Durable Bargaining Power and Stochastic Deadlines[1]

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March 10, 2009

[1]We thank Glenn Ellison and Bob Wilson for helpful comments. We also thank the seminar participants at 2008 AEA meeting, UCL and Chinese University of Hong Kong.

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Abstract

Bargaining power in real bargaining situations enter and exit periods of stability, and these dynamics may have non-trivial effects on the bargaining outcome. To capture this phenomenon, we develop a continuous-time bargaining model in which bargaining power evolves over time according to a stochastic process. We define a notion of durability for the bargaining power as a condition on the stochastic process and we show that durabilty plays a central role when agents are optimistic. In particular, when bargaining power becomes more durable after a date, optimistic parties are enticed to wait until that date, leading to a period of disagreement. This "durability effect" is closely related to a an empirical regularity in bargaining, the "deadline effect". Using the same analysis that leads to the durability effect, we show that when players are optimistic and bargaining power is durable around a period during which a (possibly stochastic) deadline is likely to arrive, the parties are enticed to delay the agreement until the deadline, as widely observed in experiments and real-world negotiations. We show that this deadline effect is stronger when the bargaining power is more durable around the likely time of deadline, or when the uncertainty about the deadline is smaller, or the parties are optimistic.

1 Introduction

In an ongoing bargaining, bargaining power of the sides may become more durable in some periods and less durable in some others. There are many examples for such varying durability of bargaining power in congressional decision making, wage bargaining, international peace negotiations, and intra-household bargaining. For example, in congressional decision making, each party's bargaining power becomes more durable immediately after a general election because the election determines how much control each party will have in the government for a while. For another example, in a wage bargaining between a union and an employer, the bargaining power of each side is tied to the price of the firm's product and the unemployment rate. The employer negotiates from a strong position in recessions and from a weak position in booms. As the economy becomes more stable, the bargaining power of each side becomes more durable. Lastly, for a recent example in policy making, consider the energy policy in the US. There are several policy alternatives, such as off-shore drilling, developing alternative energy sources, and improving the energy efficiency. The political appeal of these alternatives depends on the global oil prices. When the price of oil is stable (as in the 1990s and early 2000s), the political appeal of those policies remain stable, leading to a stable bargaining power for the proponents of each policy alternative. When the price of oil becomes volatile (as in 1973 and 2008), the bargaining power becomes less durable. For example, in the summer of 2008, the oil prices were high and off-shore drilling was a hot, winning alternative, but in the fall of 2008, the oil prices were low and off-shore drilling was no longer appealing.

In this paper, we explore the role of durability of bargaining power in bargaining patterns and outcomes. We consider two players who want to divide a dollar. We define a player's *bargaining power* as the share of the surplus she gets in addition to her continuation value. (The surplus is the total cost of not reaching an agreement at the time.) We develop a continuous-time bargaining model in which bargaining power evolves over time according to a stochastic process. We define a notion of durability for the bargaining power as a condition on the stochastic process, measuring how slowly the bargaining power changes as time passes. We show that durability plays a central role when agents are optimistic, which is reportedly common in real life negotiations (see Babcock and Loewenstein (1997)). We also provide a rationale for the deadline effect, which turns out to be closely related to durability.

We first observe that the standard model of sequential bargaining with commonly used assumptions cannot capture issues related to the durability of the bargaining power. In sequential bargaining, at any instance, the proposer gets the entire surplus, leaving the other party indifferent between agreeing and disagreeing. Hence, the proposer has all the bargaining power for that instance. While this could be a reasonable way to model bargaining power, the usual assumptions for bargaining protocol are not appropriate for modeling bargaining power. For example, Rubinstein's (1982) alternating-offer bargaining assumes that bargaining power shifts from one end to the other frequently. Similarly, random proposal model with independence (Binmore, 1987) assumes that bargaining power is so transient that the bargaining power now has no impact on the bargaining power a moment later.[1]

In order to study durability of the bargaining power, we modify the standard bargaining setup by modeling the bargaining power explicitly as a continuous-time process. More specifically, we assume that the bargaining power is exogenously given as a continuous-time stochastic process which can take values between 0 and 1 (as opposed to just 0 or 1) and which has piecewise-continuous sample paths almost surely. When this process moves more slowly, the bargaining power is more durable. We introduce a formal measure of durability in terms of the stochastic rate at which the process changes. The players have possibly heterogenous beliefs for the future values of this process but they observe and agree on the current instantaneous bargaining power. The payoffs are determined by the requirement that each player's share of the surplus is consistent with the instantaneous bargaining power, and agents' agreement/disagreement decisions are characterized by individual rationality at all times and states.

The payoffs are uniquely determined, and the behavior corresponds to the subgame perfect equilibrium of a standard bargaining game in which, at each time, each player proposes with probability equal to her current bargaining power. Under the common-prior assumption, we obtain an intuitive closed-form solution. Unlike in the case of alternating-offer bargaining, the solution has many robustness properties in the continuous-time limit. Hence, our model fits into the standard bargaining framework, but it is more general, more tractable, more flexible, and it allows us to model durability of the bargaining power.

Durability has a profound role when the players have heterogenous beliefs about the bargaining power, in particular, when they are optimistic about what the bargaining power will be in future dates. We demonstrate this with the following phenomenon, which we call *the durability effect*. We show that if players are optimistic about their bargaining powers at some time t^*, and the bargaining power will become more durable after t^*, then there will be a period of inactivity before t^* in which there will not be any agreement, and there will be an agreement afterwards with high probability.

For example, suppose that at t^*, there will be a general election that will determine how much control each party will have in the government for the next several years. In that case, starting from t^*, the bargaining powers of the parties will be highly durable for a while. On the other hand, it is natural to assume that the bargaining power is less

[1] One can take the probability of being the proposer as the bargaining power, but then the above model can only capture the case in which the bargaining powers of individuals throughout the process is known at the beginning, ruling out all the phenomena we are interested in.

durable before the election. For example, prior to the election, there may be several different events, such as a terrorist attack, a financial meltdown, or a political scandal, that affect the election results. Hence, there may be a time $\hat{t}$ before the election at which the parties have a considerable uncertainty about the election results, facing considerable uncertainty about the bargaining power after t^*. In that case, our result implies that, if the uncertainty leads parties to have highly optimistic beliefs about the election results, it will not be possible to pass any policy bill between $\hat{t}$ and the election time t^*, and those bills will pass right after the election (possibly with different terms, depending on the election results). Likewise, in the wage-bargaining example, suppose that, at t^*, the country is to join the EU, making the local economy more stable. As we explained above, that makes the bargaining power more durable. Then, our result suggest that the parties may wait the country to join the EU to sign long-term employment contracts.

The intuition for this result is as follows. At t^*, knowing that their bargaining powers in a considerable future will be similar to their current bargaining powers, the agents will hold very similar beliefs about their future bargaining powers, and they will reach an agreement, each getting a share that is in accordance with her bargaining power at t^*. That is, each party's share in the actual agreement at t^* is closely related to their bargaining power at t^*. Then, at the earlier time $\hat{t}$, if the parties are highly optimistic about their bargaining powers at t^*, they will also be optimistic about those shares. Such optimism may render an agreement at $\hat{t}$ impossible because agreement has to give each agent at least as much as what she expects to get from waiting until t^*.

One of the most robust empirical regularities in bargaining is the phenomenon called *the deadline effect*: the agreement is delayed until the very last minute before the deadline. Such behavior is commonly observed in a wide range of laboratory experiments as well as real-life negotiations (see e.g., Roth, Murnighan, and Schoumaker (1988)). For example, the unions and employers reach "eleventh-hour agreements" just before the unions' deadline to strike. Litigants often reach a "settlement on the courthouse steps."

We observe that the durability and the deadline effects are closely related. Imagine that the durability of bargaining power does not necessarily change much, but there is a deadline, after which the parties cannot agree. Suppose that the time of the deadline is stochastic. For example, it is uncertain when an arbitrator will make a decision, after which bargaining is moot. Suppose also that it is very likely that the deadline will arrive within a small interval of time that starts at t^*. In that case, the effective discount factors in that interval will be very small, depending on the conditional probability of arrival at that moment. This is similar to stretching the time in that interval, stretching more when the arrival is more likely. If the bargaining power is somewhat durable in that interval, this stretching of time amounts to making bargaining power more durable. In that case, our result about the durability effect tells us that this will lead to a period of inactivity before t^*, followed by an agreement with high probability, as in the deadline effect. Therefore, the durability and the deadline effects are two sides of the same coin, and the mechanism that leads to the durability effect can also explain the deadline effect.

In order to formalize this idea, we consider some date t^*, such that bargaining power is somewhat durable around t^*. We show that if it is sufficiently likely that the deadline arrives within a sufficiently small neighborhood of t^* and the parties are sufficiently optimistic about their bargaining power at t^*, then they will not settle until sometime around t^*, and they will reach an agreement "just before the deadline" with high probability.[2] (With some small probability they will fail to agree before the deadline.) This result provides a rationale for the deadline effect based on optimism. More importantly, we find explicit conditions under which the deadline effect occurs. We demonstrate that whether there will be a deadline effect depends on the ratio of arrival rate of deadline to the "inverse rate of durability". When this ratio exceeds a cutoff value, there is a deadline effect. That is, keeping everything else constant, if we make the deadline arrive faster, or make the bargaining power more durable around the deadline, then the deadline effect will be stronger. Moreover, the cutoff is a decreasing function of the optimism level about the bargaining power at t^*, i.e. optimism makes the deadline effect stronger.

A firm deadline is often imposed upon negotiators in order to prevent them from dragging out the negotiations indefinitely. Ironically, such deadlines themselves sometimes entice parties to delay the agreement. The above comparative statics show that, if this happens because of the parties' optimism, then one may be able to avoid such a deadline effect by making the time of deadline more uncertain, or putting the deadline in a period at which the bargaining power is less durable.

In this paper, abstracting from the details, we model the bargaining power as the share one gets from the surplus. This is consistent with both sequential bargaining, in which the bargaining power is represented as the power to propose, and the (asymmetric) Nash bargaining. It is also consistent with applied work, where one models bargaining power by ability to make a take-it-or-leave-it offer. Within this abstract model, we gain some insights into how the durability of bargaining power affects the outcome. In the specific real-world examples, however, the parties' bargaining powers are related to specific factors within the application. Of course, one needs to analyze the specific situation with its details to understand how durability affects the bargaining outcome in that specific situation.

