{\kappa\kappa}$ and $a_{c\chi}$ are (strictly) negative, since, in view of assumption 2, $\varepsilon \lesseqgtr 1 \Leftrightarrow m_2/\alpha_2 \gtreqless m_1/\alpha_1$. This establishes that $\det J(x^*) > 0$, so the steady state corresponding to the CGP is hyperbolic (i.e., all eigenvalues of the Jacobian evaluated at x^* have nonzero real parts) and thus the dynamics of the linearized system represent the local dynamics of the nonlinear system (see, e.g., Acemoglu 2008, theorem B.7). Moreover, either all the eigenvalues are positive or two of them are negative and one is positive. To determine which one of these two possibilities is the case, we look at the characteristic equation given by $\det(J(x^*) - \mathbf{v}I) = 0$, where $\mathbf{v}$ denotes the vector of the eigenvalues. This equation can be expressed as the following cubic in v, with roots corresponding to the eigenvalues:

$$(a_{\kappa\kappa} - v)[v(a_{\chi\chi} - v) + a_{\chi c}a_{c\chi}] = 0.$$

This expression implies that one of the eigenvalues is equal to $a_{\kappa\kappa}$ and thus negative, so there must be two negative eigenvalues. This establishes the existence of a unique two-dimensional manifold of solutions in the neighborhood of this CGP, converging to it. QED

This theorem establishes that the CGP is locally (saddle-path) stable, and when the initial values of capital, labor, and technology are not too far from the constant growth path, the competitive equilibrium converges to this CGP, with nonbalanced growth at the sectoral level and a constant capital share and interest rate at the aggregate.

III. A Simple Calibration

We now undertake an illustrative calibration to investigate whether the equilibrium dynamics generated by our model economy are broadly consistent with the patterns in U.S. data. For this exercise, we use data from the National Income and Product Accounts (NIPA) between 1948

TABLE 1
Industry Capital Shares

Industry	Sector	Capital Share
Educational services	1	.10
Management of companies and enterprises	1	.20
Health care and social assistance	1	.22
Durable goods	1	.27
Administrative and waste management services	1	.28
Construction	1	.32
Other services, except government	1	.33
Professional, scientific, and technical services	1	.34
Transportation and warehousing	1	.35
Accommodation and food services	2	.36
Retail trade	2	.42
Arts, entertainment, and recreation	2	.42
Finance and insurance	2	.45
Wholesale trade	2	.46
Nondurable goods	2	.47
Information	2	.53
Mining	2	.66
Utilities	2	.77

Note.—U.S. data from NIPA. Sector 1 comprises the low capital intensity industries and sector 2 comprises the high capital intensity industries. The capital intensity of each industry is the average capital share between 1987 and 2005, where capital share is computed as value added minus total compensation divided by value added.

and 2005. Industries are classified according to the North American Industrial Classification System (NAICS) at the 22-industry level of detail.[8] We use industry-level data for current-price value added (which we refer to as *value*), chain-type fixed-price quantity indices for value added (which we refer to as *quantity*), total number of employees,[9] total employee compensation, and fixed assets. To map our model to data, we classify industries into low and high capital intensity "sectors," each constituting approximately 50 percent of total value. Table 1 shows the average capital share for each industry and the sector classification.[10]

Figure 1 in the introduction depicts the evolution of relative quantity, value, and employment in these sectors and shows the more rapid

[8] Throughout, we exclude the government and private household sectors; agriculture, forestry, fishing, and hunting; and real estate and rental.

[9] In particular, we use full-time and part-time employees, since this is the only measure for which we have consistent NAICS data going back to 1948. The alternative classification system, Standard Industrial Claassification (SIC), does not enable us to extend data to 2005 and also reports the quantity indices for value added only back to 1977.

