Firms as Clubs in Walrasian Markets with Private Information

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We incorporate multiagent, principal-agent theory into general equilibrium analysis. The traded commodities are multiagent contracts that include a description of the individual's job, effort level, and state-contingent consumption. These contracts are club goods. The competitive equilibrium and the Pareto program are formulated. The contracts are identified with firms, so the market determines which firms exist and who is assigned to which firm in what capacity. An example is provided in which the internal organization of firms and the distribution of firm classes vary with the aggregate capital endowment and its distribution across agents. A simplex-based algorithm for solving the Pareto program is developed.

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I. Introduction

In this paper, we incorporate multiagent private information theory into the classical general equilibrium model of Arrow, Debreu, and McKenzie. We analyze a model with risk and moral hazard in production in which single-agent and multiagent firms mitigate incentive problems. Not only does the market set the prices of credit, insurance, and production inputs, but it determines the types of firms that form, their internal organization, their compensation structure, and the assignment of people to them. A by-product of our theory is that the distribution and level of wealth affect the industrial organization of the economy, as well as the economywide distribution of labor effort and consumption.

Agency theory has been enormously influential in the study of the firm. Alchian and Demsetz (1972) studied team production in an environment in which team members' individual efforts could not be ascertained from the team's output. This feature of the production technology created a role for a supervisor, someone who could measure, monitor, and supervise team members' efforts. Modern treatments of this problem began with the principal-agent problem of Harris and Raviv (1979) and Holmström (1979). A worker's effort is private information, and contracts and supervision are used to cope with and ameliorate this problem. Extensions to this model include the multiagent relative-performance and team production models of Holmström (1982).

A characteristic feature of these principal-agent theories is that they take as given the assignment of workers to a principal, the prices faced by the principal and agents, and the opportunity cost to an agent of not participating in the arrangement.[1] Consequently, these theories cannot answer certain substantive economic questions. For example, if workers are already assigned to firms, then there can be no endogenous size distribution of firms. If opportunity costs of workers are fixed and utility is driven to those margins, then there can be no welfare effects on the workers of changing internal organization. If the wealth distribution or a policy changes, then we cannot predict how the size and type of firms, their internal organization, and the wages of laborers and managers change.

By making these features endogenous, we bring the contractual theory of the firm into the general equilibrium tradition in industrial organization, emanating from Lucas (1978). In that seminal paper, agents decide whether to be a worker or to be a manager and hire workers. The market sets the wage and the returns to operating a firm, so the

[1] The article by Legros and Newman (1996) is an important exception. They studied incentive-based organizations in a general equilibrium economy with risk-neutral agents. Instead of using competitive analysis, they used a core equilibrium concept and took the price of the capital input as exogenous.

size distribution of firms is determined as part of a competitive equilibrium.

Our theory is broadly applicable to principal-agent problems in which the private information occurs after contracting and there are no side trades around a contract. We illustrate the analysis with four types of principal-agent relationships that regularly appear in the literature: single-agent self-employment firms with moral hazard, multiagent supervisor-worker firms, multiagent relative-performance firms, and team production. These principal-agent relationships, and others, are identified with different classes of firms in our theory.

The general equilibrium problem is to determine which firms form, the contracts for employees of these firms, and the assignment of agents to jobs within the firms. Because of moral hazard within a firm, a contract covers all of a firm's employees and is characterized by joint consumption and joint production. For example, in a relative-performance contract, compensation depends on comparing outputs across agents. In a team, production depends on each member's effort.

Arrangements with joint production and joint consumption are precisely what club theory, developed originally by Buchanan (1965), was designed to study. A club's members, be it through their characteristics or activities, can affect the production or enjoyment of the club good. Swimming pools and marriages are two other examples of clubs. We decentralize our club economies using the tools developed in Cole and Prescott (1997) and Ellickson et al. (1999).

With club goods and private information, it is natural for nonconvexities to arise in our environment. We follow E. C. Prescott and Townsend (1984*a*, 1984*b*) and eliminate the nonconvexities with lotteries. It is well known that lotteries can improve on deterministic allocations.