[2] Durability at t^* is crucial for this result. Indeed, we show that under the usual independence assumption, the deadline effect disappears in the continuous-time limit when there is any uncertainty about the deadline. This is because the independence assumption leads to an extremely discontinuous (and extremely non-durable) bargaining power in continuous time, leading to a highly discontinuous equilibrium behavior.

The outline of the paper is as follows. In the next section, we lay out our model and characterize the unique solution. In Section 3, we present the solution under the common prior assumption. In Section 4, we formally introduce our notions of durability. In Section 5, we present our main result which shows the durability effect. Section 6 establishes the relationship between the deadline and the durability effects. Section 7 concludes. We present the proofs in an appendix. Throughout, we demonstrate our general results with examples from a canonical stylized model which may be of interest in itself. In this model, the bargaining power changes only after certain events that arrive with Poisson distribution and the deadlines are exponential so that the rates of durability and deadlines are constant.

2 The Model

Two risk-neutral agents, $i, j \in \{1, 2\}$, want to divide a dollar among themselves before a deadline. (The possible divisions are of the form $(x, 1-x)$ for $x \in [0, 1]$.) The set of times t is a continuum $T = [0, \infty)$. The agents discount the future payoffs, so that the payoff of getting x at t for an agent is $e^{-rt}x$ for some constant discount rate r. The payoffs are 0 if agents never agree. The agents can strike a deal only on a grid $T^* = \{0, 1/n, 2/n, \ldots\}$, where n is a large integer. The (instantaneous, relative) *bargaining power* of an agent is defined as the portion of the surplus, or the gain from agreement, the agent gets in addition to his continuation value from delay. Here, an agent i's *continuation value from delay* at t is defined as the amount $\bar{x}^i$ such that i is indifferent between getting $\bar{x}^i$ at t and waiting until $t+1/n$ for another opportunity to strike a deal. The surplus is defined as $1-\bar{x}^1-\bar{x}^2$. Hence, assuming that the surplus is positive, the instantaneous bargaining power of an agent i is

$$\pi_t^i = \frac{x^i - \bar{x}^i}{1 - \bar{x}^1 - \bar{x}^2} \tag{1}$$

where x^i is his share. In this paper, we take the agents' bargaining powers as the primitives of the model and compute the agents' continuation values, which in turn determine whether agents agree and (if so) how they divide the dollar at any given time. (Here, we make the standard assumption that the bargaining power is independent of the continuation values.)

We allow the agents' bargaining powers to be stochastic and the agents to have subjective beliefs about how their bargaining power will evolve. We consider a state space Ω, endowed with a σ-algebra Σ and a filtration $(\mathcal{F}_t)_{t \in T}$. The instantaneous bargaining power of agent 1 is given by an adapted stochastic process $\pi_t^1 : \Omega \to [0, 1]$, $t \in T$, such that the mapping $(\omega, t) \mapsto \pi_t^1(\omega)$ is measurable. We define the process $(\pi_t^2)_{t \in T}$ by $\pi_t^2 = 1 - \pi_t^1$. If the true state of the world is ω, then the bargaining power of agent i at t is $\pi_t^i(\omega)$. We endow (Ω, Σ)

with two probability distributions P^1 and P^2, representing the beliefs of agents 1 and 2, respectively. For any random variable X and any event A, we write $E_t^i[X](\omega)$ and $\Pr_t^i(A|\omega)$ for the conditional expectation of X and the conditional probability of A, respectively, at state ω with respect to $(\Omega, \mathcal{F}_t, P^i)$. We suppress the dependence on ω in our notation and simply write $E_t^i[X]$ or $\Pr_t^i(A)$ when it does not lead to a confusion. Note that with this specification, at any t, the agents know and agree on the instantaneous bargaining power at t but may hold subjective beliefs about how the bargaining power will evolve.

The common-prior assumption (CPA) is

$$P^1 = P^2. \tag{CPA}$$

We define the optimism at time t about time $s \geq t$ (as a function of ω) as

$$y_{t,s} = E_t^1\left[\pi_s^1\right] + E_t^2\left[\pi_s^2\right] - 1. \tag{2}$$

Here, $y_{t,s}$ is the amount by which an agent over-estimates her own bargaining power with respect to the other agent, i.e. $y_{t,s} = E_t^i[\pi_s^i] - E_t^j[\pi_s^i]$ for $i \neq j$. Under the common prior assumption, $y_{t,s} = 0$ for all t and $s \geq t$. In general, when $t < s$, $y_{t,s}$ can take any value in $[-1, 1]$. We simplify our notation by using $y_t = y_{t,t+1/n}$.

We model the deadline as a positive random variable d with cumulative distribution function F. For simplicity, we assume that d is stochastically independent from $\mathcal{F}_t$, $t \in T$. At $t = d$, the negotiation automatically ends, and each agent gets payoff of 0 at any $t > d$ if they have not agreed at d or before. For $t \leq s$, $\frac{1-F(s)}{1-F(t)}$ is the conditional probability that the deadline will not arrive before s given it has not arrived by t. The deadline is said to be *deterministic* if and only if $d = d^*$ with probability 1 for some $d^* \in [0, \infty]$. The case $d^* = \infty$ corresponds to the case that there is no deadline. We define the effective discount factor between t and s as

$$\delta_{t,s} = e^{-r(s-t)} \frac{1 - F(s)}{1 - F(t)}. \tag{3}$$

For notational convenience, we write δ_t for $\delta_{t,t+1/n}$. Agents discount the $t + 1/n$ payoffs by the effective discount factor δ_t.

We let V_t^i denote the continuation value of agent i at t, as a function of ω. By individual rationality, V_t^i is restricted to be in $[0, 1]$. For any $t = k/n \geq 0$, we let

$$S_{t+1/n} \equiv E_t^1\left[V_{t+1/n}^1\right] + E_t^2\left[V_{t+1/n}^2\right]$$

denote the size of the pie at $t+1/n$ as perceived by the agents at t. Under the common-prior assumption, $S_{t+1/n} = 1$ for each $t \geq 0$. It can take any value in $[0, 2]$ in general. If

$$\delta_t S_{t+1/n} > 1,$$

there cannot be an agreement that would satisfy both agents' expectations, as the sum of their continuation values from delay exceeds 1. Assuming that the agents are individually rational, we reckon that there will not be an agreement at such t. In that case, the continuation value of i at t will be

$$V_t^i = \delta_t V_{t+1/n}^i.$$

Now consider the case that $\delta_t S_{t+1/n} \leq 1$. In that case, the surplus is $1 - \delta_t S_{t+1/n}$, and by (1), we conclude that the agents agree on the division (V_t^1, V_t^2) with

$$V_t^i = \pi_t^i \left(1 - \delta_t S_{t+1/n}\right) + \delta_t E_t^i \left[V_{t+1/n}^i\right];$$

an agent's continuation value will be simply his share in the agreement. The last two difference equations can be combined as

$$V_t^i = \pi_t^i \max\left\{1 - \delta_t S_{t+1/n}, 0\right\} + \delta_t E_t^i \left[V_{t+1/n}^i\right]. \tag{4}$$

The continuation values are determined by the difference equation (4), the boundary condition that $V_t^i = 0$ for $t > d$, and the condition that $V_t^i \in [0, 1]$.

Formally, a solution is a stochastic process $V : \Omega \times T^* \to [0, 1]$ that satisfies (4) and boundary condition, meaning that continuation values are consistent with the exogenously given bargaining power and the fact that each gets zero after the deadline. As we will show below in Proposition 1, there is a unique solution. When the time of arrival is bounded, i.e., $F(\bar{t}) = 1$ for some $\bar{t} < \infty$, the solution is computed by applying backward induction. Otherwise, the solution can be computed by applying backward induction to the finite horizon approximation, by setting $F(\bar{t}) = 1$, and letting $\bar{t} \to \infty$.

In this model, the agents' bargaining powers are taken as primitives, and the agents' behavior is derived using backward induction and individual rationality, assuming that, at a given date, the agents will divide the dollar according to the agents' instantaneous bargaining powers—when there is an individually rational division. As in Nash (1950), we take the instantaneous bargaining at a given date as a black box, hence the analysis in this paper does not necessarily refer to a game. Nevertheless, it can be formalized as a game. As an example, consider the following perfect-information game, Γ. At each $t \leq d$, Nature chooses a publicly observable number $\pi_t^1 \in [0, 1]$, and we set $\pi_t^2 = 1 - \pi_t^1$. If $t = k/n \leq d$ for some k, then one of the agents is recognized where the probability of recognition for agent

i is π_t^i. The recognized agent offers a division (x^1, x^2), with $x^1 + x^2 = 1$. If the other agent accepts the offer, then the game ends yielding a payoff vector $\delta^t x = (\delta^t x^1, \delta^t x^2)$; otherwise, the game continues. If $t > d$, the game ends with payoff vector $(0, 0)$. The agents' beliefs about π_t^i and d are as described above. Here, we introduce an additional public signal π_t^1 to the model of Yildiz (2003), a signal that represents the "true" probability of agent 1 making an offer at t. The next result states that the behavior described in this paper corresponds to the (generically unique) subgame perfect equilibrium of Γ, where the bargaining power is represented as probability of making an offer (in the present transferable utility case). It also shows that there is a unique stochastic process for agents' continuation values that solve Eq. (4). The proof is similar to that of Theorem 1 in Yildiz (2003), and hence is omitted.

Proposition 1. *There exists a unique solution V to difference equation (4) with the boundary condition $V_t = 0$, $t \geq d$. In any subgame perfect equilibrium of game Γ, for each $t \in T^*$ and each i, after π_t^1 is revealed but before the proposer at t is recognized, the continuation value of agent i is V_t^i. In any subgame perfect equilibrium and at any $t \in T^*$, if $\delta_t S_{t+1/n} > 1$, then there will be no agreement at t; if $\delta_t S_{t+1/n} < 1$, the agents agree on a division that makes the responder indifferent between accepting and rejecting the offer.*

If one restricts π_t^i to be in $\{0, 1\}$, the game Γ becomes a sequential bargaining model similar to that of Yildiz (2003),[3] who extends the usual models of sequential bargaining, such as Rubinstein (1982), by allowing the agents to have subjective beliefs about the recognition. In view of Proposition 1, this model extends Yildiz (2003) and the above models in several directions. First, the bargaining power here can take any value in $[0, 1]$, while the sequential bargaining models allocate the entire instantaneous bargaining power to one agent or the other, by restricting π_t^i to be in $\{0, 1\}$. Second, the bargaining power here is defined as a function of the real time, independent of how frequently the agents come together to negotiate. This allows us to consider more realistic and better-behaved processes of bargaining power. For example, assuming alternating offer bargaining or independent recognitions as $n \to \infty$, requires a process of bargaining power that shifts from one extreme to the other infinitely frequently, while we can focus on processes with piecewise-continuous sample paths. Third, our model allows a stochastic deadline with arbitrary distributions. When the discounting is due to the random bargaining breakdowns, the usual models allow stochastic deadlines, but it is usually assumed to have exponential distribution, as the discount rate is usually constant. We do not impose any such restriction on deadlines because that would rule out the critical class of stochastic deadlines that approximate deterministic deadlines. Finally, as Yildiz (2003), we generalize the usual models by allowing the agents to hold subjective beliefs about their bargaining power.