[10] NAICS data on compensation of employees are available only between 1987 and 2005 (all the other variables are available between 1948 and 2005). We therefore compute the capital share of each industry as the average capital share between 1987 and 2005. The average capital shares in the two sectors are relatively stable over time. In particular, the average capital share of sector 1 is 0.288 in 1987 and 0.290 in 2005, and the average capital share of sector 2 is 0.466 in 1987 and 0.499 in 2005.

growth of quantity in the capital-intensive sectors and the more rapid growth of value and employment in the less capital-intensive sectors.[11]

For our calibration, we take the initial year, $t = 0$, to correspond to the first year for which we have NIPA data for our sectors, 1948. In our model, $L(t)$ corresponds to total employment at time t, $K(t)$ to fixed assets, $Y_j(t)$ to the quantity of output in sector j, and $Y_j^N(t) \equiv p_j(t)Y_j(t)$ to the value of output in sector j.

Our model economy is fully characterized by 10 parameters, ρ, δ, θ, γ, ε, α_1, α_2, n, m_1, and m_2, and five initial values, $L(0)$, $K(0)$, $M_1(0)$, $M_2(0)$, and $\kappa(0)$. We choose these parameters and initial values as follows. First, we adopt the standard parameter values for the annual discount rate, $\rho = 0.02$, the annual depreciation rate, $\delta = 0.05$, and the annual (asymptotic) interest rate, $r^* = 0.08$.[12] We take the annual population growth rate $n = 0.018$ from the NIPA data on employment growth for 1948–2005 and choose the asymptotic growth rate to ensure that our calibration matches total output growth between 1948 and 2005 in the NIPA, which is 3.4 percent. This implies an asymptotic growth rate of $g^* = 0.033$. The initial values $L(0) = 40{,}336$ (in thousands) and $K(0) = 244{,}900$ (in millions of dollars) are also taken directly from the NIPA data for 1948. Next, our classification of industries leads to two "aggregate sectors" with average shares of labor in value added of 0.72 and 0.52. In terms of our model this implies $\alpha_1 = 0.72$ and $\alpha_2 = 0.52$.

An important parameter for our calibration is the elasticity of substitution between the two sectors. Although we do not have independent information on this variable, our model suggests a way of evaluating this elasticity. In particular, equation (11) implies the following relationship between value and quantity ratios in the two sectors:

$$\log\left[\frac{Y_1^N(t)}{Y_2^N(t)}\right] = \log\left(\frac{\gamma}{1-\gamma}\right) + \frac{\varepsilon - 1}{\varepsilon}\log\left[\frac{Y_1(t)}{Y_2(t)}\right]. \tag{39}$$

We can therefore estimate $(\varepsilon - 1)/\varepsilon$ by regressing the log of the ratio of the values of the two sectors on the log of the ratio of the quantities. Since our focus is on medium-run frequencies (rather than business cycle fluctuations), we use the Hodrick-Prescott filter to smooth both the dependent and the independent variables (with smoothing weight 1,600) and use the smoothed variables to estimate (39). This simple regression yields an estimate of $\varepsilon \simeq 0.76$ (and a two standard error

[11] Online App. B provides more details on data sources and construction. It also shows that the general patterns in the U.S. data, in particular those plotted in fig. 1, are robust to changing the cutoff between high and low capital intensity industries.

[12] These numbers are the same as those used by Barro and Sala-i-Martin (2004) in their calibration of the baseline neoclassical model.

confidence interval of [0.73, 0.79]). We therefore choose $\varepsilon = 0.76$ for our benchmark calibration. We also choose the parameter γ to ensure that equation (39) holds at $t = 0$.

Throughout, motivated by our estimate of ε reported in the previous paragraph, the existing evidence discussed in note 3, and the pattern shown in figure 1 indicating that employment and value grow more in the less capital-intensive sector, we focus on the case in which $\varepsilon < 1$ and $m_1/\alpha_1 < m_2/\alpha_2$ (case i of assumption 2). In particular, for our benchmark calibration we set $m_1 = m_2$ (even though, as we will see below, higher values of m_2 improve the fit of the model to U.S. data). Since in this case sector 1 is the asymptotically dominant sector, the asymptotic growth rate of output is $g^* = n + (m_1/\alpha_1)$. The above-mentioned values of g^*, n, and α_1 imply $m_1 = m_2 = 0.0108$. The growth rate of output also pins down the intertemporal elasticity of substitution. In particular, the Euler equation (27) together with (26) yields $g^* = [(r^* - \rho)/\theta] + n$, which implies $\theta = 4$ and results in a reasonable elasticity of intertemporal substitution of 0.25.

