8. Note that two different lump-sum charges are involved: one to recover the cost of production subsidies ($q - 1$) and another to compensate dealers for their opportunity cost of not being producers. See Hayes (1984) for an alternative justification of two-part tariffs under competition given risk-sharing externalities.

References

Diamond, P. 1982. "Aggregate Demand Management in Search Equilibrium." *Journal of Political Economy* 90:881–94.

Diamond, P., and D. Fudenberg. 1982. "An Example of Aggregate Demand Management in Search Equilibrium." Economics Department, M.I.T.

Diamond, P., and J. Yellen. 1983. "The Distribution of Inventory Holdings in a Pure Exchange Barter Search Economy." Economics Department, M.I.T.

Fuchs, G., and G. Laroque. 1974. "Continuity of Equilibria for Economies with Vanishing External Effects." *Journal of Economic Theory* 9:1–22.

Harris, M., and R. Townsend. 1981. "Resource Allocation under Asymmetric Information." *Econometrica* 49:33–64.

Hayes, B. 1984. "Competition and Two-Part Tariffs." Department of Managerial Economics and Decision Science, Kellogg Graduate School of Management, Northwestern University.

Mortensen, D. 1976. "Job Matching under Imperfect Information." In *Evaluating the Labor Market Effects of Social Programs*, edited by O. Ashenfelter and J. Blum. Princeton, N.J.: Princeton University Press.

Myerson, R. 1979. "Incentive Compatibility and the Bargaining Problem." *Econometrica* 47:61–74.

Rubinstein, A., and A. Wolinsky. 1984. "Equilibrium in a Market with Sequential Bargaining." ICERD, London School of Economics and Political Science.

Townsend, R. 1982. "Theories of Intermediated Structures." Unpublished paper, Carnegie-Mellon University.

V

Circulating Private Debt: An Example with a Coordination Problem

Robert Townsend and
Neil Wallace

We use a model of pure, intertemporal exchange with spatially and informationally separated markets to explain the existence of private securities that circulate and, hence, play a prominent role in exchange. The model, which utilizes a perfect-foresight equilibrium concept, implies that a Schelling-type coordination problem can arise. It can happen that the amounts of circulating securities that are required to support an equilibrium and that are issued at the same time in informationally separated markets must satisfy restrictions not implied by individual maximization and market clearing in each market separately.

1. Introduction

A seemingly central observation for monetary economics is that some objects—often referred to as monies—appear in exchange much more frequently than other objects.[1] In this paper, we present and study a model that generates a version of this observation for private securities; in the model, some securities get traded frequently, or circulate, whereas others do not. In this and other respects, the securities in our model resemble historically observed bills of exchange.

The model that we use to explain the existence of circulating securities is one of intertemporal trade in spatially and informationally separated markets. The assumption that trade occurs in separate markets has been used by Ostroy (1973), Ostroy-Starr (1974), and Harris (1979) to model the transaction patterns of commodities and by Townsend (1980) to model that of fiat money. We use it here to model the transaction patterns of private securities. Adopting that assumption is consistent with the general view that fruitful theories of the pattern of exchange and of media of exchange require settings in which it is somehow difficult to carry out exchange.

The difficulty of carrying out exchange under our assumptions shows up in two distinct ways. One is market incompleteness: it can happen that some physically feasible and beneficial trades cannot be accomplished. This, as we will see, is an obvious implication of our assumptions and does not, therefore, require extended discussion. The other is much less obvious: it turns out to resemble the problem in a Schelling pure coordination game.

As described by Schelling (1960), a *pure coordination game* is one in which there is no communication and no conflict and in which the problem facing the players is to choose strategies that are coordinated. In our model, it can happen that the quantities of securities that are required to support an equilibrium and that are issued by individuals at the same time in spatially and informationally separated markets must satisfy restrictions not implied by individual maximization and by market clearing in each separate market. In other words, the utility-maximizing choices of quantities of securities, the strategies of individuals, are not in general unique but must somehow be coordinated across informationally separated markets if they are to be consistent with the existence of an equilibrium.

This (coordination) problem arises only in some versions of our model. In fact, there is a close connection between its appearance and that of circulating securities; the problem does not appear unless there is a role for circulating securities. In this sense, the model is consistent with the widely held view that problems—perhaps in the form of chaotic conditions—sometimes arise in credit markets with unregulated issue of private securities that play an important role in exchange.[2] Although this view is widely held, there are few, if any, other models that provide an interpretation of it.[3]

Our presentation is organized as follows. We begin in Section 2 with an introductory description of our model and of the example we use to display the coordination problem. In Section 3, we describe a somewhat general class of environments, introduce our notation, and formally describe our equilibrium concept. In Section 4, we establish for our example equivalence between our equilibrium and that implied by complete date-location contingent markets with complete participation. In Section 5, we use that equivalence to display the transaction patterns of the securities issued. Finally, in Section 6, we use it to display the coordination problem.

2. Preliminary Description: Some Example Economies

We study setups with a finite number of finite-lived people who meet deterministically at prescribed locations and at prescribed times. The example

we focus on is an economy of four people who meet according to the pattern laid out in table V-1. Although the coordination problem arises only in the four-period version of this setup, in this section we also comment on the two- and three-period versions of it. In this example, at date 1, persons 1 and 2 are together at location 1, whereas persons 3 and 4 are together at location 2. Persons 1 and 4 always stay at those locations, whereas persons 2 and 3 switch locations each period.

As regards commodities or consumption goods, we assume there is one commodity for each location-date combination. Equivalently, we assume that there is one good that is indexed by location and date. The setup is pure exchange in the usual sense: goods indexed by one location-date combination cannot be transformed into goods indexed by another location-date combination; that is, there is no transportation, production, or storage technology for goods. Letting J denote the number of locations and T the number of dates, the commodity space has dimension JT. We assume that each person gets utility from commodities and has positive endowments of commodities in a proper subset whose elements correspond to the location-date combinations that the person visits. In figure V-1, we indicate by X's the subspace of the $2T$ commodity space that is relevant in the above sense for each of the persons in the four-person, four-period economy.