[3] Ali (2006) extends the model of Yildiz (2003) to multilateral bargaining, but as usual, the uniqueness result here cannot be extended to multilateral bargaining.

Remark 1 *(Continuous-time limit).* In this paper, to simplify the exposition, we take real-time differences between any consecutive index times equal. When the sample paths are piecewise continuous almost surely, this is irrelevant for the continuous-time limit. In that case, regardless of how one approaches continuous time, as long as one uses a deterministic grid, the continuous-time limit is the same, and our results remain intact. This is remarkable because in usual bargaining models there is a discontinuity at the continuous-time limit, and the limit depends on how we approach continuous time. For example, in the alternating-offer bargaining, if the real-time delays after the dates at which Agent 1 makes an offer are K times as long as those after Agent 2 makes offers, then Agent 1 gets K times as much as Agent 2 gets in the continuous-time limit. The critical difference here is that in those models the process of bargaining power depends crucially on the grid and is not even well-defined in the continuous-time limit.

Throughout the paper, we will illustrate our concepts and results on the following tractable and useful model that embodies certain plausible assumptions. It is reflective of the situations in which the bargaining power changes only in occurrence of certain events.

Example 1 *(The Poisson Model).* Consider a Poisson process $(N_t)_{t\in T}$ with arrival rate λ. The bargaining power changes only at the time of an arrival, and at each arrival, the new bargaining power is drawn from a fixed distribution G^i according to each agent i. That is, if $N_t(\omega) = \lim_{s\uparrow t} N_s(\omega)$, then $\pi_t^1(\omega) = \lim_{s\uparrow t} \pi_s^1(\omega)$, and conditional on $N_t \neq \lim_{s\uparrow t} N_s$, the distribution of π_t^1 is G^i, independent of the history, according to agent i.

That is, the bargaining power remains constant until some important event, such as a major terrorist attack or a financial meltdown, occur. When such an event occur, the balance of the power between the parties is completely reset. The parties agree how often such events occur (common λ), but they may disagree on these events' impact on the bargaining power. Intuitively, bargaining power becomes less durable as λ increases. In the limit $\lambda = 0$, bargaining power never changes: $\pi_t^i = \pi_s^i$ for all t, s. In the other limit $\lambda = \infty$, bargaining power is completely transient: π_t^i and π_s^i are stochastically independent for all $t \neq s$.

In this example, the expected bargaining power has a simple form:

$$E_t^i\left[\pi_s^i\right] = e^{-\lambda(s-t)}\pi_t^i + \left(1 - e^{-\lambda(s-t)}\right)\bar{\pi}^i \tag{5}$$

where $\bar{\pi}^i$ is the expected value of π^i according to agent i after each arrival, i.e. $\bar{\pi}^1 = \int \pi dG^1(\pi)$ and $\bar{\pi}^2 = \int (1-\pi)\, dG^2(\pi)$. By (2) and (5), the optimism level at date t for the bargaining power at date s is

$$y_{t,s} = \left(1 - e^{-\lambda(s-t)}\right)\bar{y} \tag{6}$$

where $\bar{y} = \bar{\pi}^1 + \bar{\pi}^2 - 1$ is the level of optimism about the bargaining power in the long run.

The level of optimism for the bargaining powers in the near future is low, as the agents know that their bargaining powers will be similar to the current ones. As they consider later dates, the level of optimism grows, for they expect the bargaining power to be shifted by the underlying reality, about which they hold optimistic beliefs. Moreover, as λ increases and the bargaining power becomes less durable, the individuals become more optimistic. As $s \to \infty$ or $\lambda \to \infty$, $y_{t,s}$ approaches $\bar{y}$.

Finally, one can easily check—as we do in the Appendix—that the solution to (4) is

$$V_t^i = \frac{b}{a + \bar{y}\,(a-b)}\pi_t^i + \frac{a-b}{a + \bar{y}\,(a-b)}\bar{\pi}^i \qquad (\forall i \in N, t \in T^*) \tag{7}$$

where $a = 1/\left(1 - e^{-r/n}\right)$ and $b = 1/\left(1 - e^{-(r+\lambda)/n}\right)$, and the agents immediately reach an agreement, agent i receiving V_0^i. As $n \to \infty$,

$$V_t^i \to \frac{r}{r + \lambda + \bar{y}\lambda}\pi_t^i + \frac{\lambda}{r + \lambda + \bar{y}\lambda}\bar{\pi}^i. \tag{8}$$

The share V_t^i of agent i is a convex combination of the current bargaining power, π_t^i, and the expected value of long-run bargaining power, $\bar{\pi}^i$. In the limit, the weights are determined by the ratio of the arrival rate λ of new events that reset the bargaining power to the discount rate r. An important aspect of Eqs. (7) and (8) is that as the bargaining power becomes more durable (i.e. λ decreases), the weight of the current bargaining power π_t^i increases.

3 Bargaining with a Common Prior

We first compute the continuation values under (CPA) to illustrate the workings of our model in this simple setting and to demonstrate that the durability of the bargaining power process does not play an important role when agents share the same beliefs. Under CPA, $S_{t+1/n}$ is identically 1 before the deadline, so that the agents agree at each date (if they have not yet agreed), and (4) simplifies to

$$V_t^i = \pi_t^i\,(1 - \delta_t) + \delta_t E_t\left[V_{t+1/n}^i\right], \tag{9}$$

where E_t denotes the common conditional expectation operator. The solution to this difference equation and its limit as $n \to \infty$ are easily computed as in the following result. The proof is omitted.

Proposition 2. *Under (CPA), for each $t \in T^*$,*

$$V_t^i = \frac{1}{1 - F(t)} E_t \left[\sum_{k=tn}^{\infty} e^{-r(k/n - t)} \left(1 - F(k/n)\right) \left(1 - \delta_{k/n}\right) \pi_{k/n}^i \right] \qquad (a.s.).$$

If in addition the sample paths $t \mapsto \pi_t^i(\omega)$ are piecewise continuous for almost all ω, then

$$\lim_{n \to \infty} V_t^i \equiv \bar{V}_t^i = \frac{1}{1 - F(t)} E_t \left[\int_{s \geq t} e^{-r(s-t)} \left(f(s) + r\left(1 - F(s)\right)\right) \pi_s^i ds \right] \qquad (\forall t)\,(a.s.)\,.$$

This proposition shows that the overall bargaining power of an agent is the expected value of a weighted sum (or integral) of his future instantaneous bargaining powers. The agents weigh their bargaining power at different situations differently. First, as expected, they discount the future bargaining powers in the same way as they discount their future payoffs, using the effective discount rate δ_t. Second, they put higher weight to the bargaining power at the times at which it is more likely that they will face a deadline. This is reflected by the term $\left(1 - \delta_{k/n}\right)$ in the discrete sum and $f(s)$ in the limiting integral. The rationale for this is somewhat deeper. When the probability of deadline is high, so is the cost of delay. Hence, there is a wide range of individually rational divisions, which can be traced as we vary the agents' relative bargaining power. In that case, the bargaining power has a large impact on the share agent would get if the negotiation continues until that day. Therefore, at the beginning of the negotiation, the agents put a high weight on their instantaneous bargaining power at that time. In particular, if there is a firm deadline, the agents' bargaining power just before the deadline will have a large impact.

By Proposition 2, the share of agent i is

$$V_0^i = \sum_{t \in T^*}^{\infty} e^{-rt} \left(1 - F(t)\right) \left(1 - \delta_t\right) E\left[\pi_t^i\right]$$

where E denotes the unconditional expectation operator. That is, the share of an agent is an additive function of the expected value of his instantaneous bargaining powers in the future, and it is independent of how the bargaining powers at different dates interact with each other, e.g., whether the instantaneous bargaining power changes frequently, or whether the agents will learn about their future bargaining power from past, and so on. In that sense, under the common-prior assumption, the durability of bargaining power per se does not play an important role in determining the outcome.

4 Durability of Bargaining Power

In this section, we formally introduce two notions of durability, one weak and one strong.

Definition 1 (Weak Durability). We say that the bargaining power is *weakly durable on* $[t_1, t_2]$ *with inverse rate of durability* $c > 0$ if for all $[t, s] \subseteq [t_1, t_2]$, for all i, and for all $\omega \in \Omega$,

$$|E_t^i[\pi_s^1] (\omega) - \pi_t^1 (\omega) | \leq c(s - t). \tag{10}$$

That is, in expectation, the bargaining power is Lipschitz-continuous in time. The inverse rate of durability c captures the rate by which agents' expectations about the bargaining power in the near future can deviate from the current bargaining power. Since the current bargaining power is known, weak durability puts the following bound on how optimistic agents can be about the bargaining power in the near future:

$$y_{t,s} = E_t^1 \left[\pi_s^1\right] + E_t^2 \left[\pi_s^2\right] - 1 \leq 2c\,(s - t)\,. \tag{11}$$

It is this aspect of weak durability that leads to the durability effect. When the bargaining power is weakly durable, agents cannot be too optimistic, and hence their continuation values are close to their instantaneous bargaining powers. This in turn leads to an ex-ante delay.

Weak durability is the crucial ingredient for our results. For some results, we will also use a stronger durability notion. Towards defining this stronger notion, let us write

$$L\,(c, t_1, t_2) = \left\{\omega \mid \left|\pi_t^1\,(\omega) - \pi_{t_1}^1\,(\omega)\right| \leq c(t - t_1),\ \forall t \in [t_1, t_2]\right\}$$

for the set of states at which the bargaining power is Lipschitz-continuous in time over the interval $[t_1, t_2]$.