This leaves us with the initial values for κ, M_1, and M_2. First, notice that equation (15) at time 0 can be rewritten as

$$\kappa(0) = \left[1 + \left(\frac{1 - \alpha_2}{1 - \alpha_1}\right)\frac{Y_2^N(0)}{Y_1^N(0)}\right]^{-1}. \tag{40}$$

This equation together with the 1948 levels of values in the two sectors from the NIPA data, $Y_1^N(0)$ and $Y_2^N(0)$, gives $\kappa(0) = 0.32$. Equation (16) then gives the initial value of relative employment as $\lambda(0) = 0.52$.[13] It is also worth noting that in addition to the parameters and the initial values necessary for our calibration, the numbers we are using also pin down the initial interest rate and the capital share in national income as $r(0) = 0.095$ and $\sigma_K(0) = 0.39$.[14] Moreover, since sector 1 is the asymptotically dominant sector, the asymptotic capital share in national income and the interest rate are determined as $\sigma_K^* = 1 - \alpha_1 = 0.28$ and $r^* = 0.08$. This implies that, by construction, both the interest rate and the capital share must decline at some point along the transition path. A key question concerns the speed of these declines. If this happened

[13] We can also compute the empirical counterparts of $\lambda(0)$ and $\kappa(0)$ from the NIPA data using employment and capital in the two-sector aggregates. These give the values of 0.49 and 0.50, which are similar, though clearly not identical, to the implied values we use. The fact that the theoretically implied values of $\lambda(0)$ and $\kappa(0)$ differ from their empirical counterparts is not surprising, since we are assuming that the U.S. economy can be represented by two sectors with Cobb-Douglas production functions and with their output being combined with a constant elasticity of substitution. Naturally, this is at best an approximation, and in the data, sectoral factor intensities are not constant over time.

[14] The reason is that the capital share and interest rate are functions of κ only; just combine (20) and (22) with (3) and (15).

at the same frequency as the change in the composition of employment and capital across the two sectors, then the model would not generate a pattern that is simultaneously consistent with nonbalanced sectoral growth and the aggregate Kaldor facts. We will see that this is not the case and that the model's implications are broadly consistent with both sets of facts.

Finally, given $\kappa(0)$ and $\lambda(0)$, the NIPA data imply values for $L_1(0)$, $L_2(0)$, $K_1(0)$, and $K_2(0)$, and we also have the values for $Y_1(0)$ and $Y_2(0)$ directly from the NIPA. We then obtain the remaining initial values, $M_1(0)$ and $M_2(0)$, from equation (5). This completes the determination of all the parameters and initial conditions of our model. We then compute the time path of all the variables in our model using two different numerical procedures (both giving equivalent results).[15]

Figure 2 shows the results of our benchmark calibration with the parameter values described above. In particular, in this benchmark we have $\varepsilon = 0.76$ and $m_1 = m_2 = 0.0108$. The four panels depict relative employment in sector 1 (λ), relative capital in sector 1 (κ), the interest rate (r), and the capital share in national income (σ_K) for the first 150 years (in terms of data, corresponding to 1948–2098).

A number of features are worth noting. First, for the first 150 years, there is significant reallocation of capital and labor away from the more capital-intensive sector toward the less capital-intensive sector, sector 1, and the economy is far from the asymptotic equilibrium with $\kappa = \lambda = 1$. In fact, the economy takes a very long time, over 5,000 years, to reach the asymptotic equilibrium. This illustrates that our model economy generates interesting and relatively slow dynamics, with a significant amount of structural change. Second, while there is nonbalanced growth at the sectoral level, the interest rate and the capital share remain approximately constant. The interest rate shows an early decline from about 9.5 percent to 9 percent, which largely reflects the initial consumption dynamics. It then remains around 9 percent. The capital share

