Core is Manipulable via Segmentation

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Any allocation rule that picks only core allocations is *manipulable via segmentation.* That is, there exists an economy with a coalition of agents such that, once this coalition splits momentarily from the rest of the economy and institutes the allocation rule within itself, no matter which individually rational sub-allocation the complementary coalition picks, when we paste all the agents back together at their new endowments and apply the allocation rule to this "collage" economy, each member of the former coalition will be strictly better off than under direct application of the allocation rule to the original economy. *Journal of Economic Literature* Classification Numbers: D41, D51.
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1. INTRODUCTION AND MOTIVATION

The Edgeworthian notion of equilibrium for a pure trade economy is one of a feasible allocation in comparison with which no coalition can do better for all its members by trading among themselves, starting from their initial endowments. We call the set of all such allocations the *core* of an economy. With the understanding that a coalition "blocks" an allocation when it could have done better for all of its members by trading within itself, the core is also described as the set of all allocations which no coalition blocks.

Postlewaite [5] showed that any selection from the core can be blocked by coalitions who can benefit from reallocating their initial endowments before core-destined trade takes place according to the given selection. Thus, a coalition can sometimes better all its members by coming to the market after strategically reallocating endowments within itself, given that the market will then be resolved at a core allocation. This shows that instituting a core allocation rule is ridden with implementation problems, since the very institution may incite manipulation by some coalition in Postlewaite's manner, resulting in an allocation other than the one which was supposed to be instituted. In its strategic internal reallocation, however, Postlewaite's heretic coalition was left free to use *any* allocation rule, not necessarily taking values in the core of its subeconomy.

What we show here is something stronger, pointing to a nastier difficulty with the internal consistency of the core concept and a worse problem with its implementation. We show that a coalition can do better for all its members by applying any *core* allocation to its subeconomy before the core allocation rule is applied to the entire economy. In particular, we provide an example where a coalition strictly improves welfare for all its members by coming to the big market *with its own internal core allocation*, as this constitutes a better initial position for its members from which to trade and end up at a (more advantageous) core allocation of the entire economy. Thus, the manipulation of the core is carried out by resort to the core of the subeconomy of the manipulating coalition. In particular, competitive trade can very well be the instrument used by the subeconomy here, whereas Postlewaite's heretic coalition was free to use some possibly esoteric allocation, which may fail to be feasible, due to informational or other constraints faced by the coalition. It seems institutionally somewhat more practicable to manipulate the core using the very same (core) allocation rule applied to a subeconomy. And its manipulability in this stronger sense exposes an even deeper inconsistency of the core concept.

In fact, we show (Theorem 4.1) an even stronger strategic manipulation of the core allocation rule again via "segmentation." In this case, a manipulative coalition gains by applying the core allocation within its own subeconomy while the complementary subeconomy applies *any individually rational allocation rule*, given that the resulting two subeconomies then rejoin[1] and their composite economy witnesses the core allocation. Incidentally, Example 3.2 shows that, so long as the two subeconomies are to then rejoin and their composite is to witness the core allocation, a coalition can also benefit by initially standing idle (at its autarky) while its complementary coalition internally applies the core allocation.

Is there an allocation rule which is immune to the types of manipulation outlined above? There are plenty. But certainly there are no refinements of the core allocation rule that fit the bill. Less ambitiously, can we find an imputational (i.e., Pareto-optimal and individually rational) allocation rule that is immune to such manipulation? In answer, defined as the maximal imputational allocation rule, *the* imputation itself is immune to manipulation via segmentation. This is because the imputations of the original economy always include those of the economy after segmentation, so no coalition can find the latter set to be unambiguously superior.

[1]The complementary coalition never rejects the manipulating coalition's request for (individually rational) trade to occur once again.

Regarding the imputation, however, we also have a further result, which we prove by use of the manipulability of the core via segmentation and a further lemma. Accordingly, the imputation is utility-wise indecomposable, meaning that its image in utility space for the economy is not invariant to segmentation. Hence, the set of imputations itself is not invariant to segmentation. In fact, all imputational allocation rules are indecomposable.
Notation: Before we start our formal analysis, we present the general notation we use throughout the paper.

- Given any set S with a subset $T \subseteq S$ and any function f defined on S, we write f_T for the restriction of f to T.
- Given any two disjoint sets S and T and any two functions $f : S \to R$ and $g : T \to R$ with a common range R, we write $f \vee g$ for the *common extension* of f and g to $S \cup T$, i.e., we define $f \vee g : S \cup T \to R$ by setting $(f \vee g)(s) = f(s)$ at each $s \in S$ and $(f \vee g)(t) = g(t)$ at each $t \in T$.
- For any $x, y \in \mathbb{R}^K$, we write $x \geq y$ (resp., $x \gg y$) iff $x_i \geq y_i$ (resp., $x_i > y_i$) for each $i \in K$, and we write $x > y$ iff $x \geq y$ and $x \neq y$.
- Given any two subsets S and T of $\mathbb{R}^K$, we write $S > T$ iff we have $s > t$ for each $(s, t) \in S \times T$.

2. PRELIMINARY NOTIONS

We take some m-dimensional Euclidean space $\mathbb{R}^m$ as commodity space, and we write X for the non-negative orthant $\mathbb{R}^m_+$. We regard X as consumption space and write $\mathcal{U}$ for the space of strictly quasi-concave, continuous and monotonically increasing *utility* functions $u : X \to \mathbb{R}$. By an economy (with m goods) we mean any function $\varepsilon : N \to \mathcal{U} \times X$ defined on some non-empty subset N of indices, identifying agents. We display such an economy also as an ordered pair $\varepsilon = (\mathbf{u}, \bar{\mathbf{x}}) \in \mathcal{U}^N \times X^N$, where $\mathbf{u} = \{u_i\}_{i \in N}$ and $\bar{\mathbf{x}} = \{\bar{x}_i\}_{i \in N}$ are the profiles, respectively, of the utility functions and the initial endowments of the economy. Given any economy $\varepsilon = (\mathbf{u}, \bar{\mathbf{x}}) \in \mathcal{U}^N \times X^N$, we assume that each agent cares only for his own consumption $x_i \in X$, so that we are able to evaluate any $\mathbf{x} \in X^N$ via $\mathbf{u}$ as $\mathbf{u}(\mathbf{x}) = \{u_i(x_i)\}_{i \in N}$.

Taking any economy $\varepsilon = (\mathbf{u}, \bar{\mathbf{x}}) \in \mathcal{U}^N \times X^N$, by a *subeconomy* we mean a restriction $\varepsilon_{\hat{N}} = (\mathbf{u}_{\hat{N}}, \bar{\mathbf{x}}_{\hat{N}})$ of ε to a non-empty subset $\hat{N}$ of N. We write $\mathbf{X}(\varepsilon) = \{\mathbf{x} \in X^N | \sum_{i \in N} x_i \leq \sum_{i \in N} \bar{x}_i\}$ for the set of all (feasible) *allocations* in economy ε. By an *imputation* we mean any allocation $\mathbf{x} \in \mathbf{X}(\varepsilon)$ which is both Pareto-optimal (i.e., $\mathbf{u}(\mathbf{x}) \not< \mathbf{u}(\mathbf{y})$ for each $\mathbf{y} \in \mathbf{X}(\varepsilon)$) and individually rational (i.e., $\mathbf{u}(\mathbf{x}) \geq \mathbf{u}(\bar{\mathbf{x}})$). The set of all imputations of ε is non-empty and denoted by $\mathbf{M}(\varepsilon)$. Likewise, $\mathbf{C}(\varepsilon)$ denotes the *core* of economy ε, i.e., the (non-empty) set of allocations $\mathbf{x} \in X(\varepsilon)$ with $\mathbf{u}_{\hat{N}}(\mathbf{x}) \not<$