Definition 2 (Strong Durability). We say that the bargaining power is *strongly durable on* $[t_1, t_2]$ *with inverse rate of durability* $c > 0$ if there exist $c', c'' \geq 0$ such that $c' + c'' \leq c$ and for all $[t, s] \subseteq [t_1, t_2]$, for all i, and for all $\omega \in \Omega$,

$$Pr_t^i\,(L\,(c', t, s)\,|\omega) \geq 1 - c''(s - t). \tag{12}$$

That is, in each agent's view, the bargaining power is Lipschitz-continuous in time with the exception of some rare events which happen with a rate slower than c''. Note that strong durability implies weak durability. The difference $|\pi_s^1 - \pi_t^1|$, which is bounded by 1, cannot exceed $c'(s - t)$ on the event $L\,(c', t, s)$. Hence, under strong durability,

$$|E_t^i[\pi_s^1] - \pi_t^i| \leq (1 - c''(s - t))c'(s - t) + c''(s - t) \leq (c' + c'')(s - t) \leq c(t - s), \tag{13}$$

implying weak durability.

Example 2 (Durability in the Poisson Model). The bargaining power in the Poisson model is strongly durable with inverse rate of λ, which is the arrival rate of the Poisson process. To see this, note that according to agent i, the conditional distribution of π_s^1 at $t < s$ is the mixture of point mass at π_t^1 with probability $e^{-\lambda(s-t)}$ and G^i with probability $1 - e^{-\lambda(s-t)}$. Thus, for any $t \leq s$, we have

$$Pr_t^i(\pi_{s'}^i = \pi_t^i,\ \forall s' \in [t,s]) \geq 1 - e^{-\lambda(t-s)} \geq \lambda(s-t),$$

establishing the condition for strong durability for $c' = 0$ and $c'' = \lambda$. This is intuitive since for higher λ, the bargaining power jumps more frequently and hence is less durable.

5 Durability Effect

We next present our main result, which demonstrates the durability effect. Consider two dates $\hat{t}, t^*$ such that $\hat{t} \leq t^*$. After t^*, the bargaining power becomes relatively more durable, and $\hat{t}$ is close to t^*. Our main result will conclude that agents will not agree at any $t < \hat{t}$.

To state our result, we need to introduce two conditions, one on the level of optimism before $\hat{t}$ (regarding the bargaining power at t^*) and one on the durability of the bargaining power after t^*. The first condition ensures that at all dates before $\hat{t}$, the agents are sufficiently optimistic about their bargaining power at date t^*.

Condition O for $\left(\hat{t}, t^*\right)$. Given dates $\left(\hat{t}, t^*\right)$, the following inequality holds

$$\bar{\beta}\left(\hat{t}, t^*\right) \equiv \inf_{t<\hat{t}} \left(\delta_{t,t^*}\left(1 + y_{t,t^*}\right) - 1\right)/\delta_{t,t^*} > 0. \tag{14}$$

Note that $\delta_{t,t^*}\left(1 + y_{t,t^*}\right)$ is the discounted size of the pie at t^* as perceived by the agents at t. When the agents are optimistic about t^*, they hold inflated expectations about t^*, and $\delta_{t,t^*}\left(1 + y_{t,t^*}\right)$ may exceed the actual size of the pie at t, which is 1. The condition O states that prior to $\hat{t}$ the normalized distance between $\delta_{t,t^*}\left(1 + y_{t,t^*}\right)$ and 1 is bounded away from zero. When $\hat{t} = t^*$, this condition implies that y_{t,t^*} is discontinuous in t at $t = t^*$. This may be plausible only when t^* corresponds to the date of a discrete event that resets the bargaining powers, such as an election. When $\hat{t} < t^*$, this condition may hold even when y_{t,t^*} is continuous. (That is why we introduced a separate date $\hat{t}$.) The next set of conditions ensure that the bargaining power becomes sufficiently durable after t^*.

Condition D for (t^*, ε)**.** Given date t^* and the parameter $\varepsilon > 0$ (which controls the required level of durability), there exists $t_e > t^*$ and $c > 0$ that satisfy

$$c(t_e - t^*) \leq \varepsilon/4 \text{ and } \delta_{t^*,t_e} \leq \varepsilon/4, \tag{15}$$

such that the bargaining power is weakly durable on $[t^*, t_e]$ with inverse rate of durability c.

Condition SD for (t^*, ε)**.** Given date t^* and the durability parameter $\varepsilon > 0$, there exists $t_e > t^*$ and $c > 0$ that satisfy Eq. (15) such that the bargaining power is strongly durable on $[t^*, t_e]$ with inverse rate of durability c.

The durability condition D requires that bargaining power is sufficiently durable for a sufficiently long while. As ε decreases, the bargaining power needs to be durable for a longer period of time and with higher rate of durability (i.e. with lower c). The strong durability condition SD requires in addition that bargaining power must be strongly durable.

The next proposition, which is our main result, establishes that under the durability and optimism conditions, agents will wait until some time near t^* to settle. Let $t^a(\omega)$ be the first date with an agreement regime, namely the *settlement date*, at state ω. We use the convention that $t^a(\omega) = \infty$ when they fail to agree before the deadline. Let

$$A^{t_e} = \{\omega \mid t^a(\omega) \leq \min(d(\omega), t_e)\}$$

denote the states at which the agents agree before the deadline and before some date t_e.

Proposition 3. *For any $\hat{t}, t^* \in T^*$ and $\varepsilon \in (0, \bar{\beta}(\hat{t}, t^*)]$ that satisfy the optimism condition O for $(\hat{t}, t^*)$ and the durability condition D for (t^*, ε), the agents disagree at each $t < \hat{t}$. Moreover, if the strong durability condition SD for (t^*, ε) holds with date t_e, then $Pr^i_{t^*}(A^{t_e}) \geq 1 - 2\varepsilon/(\pi_{t^*} + \varepsilon)$ for all $i \in \{1, 2\}$.*

That is, if the bargaining power is sufficiently durable after t^* and the agents remain sufficiently optimistic about t^* until $\hat{t}$, then they will not settle before $\hat{t}$. Moreover, with strong durability, at t^*, each agent assigns high probability that they will reach an agreement before t_e (and before the deadline arrives). Therefore, with high probability, they will settle after $\hat{t}$ and before the bargaining power ceases to be durable. The second part of the proposition also implies that, when ε is small, the individuals agree soon after t^* with high probability. This is because any discounting can be represented as arrival of deadline, and the proposition concludes that they agree before any arrival and hence before any discounting. Our proof, which is presented in the appendix, relies on the following lemma.

Lemma 1. *Assume that the bargaining power is weakly durable with inverse rate of durability c on some $[t_s, t_e]$ with $t_s, t_e \in T^*$. Then, for each $t \in [t_s, t_e] \cap T^*$ and i,*

$$\left(\pi^i_t - c(t_e - t)\right)(1 - \delta_{t,t_e}) \leq V^i_t \leq \left(\pi^i_t + c(t_e - t)\right)(1 - \delta_{t,t_e}) + \delta_{t,t_e}.$$

That is, when the bargaining power is durable, the payoffs at t are within a bounded interval around the bargaining power. If both δ_{t,t_e} and $c(t_e - t^*)$ are small, i.e. bargaining power is sufficiently durable for a long enough time, then the payoff of an agent at t will be close to her bargaining power at t. We obtain the lower bound by applying backward induction using the smallest possible values for V^i and largest possible values for V^j under the durability assumption. Upper bound is obtained similarly.

Under condition D, for $t = t^*$, the lower bound becomes $(\pi^i_{t^*} - \varepsilon/4)(1 - \varepsilon/4)$. Under condition O and $\varepsilon \leq \bar{\beta}(\hat{t}, t^*)$, this implies that at any $t < \hat{t}$, $\delta_{t,t^*}(E^1_t[V^1_{t^*}] + E^2_t[V^2_{t^*}]) > 1$, and hence it is impossible to satisfy both agents' expectations from waiting until t^*. Therefore, there cannot be an agreement at any $t < \hat{t}$. Hence the first statement in Proposition 3. There is a simple intuition for the second statement, which concludes that each individual assigns high probability to reaching an agreement soon after t^*. If the continuation value of an agent i at t^* is high, then she must assign a high probability $Pr^i_{t^*}(A^{t_e})$ on reaching an agreement in the near future because she gets 0 if they fail to agree before the deadline (or a small payoff if they wait). Unfortunately, this argument does not lead to a very high lower bound for $Pr^i_{t^*}(A^{t_e})$. It leads to a lower bound for $Pr^1_{t^*}(A^{t_e}) + Pr^2_{t^*}(A^{t_e})$ that is nearly 1. In the Appendix, using more subtle arguments based on strong durability, we show that each of these probabilities is nearly 1.

Proposition 3 suggests that certain events that lock-in the agents' bargaining powers for a long enough period may generate ex-ante settlement delay. How much durability is enough for delaying the settlement? Our next result answers this question and provides intuitive comparative statics for the durability effect.

Corollary 1. *For any $\hat{t}, t^* \in T^*$ that satisfy optimism condition O, assume that for some $\varepsilon \in (0, \bar{\beta}(\hat{t}, t^*)]$ the bargaining power is weakly durable on $[t^*, t^* + \varepsilon + \log(4/\varepsilon)/r]$ with inverse rate of durability c such that*

$$c < r\frac{\varepsilon/4}{\log(4/\varepsilon)}.$$

Then, in the continuous-time limit (i.e. for large n), the agents disagree at each $t < \hat{t}$.

Proof. Pick n large enough so that $1/n \leq \varepsilon$ and $c \leq r(\varepsilon/4)/(\log(4/\varepsilon) + r/n)$. Let $t_e = \max\{t \in T^* | t^* < t \leq t^* + 1/n + \log(4/\varepsilon)/r\}$. Then, $\delta_{t^*,t_e} \leq e^{-r(t_e - t^*)} \leq \varepsilon/4$. Moreover, since $t_e - t^* \leq \log(4/\varepsilon)/r + 1/n$,

$$c(t_e - t^*) \leq r\frac{\varepsilon/4}{\log(4/\varepsilon) + r/n}[\log(4/\varepsilon)/r + 1/n] = \varepsilon/4.$$

The result then follows from Proposition 3. □

Corollary 1 provides intuitive comparative statics for the durability effect. The higher the optimism before event date (in which case we can choose a higher ε) and the higher the discount rate, the stronger the durability effect becomes, i.e. the less durable the bargaining power could be over a shorter interval and still generate the durability effect. A higher level of optimism about bargaining power translates itself into a higher level of optimism about payoffs, which makes the durability effect stronger. The rationale for the comparative statics with respect to the discount rate is somewhat deeper. A higher discount rate increases the scope of trade at t^*, which brings the (worst case) payoffs at t^* closer to the bargaining powers at t^* (cf. Lemma 1). Hence, keeping the optimism for t^* constant, a higher discount rate makes durability effect stronger. We next demonstrate Proposition 3 in the context of the Poisson model. Interestingly, the sufficient conditions are also necessary in the Poisson model because durability is constant.