As regards private securities, we let the spatial separation limit trades in securities in what seems to be a natural way. First, at a particular time, a person can only trade securities with someone he or she meets. Second, although securities can be transported, they can move only with a person. Finally, we do not allow people to renege on their debts or to counterfeit others' debts. Securities or debts in our model take the form of promises to pay stated amounts of goods that are date and location specific. We assume that if the promise is presented at the relevant date and location, then it is honored.

To suggest how these rules and our spatial separation work, we briefly describe some of their implications for the table V-1 example.

If $T = 2$—that is, if the economy lasts only two periods—then no trade

Table V-1. Who Meets Whom When

	Location	
Date	1	2
1	(1,2)	(3,4)
2	(1,3)	(2,4)
3	(1,2)	(3,4)
4	(1,3)	(2,4)

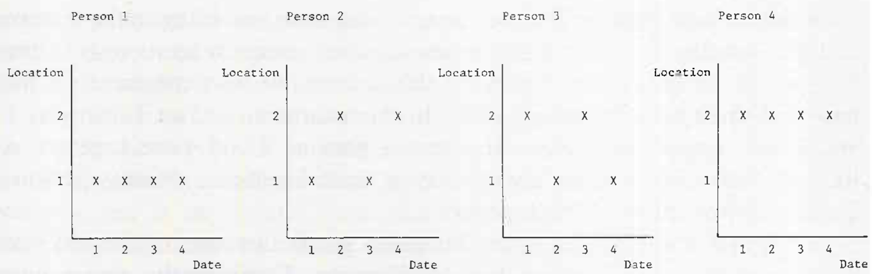

Figure V-1. Relevant commodity subspaces in the economy shown in table V-1.

is possible in the table V-1 economy under our security trading rules. For example, person 1 cannot sell a promise to person 2 because person 2 can neither redeem it at date 2 nor pass it on to person 4, who has no use for it at date 2, the assumed last date. Note in this connection that there is a complete absence of double coincidence in the table V-1, $T = 2$ example; as shown in figure V-1, for $T = 2$ no pair of persons has endowments and cares about a common two-dimensional subspace of the commodity space. From what we have just seen, the kinds of private securities we allow do not at all overcome this particular absence of double coincidence. Note, by the way, that there is potentially something to overcome in the sense that there can exist redistributions of the endowments that give rise to allocations that are Pareto-superior to the endowment allocation. Put differently, if all four people were together at some time zero and traded in complete (location and date contingent) markets, something we rule out, then the endowment would not necessarily be a competitive equilibrium.

If $T = 3$ in the table V-1 setup, our rules are consistent with some trade in private securities. It is easy to see, however, that only the following kinds of securities get traded: persons who meet at date 1 can trade debts due at date 3 when they meet again. For example, person 1 can issue a promise to pay a good at location 1, date 3, a promise that person 2 holds until he or she again meets person 1. Thus, such securities do not circulate; they do not get traded in a secondary market and are not used to make third-party payments. Corresponding to this noncirculating characteristic is the fact that such securities do no more than accomplish trades for which there is a double coincidence. For example, as is clear from figure V-1, persons 1 and 2 have a double coincidence between goods at location 1, date 1, and location 1, date 3. Note also that there remains a degree of market incompleteness in the $T = 3$ economy; in particular, the two goods at date 2 cannot be traded.

If $T = 4$ in the table V-1 economy, then our security trading rules are consistent not only with the existence of several noncirculating securities,

but also with the existence of several circulating securities. At date 1, person 1 can issue to person 2 a promise to pay a good at location 1, date 4. This promise can be redeemed by being passed from person 2 to person 4 at date 2, from person 4 to person 3 at date 3, and from person 3 to person 1—the issuer—at date 4. Similarly, each of persons 2, 3, and 4 can issue at date 1 a promise of a good at date 4 at some location. Whether such securities are in fact issued is one of the questions taken up below.

3. Debt Equilibria in the General Spatial-Separation Setup

In this section, we describe the general class of economies under consideration and define a competitive debt equilibrium.

We assume an economy with G persons, each of whom lives T periods. At each time t, each person g can be paired with some other person or with no one. These pairings occur at (isolated) locations. Thus, we assume that person g is assigned to some location i at each time t and that in that location there either is or is not a single trading partner. We let there be $J \geq G/2$ locations.

If person g is in location i at time t, then he or she is endowed with some positive number of units, w^g_{it}, of the consumption good at location i, date t. For other location-date combinations, the endowment is zero. Let w^g denote the entire JT dimensional endowment vector for person g. Also let c^g_{it} denote the nonnegative number of units of location i-date t consumption of person g and let c^g denote the entire JT dimensional consumption vector for person g. Preferences of each person g are described by a utility function $U^g(c^g)$ that is continuous, concave, and strictly increasing in the T-dimensional subspace that is relevant for g.

We restrict attention to securities that can be redeemed. Thus, if d^f_{st}, which is nonnegative, denotes securities issued by person f at time s to pay d^f_{st} units of the consumption good where f will be at time t, we consider only triplets (f,s,t) with the property that there is a path or chain of pairings leading from where f is at s to where f is at t.