$\mathbf{u}_{\hat{N}}(\hat{\mathbf{x}})$ for each $\hat{\mathbf{x}} \in \mathbf{X}(\varepsilon_{\hat{N}})$ and each non-empty subset $\hat{N}$ of N. We write $\mathcal{P}$ for the set of all prices $p = (1, p_2, \ldots, p_m) \gg 0$. A Walrasian equilibrium of economy ε is any ordered pair $(p, \mathbf{x}) \in \mathcal{P} \times \mathbf{X}(\varepsilon)$ consisting of a price $p \in \mathcal{P}$ and an allocation $\mathbf{x} = \{x_i\}_{i \in N} \in \mathbf{X}(\varepsilon)$ such that, for each $i \in N$, x_i maximizes u_i subject to $p \cdot x_i \leq p \cdot \bar{x}_i$ and $x_i \in X$. We write $\mathbf{W}(\varepsilon)$ for the set of *Walrasian allocations* of ε.

We write $\mathcal{E}$ for the set of all economies $\varepsilon : N \to \mathcal{U} \times X$ such that, for each subeconomy $\hat{\varepsilon}$ of ε, $\mathbf{W}(\hat{\varepsilon}) \neq \emptyset$. For instance, if each agent is initially endowed with some positive amount of each good, then the economy will be in $\mathcal{E}$.

By an *allocation rule* we mean any non-empty set-valued function $\mathbf{A}$ on $\mathcal{E}$ that maps each economy $\varepsilon \in \mathcal{E}$ to some non-empty set $\mathbf{A}(\varepsilon) \subseteq \mathbf{X}(\varepsilon)$ of allocations. An allocation rule $\mathbf{A}$ will be said to be *imputational* iff $\mathbf{A}(\varepsilon) \subseteq \mathbf{M}(\varepsilon)$ at each $\varepsilon \in \mathcal{E}$. It is well-known that $\mathbf{W} \subseteq \mathbf{C} \subseteq \mathbf{M}$, so that $\mathbf{W}$ and $\mathbf{C}$ are both imputational.

We write $\mathcal{E}_1$ for the set of economies $\varepsilon \in \mathcal{E}$ whose agents are all identical, i.e., $|\varepsilon(N)| = 1$ where N is the domain of ε. We restrict ourselves in this paper to the class $\mathcal{A}$ of allocation rules $\mathbf{A}$ with $\mathbf{A}(\mathbf{u}, \bar{\mathbf{x}}) = \{\bar{\mathbf{x}}\}$ at every $(\mathbf{u}, \bar{\mathbf{x}}) \in \mathcal{E}_1$. Since we have assumed utility functions to be strictly quasi-concave, any individually rational allocation rule $\mathbf{A}$ is in $\mathcal{A}$.

Naturally, an allocation rule will in general map an economy to multiple allocations.[2] This multiplicity in outcomes is natural, as the Walrasian solution, which is central to our paper, is generally multi-valued. In fact, our propositions all admit singleton-valued allocation rules as special cases.

Since our allocation rules are multiple-valued, we need to elaborate on how to evaluate their outcomes. Take any two economies $\varepsilon = (\mathbf{u}, \bar{\mathbf{x}})$ and $\varepsilon' = (\mathbf{u}, \bar{\mathbf{x}}') \in \mathcal{E}$ with a common agent set N and an agent $i \in N$. Now, if both $\mathbf{A}(\varepsilon)$ and $\mathbf{A}(\varepsilon')$ are singleton, say $\mathbf{A}(\varepsilon) = \{\mathbf{x}\}$ and $\mathbf{A}(\varepsilon') = \{\mathbf{x}'\}$, then agent i would find $\mathbf{A}(\varepsilon)$ at least as good as $\mathbf{A}(\varepsilon')$ iff $u_i(\mathbf{x}) \geq u_i(\mathbf{x}')$. When $\mathbf{A}(\varepsilon)$ and $\mathbf{A}(\varepsilon')$ are not singleton, however, such an extension might not be meaningful. For the realized outcomes at ε and ε' might be distinct from each other even when $\mathbf{A}(\varepsilon) = \mathbf{A}(\varepsilon')$. Likewise, the realized outcomes might be the same while $\mathbf{A}(\varepsilon)$ and $\mathbf{A}(\varepsilon')$ are distinct from each other. Nevertheless, we can still unambiguously infer that agent i finds $\mathbf{A}(\varepsilon)$ strictly better than $\mathbf{A}(\varepsilon')$ if $u_i(\mathbf{A}(\varepsilon)) > u_i(\mathbf{A}(\varepsilon'))$, which we can spell out as $u_i(\mathbf{x}) > u_i(\mathbf{x}')$ at each $\mathbf{x} \in \mathbf{A}(\varepsilon)$ and $\mathbf{x}' \in \mathbf{A}(\varepsilon')$. We will use only this partial information in our paper.

3. MAIN CONCEPTS

In this section we will formulate our main concepts, *manipulation via segmentation* and *decomposability*. Given any two economies $\varepsilon = (\mathbf{u}_N, \bar{\mathbf{x}}_N)$, $\varepsilon' =$

[2] Of course, one *and only one* of these allocations will be realized. That is, given any economy $\varepsilon = (\mathbf{u}, \bar{\mathbf{x}}) \in \mathcal{E}$ and an allocation $\mathbf{x} \in \mathbf{X}(\varepsilon)$, we write $\mathbf{x} \in \mathbf{A}(\varepsilon)$ iff $\mathbf{A}$ does not preclude $\mathbf{x}$ from being realized at economy ε.

$(\mathbf{u}_{N'}, \bar{\mathbf{x}}_{N'}) \in \mathcal{E}$ with disjoint agent sets, by the *collage* economy of ε and ε', we mean the common extension $\varepsilon \vee \varepsilon' = (\mathbf{u}_N \vee \mathbf{u}_{N'}, \bar{\mathbf{x}}_N \vee \bar{\mathbf{x}}_{N'})$ of ε and ε' to $N \cup N'$; i.e., $\mathbf{u}_N \vee \mathbf{u}_{N'} = \{u_i\}_{i \in N \cup N'}$ and $\bar{\mathbf{x}}_N \vee \bar{\mathbf{x}}_{N'} = \{\bar{x}_i\}_{i \in N \cup N'}$. When $\mathcal{F}$ and $\mathcal{F}'$ are sets of economies such that the agent set N of any $\varepsilon \in \mathcal{F}$ and the agent set N' of any $\varepsilon' \in \mathcal{F}'$ are disjoint, we also write $\mathcal{F} \vee \mathcal{F}' = \{\varepsilon \vee \varepsilon' \mid \varepsilon \in \mathcal{F}, \varepsilon' \in \mathcal{F}'\}$.