Example 3 *(Durability Effect in the Poisson Model).* Fix a date t^* and two arrival rates $\hat{\lambda}$ and λ such that $\hat{\lambda}+r > (\lambda+r)(1+\bar{y})/\bar{y}$. The last condition is analogous to the optimism condition (O). Assume that the arrival rate is at the higher level $\hat{\lambda}$ for all $t < t^*$, drops to λ at t^* and remains at λ for all $t \geq t$. We have already seen that, at t^* and thereafter, the agents agree on a division with shares V_t^i as in (7). In particular, in the limit $n \to \infty$,

$$V_{t^*}^i = \frac{r}{r+\lambda+\bar{y}\lambda}\pi_t^i + \frac{\lambda}{r+\lambda+\bar{y}\lambda}\bar{\pi}^i. \tag{16}$$

Now, consider any $t < t^*$. By (16), the total pie at t^* is perceived by the agents at t as

$$S_{t,t^*} = E_t^1\left[V_{t^*}^1\right] + E_t^1\left[V_{t^*}^1\right] = 1 + \frac{r}{r+\lambda+\bar{y}\lambda}y_{t,t^*}$$

where $y_{t,t^*} = \left(1-e^{-\hat{\lambda}(t^*-t)}\right)\bar{y}$ by (6). Since the agents can guarantee themselves a payoff $e^{-r(t^*-t)}E_t^1\left[V_{t^*}^1\right]$ by waiting until t^*, there cannot be an individually rational agreement at date t when $e^{-r(t^*-t)}S_{t,t^*} > 1$. After some algebraic manipulation, this condition can be written as

$$1+\frac{\lambda}{r} < \frac{\bar{y}}{1+\bar{y}}\frac{1-e^{-\left(\hat{\lambda}+r\right)(t^*-t)}}{1-e^{-r(t^*-t)}} \equiv R(t^*-t). \tag{17}$$

As t approaches t^*, $R(t^*-t)$ approaches $(1+\hat{\lambda}/r)\bar{y}/(1+\bar{y})$ and the inequality is satisfied by assumption. Hence, just before t^*, there is a period of disagreement at which the individuals cannot reach an agreement. The length of this period, denoted by Δ^*, is given by

$$R(\Delta^*) = 1+\lambda/r. \tag{18}$$

In the last date t with $t < t^* - \Delta^*$, the individuals recognize that, if they do not reach an agreement at t, they will never reach an agreement until t^*, and the perceived total payoff from disagreement is $e^{-r\Delta^*} S_{t,t^*} < 1$. Hence, at that date they reach an agreement. Note that the length Δ^* of disagreement period is increasing in the level of long-run optimism $\bar{y}$. That is, the higher the optimism, the stronger the durability effect. It is also increasing the initial arrival rate $\hat{\lambda}$ and decreasing in the final arrival rate λ. This is because a higher $\hat{\lambda}$ and a lower λ correspond to a higher drop in the arrival rate and a larger increase in durability, leading to a stronger durability effect.

6 Deadline Effect and Durability

In this section, we explore the deadline effect under optimism and durability. We first illustrate that under a firm deadline, optimism leads to a deadline effect. We then demonstrate that with continuous sample paths deadlines act as making bargaining power more durable. Hence, using the durability effect of Proposition 3, we then establish a robust deadline effect under durability for stochastic deadlines. As in Corollary 1, we further provide monotone comparative statics for the deadline effect. We finally show that the usual independence assumption leads to a discontinuity, in that the deadline effect disappears even with a vanishing amount of uncertainty about the deadline.

6.1 Deadline Effect with a Deterministic Deadline

We first show that if there is a deterministic deadline at t^*, then sufficient optimism about the bargaining power at t^* will entice agents to wait until t^* and agree at t^*. This behavior is known as *the deadline effect*.

Proposition 4. *Let $t^* \in T^*$ be such that there is a deterministic deadline at some $d^* \in (t^*, t^* + 1/n)$, and $e^{-r(t^*-t)} (1 + y_{t,t^*}) > 1$ for each $t < t^*$. Then, the agents disagree at each $t < t^*$ and agree at t^*.*

At t^*, since the deadline is imminent, the continuation values are zero, and they agree, each i getting his bargaining power $\pi^i_{t^*}$. But, $e^{-r(t^*-t)} (1 + y_{t,t^*}) > 1$ implies that agents are willing to wait until t^* to get their bargaining power at t^*. Hence, they will never agree before t^*. Note that a deterministic deadline at $(t^*, t^* + 1/n)$ is equivalent to bargaining power remaining constant between t^* and the deadline. Hence, it can be considered as a stronger version of the durability conditions D and SD in the last section. Then, the deadline effect in this proposition can be considered as an instance of the durability effect in the previous section.

6.2 Durability and Stochastic Deadlines

In fact, in our model there is a deep connection between deadlines and variations in durability. In our model, time is a relative notion that affects the behavior only through the effective discount factor $\delta_{t,s}$. Hence, any stochastic deadline can be represented as a transformation τ of time by $e^{-rt}(1-F(t)) = e^{-r\tau(t)}$, or

$$\tau(t) = t - \frac{1}{r}\log(1-F(t)).$$

Then, a high arrival rate at a given t can be replicated by stretching the time around t:

$$\tau'(t) = 1 + \frac{1}{r}\frac{f(t)}{1-F(t)}.$$

If π_t^i is continuous at t, such a transformation of time is equivalent to making bargaining power change more slowly, or more "durable." Therefore, in our analysis deadlines and variations in durability of the bargaining power play similar roles.

6.3 Robust Deadline Effect with Durable Bargaining Power

Since the deadlines are equivalent to making the bargaining power more durable, a stochastic deadline with small uncertainty leads to a *durability effect*, enticing the agents to wait until the bargaining power becomes more durable, which happens around the deadline. Therefore, under optimism and any amount of durability, such a stochastic deadline leads to a *deadline effect*. Our next result establishes this, formally.

Proposition 5. *Consider any* $\hat{t}, t^* \in T^*$ *that satisfies* $\inf_{t<\hat{t}} e^{-r(t^*-t)}(1+y_{t,t^*}) > 1$. *Assume that the bargaining power is weakly durable on* $[t^*, t^*+\gamma]$ *with inverse rate of durability* c *for some* $\gamma, c > 0$. *Then, there exist* $\varepsilon \in (0,1)$, $\tau \in (t^*, t^*+\gamma)$ *and* $\bar{n} < \infty$ *such that the agents disagree at each* $t < \hat{t}$ *whenever*

$$F(\tau) - F(t^*) > 1 - \varepsilon \tag{19}$$

and $n > \bar{n}$. *Moreover, if the bargaining power is strongly durable and* $\Pr_0^i(\pi_{t^*}^i > 0) = 1$, *then for any* $\mu > 0$, *we can pick* ε *so that agent* i *assigns at least probability* $1-\mu$ *that they will reach an agreement before the deadline arrives (and before* τ*).*

The condition (19) quantifies the notion that the deadline arrives around t^* and the uncertainty about the deadline is small. When the distribution of the deadline converges to the point mass at $t \in (t^*, \tau)$, this condition is satisfied for any ε. Under this condition, the probability that the deadline will arrive before t^* is quite small, less than ε. Since the agents remain optimistic about their bargaining power at t^* (i.e. $\inf_{t<\hat{t}} e^{-r(t^*-t)} (1 + y_{t,t^*}) > 1$), this implies that the optimism Condition O of Proposition 3 is satisfied. Moreover, under (19), $\delta_{t^*,\tau} \leq (1 - F(\tau)) / (1 - F(t^*)) < \varepsilon / (1 - \varepsilon)$ is also small. Since we can choose τ arbitrarily close to t^* so that $c(\tau - t^*)$ is as small as needed, the durability condition D of Proposition 3 is also satisfied. Proposition 3 then implies that the agents will not settle before $\hat{t}$. Likewise, the second statement follows from the second part of Proposition 3. Under the above conditions Proposition 3 states that, at t^*, an agent will assign high probability of reaching an agreement before the deadline arrives. Since the probability of arrival of a deadline before t^* is small, the Bayes rule then implies that the agent will assign a high probability of reaching an agreement.

To sum up from an agent's point of view, if the uncertainty about the deadline is sufficiently small and the agents are sufficiently optimistic about their bargaining power around the deadline, then they will wait sometime very close to the deadline but they will settle before the deadline arrives with very high probability. Hence the deadline effect.

How much uncertainty is small enough though? In order to answer this question, we introduce a standard way to measure the speed of arrival.

Definition 3. For any interval $[t_1, t_2]$, we say that *the deadline arrives faster than the rate* ρ *on average over* $[t_1, t_2]$ if $(1 - F(t)) / (1 - F(t_1)) \leq e^{-\rho(t-t_1)}$ for all $t \in [t_1, t_2]$.

Note that our requirement is slightly weaker than the requirement that d is first-order stochastically dominated by the exponential distribution with rate ρ conditional on $d \geq t_1$, which would have imposed the above inequality for all $t \geq t_1$. It is also weaker than the requirement that its hazard rate exceeds ρ over the interval. Our next result, under regularity assumptions, quantifies how fast the deadline arrival rate should be to generate the deadline effect. The deadline effect prevails when deadline arrives sufficiently faster than the inverse rate of durability.

Proposition 6. *Let $\hat{t}, t^* \in T^*$ be such that $\beta \equiv \inf_{t<\hat{t}} \left[1 + y_{t,t^*} - e^{r(t-t^*)} / (1 - F(t^*))\right] > 0$. For some $\varepsilon \in (0, \beta]$, assume that the bargaining power is weakly durable on $[t^*, t^* + \varepsilon + \frac{\log(4/\varepsilon)}{r}]$ with the inverse rate of durability c. If the deadline arrives faster than $\hat{r} - r \geq 0$ on average over the interval $[t^*, t^* + \varepsilon + \frac{\log(4/\varepsilon)}{r}]$ for some $\hat{r}$ with*

$$\hat{r} > \frac{c \log(4/\varepsilon)}{\varepsilon/4},$$

then in the continuous-time limit the agents disagree at each $t < \hat{t}$. Moreover, if the bargaining power is strongly durable, then at t^* each agent assigns at least probability $1 - 2\varepsilon / (\pi_{t^*} + \varepsilon)$ that they will reach an agreement before the deadline arrives.