[15] In particular, we first return to the two-dimensional, nonautonomous system of equations in c and χ (rather than the three-dimensional representation used for theoretical analysis in proposition 3). This two-dimensional system has one state and one control variable. Following Judd (1998, chap. 10), we discretize these differential equations using the Euler method to obtain a system of first-order difference equations in $c(t)$ and $\chi(t)$. This system can in turn be transformed into a second-order, nonautonomous system in $\chi(t)$, which is easier to work with. We then compute the numerical solution to this second-order difference equation either by using a shooting algorithm or by minimizing the squared residuals. Our main numerical method is to use a basic shooting algorithm starting with the initial value of $\chi(0)$ and then guess and adjust the value for $\chi(1)$ to ensure convergence to the asymptotic CGP. The second numerical procedure is to choose a polynomial form for the normalized capital stock as $\chi(t) = \Gamma(t; \boldsymbol{\theta})$, where $\boldsymbol{\theta}$ represents the parameters of the polynomial. We estimate $\boldsymbol{\theta}$ by minimizing the squared residuals of the difference equations using nonlinear least squares (see, e.g., Judd 1998, chap. 4) and then compute $c(t)$ and $\kappa(t)$. The two procedures give almost identical results, and throughout we report the results from the shooting algorithm.

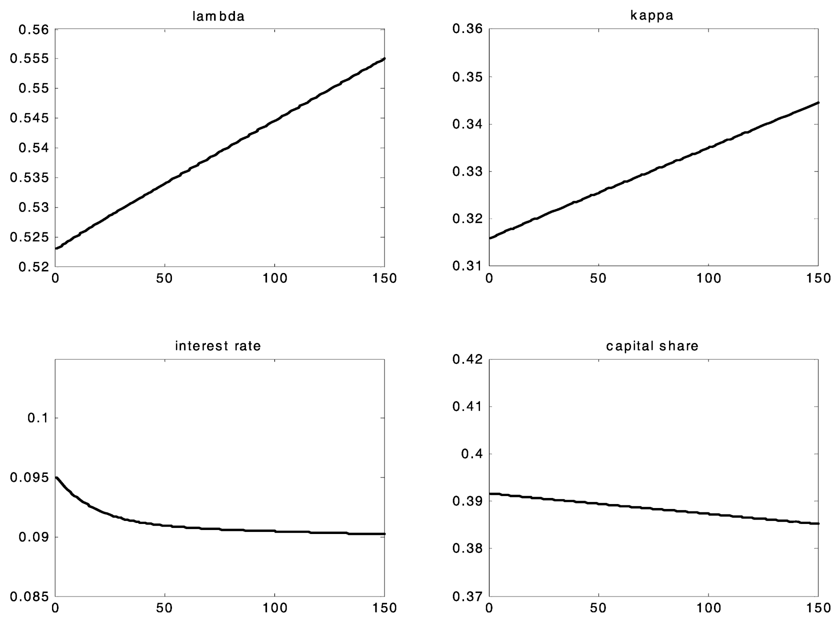

FIG. 2.—Behavior of κ, λ, r, and σ_K in the benchmark calibration with $\varepsilon = 0.76$ and $m_2 = 0.0108$. See the text for further details.

shows a very slight decline over the 150 years. This relative constancy of the interest rate and the capital share is particularly interesting since, as noted above, we know that both variables have to decline at some point to achieve their asymptotic values of $r^* = 0.08$ and $\sigma_K^* = 0.28$. Nevertheless, our model calibration implies more rapid structural change than the speed of these aggregate changes, and thus over the horizon of about 150 years, there is little change in the interest rate and the capital share, whereas there is significant reallocation of labor and capital across sectors.

These patterns are further illustrated in table 2, which shows the U.S. data and the numbers implied by our benchmark calibration between 1948 and 2005 for the relative quantity and relative employment of the high versus low capital intensity sectors as well as the aggregate capital share in national income (in terms of the model, $t = 0$ is taken to correspond to 1948; thus $t = 57$ gives the values for 2005). Columns 1 and 2 of this table confirm the patterns shown in figure 1 in the introduction: quantity grows faster in high capital intensity industries and employment grows faster in low capital intensity industries. Columns 3 and 4 show that the benchmark calibration is broadly consistent with this pattern. In particular, while in the data Y_2/Y_1 increases by about 19