Now we are ready to define our main concept, namely manipulation via segmentation. Let $\varepsilon = (\mathbf{u}, \bar{\mathbf{x}}) \in \mathcal{E}$ be an economy with agent set N, and take any proper non-empty subset $\hat{N}$ of N. We say that $\mathbf{A}$ is *manipulable via segmentation by* $\hat{N}$ *at* $\varepsilon = (\mathbf{u}, \bar{\mathbf{x}})$ iff

$$\mathbf{u}_{\hat{N}}\left(\mathbf{A}(\mathbf{u}, \mathbf{A}(\varepsilon_{\hat{N}}) \vee \mathbf{B}(\varepsilon_{N \setminus \hat{N}}))\right) \gg \mathbf{u}_{\hat{N}}(\mathbf{A}(\varepsilon)) \qquad (MS)$$

for every $\mathbf{B} \in \mathcal{A}$. Note that MS requires that, once the coalition $\hat{N}$ of agents split from $N \setminus \hat{N}$ and institute the allocation rule $\mathbf{A}$ among themselves, no matter which allocation rule $\mathbf{B} \in \mathcal{A}$ the remaining coalition $N \setminus \hat{N}$ applies in the rest of the economy, when we paste them back together at their new individual allocations and apply $\mathbf{A}$ to this collage economy, each member of the coalition $\hat{N}$ will be strictly better off than with what she got when $\mathbf{A}$ was applied to the original economy. We require this for all realizable allocations $\mathbf{x}_{\hat{N}} \in \mathbf{A}(\varepsilon_{\hat{N}})$, $\mathbf{x}_{N \setminus \hat{N}} \in \mathbf{B}(\varepsilon_{N \setminus \hat{N}})$, $\mathbf{y} \in \mathbf{A}(\mathbf{u}, \mathbf{x})$ and $\mathbf{z} \in \mathbf{A}(\varepsilon)$. (See the last paragraph of the previous section.)

Consider the special case where $\mathbf{B}$ in MS is the autarkic allocation rule leaving the endowments intact. MS then becomes

$$\mathbf{u}_{\hat{N}}\left(\mathbf{A}((\mathbf{u}_{\hat{N}}, \mathbf{A}(\varepsilon_{\hat{N}})) \vee \varepsilon_{N \setminus \hat{N}})\right) \gg \mathbf{u}_{\hat{N}}(\mathbf{A}(\varepsilon)), \qquad (SCM)$$

describing a stronger version of Postlewaite's [5] coalitional manipulability, as here the manipulating coalition $\hat{N}$ applies the same allocation rule $\mathbf{A}$ to their subeconomy $\varepsilon_{\hat{N}}$ before joining the rest of the economy while $N \setminus \hat{N}$ remains idle, $\varepsilon_{N \setminus \hat{N}}$ staying as is. We say that $\mathbf{A}$ *is coalitionally manipulable in the strong sense* iff SCM holds for some economy $\varepsilon = (\mathbf{u}, \bar{\mathbf{x}}) \in \mathcal{E}$ and $\hat{N} \subseteq N$.

It may be too strong to require that the members of the manipulating coalition end up better off no matter which allocation rule the remaining subeconomy applies. For certain allocation rules might not be feasible for the remaining subeconomy. Assuming that $\mathbf{A}$ is feasible for all subcoalitions, we define a weaker form of manipulation via segmentation by fixing $\mathbf{B} = \mathbf{A}$ in MS: We say that $\mathbf{A}$ is *weakly manipulable via segmentation by* $\hat{N}$ *at* ε iff

$$\mathbf{u}_{\hat{N}}\left(\mathbf{A}(\mathbf{u}, \mathbf{A}(\varepsilon_{\hat{N}}) \vee \mathbf{A}(\varepsilon_{N \setminus \hat{N}}))\right) \gg \mathbf{u}_{\hat{N}}(\mathbf{A}(\varepsilon)). \qquad (WMS)$$

We say that allocation rule $\mathbf{A}$ is *(weakly) manipulable via segmentation* iff it is (weakly) manipulable via segmentation by some proper (non-empty) sub-coalition $\hat{N} \subset N$ at some economy $\varepsilon \in \mathcal{E}$ with agent set N.

Now we will present an economy where the Walrasian solution $\mathbf{W}$ is manipulable via segmentation. Since MS implies both WMS and SCM, this shows that the Walrasian solution is both coalitionally manipulable in the stronger sense and weakly manipulable via segmentation.

Example 3.1. Consider the economy $\varepsilon = (\mathbf{u}_N, \bar{\mathbf{x}}_N)$, where $N = \{1, 2, 3\}$, $u_1(x, y) = \min\{25(4x + y), 40(x + y), 16(x + 4y)\}$, $u_2(x, y) = \min\{4(4x + y), x + 4y + 10\}$, $u_3(x, y) = \min\{7(4x + y), 3\,(x + 4y) + 5\}$ at each $(x, y) \in X = \mathbb{R}^2_+$, while $\bar{x}_1 = (0, 2)$ and $\bar{x}_2 = \bar{x}_3 = (1, 0)$. The indifference curves of the agents are plotted in Figure 1. We also compute the excess demand for Good 2 in ε and the two economies below as functions of the price p_2 of this good in the Appendix, and plot them in Figure 2; Good 1 is taken as the numéraire. As shown in Figure 2, the excess demand $\hat{y}_N$ in ε is zero only at $p_2 = 1$. Hence, ε has a unique Walrasian equilibrium, where the price is $p = (1, 1)$. The Walrasian allocation is

$$\mathbf{W}(\varepsilon) = \left\{ \left((1, 1), (\frac{2}{3}, \frac{1}{3}), (\frac{1}{3}, \frac{2}{3}) \right) \right\}.$$

Let $\hat{N} = \{1, 2\}$. The excess demand $\hat{y}_{\hat{N}}$ for Good 2 in economy $\varepsilon_{\hat{N}}$ becomes zero uniquely at $p_2' = 1/2$. Hence, $\varepsilon_{\hat{N}}$ also has a unique Walrasian equilibrium, in which Good 2 is cheaper. In fact,

$$\mathbf{W}(\varepsilon_{\hat{N}}) = \left\{ \left((\frac{1}{3}, \frac{4}{3}), (\frac{2}{3}, \frac{2}{3}) \right) \right\}.$$

Here Agent 2 already gets a better allocation than that of the Walrasian equilibrium at the original economy. Let us paste $\mathbf{W}(\varepsilon_{\hat{N}})$ back to the unaltered allocation of complementary subeconomy $\varepsilon_{N \setminus \hat{N}} \in \mathcal{E}_1$. The excess demand $\hat{y}'_N$ for Good 2 in this new economy is nil only at $p_2'' = 3$ (see Figure 2). Therefore, we have again a unique Walrasian equilibrium where the price is now $p'' = (1, 3)$, rendering Good 2 dearer. The new Walrasian allocation is

$$\mathbf{W}(\mathbf{u}_N, \mathbf{W}(\varepsilon_{\hat{N}}) \vee \mathbf{B}(\varepsilon_{N \setminus \hat{N}})) = \left\{ \left((\frac{13}{12}, \frac{13}{12}), (\frac{2}{3}, \frac{2}{3}), (\frac{1}{4}, \frac{1}{4}) \right) \right\}$$

at each $\mathbf{B} \in \mathcal{A}$, which leaves the endowments of $\varepsilon_{N \setminus \hat{N}}$ intact. At the new composite economy $(\mathbf{u}_N, \mathbf{W}(\varepsilon_{\hat{N}}) \vee \mathbf{B}(\varepsilon_{N \setminus \hat{N}}))$, Agent 1 faces a unique Walrasian price which compensates him against what would have been his loss at $\mathbf{W}(\varepsilon_{\hat{N}})$ (see Figure 1). As the sub-coalition $N \setminus \hat{N}$ is singleton, each allocation rule $\mathbf{B} \in \mathcal{A}$ leaves the endowments of $\varepsilon_{N \setminus \hat{N}}$ unaltered.