We let $p^f_{st}(i,u)$ be the price per unit of d^f_{st} at location i, date u, in units of good (i,u). However, we define such a price only for pairs (i,u) that potentially admit of a nontrivial trade in d^f_{st}. (This allows us to avoid having to determine a price for d^f_{st} in a market where demand and supply are identically zero and also allows us to restrict attention to positive prices.) Thus, suppose h and g meet at (i,u). We say that h is a *potential demander* of d^f_{st} at (i,u) if there is a route from h at u to f at t. We say that h is a *potential supplier* of d^f_{st} at (i,u) if there is a route from f at s to h at u. We say there is a *market* in d^f_{st} at (i,u) if and only if h is a potential demander and g is a potential supplier at (i,u), or vice versa.

We let $d_{st}^{fg}(i,u)$ be the excess demand by g at (i,u) for d_{st}^f. In terms of this notation, our debt trading rules are

$$\begin{aligned} &\sum_{u=s}^{t} d_{st}^{ff}(\cdot,u) \geq 0 \text{ for each } f \\ &\sum_{u=s}^{t'} d_{st}^{fg}(\cdot,u) \geq 0 \text{ for each } t' \geq s \text{ and } g \neq f \end{aligned} \tag{1}$$

where, in each case, the locations range over those that the demander visits. The first inequality says that f must end up demanding as much as f issues, which expresses our no-reneging rule. The second says that $g \neq f$ cannot supply d_{st}^f without having previously acquired it. Finally, as a convention, if there is not a market in d_{st}^f at (i,u), we set $d_{st}^{fg}(i,u) \equiv 0$.

Then, as budget constraints for any person g, we may write

$$w_{iu}^g \geq c_{iu}^g + \sum d_{st}^{fg}(i,u) p_{st}^f(i,u), \tag{2}$$

there being T such constraints, one for each (i,u) that g visits. The summation in (2) is over all securities—all (f,s,t)—for which a market exists at (i,u).

Now, letting d^g denote the vector of debt demands of g over *all* securities that can be issued, a vector that has many zeros, we can now give the following definition of a debt equilibrium or of a competitive, perfect-foresight equilibrium under our security trading rules.

DEFINITION. A debt equilibrium is a specification of consumption and debt demands—c^g and d^g for each $g = 1, 2, \ldots, G$—and positive security prices, $p_{st}^f(i,u)$, such that

(i) c^g and d^g maximize $U^g(c^g)$ subject to (1) and (2)

(ii) $\Sigma_g (c_{iu}^g - w_{iu}^g) = 0$ for each (i,u) and $\Sigma_g d_{st}^{fg}(i,u) = 0$ for each (i,u) and all potentially redeemable d_{st}^f.

Although it may seem strange to be considering competitive (price-taking) equilibrium in markets with only two traders, everything we do also holds for setups in which each of our persons is a trader type and in which there are many traders of each type.

Although we do not appeal to a general existence result in what follows, we include, in Appendix 1, a proof that every economy in our class of spatial-separation setups has a debt equilibrium. The proof draws heavily on Debreu (1959), although with modifications connected with the fact that the objects traded—the securities—are not ultimate consumption goods and are not bounded in an obvious way.

4. Debt Equilibria and Complete-Markets Equilibria

In this section we establish equivalence for the table V-1, $T = 4$ economy between the equilibrium allocations and prices of complete date-location contingent markets and the allocations and prices of debt equilibria. This proves useful in describing the transaction pattern implications of the theory and the coordination problem. We begin by showing that any complete-markets equilibrium (CME) consumption allocation can be supported by a debt equilibrium (DE).

To show that any CME can be supported by a DE in the table V-1, $T = 4$ economy, we start with a given CME. This we describe by individual consumption excess demands, $e_{it}^g \equiv c_{it}^g - w_{it}^g$, and by associated prices, s_{it} (in terms of an abstract unit of account). These constitute a CME if they satisfy:

$$\sum_i \sum_t e_{it}^g s_{it} = 0 \text{ for each } g \tag{3}$$

$$\sum_g e_{it}^g = 0 \text{ for each } (i,t) \tag{4}$$

and if, in addition, for each g, the e_{it}^g's are utility maximizing for g subject to (3).

A corresponding DE consists of positive debt prices and nonnegative, market-clearing debt quantities such that (a) the debt quantities and the given CME e_{it}^g's satisfy each person's debt budget constraints and (b) the debt quantities and the given CME e_{it}^g's are utility maximizing for each person given those debt prices.

Our first step is to produce candidate debt prices for the table V-1, $T = 4$ economy. This candidate is produced by matching the terms of trade between consumption goods implied by unconstrained trades in debts to the corresponding terms of trade given by the CME prices. Thus, for example, for person 1, $p_{14}^1(1,1)$ implies a trade between goods at location 1, date 1, and location 1, date 4. [Recall that, given our way of measuring debt quantities, $p_{14}^1(1,4) = 1$.] Thus, we let $p_{14}^1(1,1) = s_{14}/s_{11}$. In general, then, each debt price is taken to be a ratio of CME prices with the numerator corresponding to the redemption location-date and the denominator to the location-date of the current trade.

For noncirculating debts, then, our candidate is

$$\begin{aligned} &(p_{13}^1(1,1), p_{13}^3(2,1), p_{24}^1(1,2), p_{24}^2(2,2)) \\ &= (p_{13}^2(1,1), p_{13}^4(2,1), p_{24}^3(1,2), p_{24}^4(2,2)) \\ &= (s_{13}/s_{11}, s_{23}/s_{21}, s_{14}/s_{12}, s_{24}/s_{22}), \end{aligned} \tag{5}$$

whereas, for circulating debts, it is

$$\begin{bmatrix} p^1_{14}(1,1), p^1_{14}(2,2), p^1_{14}(2,3) \\ p^2_{14}(1,1), p^2_{14}(1,2), p^2_{14}(2,3) \\ p^3_{14}(2,1), p^3_{14}(2,2), p^3_{14}(1,3) \\ p^4_{14}(2,1), p^4_{14}(1,2), p^4_{14}(1,3) \end{bmatrix} = \begin{bmatrix} s_{14}/s_{11}, s_{14}/s_{22}, s_{14}/s_{23} \\ s_{24}/s_{11}, s_{24}/s_{12}, s_{24}/s_{23} \\ s_{14}/s_{21}, s_{14}/s_{22}, s_{14}/s_{13} \\ s_{24}/s_{21}, s_{24}/s_{12}, s_{24}/s_{13} \end{bmatrix}. \tag{6}$$