TABLE 2
DATA AND MODEL CALIBRATION, 1948–2005

	U.S. DATA		BENCHMARK CALIBRATION: $\varepsilon = .76$, $m_2 = .0108$	
	1948 (1)	2005 (2)	1948 (3)	2005 (4)
Y_2/Y_1	.85	1.01	.85	1.00
L_2/L_1	1.03	.80	.91	.87
σ_K	.40	.40	.39	.39

NOTE.—U.S. data from NIPA. Classifications and calibration described in the text.

percent between 1948 and 2004, the model leads to an increase of about 17 percent. In the data, L_2/L_1 declines by about 33 percent.[16] In the model, the implied decline is in the same direction but is considerably smaller, about 5 percent.[17] However, the evolutions of the capital share in the data and in the model are very similar. In the data, the capital share declines from 0.398 to 0.396, whereas in the model it declines from 0.392 to 0.389.[18]

Tables 3 and 4 show alternative calibrations of our model economy. In table 3, we consider different values for the elasticity of substitution, ε, and table 4 considers the implications of different growth rates of the capital-intensive sector, m_2.

The results for different values of ε in table 3 are generally similar to those of the benchmark model. The most notable feature is that when ε is smaller, for example, $\varepsilon = 0.56$ or $\varepsilon = 0.66$ instead of the benchmark value of $\varepsilon = 0.76$, there are greater changes in relative employments. With $\varepsilon = 0.56$, the decline in the relative employment of the capital-intensive sector is approximately 9 percent instead of 5 percent in the benchmark. The opposite happens when ε is larger and there is even less change in relative employment. This is not surprising in view of the fact that, as noted in note 6, with $\varepsilon = 1$ there would be no reallocation of capital and labor.

The broad patterns implied by different values of m_2 in table 4 are

[16] Equivalently, in the data the share of sector 1 in total employment, L_1/L, increases from 0.49 in 1948 to 0.56 in 2005. The corresponding increase in our benchmark calibration is from 0.52 to 0.54.

[17] Note that the initial values of L_2/L_1 are not the same in the data and in the benchmark model since, as remarked above, we chose the sectoral allocation of labor implied by the model given the relative values of output in the two sectors (recall eq. [40]).

[18] One reason why our model accounts for only a fraction of the structural change in the U.S. economy may be that it focuses on a specific dimension of structural change: the reallocation of output between sectors with different capital intensities. In practice, a significant component of structural change is associated with the reallocation of output across agriculture, manufacturing, and services. In addition, our model does not allow for changes in sectoral factor intensities over time.

TABLE 3
Data and Model Calibration, 1948–2005 (Robustness I)

	U.S. Data		Model $\varepsilon = .56$, $m_2 = .0108$		Model $\varepsilon = .66$, $m_2 = .0108$		Model $\varepsilon = .86$, $m_2 = .0108$	
	1948 (1)	2005 (2)	1948 (3)	2004 (4)	1948 (5)	2004 (6)	1948 (7)	2004 (8)
Y_2/Y_1	.85	1.01	.85	.96	.85	.98	.85	1.02
L_2/L_1	1.03	.80	.91	.83	.91	.85	.91	.89
σ_K	.40	.40	.39	.39	.39	.39	.39	.39

Note.—U.S. data from NIPA. Classifications and calibration described in the text.

TABLE 4
Data and Model Calibration, 1948–2005 (Robustness II)

	U.S. Data		Model $\varepsilon = .76$, $m_2 = .0098$		Model $\varepsilon = .76$, $m_2 = .0118$		Model $\varepsilon = .76$, $m_2 = .0128$	
	1948 (1)	2005 (2)	1948 (3)	2004 (4)	1948 (5)	2004 (6)	1948 (7)	2004 (8)
Y_2/Y_1	.85	1.01	.85	.96	.85	1.05	.85	1.09
L_2/L_1	1.03	.80	.91	.88	.91	.86	.91	.85
σ_K	.40	.40	.39	.39	.39	.39	.39	.39

Note.—U.S. data from NIPA. Classifications and calibration described in the text.

also similar to the results of the benchmark calibration. It is noteworthy, however, that if m_2 is taken to be greater, for example, $m_2 = 0.0128$, the fit of the model to the data is improved. For example, in this case, there is a somewhat larger change in relative employment levels and also a larger decline in the relative quantity of the capital-intensive sector. In contrast, when m_2 is smaller than the benchmark, the changes in Y_2/Y_1 and L_2/L_1 are somewhat less pronounced.