Our identification of firms as clubs is close in spirit to McKenzie's treatment of firms. In McKenzie (1959, 1981), firms are identified with entrepreneurial factors supplied to the market by individuals. In our theory, firms are identified with groups of individuals who supply participation in a contract. In both cases the formation of firms is a linear activity in the aggregate production set.[2]

Our theory is applicable to problems in which the economy needs to assign individuals to jobs and incentives within these jobs are important. For example, executives are an important input into production. They are well paid, and it matters how they are paid. A study of executive pay should benefit from a theory that assigns executives to firms, as in Rosen (1992). Another application is the role of occupational choice

[2] For more on this perspective, see the discussions contained in McKenzie (1981) and Hornstein and Prescott (1993).

in the development process, as in Banerjee and Newman (1993) and Aghion and Bolton (1997).

Section II defines the competitive equilibrium and the Pareto program. Section III maps four common private-information problems into the framework. Section IV discusses the welfare and existence theorems. Section V analyzes prices. Section VI provides numerical examples that demonstrate how the distribution of income and the aggregate capital endowment can affect the industrial organization of the economy. Section VII extends the model to include limited commitment and heterogeneity in agents' abilities and preferences. Section VIII contains some concluding comments. Finally, the Appendix develops a simplex-based algorithm for solving the Pareto program.

II. Competitive Equilibria and Pareto Optima

There are L classes of firms, indexed by $l = 1, \ldots, L$. Each class of firm uses a fixed number of employees, and these employment positions may differ from each other. Let n_l denote the number of positions in a firm of class l.

A firm uses capital and its employees' efforts to create output, possibly stochastically. It also distributes consumption to its employees. For each class of firm, there is a finite set of incentive-compatible contracts. A contract specifies levels of capital and effort and the state-dependent consumption of its employees. An incentive-compatible contract guarantees that state-dependent consumption induces employees to take desired effort levels. The precise characteristics of a contract depend on the class of firm, so it will be formally defined later when we give examples. Let $m = 1, \ldots, M_l$ index the set of incentive-compatible contracts for a firm of class l. Let b_{lm} denote contract m at a firm of class l.

Consumers.—There is a continuum of agents of measure one. The agents are divided into a finite number of types, indexed by $i = 1, \ldots, I$. For each type i, the number of agents is a positive fraction $\alpha_i > 0$ of the population. Types differ only in their nonnegative endowment of capital κ_i. The total endowment of capital is $\kappa = \sum_i \alpha_i \kappa_i$.

Agents receive utility from consumption $c \geq 0$, effort $a \geq 0$, the class of firm they work for, and their position. Let

$$U(c, a, j, l), \quad c \geq 0, \ a \geq 0, \ l = 1, \ \ldots, L, \ j = 1, \ \ldots, n_l,$$

denote the utility of an agent who consumes c, provides effort a, and works in position j in a firm of class l. Utility is strictly increasing and concave in c and decreasing in a. An agent's consumption and effort are specified in a contract. The *indirect* utility of an agent who participates in contract b_{lm} and works in position j of firm l is $u(b_{lm}, j)$. Indirect utility will be formally defined in terms of the underlying utility function later, when the examples are presented.

People are assigned to firms through a market in which agents purchase *probabilities* of being assigned to contracts and positions in a firm. We use lotteries for two reasons. First, as Hansen (1985) and Rogerson (1988) have demonstrated, lotteries can improve on deterministic allocations in environments with indivisibilities. Second, as E. C. Prescott and Townsend (1984*a*) have shown in environments without indivisibilities, lotteries may also be valuable because sets of incentive-compatible contracts need not be convex.

Agents sell their capital at price p_k and purchase probabilities of being assigned to position j in firm l under contract b_{lm} at price $p(b_{lm}, j)$. Let $x_i(b_{lm}, j)$ be the purchase of a type i agent. The consumption set is

$$X = \left\{ x_i(b_{lm}, j) \geq 0 \,\middle|\, \sum_{l=1}^{L} \sum_{m=1}^{M_l} \sum_{j=1}^{n_l} x_i(b_{lm}, j) = 1 \right\}, \tag{1}$$

which guarantees that an agent's choice is a probability measure.