The sufficient condition for the deadline effect depends only on the ratio of the normalized arrival rate of deadline ($\hat{r}$) to the inverse rate of durability of the bargaining power (c). When this ratio is above a cutoff value (i.e. deadline arrives sufficiently faster than the inverse rate of durability), the deadline effect prevails. How much faster should the deadline arrive than the inverse rate of durability is determined by the optimism about t^*. When the optimism for t^* is higher, we can choose ε higher and the cutoff becomes lower. In that case, a less durable bargaining power or a slower deadline arrival rate suffices to generate a deadline effect. In summary, deadline effect is stronger (i) when players are initially more optimistic about the deadline (in particular, bargaining power is expected to vary enough until the deadline), (ii) when the bargaining power is more durable around the likely time of deadline, and (iii) the uncertainty about the deadline is less.

Example 4 *(Deadline Effect in the Poisson Model).* Suppose that the arrival rate of the Poisson process is $\hat{\lambda}$ before some t^* and λ after t^*, where $\hat{\lambda}$ can be higher or lower than λ. Moreover, a stochastic deadline arrives with constant hazard rate $\hat{r} - r$ starting at t^* (i.e. $F(t) = 1 - e^{-(\hat{r}-r)(t-t^*)}$ for $t \geq t^*$). Starting at t^*, the effective discount factor is $\delta_{t,s} = e^{-\hat{r}(s-t)}$. Hence, in the limit $n \to \infty$, at t^*, the agents agree on a division with shares

$$V_t^i = \frac{\hat{r}}{\hat{r} + \lambda + \overline{y}\lambda}\pi_t^i + \frac{\lambda}{\hat{r} + \lambda + \overline{y}\lambda}\overline{\pi}^i. \tag{20}$$

Once again, as in Example 4, at any $t < t^*$, the the agents will disagree at t if $e^{-r(t^*-t)}\left(E_t^1\left[V_{t^*}^1\right] + E_t^1\left[V_{t^*}^1\right]\right) > 1$. By (6) and (20), the latter condition can be written as

$$1 + \frac{\lambda}{\hat{r}} < \frac{\overline{y}}{1+\overline{y}} \frac{1 - e^{-\left(\hat{\lambda}+r\right)(t^*-t)}}{1 - e^{-r(t^*-t)}} \equiv R\left(t^* - t\right). \tag{21}$$

The expression on the left-hand side is monotonically decreasing in $\hat{r}$, the arrival rate of deadline. In particular, there is a cutoff value

$$r_c = \sup_{0<t\leq t^*} \lambda / \left(R\left(t^* - t\right) - 1\right),$$

such that when $\hat{r} > r_c$ the agents will never agree until t^* and reach an agreement at t^*. Note that the condition $R\left(t^* - t\right) > 1$ for all $t < t^*$ is equivalent to Condition O of Proposition 3 for this model (with $\hat{t} = t^*$). When this condition is satisfied and the bargaining power is not completely transient ($\lambda \neq \infty$), the Poisson model features the deadline effect when the deadline arrives sufficiently fast (i.e. if $\hat{r} > r_c$), in line with Propositions 5 and 6. On the other hand, when $\hat{r} < r_c$, we will not observe the deadline effect: the parties will reach an agreement before t^*. Finally, as in Example 4, there is a period of disagreement right before t^* with length Δ^*, given by $R\left(\Delta^*\right) = 1 + \lambda/\hat{r}$. We can measure the strength of the deadline effect by Δ^*. For fixed r, Δ^* is increasing with $\hat{r}-r$, the arrival rate of the deadline, decreasing with λ, the inverse rate of durability before the deadline, and increasing with $\hat{\lambda}$, the inverse rate of durability during the deadline. Therefore, the deadline effect is stronger when (i) the uncertainty about the deadline is smaller, (ii) the bargaining power before the deadline is less durable, and (iii) the bargaining power during the deadline is more durable.

Spier (1992) shows that, in a pre-trial negotiation with incomplete information, the settlement probability will be a U-shaped function of time, consistent with the deadline effect. Ma and Manove (1993) also develop a model in order to explain the deadline effect. In that model, the delay is not costly and a party can wait as much as she wants before making an offer, and his opponent has to wait for his offer. Then, the party who is to make an offer waits until the deadline and makes a last minute take-it-or-leave it offer. Roth, Murnighan, and Schoumaker (1988) informally discuss a possible explanation based on the idea that there is no cost of delay except for a cost at the end due to a slight uncertainty about the deadline.[4] Like Spier, our model is not geared towards explaining the deadline effect, but it naturally yields the deadline effect even when the delay is costly, and it suggests possible ways to avoid the deadline effect.

6.4 Discontinuity of Deadline Effect under Independence

We now show that durability is crucial for the deadline effect under stochastic deadlines: when the beliefs about the future are independent of the past and the deadline is stochastic, there is an immediate agreement in the continuous time limit. We need the following *independence assumption* to state our result.

Assumption IND. For any i, $t < s$ and $\pi^i \in [0,1]$, the conditional probability $F_s^i(\pi^i) \equiv Pr_t^i(\pi_s^i \leq \pi^i|\omega)$ is independent of ω.

That is, the beliefs about the future bargaining powers do not depend on the past observations, including the past bargaining powers. Assumption IND implies that the optimism at $t < s$ for s is a function of s alone, i.e.

$$y_{t,s} = E_t[\pi_s^1] + E_t[\pi_s^2] - 1 = \int_{\pi^1 \in [0,1]} \pi^1 dF_s^1(\pi^1) + \int_{\pi^2 \in [0,1]} \pi^2 dF_s^2(\pi^2) - 1 \equiv Y(s).$$

This further implies that whether there will be agreement at date t is deterministic. The following proposition provides conditions under which the deadline effect disappears when the deadline becomes stochastic.

[4] Roth and Ockenfels (2002) consider a similar model in which the delay is motivated by the possibility that the last offer may not go through. They use this model to explain why the deadline effect is observed in e-Bay auctions but not in Amazon auctions.

Proposition 7. *Assume that the deadline is stochastic with continuous distribution (i.e. F is continuous). Under Assumption IND, assume also that either (i) $Y(t) \geq 0$ for all t, or (ii) $Y(t)$ is a continuous function. Then, for any $\varepsilon > 0$, there exists $\overline{n}$ such that if $n > \overline{n}$, the agents agree before ε (in states in which the deadline does not arrive before date ε).*

That is, under Assumption IND, if the deadline is stochastic, then there will be an immediate agreement in continuous-time limit, so long as agents are optimistic or the level of optimism is a continuous function of time. Under the independence assumption, any uncertainty about the deadline makes the deadline effect disappear and leads to an immediate agreement. More generally it shows that, under Assumption IND, the equilibrium outcome is discontinuous with respect to the distribution of the deadline (as the uncertainty of deadline vanishes). This discontinuity is due to the extreme discontinuity assumption inherent in the independence assumption: the bargaining power is completely transient in the sense that the bargaining power now has no effect on the bargaining power a moment later.

In order to relate this result to the Poisson model, notice in Example 4 that if we increase λ, keeping everything else constant (and $\hat{r} < \infty$), the deadline effect eventually disappears. Intuitively, Assumption IND corresponds to the case $\lambda = \infty$. In that case, (21) necessarily fails and there is no deadline effect in the continuous-time limit, as long as the deadline is stochastic (i.e. $\hat{r} < \infty$).

7 Conclusion

In an ongoing bargaining, bargaining power may be more durable in some periods and less durable in some others. We model the bargaining power as a stochastic process defined on continuous time and focus on the realistic and tractable case that bargaining power is continuous except for rare events, i.e. with piecewise-continuous sample paths. We formally introduce a notion of durability, as Lipschitz continuity, and quantify the level of durability, in terms of the Lipschitz coefficient. We show that durability plays a profound role, especially when the parties are optimistic. In particular, if the bargaining power is anticipated to become more durable at some given time t^* (e.g. after an election), then there will be a period in which optimistic parties will rather delay their agreement to that time t^*.

We observe that, under piecewise-continuous sample paths for bargaining power, arrival of a stochastic deadline in a period is mathematically equivalent to making the bargaining power more durable, making deadlines and durability two faces of the same coin. Using the same logic that leads to the durability effect, we show that if a (possibly stochastic) deadline is likely to arrive around a given time, then there will be a period in which optimistic parties will delay agreement sometime very near the arrival of deadline and reach an agreement just before the deadline with high probability, as in the real-life negotiations. This deadline effect becomes stronger (i) as the deadline becomes firmer in the sense having higher arrival rate, or (ii) as the bargaining power around the deadline becomes more durable, or (iii) as the initial optimism about the deadline effect gets larger (which is intuitively inversely related to the durability of bargaining power until the time of deadline). We hope that such comparative statics will be helpful for practitioners who want to design procedures of conflict resolution, or formal negotiation rules.