Overall, our calibration exercises indicate that the mechanism proposed in this paper can generate changes in the sectoral composition of output that are broadly comparable with the changes we observe in the U.S. data and changes in relative employment levels that are in the same direction as in the data, though quantitatively smaller. It is notable that during this process of structural change the capital share in national income remains approximately constant.

IV. Conclusion

We proposed a model in which the combination of factor proportion differences across sectors and capital deepening leads to a nonbalanced pattern of economic growth. We illustrated the main economic forces using a tractable two-sector growth model, where there is a constant elasticity of substitution between the two sectors and Cobb-Douglas production technologies in each sector. We characterized the constant growth path and equilibrium (Pareto-optimal) dynamics in the neighborhood of this growth path. We showed that even though sectoral growth is nonbalanced, the behavior of the interest rate and the capital share in national income are consistent with the Kaldor facts. In particular, asymptotically the two sectors still grow at different rates, whereas the interest rate and the capital share are constant.

The main contribution of the paper is theoretical, demonstrating that the interaction between capital deepening and factor proportion differences across sectors will lead to nonbalanced growth, while being still consistent with the aggregate Kaldor facts. We also presented a simple calibration of our baseline model, which showed that the equilibrium path exhibits sectoral employment and output shares changing significantly, whereas the aggregate capital share and the interest rate remain approximately constant. Moreover, the magnitudes implied by this simple calibration are comparable to, though somewhat smaller than, the sectoral changes observed in postwar U.S. data. A full investigation of whether the mechanism suggested in this paper plays a first-order role in nonbalanced growth in practice is an empirical question left for future research. It would be particularly useful to combine the mechanism proposed in this paper with nonhomothetic preferences and estimate a structural version of the model with multiple sectors using data from the United States or the Organization for Economic Cooperation and Development.

Appendix A

Proof of Proposition 3

The equivalence of the solutions to (SP) and (SP′) follows from the discussion in the text, whereas the equivalence between (SP) and competitive equilibria follows from the first and second welfare theorems. Next, consider the maximization (SP′). This corresponds to a standard optimal control problem. Moreover, the objective function is strictly concave, the constraint set is convex, and the state variable, $K(t)$, is nonnegative, so that the Arrow sufficiency theorem (e.g., Acemoglu 2008, theorem 7.14) implies that an allocation that satisfies the Pontryagin maximum principle and the transversality condition (29) uniquely achieves the maximum of (SP′). Thus we need to show only that equations (27)–(29) are equivalent to the maximum principle and the transversality condition. The Hamiltonian for (SP′) takes the form

$$H(\tilde{c},\ K,\ \mu) = \exp\left[-(\rho - n)t\right]\frac{\tilde{c}(t)^{1-\theta} - 1}{1-\theta}$$
$$+ \mu(t)[\Phi(K(t),\ t) - \delta K(t) - \exp(nt)L(0)\tilde{c}(t)],$$

with $\mu(t)$ denoting the costate variable. Inspection of (SP′) shows that paths that reach zero consumption or zero capital stock at any finite t cannot be optimal; thus we can focus on interior solutions and write the maximum principle as

$$H_{\tilde{c}}(\tilde{c},\ K,\ \mu) = \exp[-(\rho - n)t]\,\tilde{c}(t)^{-\theta} - \mu(t)\exp(nt)L(0) = 0,$$

$$H_K(\tilde{c},\ K,\ \mu) = \mu(t)[\Phi_K(K(t),\ t) - \delta] = -\dot{\mu}(t) \tag{A1}$$

whenever the optimal control $\tilde{c}(t)$ is continuous. Combining these two equations, we obtain the Euler equation for consumption growth as

$$\frac{d\tilde{c}(t)/dt}{\tilde{c}(t)} = \frac{1}{\theta}[\Phi_K(K(t),\ t) - \delta - \rho]. \tag{A2}$$