We can immediately indicate that this implies that satisfaction of (a) implies satisfaction of (b). To see this, multiply the debt constraint for e^g_{it} [equation (2)] by s_{it} and sum over i and t. Using (5) and (6), the result is (3), in which debt quantities do not appear. Thus, at prices given by (5) and (6), the debt constraints for any person are at least as constraining as (3). Therefore, if we can produce market-clearing debt quantities, d^f_{st}'s, which make the CME e^g_{it}'s feasible choices subject to the budget constraints (2), then they are certainly utility-maximizing choices. That is, (a) implies (b).

To motivate how we produce debt quantities, recall that a CME consists of arbitrary s_{it}'s and e^g_{it}'s that satisfy (3), (4), and zero restrictions for those e^g_{it}'s that correspond to (i,t)'s that g does not visit. For the table V-1, $T = 4$ economy, there are 3 + 8 + 16 *independent* constraints on the 32 e^g_{it}'s. This leaves us free to choose 5 e^g_{it}'s arbitrarily, but not any 5. For example, e^1_{11} and e^2_{11} cannot both be chosen arbitrarily because (4) and the zero restrictions imply that these sum to zero. Similarly, e^1_{11}, e^1_{12}, e^1_{13}, e^1_{14} cannot each be chosen arbitrarily since (3) must be satisfied. We arrive at candidates for equilibrium debt quantities by finding some that satisfy constraints (1) and (2) and the relevant debt market clearing conditions for a set of e^g_{it}'s that can be chosen arbitrarily.

For the table V-1, $T = 4$ economy, the following equations are the debt budget constraints, at prices satisfying (5) and (6), for five e^g_{it}'s that *can* be chosen arbitrarily:

$$(e^4_{21}, e^4_{22}, e^1_{11}, e^1_{12}, e^1_{13})' = Ad \tag{7}$$

where

$$A = \begin{bmatrix} s_{23}/s_{21} & 0 & 0 & 0 & 0 & 0 & -s_{14}/s_{21} & s_{24}/s_{21} \\ 0 & s_{24}/s_{22} & 0 & 0 & -s_{14}/s_{22} & 0 & s_{14}/s_{22} & 0 \\ 0 & 0 & s_{13}/s_{11} & 0 & s_{14}/s_{11} & -s_{24}/s_{11} & 0 & 0 \\ 0 & 0 & 0 & s_{14}/s_{12} & 0 & s_{24}/s_{12} & 0 & -s_{24}/s_{12} \\ 0 & 0 & -1 & 0 & 0 & 0 & -s_{14}/s_{13} & s_{24}/s_{13} \end{bmatrix}$$

and $d = (d^4_{13} - d^3_{13}, d^4_{24} - d^2_{24}, d^1_{13} - d^2_{13}, d^1_{24} - d^3_{24}, d^1_{14}, d^2_{14}, d^3_{14}, d^4_{14})'$. Note that zeros in the A matrix do not denote zero debt prices, but rather that the particular debt cannot be traded at the relevant location-date combination.

Note also that the relevant debt market clearing conditions are imposed in equation (7). Thus, for example, the equilibrium demands for d^3_{14} at dates 1, 2, and 3 are imposed in the first, second, and fifth equations of (7), respectively.

To see that there are *nonnegative* debt quantities that satisfy (7) for arbitrary s_{it}'s and an arbitrary left-hand side (LHS) of (7), consider an equivalent set of equations obtained by replacing the last equation of (7) by itself plus a multiple (s_{11}/s_{13}) of the third equation, namely

$$[e^4_{21}, e^4_{22}, e^1_{11}, e^1_{12}, e^1_{13} + (s_{11}/s_{13})e^1_{11}]' = [A_1, A_2, A_3, A_4, A_5 + (s_{11}/s_{13})A_3]'d\,, \tag{8}$$

where A_i denotes the ith row of the matrix A. Note that in each of the first four equations of (8), there appears (with a nonzero coefficient) a difference between noncirculating debts that do not appear in any other equation. Thus, for any quantities of the other debts, each of the first four equations can be satisfied by choosing *nonnegative* quantities of the noncirculating debts that appear in that equation only. This allows us to choose nonnegative quantities of the circulating debts in any way that satisfies the last equation of (8), namely

$$e^1_{13} + (s_{11}/s_{13})e^1_{11} = (s_{14}/s_{13})(d^1_{14} - d^3_{14}) - (s_{24}/s_{13})(d^2_{14} - d^4_{14}). \tag{9}$$

Equation (9) is easily satisfied; if the LHS is positive (negative), it can be satisfied by setting at zero all but $d^1_{14}(d^3_{14})$.

Given debt quantities that satisfy (8), all that remains is to show that they, (5) and (6), and the 11 other potentially nonzero CME e^g_{it}'s satisfy the associated debt budget constraints. Two facts imply that they do. First, if, for any g, three debt budget constraints are satisfied at equality (as they are for person 1), then the fourth is also; note that we have already referred to the fact that (5) and (6) imply that the debt budget constraints satisfy (3). Second, we know that if g and h meet at (i,t), then the debt budget constraint for e^g_{it} is minus that for e^h_{it}. Thus, if debt prices and quantities are such that the debt budget constraint for g implies the CME e^g_{it}, then the debt budget constraint for h implies minus the CME e^g_{it}. But, by (4), this is the CME value of e^h_{it}. This concludes our argument that any CME for the table V-1, $T = 4$ economy can be supported by a DE.