A type i consumer chooses $x_i(b_{lm}, j)$, $m = 1, \ldots, M_l$, $l = 1, \ldots, L$, $j = 1, \ldots, n_l$, to solve

$$\max \sum_{l=1}^{L} \sum_{m=1}^{M_l} \sum_{j=1}^{n_l} x_i(b_{lm}, j) u(b_{lm}, j) \tag{2}$$

subject to $x_i \in X$ and the budget constraint

$$\sum_{l=1}^{L} \sum_{m=1}^{M_l} \sum_{j=1}^{n_l} x_i(b_{lm}, j) p(b_{lm}, j) \leq p_k \kappa_i, \tag{3}$$

where the capital endowment, κ_i, is inelastically supplied for income at price p_k. Note that preferences are linear in x_i.

The budget constraint does not explicitly list a return on labor. Instead, any return on labor, as well as the agent's capital, is bundled implicitly into the contractual terms of b_{lm}, because these terms specify the efforts and the consumptions of employees.

Production sector.—The production sector creates firms. It buys the capital and net consumption needed to operate firms and it sells positions in them. The production sector faces constant returns to scale, so it does not matter how many profit-maximizing entities there are. For convenience, we assume that the sector is characterized by a single such representative entity.

The production sector buys capital at price p_k and consumption at price p_c. Let $r_k(b_{lm})$ denote the capital used by a firm of class l with contract b_{lm} and let $r_c(b_{lm})$ denote the *expected net* consumption of firm l with contract b_{lm}. Each firm can produce a different amount than it distributes in consumption to its employees. This feature is necessary because, in general, a firm's contract solves moral hazard problems for its employees. Thus the firm needs to control all consumption of its employees.

Let $\delta(b_{lm})$ be the *number* of firms of class l with contract b_{lm} that are created. The production sector maximizes profits by choosing $\delta(b_{lm}) \geq 0$, $m = 1, \ldots, M_l$, $l = 1, \ldots, L$, to solve

$$\max \sum_{l=1}^{L} \sum_{m=1}^{M_l} \delta(b_{lm}) \left[\sum_{j=1}^{n_l} p(b_{lm}, j) - p_k r_k(b_{lm}) - p_c r_c(b_{lm}) \right]. \tag{4}$$

Market clearing.—The market-clearing conditions for positions and contracts in firms are

$$\delta(b_{lm}) = \sum_{i=1}^{I} \alpha_i x_i(b_{lm}, j) \quad \forall j, m, l. \tag{5}$$

Equation (5) is a club condition. It guarantees that for each firm, or club good, created, there is one person in each position.

The market-clearing conditions for capital and consumption are

$$\sum_{l=1}^{L} \sum_{m=1}^{M_l} \delta(b_{lm}) r_k(b_{lm}) \leq \sum_{i=1}^{I} \alpha_i \kappa_i \tag{6}$$

and

$$\sum_{l=1}^{L} \sum_{m=1}^{M_l} \delta(b_{lm}) r_c(b_{lm}) \leq 0. \tag{7}$$

Each firm consists of a finite number of agents that is infinitesimally small relative to the entire economy.

DEFINITION 1. A competitive equilibrium is an allocation $\{x_i(b_{lm}, j), \delta(b_{lm})\}$ and prices $\{p(b_{lm}, j), p_c, p_k\}$ such that, for each i, $x_i(b_{lm,\ j})$ solves (2) subject to $x_i \in X$ and (3), $\delta(b_{lm})$ solves (4), and (5)–(7) hold.

Let $\lambda_i > 0$, $i = 1, \ldots, I$. Pareto optimum can be found by choosing $\delta(b_{lm}) \geq 0$ and $x_i(b_{lm}, j)$, $m = 1, \ldots, M_l$, $l = 1, \ldots, L$, $j = 1, \ldots, n_l$, to solve

$$\max \sum_{l=1}^{L} \sum_{m=1}^{M_l} \sum_{j=1}^{n_l} \lambda_i \alpha_i x_i(b_{lm}, j) u(b_{lm}, j)$$

subject to $x_i \in X$ for $i = 1, \ldots, I$ and (5)–(7). The Pareto program is a linear program. It has a finite number of variables, a linear objective function, and a finite number of linear constraints.