Appendix

Proof of Equation (7). By Proposition 1, it suffices to verify that V is indeed a solution to (4). We check it using mathematical induction backwards on time. Check that, by (7) for $t + 1/n$ and (5),

$$E_t^i \left[V_{t+1/n}^i \right] = \frac{be^{-\lambda/n}}{a + \bar{y}(a-b)} \pi_t^i + \frac{a - be^{-\lambda/n}}{a + \bar{y}(a-b)} \bar{\pi}^i. \tag{22}$$

Since $\pi_t^1 + \pi_t^2 = 1$ and $\bar{\pi}^1 + \bar{\pi}^2 = \bar{y} + 1$, this yields

$$S_{t+1/n} = E_t^1 \left[V_{t+1}^1 \right] + E_t^2 \left[V_{t+1}^2 \right] = \frac{a + \bar{y}\left(a - be^{-\lambda/n}\right)}{a + \bar{y}(a-b)}.$$

Thus,

$$1 - \delta_t S_{t+1/n} = 1 - e^{-r/n} S_{t+1/n} = (a + \bar{y}(a-b))^{-1} > 0. \tag{23}$$

Hence, the agents agree at t. In order to verify (4), write

$$\begin{aligned} V_t^i &= \frac{b}{a + \bar{y}(a-b)} \pi_t^i + \frac{a-b}{a + \bar{y}(a-b)} \bar{\pi}^i \\ &= \frac{1}{a + \bar{y}(a-b)} \pi_t^i + \frac{be^{-(r+\lambda)/n}}{a + \bar{y}(a-b)} \pi_t^i + \frac{\left(a - be^{-\lambda/n}\right) e^{-r/n}}{a + \bar{y}(a-b)} \bar{\pi}^i \\ &= \pi_t^i \max\left\{1 - \delta_t S_{t+1/n}, 0\right\} + \delta_t E_t^i \left[V_{t+1/n}^i \right], \end{aligned}$$

where the first equality by (7), the second equality by definitions of a and b, and the last equality by (23) and (22). □

Proof of Lemma 1. For $t = t_e$, the inequalities in the lemma are trivially satisfied because $\delta_{t,t_e} = 1$. Towards an induction, assume that for each i,

$$\begin{aligned} &\left(\pi_{t+1/n}^i - c(t_e - (t + 1/n))\right) \left(1 - \delta_{t+1/n,t_e}\right) \leq V_{t+1/n}^i \\ &\quad \leq \left(\pi_{t+1/n}^i + c(t_e - (t + 1/n))\right) \left(1 - \delta_{t+1/n,t_e}\right) + \delta_{t+1/n,t_e}. \end{aligned}$$

By the weak durability assumption, $E_t^i[\pi_{t+1/n}^i] \in [\pi_t^i - c/n, \pi_t^i + c/n]$. Hence,

$$\begin{aligned} LHS^i &\equiv \left(\pi_t^i - c(t_e - t)\right)\left(1 - \delta_{t+1/n,t_e}\right) \le E_t^i\left[V_{t+1/n}^i\right] \\ &\le \left(\pi_t^i + c(t_e - t)\right)\left(1 - \delta_{t+1/n,t_e}\right) + \delta_{t+1/n,t_e} \equiv RHS^i. \end{aligned} \tag{24}$$

Recall that

$$V_t^i = \pi_t^i \max\left\{1 - \delta_t E_t^j\left[V_{t+1/n}^j\right], \delta_t E_t^i\left[V_{t+1/n}^i\right]\right\} + \left(1 - \pi_t^i\right)\delta_t E_t^i\left[V_{t+1/n}^i\right].$$

Combining the last two statements we obtain the necessary bounds. To find an upper bound, we write

$$\begin{aligned} V_t^i &\le \pi_t^i \max\left\{1 - \delta_t LHS^j, \delta_t RHS^i\right\} + \left(1 - \pi_t^i\right)\delta_t RHS^i = \pi_t^i\left(1 - \delta_t\right) + \delta_t RHS^i \\ &= \left(\pi_t^i + c(t_e - t)\right)\left(1 - \delta_{t,t_e}\right) + \delta_{t,t_e} - c(t_e - t)(1 - \delta_t) \\ &\le \left(\pi_t^i + c(t_e - t)\right)\left(1 - \delta_{t,t_e}\right) + \delta_{t,t_e}. \end{aligned}$$

Here, the first inequality is by (24), the next equality is by $RHS^i + LHS^j = 1$, and the last equality and inequality are by simple algebra. Similarly,

$$\begin{aligned} V_t^i &\ge \pi_t^i \max\left\{1 - \delta_t RHS^j, \delta_t LHS^i\right\} + \left(1 - \pi_t^i\right)\delta_t LHS^i = \pi_t^i\left(1 - \delta_t\right) + \delta_t LHS^i \\ &= \left(\pi_t^i - c(t_e - t)\right)\left(1 - \delta_{t,t_e}\right) + c(t_e - t)(1 - \delta_t) \\ &\ge \left(\pi_t^i - c(t_e - t)\right)\left(1 - \delta_{t,t_e}\right). \end{aligned}$$

□

Proof of Proposition 3. To prove the first part, substituting $t = t^*$ in Lemma 1, we obtain

$$\begin{aligned} V_{t^*}^i &\ge (1 - \delta_{t^*,t_e})(\pi_{t^*}^i - c(t_e - t^*)) \\ &\ge (1 - \bar{\beta}\left(\hat{t}, t^*\right)/4)(\pi_{t^*}^i - \bar{\beta}\left(\hat{t}, t^*\right)/4) \end{aligned} \tag{25}$$

for each i. Then, we have,

$$\begin{aligned} \delta_{t,t^*}\left(E_t^1[V_{t^*}^i] + E_t^2[V_{t^*}^2]\right) &\ge \delta_{t,t^*}(1 - \bar{\beta}\left(\hat{t}, t^*\right)/4)\left(E_t^1\left[\left(\pi_{t^*}^1 - \bar{\beta}\left(\hat{t}, t^*\right)/4\right)\right] + E_t^2\left[\left(\pi_{t^*}^2 - \bar{\beta}\left(\hat{t}, t^*\right)/4\right)\right]\right) \\ &= \delta_{t,t^*}\left[(1 - \bar{\beta}\left(\hat{t}, t^*\right)/4)(1 + y_{t,t^*} - \bar{\beta}\left(\hat{t}, t^*\right)/2)\right] \\ &> \delta_{t,t^*}\left[1 + y_{t,t^*} - \bar{\beta}\left(\hat{t}, t^*\right)\right] \ge 1, \end{aligned}$$

where the first inequality uses (25), the second inequality uses $y_{t,t^*} \le 1$ and the last inequality uses the definition of $\bar{\beta}\left(\hat{t}, t^*\right)$. Hence, the agents expect to be better off waiting until t^*. Therefore, they disagree at t, proving the first part of the proposition.

To prove the second part, recall that $L\left(c', t^*, t_e\right)$ is the set of states in which the bargaining power "does not jump" between t^* and t_e. By the strong durability assumption, $Pr^i_{t^*}(\Omega \setminus L(c', t^*, t_e)) \leq c''(t_e - t^*)$. We will find a lower bound for $Pr^i_{t^*}(A^{t_e} \cap L(c', t^*, t_e))$. Towards this end, write also $\Omega^A = \{\omega | t_a(\omega) < \infty\}$ for the set of states with agreement. Note that

$$\begin{aligned}(\pi^i_{t^*} - c(t_e - t^*))(1 - \delta_{t^*,t_e}) &\leq E^i_{t^*}[V^i_{t^*}] = E^i_{t^*}[V^i_{t^*} \mid \Omega^A] \\ &= E^i_{t^*}[V^i_{t^*} | A^{t_e} \cap L\left(c', t^*, t_e\right)] Pr(A^{t_e} \cap L\left(c', t^*, t_e\right)) \\ &\quad + E^i_{t^*}[V^i_{t^*} | A^{t_e} \setminus L\left(c', t^*, t_e\right)] Pr(A^{t_e} \setminus L\left(c', t^*, t_e\right)) \\ &\quad + E^i_{t^*}[V^i_{t^*} | \Omega^A \setminus A^{t_e}] Pr(\Omega^A \setminus A^{t_e}) \\ &\leq E^i_{t^*}[V^i_{t^*} | A^{t_e} \cap L\left(c', t^*, t_e\right)] Pr(A^{t_e} \cap L\left(c', t^*, t_e\right)) + c''(t_e - t^*) + \delta_{t^*,t_e}, \quad (26)\end{aligned}$$

where the first inequality follows from Lemma 1 and the last inequality follows since $Pr^i_{t^*}(A^{t_e} \setminus L(c', t^*, t_e)) \leq Pr^i_{t^*}(\Omega \setminus L(c', t^*, t_e)) \leq c''(t_e - t^*)$, $V^i_{t^*} \in [0, 1]$, and

$$E_{t^*}[V^i_{t^*} \ |\Omega^A \setminus A^{t_e}] = E_{t^*}[\delta_{t^*,t^a(\omega)} V^i_{t^a(\omega)} \mid \Omega^A \setminus A^{t_e}] \leq \delta_{t^*,t_e}.$$

Therefore, towards finding a lower bound for $Pr(A^{t_e} \cap L(c', t^*, t_e))$, we will find an upper bound for $E^i_{t^*}[V^i_{t^*} | A^{t_e} \cap L(c', t^*, t_e)]$.

Note that, we have

$$\begin{aligned}E^i_{t^*}[V^i_{t^*} | A^{t_e} \cap L\left(c', t^*, t_e\right)] &= E^i_{t^*}[\delta_{t^*,t^a(w)} V^i_{t^a(w)} | A^{t_e} \cap L\left(c', t^*, t_e\right)] \\ &\leq E^i_{t^*}[\delta_{t^*,t^a}\left(\left(\pi^i_{t^a} + c(t_e - t^a)\right)(1 - \delta_{t^a,t_e}) + \delta_{t^a,t_e}\right) | A^{t_e} \cap L\left(c', t^*, t_e\right)] \\ &= E^i_{t^*}[\left(\pi^i_{t^a} + c(t_e - t^a)\right)(\delta_{t^*,t^a} - \delta_{t^*,t_e}) | A^{t_e} \cap L\left(c', t^*, t_e\right)] + \delta_{t^*,t_e} \\ &\leq E^i_{t^*}[\left(\pi^i_{t^a} + c(t_e - t^a)\right)(1 - \delta_{t^*,t_e}) | A^{t_e} \cap L\left(c', t^*, t_e\right)] + \delta_{t^*,t_e} \\ &\leq E^i_{t^*}[\left(\pi^i_{t^*} + c'(t^a - t^*) + c(t_e - t^a)\right)(1 - \delta_{t^*,t_e}) | A^{t_e} \cap L\left(c', t^*, t_e\right)] + \delta_{t^*,t_e} \\ &\leq E^i_{t^*}[\left(\pi^i_{t^*} + c(t_e - t^*)\right)(1 - \delta_{t^*,t_e}) | A^{t_e} \cap L\left(c', t^*, t_e\right)] + \delta_{t^*,t_e} \\ &= \left(\pi^i_{t^*} + c(t_e - t^*)\right)(1 - \delta_{t^*,t_e}) + \delta_{t^*,t_e},\end{aligned}$$