The converse—that any DE consumption excess demands are also CME excess demands in the table V-1, $T = 4$ economy—is established in Appendix 2.

5. Debt Equilibrium Transaction Patterns

We now show that most table V-1, $T = 4$ environments imply, rather than just permit, the existence of several private securities that play different roles in exchange. We then describe two ways of summarizing these different roles.

The coexistence of circulating and noncirculating private debts can be demonstrated using the equivalence results of Section 4. That is, for some setups for which DEs and CMEs coincide, we now show that some CMEs can be supported *only* by DEs with both circulating and noncirculating securities.

From (9), if $e_{13}^1 + (s_{11}/s_{13})e_{11}^1 \neq 0$, then some circulating debt, d_{14}^h for some h, must be positive. Notice also that by multiplying the fifth equation of (7) by s_{13}/s_{21} and subtracting it from the first, we get an equation for $e_{21}^4 - (s_{13}/s_{21})e_{13}^1$ that contains only noncirculating debts. Thus, if a table V-1, $T = 4$ setup is such that any CME satisfies $e_{13}^1 + (s_{11}/s_{13})e_{11}^1 \neq 0$ and $e_{21}^4 - (s_{13}/s_{21})e_{13}^1 \neq 0$, then every DE for that setup displays positive amounts of both circulating and noncirculating debts.[4]

We now describe two ways of summarizing the different exchange roles played by the different objects in debt equilibria in our setups. One way is in terms of a payments matrix (see Clower 1967); the other is in terms of transaction velocities.

By a payments matrix we mean an N by N matrix, where N is the number of objects observed in a debt equilibrium, in which the (i,j)-th element is one if object i is observed to trade for object j and is zero otherwise. Thus, for a table V-1, $T = 4$ economy, N equals the number of distinct consumption goods—eight—plus the number of distinct private securities issued in an equilibrium. And, if the transaction pattern is such that each consumption good gets traded for one circulating security and one noncirculating security, then there are two nonzero elements in each row corresponding to a consumption good or to a noncirculating debt, and there are four in each row corresponding to a circulating debt. Note, by the way, that nontrivial spatial setups seem not to produce equilibria in which *one* object trades for *every* other object.

By the transaction velocity of an object, we mean the ratio of the average amount traded per date to the average stock, a pure number per unit time. For example, for a table V-1, $T = 4$ economy, the following transaction velocity pattern among objects shows up in a debt equilibrium. For a consumption good at location i, date t, the *average* stock outstanding may be taken to be the total endowment divided by 4 (at dates other than t, the stock of this good is zero), whereas the average amount traded per date is the amount traded at t divided by 4. Thus, the transaction velocity is in the interval (0,1). Computed in a similar way, the transaction velocity of noncirculating debt in such an economy is 2/3 (such debt is outstanding for three dates and the entire stock is traded at two of those dates), whereas that of circulating debt is unity (the maximum possible velocity given our choice of time unit).

Thus, either in terms of a payments matrix or in terms of the pattern of transaction velocities across objects, our setups can imply different exchange roles for different objects and, in particular, a relatively prominent exchange role for what we have been calling circulating debt.

6. The Coordination Problem

Although a debt equilibrium exists in a table V-1, $T = 4$ economy, there is a difficulty in arriving at debt quantities that achieve it. The difficulty can be described as follows.

Individual equilibrium debt demands are correspondences, not functions; that is, when faced with equilibrium debt prices, each of many vectors of debt quantities achieves a given equilibrium vector of consumption for each individual. Of course, not surprisingly, in order to be a debt equilibrium, the vectors chosen from these individual correspondences must satisfy a restriction. In particular, the quantities of circulating debts must satisfy equation (9), which we rewrite here as

$$d_{14}^1 - (s_{24}/s_{14})d_{14}^2 = b + d_{14}^3 - (s_{24}/s_{14})d_{14}^4, \tag{10}$$

where $b = (s_{13}/s_{14})[e_{13}^1 + (s_{11}/s_{13})e_{11}^1]$. Equation (10) says that a linear function of the circulating debts issued in location 1 must equal a linear function of those issued in location 2. What distinguishes this situation from others in which demands are correspondences is that if people in one location at date 1 do not observe the quantities issued in the other location at date 1, then whether a vector of debt quantities is consistent or inconsistent with equation (10) is not revealed at date 1. Corresponding to any nonnegative pair of circulating debts issued in location 1 at date 1 is a net trade of noncirculating debts in location 1 at date 1, a magnitude of $d_{13}^1 - d_{13}^2$, consistent with the equilibrium date 1, location 1 consumption trade and with market clearing in the debts traded in location 1 at date 1 [see the third row of equation (8)]. A similar situation prevails in location 2 at date 1 [see the first row of equation (8)]. Only at date 2 and thereafter is it revealed whether the quantities chosen at date 1 are consistent with equation (10) and, hence, with equilibrium consumption trades at subsequent dates.

We call this difficulty a coordination problem because arriving at quantities that satisfy equation (10) calls for communication across locations, which is precluded by assumption. In that respect, our situation resembles those described by Schelling (1960) as giving rise to coordination problems. The absence of communication across locations is, by the way, consistent with one interpretation of our concept of a perfect-foresight equilibrium. That concept can be interpreted to mean that each person knows the endowments and preferences of each other person and, hence, knows the equilibrium consumption excess demands and debt prices. Such knowledge is consistent with people in one location not knowing the debt quantities issued in the other location.