Moreover, equations (13) and (14) imply

$$\Phi(K(t),\ t) = \gamma\eta(t)^{1/\varepsilon}M_1(t)L(t)^{\alpha_1}\lambda(t)^{\alpha_1}K(t)^{1-\alpha_1}\kappa(t)^{1-\alpha_1} \tag{A3}$$

and

$$\Phi_K(K(t),\ t) = (1-\alpha_1)\gamma\eta(t)^{1/\varepsilon}M_1(t)L(t)^{\alpha_1}\lambda(t)^{\alpha_1}K(t)^{-\alpha_1}\kappa(t)^{-\alpha_1}. \tag{A4}$$

The law of motion of technology in (6) together with the normalization in (26) implies $\dot{c}(t)/c(t) = [d\tilde{c}(t)/dt]/\tilde{c}(t) - (m_1/\alpha_1)$. Also from (26), we have $\chi(t)^{-\alpha_1} \equiv M_1(t)L(t)^{\alpha_1}K(t)^{-\alpha_1}$. Using the previous two expressions and substituting (A4) into (A2), we obtain the first equation in (27). Next, again using (26) to write

$$\frac{\dot{\chi}(t)}{\chi(t)} = \frac{\dot{K}(t)}{K(t)} - n - \frac{m_1}{\alpha_1}$$

and substituting for $\dot{K}(t)$ from (25) and for $\Phi(K(t),\ t)$ from (A3), we obtain the second equation in (27). Notice also that both of these equations depend on $\kappa(t)$. To obtain the law of motion of $\kappa(t)$, differentiate (15) and then use (5) and (16). Here $\kappa(0)$ is also taken as given because for given $K(0)$, (15) uniquely pins down $\kappa(0)$.

Finally, the transversality condition of (SP′) requires

$$\lim_{t\to\infty}\{\exp[-(\rho - n)t]\mu(t)K(t)\} = 0.$$

Combining (26) with (A1) shows that this condition is equivalent to (29).

Results with the Converse of Assumption 2

We stated and proved proposition 3 and theorems 1 and 2 under assumption 2. This assumption was imposed only to reduce notation. When it is relaxed (and its converse holds), sector 1 is no longer the asymptotically dominant sector, and a different type of normalization than that in (26) is necessary. In particular, the converse of assumption 2 is as follows.

ASSUMPTION 2′. Either (i) $m_1/\alpha_1 > m_2/\alpha_2$ and $\varepsilon < 1$, or (ii) $m_1/\alpha_1 < m_2/\alpha_2$ and $\varepsilon > 1$.

It is straightforward to see that in this case sector 2 will be the asymptotically dominant sector. We therefore adopt a parallel normalization with

$$c(t) \equiv \frac{\tilde{c}(t)}{M_2(t)^{1/\alpha_2}},\quad \chi(t) \equiv \frac{K(t)}{L(t)M_2(t)^{1/\alpha_2}}. \tag{A5}$$

Given this normalization, it is straightforward to generalize proposition 3 and theorems 1 and 2.

PROPOSITION 4. Suppose that assumptions 1 and 2′ hold. Then a competitive equilibrium satisfies the following three differential equations:

$$\frac{\dot{c}(t)}{c(t)} = \frac{1}{\theta}[(1-\alpha_2)\gamma\eta(t)^{1/\varepsilon}\lambda(t)^{\alpha_2}\kappa(t)^{-\alpha_2}\chi(t)^{-\alpha_2} - \delta - \rho] - \frac{m_2}{\alpha_2},$$

$$\frac{\dot{\chi}(t)}{\chi(t)} = \lambda(t)^{\alpha_2}\kappa(t)^{1-\alpha_2}\chi(t)^{-\alpha_2}\eta(t) - \chi(t)^{-1}c(t) - \delta - n - \frac{m_2}{\alpha_2},$$

$$\frac{\dot{\kappa}(t)}{\kappa(t)} = \frac{[1-\kappa(t)]\{-\Delta[\dot{\chi}(t)/\chi(t)] + m_1 - (\alpha_1/\alpha_2)m_2\}}{(1-\varepsilon)^{-1} - \Delta[\kappa(t) - \lambda(t)]}, \tag{A6}$$