Notice in Figure 2 that the excess demand function $\hat{y}_N$ is very flat around the Walrasian equilibrium of the economy ε. Equivalently, the aggregate inverse-demand function is very steep around the equilibrium. Agents 1 and 2 manipulate the market by increasing their total demand

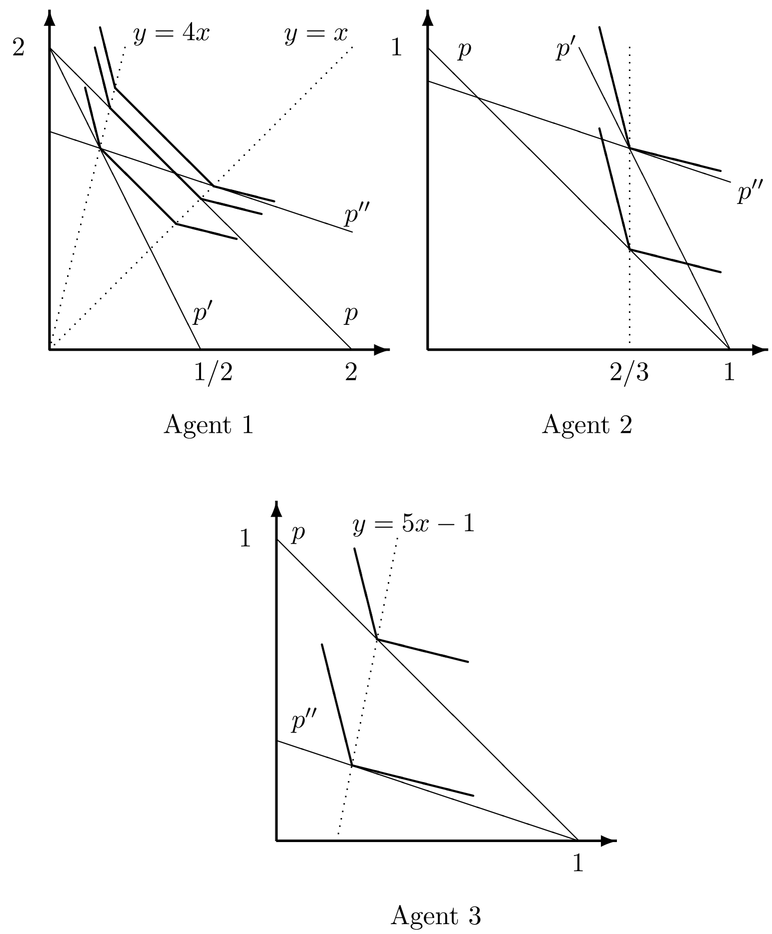

FIG. 1 The indifference curves of the agents. (The budget lines at prices $p = (1, 1)$, $p' = (1, 1/2)$, and $p'' = (1, 3)$ are indicated by the prices.)

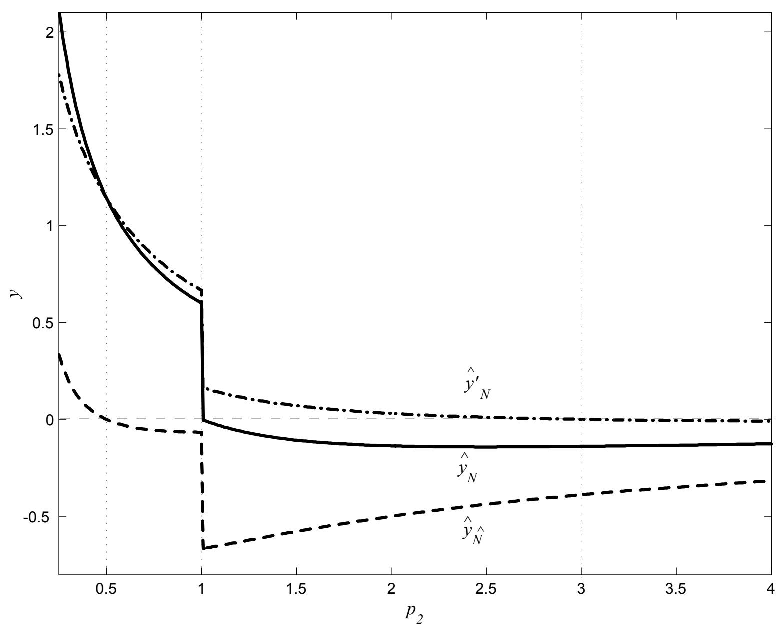

FIG. 2 The excess demand functions $\hat{y}_N$, $\hat{y}_{\hat{N}}$, and $\hat{y}'_N$ for Good 2 in economies ε, $\varepsilon_{\hat{N}}$, and $\left(\mathbf{u}_N, \mathbf{W}\left(\varepsilon_{\hat{N}}\right) \vee \mathbf{B}(\varepsilon_{N\setminus\hat{N}})\right)$, respectively. (When $p_2 > 4$, $\hat{y}_N(p_2) = \hat{y}_{\hat{N}}(p_2) = \hat{y}'_N(p_2) = -2$.)

for Good 2 slightly, which is translated to a dramatic increase in the price of Good 2. It is well-known that the market is vulnerable to such manipulations when the aggregate inverse-demand function is very steep, and the recent price hikes in Californian electricity market have been attributed to such an inverse-demand function (see [1]).

Note that the utility functions in Example 3.1 are not *strictly* quasi-concave (although they are quasi-concave). Nevertheless, we can construct strictly quasi-concave utility functions that yield the same Walrasian allocations on which agents' preferences remain the same as in Example 3.1. Note also that in this example the manipulating coalition $\hat{N}$ gains by trading before joining the remaining sub-economy, which stands idle. In the following example a coalition gains by standing idle while the remaining sub-economy trades.

Example 3.2. Consider the economy $\varepsilon = (\mathbf{u}_N, \bar{\mathbf{x}}_N)$, where $N = \{1, 2, 3\}$, $u_1(x, y) = (x+1)^3(y+1)$, $u_2(x, y) = u_3(x, y) = (x+1)(y+1)^3$ at each

$(x, y) \in X = \mathbb{R}^2_+$, $\bar{x}_1 = \bar{x}_3 = (16, 0)$ and $\bar{x}_2 = (0, 16)$. We compute that

$$\mathbf{W}(\varepsilon) = \left\{ \left((\frac{100}{7}, \frac{36}{71}), (\frac{286}{21}, \frac{850}{71}), (\frac{86}{21}, \frac{250}{71}) \right) \right\}$$

and that

$$\mathbf{W}(\mathbf{u}_N, \mathbf{W}(\varepsilon_{\hat{N}}) \vee \mathbf{W}(\varepsilon_{N \setminus \hat{N}})) = \left\{ \left((\frac{2075}{112}, \frac{139}{104}), (\frac{1067}{112}, \frac{1075}{104}), (\frac{221}{56}, \frac{225}{52}) \right) \right\},$$

where $\hat{N} = \{3\}$. Note that $u_3(\mathbf{W}(\mathbf{u}_N, \mathbf{W}(\varepsilon_{\hat{N}}) \vee \mathbf{W}(\varepsilon_{N \setminus \hat{N}}))) > 5 > u_3(\mathbf{W}(\varepsilon))$.