Our coordination problem bears some resemblance to a result obtained by Ostroy (1973) and Ostroy-Starr (1974) in their study of the decentralization of exchange. In their model, knowledge of equilibrium prices of commodities is not enough to guide people to the trades that produce the equilibrium allocation in one round of bilateral trading if the trading rules are informationally decentralized. In our model, knowledge of current and future equilibrium prices of securities is not enough to guide people to the quantities of securities required to support an equilibrium if security transactions in other markets are not observed. Of course, both private debt in our model and money in their model alleviate a quid pro quo requirement and facilitate the attainment of equilibrium. There is a sense, though, in which the monetary exchange process is informationally centralized in their model. It requires that budget balance information be transmitted to a monetary authority or requires that there be implicit agreement about which commodity is to be used to cover budget deficits and surpluses. One interpretation of our coordination problem is that a debt equilibrium also requires centralization.

7. Concluding Remarks

As promised, we have described a class of environments that, in general, give rise to private securities that circulate. The crucial feature in our environments is separated or segmented markets among which people move over time. We have also demonstrated that in some of the environments in which circulating securities appear, a problem also appears—a problem that resembles a Schelling coordination problem. Since there is nothing particularly strange about our physical environment or our security trading rules and equilibrium concept, we think the model shows promise as an explanation for why actual credit markets appear not to work well at times. However, except for breaking down barriers to communication, the model does not suggest to us any way of solving the coordination problem.

Appendix 1

Here we prove that the debt equilibrium defined in Section 2 exists. We first prove existence with imposed arbitrary bounds on debt demands. We then argue by taking the appropriate limit that there exists some equilibrium.

Let C be the space of JT-tuples, each element of which is in $[0,w]$, where $w > w_{iu}$ for all (i,u) and where w_{iu} is the social endowment of good (i,u). Also let D be the space of n-tuples, each element of which is in $[-d,d]$, where $d > 0$ and where n is the number of elements in d^g. Note that d is the arbitrary bound on debt demands. Let P^h_{st} be the n' dimensional simplex for d^h_{st} where n' is the number of location-date combinations where there is a market for d^h_{st} and let P be the product space of the P^h_{st}'s, a finite product of finite dimensional simplexes. Finally, for each $p \in P$, let $\gamma^g(p,w^g) = \{(c^g,d^g) \in C \times D$ that satisfy constraints (1) and (2)$\}$. We are now in a position to follow Debreu's (1959) proof of the existence of a standard competitive equilibrium.

Since $C \times D$ restricted by (1) and (2) is compact and convex, it follows from (1) of 4.8 of Debreu (p. 63), modified for vector constraints, that $\gamma^g(p,w^g)$ is a continuous correspondence in p. The key to this assertion is that the RHS of (2) takes on the value zero at zero consumption and debt demands. Since zero is strictly less than w^g_{iu}, the exceptional case of minimum wealth does not occur.

Now, since the bounded competitive maximization problem of g involves maximizing a continuous function on a compact set, there exists a nonempty maximizing correspondence of security demands denoted $\phi^g(p)$. By theorem 4 of Section 1.8 of Debreu (p. 19), $\phi^g(p)$ is upper semicontinuous; it is also convex. Now let $\Phi(p) = \Sigma_g \phi^g(p)$. Clearly, $\Phi(p)$ is in the space of n-tuples, each element of which is in the interval $[-Gd,Gd]$. We denote this compact, convex set by Z.

Now consider the correspondence ρ from $P \times Z$ into itself defined by $\rho(p,z) = \mu(z) \times \Phi(p)$, where $\mu(z) = \{p \in P$ that maximize $p \cdot z$ for $z \in Z\}$. Following Debreu (p. 82), $\mu(z)$ is an upper semicontinuous correspondence from Z to P with $\mu(z)$ nonempty and convex. It follows that $\rho(p,z)$ is a nonempty, upper semicontinuous, and convex correspondence on $P \times Z$, which is a nonempty, compact, and convex set. So ρ has a fixed point: namely, (p^*,z^*) such that $p^* \in \mu(z^*)$ and $z^* \in \Phi(p^*)$.

We now establish that $\Phi(p^*) = 0$. Consider the subvector of $\Phi(p^*)$ associated with a particular debt, d^h_{st}. If this subvector is not zero, then some element must be positive because constraint (1) does not permit the sum of these elements to be negative. So suppose some elements of the subvector are positive. The correspondence μ sets debt prices at positive levels for at least one of these positive elements and for no nonpositive element. This, in turn, implies that more of d^h_{st} is being demanded at positive prices over all location-dates than is being supplied at positive prices. This contradicts individual maximization for someone and, hence, implies that the subvector of $\Phi(p^*)$ associated with d^h_{st} is zero. It follows that $\Phi(p^*) = 0$. Moreover, a similar argument implies that no component of p^* can be zero.

We can now show that the consumptions implied by z^* and p^* satisfy market clearing for each (i,u). For each g, individual maximization implies that (2) holds with equality. So suppose we sum (2) at a given (i,u) over g. Then market clearing in consumptions follows from $\Phi(p^*) = 0$.

We have thus established the existence of a debt equilibrium with arbitrary, imposed debt bounds d. Doubling these bounds, one can again establish existence. Continuing in this way, one can construct a sequence of debt equilibria for economies with larger and larger debt bounds. As the associated sequence of debt equilibrium prices and consumptions has elements in the same compact space, there exists a convergent subsequence—say, with limit prices and consumptions, $\bar{p}$ and $\bar{c}^g$, respectively.

