An Elementary Proof of Positive Optimal Marginal Taxes

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1 Introduction

The most important theoretical result for the optimal non-linear taxation problem is that optimal marginal tax rates – at interior solutions – are non-negative and zero at the top and bottom of the skill distribution. Mirrlees (1971) first stated the positivity result but his proof relied implicitly on restrictive assumptions of separability. Seade (1982) provided an alternative proof using non-inferiority of leisure.

Unfortunately, Seade's proof, while mathematically simple, relies on a lemma outside the standard economic tool-kit which also makes intuition for the result more difficult. This note provides a simpler proof of Seade's result using methods entirely familiar to economists.

The modest purpose of this note is to simplify the proof of the result in Seade (1982).[1] The main argument in our proof is as follows. Manipulating the standard well-known optimality conditions we show that if marginal tax rates are not positive then there exist two productivity levels $n^0 < n^1$ such that: (i) marginal tax rates are zero at n^0 and n^1; (ii) marginal utility of consumption is lower at n^0. These two properties together with normality of leisure imply that utility must be higher at n^0 than at n^1. However, this violates a necessary condition for incentive compatibility: more productive agents must be better-off! We conclude that negative marginal tax rates cannot be optimal.

The next section contains the main argument borrowing heavily on Mirrlees' (1976) exposition. The appendix derives the main optimality conditions required for a self-contained exposition.

[1] The result in Seade's paper is restrictive in that it assumes that bunching does not take place at the optimum. As Ebert (1992) and others has stressed this may be an important qualification.

2 Positive Taxes

We follow Mirrlees' (1976) notation closely. There is a continuum of agents that are heterogeneous with respect to their productivity n, distributed with density $f(n)$. Agents preferences are summarized by a concave utility function over consumption, c, and leisure, l, given by $U(c,l)$. We define an indirect utility over consumption, c, output, y, and productivity, n : $u(c,y,n) \equiv U(c, 1-y/n)$. Let $e(v,y,n)$ denote the 'inverse' of $v = u(c,y,n)$ with respect to c.

The marginal tax rate for an agent with productivity n has the same sign as the Lagrangian multiplier on the incentive constraint $\mu(n)$, see equation (34) from Mirrlees (1976) reproduced as equation (34-M76) in the appendix below. We will prove that this multiplier is positive for $n \in (\underline{n}, \bar{n})$ as long as output $y(n)$ is strictly positive. Optimality requires that the multiplier must be zero at the boundaries, $\mu(\underline{n}) = \mu(\bar{n}) = 0$.

From Mirrlees' equation (33) we have the optimality condition

$$f - \mu' - \mu u_{nc} e_v - f \lambda e_v = 0 \tag{1}$$

where $\lambda \geq 0$ is the multiplier on the economy-wide resource constraint.

If $\mu(n) < 0$ for some $n \in (\underline{n}, \bar{n})$ then there must be some "maximal interval" $[n^0, n^1]$ with $n^0 < n^1$ such that $\mu(n) \leq 0$ for $n \in [n^0, n^1]$, with $\mu(n^0) = \mu(n^1) = 0$ and $\mu'(n^0) \leq 0$ and $\mu'(n^1) \geq 0$. Define the expenditure function (without distortions) for agent n by:

$$m(v,n) \equiv \min_{c,l} (c + nl) \ \text{ s.t. } \ U(c,l) = v$$

Because the marginal tax is zero it follows that the allocation $c(n)$ and $l(n)$ at n^0 and n^1 solve this minimum expenditure problem – i.e. agents n^0 and n^1 are not constrained. Using this and the envelope condition we obtain: $m_v = 1/U_c = 1/u_c = e_v$ (see the appendix for details).

Combining (1) with these facts we obtain:

$$m_v^0 = \frac{1}{\lambda}\left(1 - \frac{\mu'^0}{f^0}\right) \geq \frac{1}{\lambda} \geq \frac{1}{\lambda}\left(1 - \frac{\mu'^1}{f^1}\right) = m_v^1.$$

More explicitly: $m_v(v(n^0), n^0) \geq m_v(v(n^1), n^1)$ with $n^0 < n^1$.

We now show that $m_v^0 \geq m_v^1$ with $n^0 < n^1$ implies that $v(n^0) \geq v(n^1)$. Towards a contradiction, assume instead that $v(n^0) < v(n^1)$. Strict concavity of $U(c,l)$ implies that $m_{vv} > 0$ and normality of leisure implies that $m_{vn} = m_{nv} = l_v \geq 0$. Thus, $n^0 < n^1$ and $v(n^0) < v(n^1)$ imply the opposite inequality, $m_v(n^0, v(n^0)) < m_v(n^1, v(n^1))$, a contradiction.