where the first inequality uses Lemma 1 to bound $V^i_{t^a}$ from above, the next equality follows by simple algebra using the fact that $\delta_{t_1,t_2}\delta_{t_2,t_3} = \delta_{t_1,t_3}$, the second inequality follows since $\delta_{t^*,t^a} \leq 1$, the third inequality follows by definition of $L(c', t^*, t_e)$, and the last inequality follows by simple algebra and the fact that $c' \leq c$. Then, using (26), we obtain

$$\begin{aligned} Pr^i_{t^*}(A^{t_e} \cap L(c', t^*, t_e)) &\geq \frac{(\pi^i_{t^*} - c(t_e - t^*))(1 - \delta_{t^*,t_e}) - c''(t_e - t^*) - \delta_{t^*,t_e}}{\left[\pi^i_{t^*} + c(t_e - t^*)\right](1 - \delta_{t^*,t_e}) + \delta_{t^*,t_e}} \\ &\geq \frac{\pi^i_{t^*} - \varepsilon/4 - (\pi^i_{t^*} - c(t_e - t^*))\delta_{t^*,t_e} - \varepsilon/4 - \varepsilon/4}{\left[\pi^i_{t^*} + \varepsilon/4\right] + \varepsilon/4} \\ &\geq \frac{\pi^i_{t^*} - \varepsilon/4 - \varepsilon/4 - \varepsilon/4 - \varepsilon/4}{\pi^i_{t^*} + \varepsilon/4 + \varepsilon/4} \\ &\geq \frac{\pi^i_{t^*} - \varepsilon}{\pi^i_{t^*} + \varepsilon}, \end{aligned}$$

where the inequalities after the first one follow since $c(t_e - t^*) \leq \varepsilon/4$, $\pi^i_{t^*} \leq 1$ and $\delta_{t^*,t_e} < \varepsilon/4$. Since $Pr^i_{t^*}(A^{t_e}) \geq Pr^i_{t^*}(A^{t_e} \cap L(c', t^*, t_e))$, this completes the proof of the second part. □

Proof of Proposition 4. Since $\delta_{t^*} = 0$, $\delta_{t^*} S_{t^*+1/n} = 0 < 1$. Hence, the agents agree at t^* with $V^i_{t^*} = \pi^i_{t^*}$. Thus, at any $t < t^*$,

$$\begin{aligned} e^{-r(t^*-t)}\left(E^1_t[V^1_{t^*}] + E^2_t[V^2_{t^*}]\right) &= e^{-r(t^*-t)}\left(E^1_t[\pi^1_{t^*}] + E^2_t[\pi^2_{t^*}]\right) \\ &= e^{-r(t^*-t)}(1 + y_{t,t^*}) > 1. \end{aligned}$$

Therefore, expecting to be better of waiting until t^*, the agents disagree at $t < t^*$. □

Proof of Proposition 5. Define

$$\alpha = \inf_{t<\hat{t}} e^{-r(t^*-t)}(1 + y_{t,t^*}) - 1 > 0.$$

Pick any $\varepsilon \in (0, \alpha/(5+\alpha))$ with

$$4\varepsilon/(1-\varepsilon) < (1-\varepsilon)(1+\alpha) - 1. \tag{27}$$

This is possible because the strict inequality is satisfied at $\varepsilon = 0$. Let

$$\begin{aligned} \tau &= t^* + \min\{0.8\varepsilon/c, \gamma/2\} \in (t^*, t^* + \gamma) \\ \bar{n} &= 1/(\min\{\varepsilon/c, \gamma\} - \min\{0.8\varepsilon/c, \gamma/2\}) < \infty. \end{aligned}$$

Define also

$$t_e = \min\{t \in T^* | \tau \leq t \leq t^* + \min\{\varepsilon/c, \gamma\}\},$$

which is well defined since we consider $n > \bar{n}$. Assume also that $F(\tau) - F(t^*) > 1 - \varepsilon$. We will check that conditions O and D are satisfied for $\varepsilon' \equiv 4\varepsilon/(1-\varepsilon)$ (as the value of ε in Condition D) and that $\varepsilon' < \bar{\beta}(\hat{t}, t^*)$. The first statement then follows from Proposition 3. To this end, first note that

$$F(t_e) \geq F(\tau) > 1 - \varepsilon \text{ and } F(t^*) < \varepsilon.$$

By definition (see Condition O),

$$\bar{\beta}\left(\hat{t},t^*\right) \geq (1-F(t^*))(1+\alpha)-1 > (1-\varepsilon)(1+\alpha)-1 \equiv \beta^* > 0,$$

where the last inequality follows from $\varepsilon < \alpha/(5+\alpha)$. Hence, condition O is satisfied. Moreover, by (27), $\varepsilon' < \beta^* \leq \bar{\beta}\left(\hat{t},t^*\right)$. Finally, we check that condition D is satisfied by t^* and ε'. Firstly,

$$\delta_{t^*,t_e} \leq (1-F(t_e))/(1-F(t^*)) < \varepsilon/(1-\varepsilon) = \varepsilon'/4.$$

Secondly,

$$c(t_e-t^*) \leq c\min\{\varepsilon/c,\gamma\} \leq \varepsilon < \varepsilon/(1-\varepsilon) = \varepsilon'/4.$$

Therefore, t^* and ε' satisfies condition D.

The last statement in the proposition also follows from Proposition 3. We have

$$\begin{aligned}\Pr_0^i\left(A^{t_e}\right) &\geq (1-F(t^*))\,E_0^i\left(\Pr_{t^*}^i\left(A^{t_e}\right)\right)\\ &\geq (1-\varepsilon)\,E_0^i\left[\frac{\pi_{t^*}^i-\varepsilon'}{\pi_{t^*}^i+\varepsilon'}\right].\end{aligned}$$

As $\varepsilon \to 0$, we also have $\varepsilon' \to 0$, and the last expression goes to 1 by the bounded convergence theorem. □

Proof of Proposition 6. Take n large enough so that $1/n < \varepsilon$ and $c\left(1/n + (\log(4/\varepsilon))/\hat{r}\right) \leq \varepsilon/4$, which is possible by the condition on $\hat{r}$. First note that

$$\overline{\beta}\left(\hat{t},t^*\right) = \inf_{t<\hat{t}}\left[1+y_{t,t^*}-e^{r(t-t^*)}(1-F(t))/(1-F(t^*))\right] > \beta,$$

hence $\hat{t}$,t^* satisfy condition O. Let $t_e = \min\left\{t \in T^* \mid t \geq t^* + \frac{\log(4/\varepsilon)}{\hat{r}}\right\}$. We have

$$\delta_{t^*,t_e} = e^{-r(t-t^*)}(1-F(t))/(1-F(t^*)) \leq e^{-\hat{r}(t_e-t^*)} \leq \varepsilon/4.$$

Also, since $t_e \leq t^* + 1/n + \frac{\log(4/\varepsilon)}{r}$, we have that the bargaining power is weakly durable on $[t^*, t_e]$ with rate c that satisfies

$$c\,(t_e-t^*) \leq c\left(1/n + \frac{\log(4/\varepsilon)}{\hat{r}}\right) \leq \varepsilon/4.$$

Hence, t^* and ε satisfy condition D with c and t_e, and the first part of the result follows by Proposition 3. If, in addition, the bargaining power is strongly durable, then condition SD also holds and the second part of the result also follows from Proposition 3. □

Proof of Proposition 7. Under Assumption IND, $S_{t+1/n}$ is deterministic. Hence,

$$S_t = \delta_t S_{t+1/n} \qquad \text{if } \delta_t S_{t+1/n} > 1 \tag{28}$$

$$S_t = 1 + Y(t)(1 - \delta_t S_{t+1/n}) \qquad \text{if } \delta_t S_{t+1/n} \le 1. \tag{29}$$

For each n, let t_n be the first date on which the agents agree, which is deterministic by Assumption IND. Whenever $t_n > 0$, we have

$$1 < \delta_{0,t_n} \left(E_0^1[V_{t_n}^1] + E_0^2[V_{t_n}^2] \right) = \delta_{0,t_n} S_{t_n}, \tag{30}$$

where the first inequality follows from the definition of t_n and the second equality follows from Assumption IND. Since $S_{t_n} \le 2$, (30) implies that $t_n \le [0, (\log 2)/r]$. Thus, t_n is a bounded sequence. Hence, it suffices to show that 0 is the only limit point of the sequence t_n. Let t^* be any limit point of this sequence and consider a subsequence that converges to t^*. We will show that $t^* = 0$.

Since there is agreement at t_n, by Assumption IND, (29) holds at $t = t_n$. We consider the two conditions separately. First assume (i). Then, if $\delta_{t_n+1/n} S_{t_n+2/n} \le 1$, then (29) and $Y(t_n + 1/n) \ge 0$ imply that $S_{t_n+1/n} \ge 1$. If $\delta_{t_n+1/n} S_{t_n+2/n} > 1$, then this time (28) implies that $S_{t_n+1/n} \ge 1$. Substituting this and $Y(t) \in [0,1]$ in (29), we obtain

$$S_{t_n} \le 2 - \delta_{t_n}.$$

Together with (30), this implies that

$$\delta_{0,t_n}(2 - \delta_{t_n}) > 1.$$

By taking the limit $n \to \infty$ on this inequality, we obtain

$$\delta_{0,t^*} \ge 1,$$

which implies that $t^* = 0$, as desired. Here, since F is continuous, $\delta_{0,t} = \delta^t (1 - F(t))$ is continuous in t, and $\delta_{0,t_n} \to \delta_{0,t^*}$ as $t_n \to t^*$, and also $\delta_{t_n} = e^{-r/n} (1 - F(t_n + 1/n)) / (1 - F(t_n)) \to 1$.[5]

Next assume (ii) without assuming (i). If $Y(t_n) \le 0$, then $S_{t_n} \le 1$ by (29), which, by (30), implies $t_n = 0$. Now, consider the subsequence of (t_n) with $Y(t_n) > 0$. If $Y(t_n + 1/n) \ge 0$ for all but finitely many n, then the above proof shows that $t^* = 0$. So assume $Y(t_n + 1/n) < 0$ for infinitely many values of n. This subsequence, which we denote by $t_{n'}$, also converges to t^*, and satisfies

$$Y(t_{n'} + 1/n') < 0 < Y(t_{n'}).$$

Taking limits in the previous inequality, we obtain $Y(t^*) = 0$. By (29), we have $S_{t_n} \le 1 + Y(t_n) \to$

[5] This is where the proof fails under a deterministic deadline.

$1 + Y(t^*) = 1$, which, by (30) implies that $t^* = 0$, as desired. □

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