Now consider the problem confronting a typical person g in the limit economy, at prices $\bar{p}$ and no imposed debt bounds whatever. The space of feasible consumptions for such a person (the budget set at prices $\bar{p}$ projected onto the space of consumptions) is compact, and the objective function is continuous, so there exists a solution, some maximizing choice of consumptions $\hat{c}^g$. Also, by construction, $\bar{c}^g$ is feasible in the limit economy. Suppose $U^g(\hat{c}^g) > U^g(\bar{c}^g)$. Then for some kth economy associated with the convergent subsequence, with debt bounds sufficiently large and prices sufficiently close to $\bar{p}$, one can find a feasible consumption vector with utility arbitrarily close to $U^g(\hat{c}^g)$. But along the convergent subsequence, utility must converge to $U^g(\bar{c}^g)$. We have thus contradicted maximization for person g in the kth economy. Thus $U^g(\hat{c}^g) = U(\bar{c}^g)$, and $\bar{c}^g$ solves the maximization problem of person g in the limit economy. Recall that person g was arbitrary.

We have thus established that consumptions $\bar{c}^g$ are all maximizing in the limit economy. Also, by construction, the $\bar{c}^g$ satisfy market clearing in the limit economy (this was a property of each economy in the convergent subsequence, by virtue of equilibrium). It only remains then to specify market-clearing debt demands for each person g in the budget set of person g and consistent with the choice of $\bar{c}^g$. This is done as follows. First, specify person 1's debt demands in his or her budget set consistent with $\bar{c}^1$. For market clearing, let these determine the debt demands of each person with whom person 1 trades, at specified dates and locations. Next, consider person 2. If there are no debt demands that remain to be determined for this person, then we are done. To suppose otherwise is to contradict the fact that $\bar{c}^1$ and $\bar{c}^2$ are market clearing and the fact that person 1's debt demands are in the budget set of person 1. If there do remain any debts to be determined, choose these consistent with $\bar{c}^2$. Continue in this way for person 3, and so on through person G. In the end, then, we have constructed an equilibrium in the limit economy, with no bounds on debts. Recall also that the (fixed) bounds on consumptions need not be imposed in the limit economy, by the choice of the bound w.

Appendix 2

Here we prove for the table V-1, $T = 4$ economy that any debt equilibrium (DE) consumption excess demands are also complete-markets equilibrium (CME) consumption excess demands.

Since the DE e_{it}^g's are market clearing—that is, satisfy (4)—we have to show only that the debt budget constraints are equivalent to (3) for some choice of s_{it}; if we can establish that equivalence, then it follows that the DE e_{it}^g's are utility maximizing subject to (3).

The debt budget constraints for person 1 in the table V-1, $T = 4$ economy can be written

$$\begin{aligned}
e_{11}^1 &= -d_{13}^{11}(1,1)p_{13}^1(1,1) - d_{13}^{21}(1,1)p_{13}^2(1,1) \\
&\quad - d_{14}^{11}(1,1)p_{14}^1(1,1) - d_{14}^{21}(1,1)p_{14}^2(1,1) \\
e_{12}^1 &= -d_{24}^{11}(1,2)p_{24}^1(1,2) - d_{24}^{31}(1,2)p_{24}^3(1,2) \\
&\quad - d_{14}^{21}(1,2)p_{14}^2(1,2) - d_{14}^{41}(1,2)p_{14}^4(1,2) \\
e_{13}^1 &= -d_{13}^{11}(1,3) - d_{13}^{21}(1,3) - d_{14}^{31}(1,3)p_{14}^3(1,3) - d_{14}^{41}(1,3)p_{14}^4(1,3) \\
e_{14}^1 &= -d_{24}^{11}(1,4) - d_{24}^{31}(1,4) - d_{14}^{11}(1,4) - d_{14}^{31}(1,4).
\end{aligned}$$

Let us add and subtract $d_{13}^{21}(1,1)p_{13}^1(1,1)$ on the RHS of the first equation, so that the sum $-[d_{31}^{11}(1,1) + d_{13}^{21}(1,1)]$ appears. Note that the sum $[d_{13}^{11}(1,3) + d_{13}^{21}(1,3)]$ appears in the third equation and that these sums are equal to each other because, at any debt prices, debt demands satisfy $d_{13}^{11}(1,1) = -d_{13}^{11}(1,3)$ and $d_{13}^{21}(1,1) = -d_{13}^{21}(1,3)$. Moreover, the sum $[d_{13}^{11}(1,1) + d_{13}^{21}(1,1)]$ is unconstrained (as to sign). These facts imply that the first and third equations are no more constraining than the single equation that results from substituting for that sum from the third equation into the first to produce

$$\begin{aligned}
e_{11}^1 + p_{13}^1(1,1)e_{13}^1 = {} & d_{13}^{21}(1,1)[p_{13}^1(1,1) - p_{13}^2(1,1)] \\
& - d_{14}^{11}(1,1)p_{14}^1(1,1) - d_{14}^{21}(1,1)p_{14}^2(1,1) \\
& - d_{14}^{31}(1,3)p_{14}^3(1,3)p_{13}^1(1,1) - d_{14}^{41}(1,3)p_{14}^4(1,3)p_{14}^4(1,3)p_{13}^1(1,1). \qquad \text{(i)}
\end{aligned}$$

An exactly analogous procedure allows us to combine the second and fourth equations into the following single equation, which is no less constraining than those separate equations:

$$\begin{aligned}
e_{12}^1 + p_{24}^1(1,2)e_{14}^1 = {} & d_{24}^{31}[p_{24}^1(1,2) - p_{24}^3(1,2)] - d_{14}^{21}(1,2)p_{14}^2(1,2) \\
& - d_{14}^{41}(1,2)p_{14}^4(1,2) - p_{24}^1(1,2)[d_{14}^{11}(1,4) + d_{14}^{31}(1,4)]. \qquad \text{(ii)}
\end{aligned}$$

Now consider the first term on the RHS of (i). If the price difference that multiplies $d_{13}^{21}(1,1)$ is positive, then $d_{13}^{21}(1,1)$ is infinite. Since that cannot be an equilibrium choice, it follows that the DE prices satisfy $p_{13}^1(1,1) \leq p_{13}^2(1,1)$; that is, arbitrage is not possible for person 1 in the debts d_{13}^1 and d_{13}^2. And since an analogous manipulation of the debt budget constraint for person 2 implies the reverse inequality, it follows that DE prices satisfy $p_{13}^1(1,1) = p_{13}^2(1,1)$. In addition, exactly the same reasoning allows us to conclude that the DE prices satisfy the entire first equality of equation (5).