Thus, $v(n^0) \geq v(n^1)$.[2]

Finally, $v(n^0) \geq v(n^1)$ violates incentive compatibility constraints – since output is positive more productive agents must be better off! It follows that, $\mu(n) < 0$ is not possible and thus that tax rates are non-negative.

A Optimality Conditions

After a change in variables $v(n) = u(c(n), y(n), n)$ the problem Mirrlees (1976) sets up is to maximize

$$\int v(n) f(n) \, dn$$

subject to,

$$\int (y(n) - e(v(n), y(n), n)) f(n) = G$$

and,

$$v'(n) = u_n (e(v(n), y(n), n), y(n), n).$$

where $e(v, y, n)$ is the solution of $v = u(c, y, n)$ for c. Once this problem is solved we can recover consumption and leisure using $c(n) = e(v(n), y(n), n)$ and $l(n) = 1 - y(n)/n$.

The first order conditions are:

$$f - \lambda e_v f - \mu' - \mu u_{nc} e_v = 0 \qquad \text{(33-M76)}$$

and,

$$\lambda f (1 - e_y) = \mu (u_{ny} + u_{nc} e_y)$$

where we omit the arguments where functions are evaluated for simplicity (we will be careful below where it counts).

Define the marginal rate of substitution function $s(c, y, n) \equiv u_y(c, y, n) / u_c(c, y, n)$. Then:

$$s_n(c, y, n) = \frac{1}{u_c(c, y, n)} (u_{yn}(c, y, n) - s(c, y, n) u_{cn}(c, y, n))$$

and $s_n > 0$ by the single crossing condition. Since, $e_y(v, y, n) = -s(e(v, y, n), y, n)$ the second optimality condition can be written as:

$$\lambda \; f \;\; (1 + s) = \mu \; s_n \; u_c \qquad \text{(34-M76)}$$

[2] This observation underlies Theorem 3.1 in Mirrlees (1986, pg. 1212) – that without asymmetric information and lump-sum taxation more productive agents are worse off.

Note that $\tau(n) = 1 + s(e(v(n), y(n), n), y(n), n)$ represents the marginal tax rate faced by individuals with productivity n. The sign of the marginal tax rate is equal to the sign of $\mu(n)$ since $s_n > 0$ (single crossing) and $u_c > 0$. Equations (33-M76) and (34-M76) are exactly the same numbered equations in Mirrlees (1976).

From now on we discuss equations (33-M76) and (34-M76) for productivity levels where $\mu(n) = 0$. Letting $c(n) = e(v(n), y(n), n)$ and $l(n) = 1 - y(n)/n$ we have that

$$\begin{aligned} 1 - \lambda e_v(v(n), y(n), n) - \mu' &= 0 \\ s(c(n), y(n), n) &= -1 \end{aligned} \tag{2}$$

Note that,

$$e_v(v(n), y(n), n) = 1/u_c(c(n), y(n), n) = 1/U_c(c(n), l(n)). \tag{3}$$

The second optimality condition $-s = 1$ is by definition equal to:

$$\frac{U_l(c(n), l(n))}{U_c(c(n), l(n))} = n$$

which implies that $c(n)$ and $l(n)$ solve $\min_{l,c}(nl + c)$ s.t. $U(c,l) = v(n)$. Defining the expenditure function $m(v,n) = \min(nl + c)$ s.t. $U(c,l) = v$ we have shown that $E_v(v,n) = 1/U_c(c(n), l(n))$ and thus, using (3), we obtain that

$$e_v(v(n), y(n), n) = E_v(v(n), n)$$

The argument in Section 2 proceeds by substituting this equality into (2).

References

[1] Ebert, Udo (1992) "A Reexamination of the Optimal Nonlinear Income Tax", *Journal of Public Economics* 49, p. 47-73.

[2] Mirrlees, James A. (1971), "An Exploration in the Theory of Optimal Income Taxation," *Review of Economic Studies* 38, 175-208.

[3] Mirrlees, James A. (1976), "Optimal Tax Theory: A Synthesis," *Journal of Public Economics, v.* 6, n. 4, 327-358.

[4] Mirrlees, James A. (1986) “The Theory of Optimal Taxation”, *in Handbook of Mathematical Economics*, edited by K.J. Arrow and M.D. Intrilligator, vol. III, ch. 24, 1198-1249.

[5] Seade, Jesus (1982) “On the Sign of the Optimum Mariginal Income Tax”, *Review of Economic Studies* 49, 637-643.