where

$$\eta(t) \equiv \gamma^{\varepsilon/(\varepsilon-1)}\left\{1 + \left(\frac{1-\alpha_2}{1-\alpha_1}\right)\left[\frac{1-\kappa(t)}{\kappa(t)}\right]\right\}^{\varepsilon/(\varepsilon-1)}, \tag{A7}$$

with initial conditions $\chi(0)$ and $\kappa(0)$, and also satisfies the transversality condition

$$\lim_{t\to\infty}\exp\left\{-\left[\rho - \frac{(1-\theta)m_2}{\alpha_2} - n\right]t\right\}\chi(t) = 0. \tag{A8}$$

Moreover, any allocation that satisfies these conditions is a competitive equilibrium.

Proof. The proof is analogous to that of proposition 3 and is omitted.

THEOREM 3. Suppose that assumptions 1, 2′, and 3 hold. Then there exists a unique CGP in which consumption per capita grows at the rate $g_c^* = m_2/\alpha_2$, and $\kappa^* = 0$,

$$\chi^* = \left[\frac{(\theta m_2/\alpha_2) + \rho + \delta}{(1-\gamma)^{\varepsilon/(\varepsilon-1)}(1-\alpha_2)}\right]^{-1/\alpha_2},$$

and

$$c^* = (1-\gamma)^{\varepsilon/(\varepsilon-1)}(\chi^*)^{1-\alpha_2} - \chi^*\left(\delta + n + \frac{m_2}{\alpha_2}\right).$$

Moreover, the growth rates of output, capital, and employment in the different sectors are given by

$$g^* = g_2^* = z_2^* = n + \frac{m_2}{\alpha_2}, \quad z_1^* = g^* - (1-\varepsilon)\tilde{\omega},$$

$$g_1^* = g^* + \varepsilon\tilde{\omega}, \quad n_2^* = n, \quad n_1^* = n - (1-\varepsilon)\tilde{\omega},$$

where

$$\tilde{\omega} \equiv m_1 - \alpha_1\frac{m_2}{\alpha_2} > 0.$$

Proof. The proof is analogous to that of theorem 1 and is omitted.

It can also be easily verified that in this case $\sigma_K^* = 1 - \alpha_2$ and $r^* = (1 - \alpha_2)\gamma^{\varepsilon/(\varepsilon-1)}(\chi^*)^{-\alpha_2} - \delta$.

THEOREM 4. Suppose that assumptions 1, 2′, and 3 hold. Then the nonlinear system (27) is locally (saddle-path) stable, in the sense that in the neighborhood of c^*, χ^*, and κ^*, there is a unique two-dimensional manifold of solutions that converge to c^*, χ^*, and κ^*.

Proof. The proof follows that of theorem 2. Once again linearizing the dynamics around the CGP, we obtain $\dot{z} = J(x^*)z$, with $z \equiv x - x^*$ and x^* such that $f(x^*) = 0$, where $J(x^*)$ is the Jacobian of $f(x)$ evaluated at x^*. The determinant of the Jacobian is again $\det J(x^*) = -a_{\kappa\kappa}a_{c\chi}$, where

$$a_{\kappa\kappa} = -(1-\varepsilon)\left(m_1 - \alpha_1\frac{m_2}{\alpha_2}\right),$$

$$a_{c\chi} = -(1-\gamma)^{\varepsilon/(\varepsilon-1)}(\chi^*)^{-\alpha_2-1}\frac{\alpha_2(1-\alpha_2)}{\theta}.$$

Once again $a_{\kappa\kappa}$ and $a_{c\chi}$ are strictly negative, since, under assumption 2′, $\varepsilon \gtrless 1 \Leftrightarrow m_2/\alpha_2 \gtrless m_1/\alpha_1$. Therefore, as in the proof of theorem 2, $\det J(x^*) > 0$ and the steady state is hyperbolic. The same argument as in that proof shows that there must be two negative eigenvalues and establishes the result. QED

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