We now proceed to combine (i) and (ii) into a single constraint that is no less constraining than both (i) and (ii). First, add and subtract $d_{14}^{31}(1,3)p_{14}^1(1,1)$ on the RHS of (i) so that the sum $d_{14}^{11}(1,1) + d_{14}^{31}(1,3)$ appears in (i). This sum of demands is equal at any debt prices to $-[d_{14}^{11}(1,4) + d_{14}^{31}(1,4)]$, which appears in (ii). Moreover, this sum is unconstrained, implying that the equation that results from eliminating it between (i) and (ii) is no less constraining than both (i) and (ii). This single equation, which we will not write out, has the following form: a linear combination of person 1's excess demands for consumption is equal to a linear combination of person 1's debt demands.

By an argument similar to that used above to establish that DE prices satisfy the first equality of equation (5)—an argument that uses the analogues of (i) and (ii) for persons 2, 3, and 4—it follows that the DE prices must be such that the coefficient of each debt demand is zero; that is, intertemporal arbitrage among the various debts must not be possible for anyone. These restrictions on coefficients of debt demands are the ones needed to be able to choose s_{it}'s to satisfy equation (6). And such choices for s_{it}'s imply equivalence between the debt constraints and (3).

To summarize, we have indicated how to manipulate the debt budget constraints for the table V-1, $T = 4$ economy so as to establish two results. The first is that security prices in a DE for that economy are constrained so that we can choose s_{it}'s to satisfy (5) and (6). Second, with that choice of s_{it}'s, debt constraints are equivalent to (3) so that the e_{it}^h's that are utility maximizing subject to the debt constraints are also utility maximizing subject to (3). These results imply that any DE is a CME in the table V-1, $T = 4$ economy.

Notes

1. A version of this paper entitled "A Model of Circulating Private Debt" was presented at the Econometric Society Summer Meeting, Cornell University, 16–19 June 1982, and at seminars at several universities. Helpful comments from the participants of these seminars and from Alvin Roth and Allan Drazen are gratefully acknowledged. We are indebted to the Federal Reserve Bank of Minneapolis for financial support. However, the views expressed are those of the authors and not those of the Bank or the Federal Reserve System.
2. See, for example, Friedman's comments about private bank note issue and unfettered intermediation (1960, 21 and 108).
3. For two recent attempts much different from ours, see Bryant (1981) and Diamond and Dybvig (1983).
4. Coexistence of circulating and noncirculating securities can occur even if there is not a coincidence between DEs and CMEs. See Townsend and Wallace (1982) for an example.

References

Bryant, J. 1981. "Bank Collapse and Depression." *Journal of Money, Credit, and Banking* 13:454–64.
Clower, R. 1967. "A Reconsideration of the Microfoundations of Monetary Theory." *Western Economic Journal* 6:1–8.
Debreu, G. 1959. *The Theory of Value*. New York: John Wiley.
Diamond, D. W., and P. H. Dybvig. 1983. "Bank Runs, Deposit Insurance, and Liquidity." *Journal of Political Economy* 91:401–19.
Friedman, M. 1960. *A Program for Monetary Stability*. New York: Fordham University Press.
Harris, M. 1979. "Expectations and Money in a Dynamic Exchange Economy." *Econometrica* 47:1403–20.
Ostroy, J. M. 1973. "The Informational Efficiency of Monetary Exchange." *American Economic Review* 63:597–610.
Ostroy, J. M., and R. M. Starr. 1974. "Money and the Decentralization of Exchange." *Econometrica* 42:1093–113.
Schelling, T. 1960. *The Strategy of Conflict*. Cambridge, Mass.: Harvard University Press.
Townsend, R. 1980. "Models of Money with Spatially Separated Agents." In *Models of Monetary Economies*, edited by J. Kareken and N. Wallace. Federal Reserve Bank of Minneapolis.
Townsend, R., and N. Wallace. 1982. "A Model of Circulating Private Debt." Staff report 83, Federal Reserve Bank of Minneapolis.

VI

Incomplete Market Participation and the Optimal Exchange of Credit

Lawrence M. Benveniste

This paper contains a generalized version of the Townsend and Wallace model that appears in this volume. Participants in the economy trade IOUs with redemption value denoted in units of account, and the potential for traders to fully value the contracts depends on the patterns of future market participation. Each participation pattern establishes a feasible set of trading patterns through which the credit instruments must circulate in equilibrium. These patterns, and the implied credit trades that result from the simplest market-clearing trades, are used to establish whether or not the economy can carry out a complete set of credit trades. The target set of trades describes a generalized spanning condition that characterizes complete credit markets when participation is incomplete.

1. Introduction

In our modern economic system, financial intermediaries (with the aid of computers and modern communications) coordinate the exchange of credit between large numbers of seemingly anonymous individuals.[1] We are moving ever more closely to the paradigm of the Walrasian auctioneer.

However, there are limits to the information capacity of any one organization. Thousands of financial intermediaries exchange excess credit balances among themselves, and these transactions are not coordinated by a central auctioneer. In essence, we have refined the historical financial institution in which individuals traded personalized IOUs whose value was backed by a promise to repay the bearer a predetermined amount at some time in the future.

At its origin, an IOU was a promise to pay some quantity of goods or services at some future date and location. Such an outstanding debt instrument could then be exchanged by its owner for commodities so long as it was acceptable to the merchant. Of course, this would require the merchant