Now we define the second main concept of our paper. We say that allocation rule $\mathbf{A}$ is *decomposable* iff the functional equation

$$\mathbf{A}(\mathbf{u}_N \vee \mathbf{u}_{N'}, \mathbf{A}(\varepsilon) \vee \mathbf{A}(\varepsilon')) = \mathbf{A}(\varepsilon \vee \varepsilon') \qquad (D)$$

holds whenever $\varepsilon \vee \varepsilon'$ is defined, where $\mathbf{u}_N$ and $\mathbf{u}_{N'}$ are the utility function profiles of economies ε and ε', respectively. We will say that allocation rule $\mathbf{A}$ is *indecomposable* iff D fails for some pair $\varepsilon, \varepsilon' \in \mathcal{E}$ where $\varepsilon \vee \varepsilon' \in \mathcal{E}$ is defined. We will say that allocation rule $\mathbf{A}$ is *utility-wise decomposable* iff we have

$$u_i\left(\mathbf{A}(\mathbf{u}_N \vee \mathbf{u}_{N'}, \mathbf{A}(\varepsilon) \vee \mathbf{A}(\varepsilon'))\right) = u_i\left(\mathbf{A}(\varepsilon \vee \varepsilon')\right) \quad (i \in N \cup N') \qquad (UD)$$

for every two economies $\varepsilon = (\mathbf{u}_N, \bar{\mathbf{x}}), \varepsilon' = (\mathbf{u}_{N'}, \bar{\mathbf{x}}') \in \mathcal{E}$ where $\varepsilon \vee \varepsilon' \in \mathcal{E}$ is defined. When allocation rule $\mathbf{A}$ is singleton-valued, UD unambiguously states that all the agents are indifferent between the outcomes $\mathbf{A}(\mathbf{u}_N \vee \mathbf{u}_{N'}, \mathbf{A}(\varepsilon) \vee \mathbf{A}(\varepsilon'))$ and $\mathbf{A}(\varepsilon \vee \varepsilon')$. When $\mathbf{A}$ is not singleton-valued, however, it merely states that their images under $\mathbf{u}_N \vee \mathbf{u}_{N'}$ are the same. (See the last paragraph of the last section.)

Clearly, decomposability implies utility-wise decomposability. For singleton-valued Pareto-optimal allocation rules, under our strict quasi-concavity assumption, utility-wise decomposability also implies decomposability, since all agents cannot be indifferent between two distinct Pareto-optimal allocations if their utility functions are strictly quasi-concave. Hence, for singleton-valued Paretian allocation rules, decomposability and utility-wise decomposability are the same.

Clearly, if $\mathbf{A}$ is (weakly) manipulable via segmentation at some economy $\varepsilon = \varepsilon_{\hat{N}} \vee \varepsilon_{N \setminus \hat{N}} \in \mathcal{E}$, then UD fails. Thus, manipulability via segmentation implies (utility-wise) indecomposability. Since the Walrasian solution $\mathbf{W}$ is manipulable via segmentation as we saw above in Example 3.1, it is therefore (utility-wise) indecomposable as well.

Remark 3.3. Every allocation rule $\mathbf{A}$ generates a set $\alpha(\varepsilon) = \{\mathbf{u}\} \times \mathbf{A}(\varepsilon)$ of economies at every economy $\varepsilon = (\mathbf{u}, \bar{\mathbf{x}})$. With this notation, our conditions take algebraically more transparent forms. For instance, allocation rule $\mathbf{A}$ is decomposable iff

$$\alpha(\alpha(\varepsilon) \vee \alpha(\varepsilon')) = \alpha(\varepsilon \vee \varepsilon') \qquad (D')$$

holds whenever $\varepsilon \vee \varepsilon' \in \mathcal{E}$ is defined; it is manipulable by $\hat{N} \subset N$ via segmentation at $\varepsilon = (\mathbf{u}, \bar{\mathbf{x}})$ iff

$$\mathbf{u}_{\hat{N}} \left(\alpha(\alpha(\varepsilon_{\hat{N}}) \vee \beta(\varepsilon_{N \setminus \hat{N}})) \right) \gg \mathbf{u}_{\hat{N}}(\alpha(\varepsilon)) \qquad (MS')$$

for every β defined through $\beta(\varepsilon_{N \setminus \hat{N}}) = \{\mathbf{u}_{N \setminus \hat{N}}\} \times \mathbf{B}(\varepsilon_{N \setminus \hat{N}})$ with $\mathbf{B} \in \mathcal{A}$. We could also say that allocation rule $\mathbf{A}$ is *strongly decomposable* iff the functional equation[3]

$$\alpha(\alpha(\varepsilon) \vee \varepsilon') = \alpha(\varepsilon \vee \varepsilon') \qquad (SD)$$

holds whenever $\varepsilon \vee \varepsilon' \in \mathcal{E}$ is defined. Observe that strong decomposability implies decomposability: Given any two economies $\varepsilon, \varepsilon' \in \mathcal{E}$ with disjoint agent sets, SD gives $\alpha(\varepsilon \vee \varepsilon') = \alpha(\alpha(\varepsilon) \vee \varepsilon') = \alpha(\varepsilon' \vee \alpha(\varepsilon)) = \alpha(\alpha(\varepsilon') \vee \alpha(\varepsilon)) = \alpha(\alpha(\varepsilon) \vee \alpha(\varepsilon'))$, hence D'.

4. RESULTS

Our central aim is to show that any allocation rule which picks only core allocations is manipulable via segmentation (Theorem 4.1). In demonstrating this, we will rely on the fact that the Walrasian solution is manipulable via segmentation and use the fact that in replica economies the core converges to the set of Walrasian allocations.

In an atomless economy, such as an economy with uncountably many agents of each type, the set of core allocations is just the set of Walrasian allocations. In addition, given an economy, we can construct, as Kannai [4] does, an atomless economy which has the same preference-endowment distribution and, thus, the same Walrasian prices as the original economy. Therefore, we can generalize the manipulability via segmentation (and hence also utility-wise indecomposability) of the Walrasian solution to all allocation rules that pick only the core allocations. In Theorem 4.1, we show that this generalizes even to economies with only *finitely* many agents.

Theorem 4.1. *Every allocation rule $\mathbf{A} \subseteq \mathbf{C}$ (where $\mathbf{A}(\varepsilon) \subseteq \mathbf{C}(\varepsilon)$ at each economy $\varepsilon \in \mathcal{E}$) is manipulable via segmentation.*

Since the proof of Theorem 4.1 is rather technical, requiring certain core-convergence results, we relegate it to the Appendix.

While refinements of the core are all thus manipulable via segmentation, imputational allocation rules need not be. For instance, the Imputation $\mathbf{M}$ satisfies

$$\mathbf{M}(\mathbf{u}_N \vee \mathbf{u}_{N'}, \mathbf{M}(\varepsilon) \vee \mathbf{M}(\varepsilon')) \subseteq \mathbf{M}(\varepsilon \vee \varepsilon')$$

[3] Cf. *sequential fidelity* (or "path independence") of choice functions in [6] for a similar functional equation.

at every $\varepsilon = (\mathbf{u}_N, \bar{\mathbf{x}}_N), \varepsilon' = (\mathbf{u}_{N'}, \bar{\mathbf{x}}_{N'}) \in \mathcal{E}$ where $\varepsilon \vee \varepsilon' \in \mathcal{E}$ is defined, and hence is clearly non-manipulable via segmentation. As our next theorem (4.3) tells us, however, imputational allocation rules are *utility-wise* indecomposable. To prove this, we first establish a lemma:

Lemma 4.2. *Let* $\mathbf{A}$ *be an imputational allocation rule. If* $\mathbf{A}$ *is utility-wise decomposable, then* $\mathbf{A}(\varepsilon) \subseteq \mathbf{C}(\varepsilon)$ *at each economy* $\varepsilon \in \mathcal{E}$.

Proof: Take any economy $\varepsilon = (\mathbf{u}, \bar{\mathbf{x}}) \in \mathcal{E}$ and any Pareto-optimal allocation $\mathbf{x} \notin \mathbf{C}(\varepsilon)$. Take also any $\mathbf{y} \in \mathbf{A}(\varepsilon)$. Assuming that $\mathbf{A}$ is decomposable, we will show that there exists some $j \in N$ such that $u_j(\mathbf{x}) < u_j(\mathbf{y})$, and therefore $\mathbf{x} \neq \mathbf{y}$. This will show that $\mathbf{x} \notin \mathbf{A}(\varepsilon)$, establishing our Lemma. (If $\mathbf{x}$ is not Pareto-optimal, then $\mathbf{x} \notin \mathbf{A}(\varepsilon)$ by definition.)

Since $\mathbf{x}$ is not a core allocation, there exists a non-empty coalition $\hat{N}$ of N such that $\mathbf{u}_{\hat{N}}(\mathbf{x}) \ll \mathbf{u}_{\hat{N}}(\hat{\mathbf{x}}_{\hat{N}})$ for some $\hat{\mathbf{x}}_{\hat{N}} \in \mathbf{X}(\varepsilon_{\hat{N}})$. Since $\mathbf{x}$ is Pareto-optimal, $\hat{N} \neq N$. Now, assume that $\mathbf{A}$ is utility-wise decomposable. Then, $\mathbf{u}(\mathbf{A}(\mathbf{u}, \mathbf{A}(\varepsilon_{\hat{N}}) \vee \mathbf{A}(\varepsilon_{N\setminus\hat{N}}))) = \mathbf{u}(\mathbf{A}(\varepsilon))$. Hence, there exists some $\mathbf{z} \in \mathbf{A}(\mathbf{u}, \mathbf{A}(\varepsilon_{\hat{N}}) \vee \mathbf{A}(\varepsilon_{N\setminus\hat{N}}))$ with $\mathbf{u}(\mathbf{y}) = \mathbf{u}(\mathbf{z})$. Of course, there also exists some $\hat{\mathbf{z}} \in \mathbf{A}(\varepsilon_{\hat{N}}) \vee \mathbf{A}(\varepsilon_{N\setminus\hat{N}})$ such that $\mathbf{z} \in \mathbf{A}(\mathbf{u}, \hat{\mathbf{z}})$. Now, since $\mathbf{A}$ is individually rational, we have $\mathbf{u}(\mathbf{z}) \geq \mathbf{u}(\hat{\mathbf{z}})$. Since $\mathbf{A}$ is also Pareto-optimal and $\hat{\mathbf{z}}_{\hat{N}} \in \mathbf{A}(\varepsilon_{\hat{N}})$, we also have $\mathbf{u}_{\hat{N}}(\hat{\mathbf{z}}_{\hat{N}}) \not\ll \mathbf{u}_{\hat{N}}(\hat{\mathbf{x}}_{\hat{N}})$, i.e., there exists some $j \in \hat{N}$ such that $u_j(\hat{\mathbf{z}}_{\hat{N}}) \geq u_j(\hat{\mathbf{x}}_{\hat{N}})$. Now observe that $u_j(\mathbf{y}) = u_j(\mathbf{z}) \geq u_j(\hat{\mathbf{z}}) \equiv u_j(\hat{\mathbf{z}}_{\hat{N}}) \geq u_j(\hat{\mathbf{x}}_{\hat{N}}) > u_j(\mathbf{x})$, so $u_j(\mathbf{y}) > u_j(\mathbf{x})$, as we wanted to show. ■

Theorem 4.3. *Every imputational allocation rule is (utility-wise) indecomposable.*

Proof: Suppose that allocation rule $\mathbf{A}$ is both imputational and utility-wise decomposable. This leads to a contradiction. For then, by Lemma 4.2, we have $\mathbf{A}(\varepsilon) \subseteq \mathbf{C}(\varepsilon)$ at each economy $\varepsilon \in \mathcal{E}$, thereby Theorem 4.1 tells us that it is manipulable via segmentation, and so it is utility-wise indecomposable. ■

The hypothesis of Theorem 4.3 is that the allocation rule be imputational (i.e., individually rational and Pareto-optimal). In looking for a decomposable allocation rule, if we had not insisted on individual rationality, then the Walrasian allocation from equal division would have fit the bill, since it is Pareto-optimal and clearly decomposable. If instead, we had lifted the Pareto-optimality requirement, then the autarkic solution would have fit the bill, as it is obviously individually rational and decomposable. That is, nothing in the hypothesis of Theorem 4.3 is superfluous. Actually, the Walrasian solution from equal division is a representative of a general class of allocation rules satisfying our decomposability property trivially. This class is the family of "division rules", i.e., allocation rules whose values depend only on preferences and the total endowment in the economy, exhausting the total endowment.

Now we describe a family of individually rational allocation rules that exhaust the total endowment and are indecomposable, yet non-manipulable via segmentation. This family consists of the fixed-price allocation rules that allow maximal possible trade at some fixed price $p \in \mathcal{P}$. Consider a consumption space with only two goods (i.e., $X = \mathbb{R}^2_+$). Take any price p, and any economy $\varepsilon \in \mathcal{E}$. We consider individuals' demands at the fixed price p. If there is no excess demand (at p), we give every agent what he demands. When there is excess demand for some good, we classify the agents of our economy into two disjoint groups. We put the agents with non-positive net demands for the excessively demanded good in Group 1, and the rest in Group 2. We then allow Group 1 to get what they demand, and we let the agents in Group 2 buy the excessively demanded good only in such non-negative rations that no individual buys more than he wants and the market clears. Restrict each agent's ration of a given good to depend only on the total endowment of the good, the excess demand for the good, and the agent's own demand for the good. It is easy to check that no such allocation rule is decomposable[4]. Any allocation rule in this family is, however, non-manipulable via segmentation. To see this, first note that no subcoalition containing any agents in Group 1 can manipulate, as such agents are already getting their best outcomes at the fixed price p. On the other hand, for subeconomies whose agent set is contained in Group 2, our allocation rule prescribes the autarkic allocation. Therefore, the final allocation is not altered by segmentation of such coalitions. Thus, such a subeconomy's agents will find themselves, at the final collage economy to which our fixed-price allocation rule is to be applied, facing a state where the total endowments, the individual demands, and hence the excess demand are all just the same as in the original economy. Hence, they will come out with the same consumption bundles, and so they have no incentive to segment. This shows that our fixed-price allocation rule is non-manipulable via segmentation. Note that this same allocation rule is "manipulable via segmentation" in the weaker sense by a coalition owning some agent who benefits from segmentation without hurting any member of the coalition.[5]

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[4] Fix $p = (1, 1)$ and consider an economy $\varepsilon = ((u, u, u), ((0, 2), (2, 0), (2, 0))$ such that each individual demands $(1, 1)$ at price p. As there is excess demand in the second good and Agents 2 and 3 have positive net demand for that good, one of these agents, say 2, will receive $(2 - a, a)$ for some $a \in [0, 1)$. On the other hand, to observe a segmentation, consider the subeconomy $((u, u), ((0, 2), (2, 0))$ with the coalition $\{1, 2\}$ as its agent set. For this subeconomy there is no excess demand at p, hence each agent here gets $(1, 1)$. When we paste the subeconomies at their new allocations we obtain the economy $\varepsilon' = ((u, u, u), ((1, 1), (1, 1), (2, 0))$. Each agent here again demands $(1, 1)$, but now Agents 1 and 2 both have zero net demand, and so the allocation rule will leave the new composite economy unaltered. Thus, segmentation results in economy ε', while direct application of our allocation rule to ε gives $\mathbf{A}(\varepsilon) = \{((1, 1), (2-a, a), (1+a, 1-a))\}$, disagreeing with the endowments of ε'.

[5] The example of Footnote 4 displays an economy at which our fixed-price allocation rule is "manipulable via segmentation" (by coalition $\{1, 2\}$) in this weaker sense.

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APPENDIX A: OMITTED PROOFS

In this appendix, we will show that the Walrasian equilibria in Example 3.1 are unique, and then prove Theorem 4.1.

Uniqueness in Example 3.1 We have a Walrasian equilibrium iff the excess demand for Good 2 is nil. It is exhibited in Figure 2 that excess demand functions become zero uniquely at the Walrasian price-vectors in Example 3.1. This is especially clear for the excess demand functions $\hat{y}_N$ and $\hat{y}_{\hat{N}}$ in economies ε and $\varepsilon_{\hat{N}}$, respectively. Here, we will compute the excess demand functions, and show that the excess demand function $\hat{y}'_N$ in economy $(\mathbf{u}_N, \mathbf{w}\left(\varepsilon_{\hat{N}}\right) \vee \bar{\mathbf{x}}_{N\setminus\hat{N}})$ is strictly decreasing, thus can be zero at most one price, where $\mathbf{w}\left(\varepsilon_{\hat{N}}\right)$ is the only member of $\mathbf{W}\left(\varepsilon_{\hat{N}}\right)$. Towards this goal, we compute the agents' net demands for Good 2 as functions of the price p_2 of this good. We focus on the case $p_2 \in (1/4, 4)$, because the slopes of the indifference curves are bounded by -4 and $-1/4$, and hence there cannot be any market-clearing price outside this region. (The excess demand is non-zero when $p_2 \in \{1/4, 4\}$.) At any $p_2 \in (1/4, 4)$, agents 2 and 3 demand consumption bundles on the lines defined by $x = 2/3$ and $y = 5x - 1$, respectively. Hence, their (net) demands for Good 2 are $\hat{y}_2(p_2) = \frac{1}{3p_2}$ and $\hat{y}_3(p_2) = 4/(1 + 5p_2)$. Similarly, the net demand $\hat{y}_1(p_2)$ of Agent 1 is $-2/(1 + 4p_2)$ if $p_2 \in (1/4, 1)$, $-2/(1 + p_2)$ if $p_2 \in (1, 4)$, and can take any value in $[-1, -2/5]$ if $p_2 = 1$. The excess demand functions for Good 2 in economies ε and $\varepsilon_{\hat{N}}$ are $\hat{y}_N = \hat{y}_1 + \hat{y}_2 + \hat{y}_3$ and $\hat{y}_{\hat{N}} = \hat{y}_1 + \hat{y}_2$, respectively. These functions are plotted in Figure 2.

Now consider the economy $(\mathbf{u}_N, \mathbf{w}\left(\varepsilon_{\hat{N}}\right) \vee \bar{\mathbf{x}}_{N\setminus\hat{N}})$. The net demand $\hat{y}'_1(p_2)$ of Agent 1 is 0 if $p_2 \in (1/4, 1)$, $-1/(1 + p_2)$ if $p_2 \in (1, 4)$, and can take any value in $[-1/2, 0]$ if $p_2 = 1$. The net demand $\hat{y}'_2$ of Agent 2 for

Good 2 is identically zero, and the net demand $\hat{y}_3$ of Agent 3 is unaltered. If $p_2 \in (1/4, 1)$, the excess demand $\hat{y}'_N(p_2)$ is simply $\hat{y}_3(p_2) = 4/(1+5p_2)$, a decreasing function. If $p_2 \in (1/4, 1)$, we have $\hat{y}'_N(p_2) = 4/(1+5p_2) - 1/(1+p_2)$, yielding

$$\frac{d\hat{y}'_N}{dp_2} = \frac{5p_2^2 - 30p_2 - 19}{(1+5p_2)^2(1+p_2)^2},$$

which is negative whenever $p_2 \in \left(3 - 8/\sqrt{5}, 3 + 8/\sqrt{5}\right) \simeq (-0.6, 6.6)$. Therefore, $\hat{y}'_N$ is decreasing when $p_2 \in (1/4, 1)$, too.

Proving Theorem 4.1 First, we need to develop some new notation and present the basic results we use in the proof. $\mathbb{N}$ will denote the set of positive integers. Given any function f defined on a finite set S and any $k \in \mathbb{N}$, by a k-replica of f we mean a function f^k defined on a set S^k with $|S^k| = k|S|$ such that $\{(s, f(s))\}_{s \in S}$ has exactly k copies in $\{(s^k, f^k(s^k))\}_{s^k \in S^k}$. Given any subset T of S and $k \in \mathbb{N}$, we write f_T^k for the k-replica of the restriction f_T, i.e., $f_T^k = (f_T)^k$. For instance, we write $\varepsilon_{\hat{N}}^k$ for the k-replica of subeconomy $\varepsilon_{\hat{N}}$. Finally, we let $\|\cdot\|_\infty$ be the supremum norm.

We will use the facts A.1, A.2 and A.3 below in our proof. The first one (A.1) is the core-convergence result of Debreu and Scarf [2], which we use in our proof repeatedly. The second one (A.2) is the upper semi-continuity of the core correspondence, for which we give a simple proof right away. The third one (A.3.) is a weaker form of the "equal treatment property" of the core, which simplifies our arguments substantially.

Fact A.1. *Let $\{\varepsilon^k\}_{k \in \mathbb{N}}$ be a sequence of replica economies, where every ε^k is a k-replica of ε^1. Then, for every $\epsilon > 0$, there exists an integer $\bar{k}$ such that for every $k > \bar{k}$ and for every allocation $\mathbf{x} \in \mathbf{C}(\varepsilon^k)$, there is a Walrasian equilibrium $(p, \mathbf{q})$ of ε^k with $\|\mathbf{x} - \mathbf{q}\|_\infty \le \epsilon$.*

Fact A.2. *For fixed N and $\mathbf{u}$, the core correspondence $\Gamma : \mathbf{x} \mapsto \mathbf{C}(\mathbf{u}, \mathbf{x})$ is upper semi-continuous on X^N, i.e., given any $\epsilon > 0$ and any $\mathbf{x} \in X^N$, there exists a positive real number $\delta > 0$ such that, for every $\mathbf{x}' \in X^N$ with $\|\mathbf{x}' - \mathbf{x}\|_\infty < \delta$ and for every $\mathbf{y}' \in \mathbf{C}(\mathbf{u}, \mathbf{x}')$, there exists some $\mathbf{y} \in \mathbf{C}(\mathbf{u}, \mathbf{x})$ with $\|\mathbf{y}' - \mathbf{y}\|_\infty < \epsilon$.*

To prove this familiar assertion, first note that Γ is non-empty-valued and, as a sub-correspondence of a locally bounded correspondence $\mathbf{x} \mapsto \mathbf{X}(\mathbf{u}, \mathbf{x})$, also locally bounded. Take any sequence $\{\mathbf{x}(k)\}_{k \in \mathbb{N}}$ and any sequence $\{\mathbf{y}(k)\}_{k \in \mathbb{N}}$ with $\mathbf{y}(k) \in \mathbf{C}(\mathbf{u}, \mathbf{x}^k)$ for each $k \in \mathbb{N}$. Assume that $\mathbf{x}(k) \to \mathbf{x}$ and $\mathbf{y}(k) \to \mathbf{y}$. Suppose that $\mathbf{y} \notin \mathbf{C}(\mathbf{u}, \mathbf{x})$. Since $\mathbf{u}$ is continuous, there then exist some $\mathbf{z} \in \mathbf{X}(\mathbf{u}, \mathbf{x})$, a neighborhood $\eta(\mathbf{z})$ of $\mathbf{z}$ and a coalition $N' \subseteq N$ such that $\mathbf{u}_{N'}(\mathbf{z}') > \mathbf{u}_{N'}(\mathbf{y}')$ whenever $\mathbf{z}' \in \eta(\mathbf{z})$ and $\mathbf{y}' \in \eta(\mathbf{y})$ for some neighborhood $\eta(\mathbf{y})$ of $\mathbf{y}$. Since $\mathbf{x}(k) \to \mathbf{x}$ and $\mathbf{y}(k) \to \mathbf{y}$, this implies that for some sufficiently large k, $\mathbf{y}(k) \notin \mathbf{C}(\mathbf{u}, \mathbf{x}(k))$, a contradiction. This completes the proof of (A.2).

Fact A.3. *Given any k-replica economy ε^k, both $\mathbf{W}(\varepsilon^k)$ and $\mathbf{C}(\varepsilon^k)$ are sets of k-replica allocations. To be precise, $\mathbf{y} \in \mathbf{W}(\varepsilon^k)$ iff $\mathbf{y} = \mathbf{x}^k$ for some $\mathbf{x} \in \mathbf{W}(\varepsilon)$; and, given any $\mathbf{y} \in \mathbf{C}(\varepsilon^k)$, we have $\mathbf{y} = \mathbf{x}^k$ for some $\mathbf{x} \in \mathbf{C}(\varepsilon)$.*

Proof of Theorem 4.2. Let $\varepsilon = (\mathbf{u}, \bar{\mathbf{x}})$ be the economy in Example 3.1. Note that, since $\varepsilon^k_{N \setminus \hat{N}} \in \mathcal{E}_1$, we have $\mathbf{B}(\varepsilon^k_{N \setminus \hat{N}}) = \{\bar{x}^k_3\}$ at each $\mathbf{B} \in \mathcal{A}$. For each $k \in \mathbb{N}$, write $\mathbf{x}^k$ for generic elements of $\mathbf{C}(\varepsilon^k)$ and $\tilde{\mathbf{x}}^k$ for generic elements of $\mathbf{C}(\mathbf{u}^k, \mathbf{C}(\varepsilon^k_{\hat{N}}) \vee \{\bar{x}^k_3\})$. We **claim** that there exists $\bar{k} \in \mathbb{N}$ such that, for every integer $k > \bar{k}$, we have $\mathbf{u}^k_{\hat{N}}(\mathbf{x}^k) \ll \mathbf{u}^k_{\hat{N}}(\tilde{\mathbf{x}}^k)$ at each such $\mathbf{x}^k$ and $\tilde{\mathbf{x}}^k$. Under the assumption that $\mathbf{A} \subseteq \mathbf{C}$, this will imply that, for every $k > \bar{k}$, we have $\mathbf{u}^k_{\hat{N}}(\mathbf{A}(\varepsilon^k)) \ll \mathbf{u}^k_{\hat{N}}(\mathbf{A}(\mathbf{u}^k, \mathbf{A}(\varepsilon^k_{\hat{N}}) \vee \{\bar{x}^k_3\}))$, whence the coalition $\hat{N}^k$ gains by segmentation, which will suffice to prove our Theorem.

By (A.3), we know from Example 3.1 that $\mathbf{u}^k_{\hat{N}}(\mathbf{W}(\varepsilon^k)) \ll \mathbf{u}^k_{\hat{N}}(\mathbf{W}(\mathbf{u}^k, \mathbf{W}(\varepsilon^k_{\hat{N}}) \vee \{\bar{x}^k_3\}))$. Now, since $\mathbf{u}^k_{\hat{N}}$ is continuous, there exists some $\epsilon > 0$ such that $\mathbf{u}^k_{\hat{N}}(\mathbf{x}^k) \ll \mathbf{u}^k_{\hat{N}}(\tilde{\mathbf{x}}^k)$ whenever

$$\left\|\mathbf{x}^k - \mathbf{w}(\varepsilon^k)\right\|_\infty \leq \epsilon \tag{A.4}$$

and

$$\left\|\tilde{\mathbf{x}}^k - \mathbf{w}(\mathbf{u}^k, \mathbf{w}(\varepsilon^k_{\hat{N}}) \vee \bar{x}^k_3)\right\|_\infty \leq \epsilon, \tag{A.5}$$

where $\mathbf{w}(\cdot)$ is the unique member of $\mathbf{W}(\cdot)$. By (A.1), there exists some $k_1 \in \mathbb{N}$ such that, for every $k > k_1$, (A.4) holds at each $\mathbf{x}^k$. Now we show that there exists some $k_2 \in \mathbb{N}$ such that, for every $k > k_2$, (A.5) holds at each $\tilde{\mathbf{x}}^k$. Setting $\bar{k} = \max\{k_1, k_2\}$ will then establish our above **claim**, which is all we need. Now, by (A.3), we have $\mathbf{w}(\varepsilon^k_{\hat{N}}) \vee \bar{x}^k_3 = \left[\mathbf{w}(\varepsilon_{\hat{N}}) \vee \bar{x}_3\right]^k$. Hence, by (A.1), there exists some positive integer k_3 such that

$$\left\|\mathbf{y} - \mathbf{w}(\mathbf{u}^k, \mathbf{w}(\varepsilon^k_{\hat{N}}) \vee \bar{x}_3)\right\|_\infty \leq \epsilon/2 \tag{A.6}$$

at each $\mathbf{y} \in \mathbf{C}(\mathbf{u}^k, \mathbf{w}(\varepsilon^k_{\hat{N}}) \vee \bar{x}^k_3)$ whenever $k > k_3$. On the other hand, by (A.2), there exists some $\delta' > 0$ such that, at each $\tilde{\mathbf{x}}^k$, we have

$$\left\|\tilde{\mathbf{x}}^k - \mathbf{y}\right\|_\infty \leq \epsilon/2 \tag{A.7}$$

for some $\mathbf{y} \in \mathbf{C}(\mathbf{u}^k, \mathbf{w}(\varepsilon^k_{\hat{N}}) \vee \bar{x}^k_3)$ whenever we have

$$\left\|\mathbf{z} - \mathbf{w}(\varepsilon^k_{\hat{N}})\right\|_\infty \leq \delta' \tag{A.8}$$

at each $\mathbf{z} \in \mathbf{C}(\mathbf{u}^k, \varepsilon^k_{\hat{N}})$. We can apply (A.1) to the sequence $(\varepsilon^k_{\hat{N}})$, however, to find some positive integer k_4 such that (A.8) (hence (A.7)) holds for every $k > k_4$. By the triangle inequality, (A.6) and (A.7) imply that (A.5) holds for every $k > k_2 \equiv \max\{k_3, k_4\}$, establishing our **claim** and thus completing the proof. ■

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