III. Examples of Firms

In this section, we map several classic private-information problems into the firms of our theory. For each private-information problem, we define an incentive-compatible contract, the resources the contract uses, and agents' indirect utilities. Other types of firms, or principal-agent relationships, also fit into our framework, and several of these are discussed later.

Defining a commodity by the contractual terms faced by all of a firm's employees may seem unusual, but it is really an extension of commonly used commodity spaces. For example, it is standard to consider a job as a bundle of labor effort and working conditions. Spending eight hours on a garbage truck is different from spending eight hours behind a desk. Similarly, a 40-hour workweek is a different commodity than a 35-hour workweek. With private information, it also matters *how* each employee is paid, that is, whether a labor contract includes bonuses and other incentive pay features that depend on both the performance of the individual and anyone else in the firm. For this reason, the contractual terms have to be included in the definition of a commodity.

We also assume that a contract is *exclusive*; that is, the agent cannot make unobservable trades that would undo the contract. When agents are partially insured, as is typically the case with private information, they desire additional insurance. If agents were to obtain extra insurance, they would work less, which would undo their original contracts. Our exclusivity assumption precludes this possibility. Implicitly, market trades are observable and exclusivity is enforceable.[3]

Each private-information problem uses the following common notation. As defined earlier, preferences are $U(c, a, j, l)$, though because agents typically do not receive any intrinsic utility from participating in a principal-agent relationship, we drop the l index from the utility function. Each firm produces output q as a stochastic function of its employees' efforts and capital k. Shocks are uncorrelated across firms. As is commonly assumed in the moral hazard literature, we assume that q can take on only a finite number of values.

[3] When trade is restricted to exclusive, incentive-compatible contracts, the informational requirements in this economy are implicit in the commodity space. This is a stronger informational assumption than in a standard model. For example, typically if utility functions change, the commodity space does not. That is not true in our model. We need the set of feasible commodities to depend on preferences because preferences affect incentive-compatible allocations. Otherwise, in moral hazard economies in which agents trade only in lotteries without the restriction that their trades be incentive compatible, competitive equilibria exist but there is no guarantee that they are constrained efficient (see Rustichini and Siconolfi 2005).

A. Self-Employment Firms

A self-employment firm has only one position. It has no joint aspects, but it is still a club in a degenerate sense. The agent's utility depends only on consumption and effort, so we simplify the utility function to $U(c, a)$. Let $f_s(q|a, k)$ be the conditional probability distribution of output q given effort a and capital input k. The agent's effort is private information, whereas the capital input and the output are public information. A *contract* for this class of firm consists of a compensation schedule $c(q)$, an effort level a, and a capital input level k.

This problem is the classic moral hazard problem of Harris and Raviv (1979) and Holmström (1979), though with the addition of a publicly observed input in production. The agent chooses effort a, given the capital input k and compensation schedule $c(q)$. Because of moral hazard, not all self-employment contracts are incentive compatible. For example, an arbitrary schedule $c(q)$ might induce a level of effort a other than the one specified in the contract. By the revelation principle, however, we can map any such contract into another contract that is incentive compatible with the associated, induced effort. Formally, an incentive-compatible self-employment contract is one that satisfies the constraints that the actual action a be the same as the one recommended in the contract, that is,

$$\sum_q f_s(q|a, k)U(c(q), a) \geq \sum_q f_s(q|\hat{a}, k)U(c(q), \hat{a}) \quad \forall \hat{a}. \tag{8}$$

Contracts satisfying (8) are characterized by variation in consumption with output, unless the lowest effort level is taken.

DEFINITION 2. An incentive-compatible, self-employment contract b_{lm} is a vector $(c(q), a, k)$ that satisfies (8).

The agent's indirect utility from contract $b_{lm} = (c(q), a, k)$ is

$$u(b_{lm}) = \sum_q f_s(q|a, k)U(c(q), a).$$

Contracts require resources. The capital used by the contract is

$$r_k(b_{lm}) = k,$$

and the expected *net* consumption used is

$$r_c(b_{lm}) = \sum_q f_s(q|a, k)[c(q) - q].$$

Firms may consume more, or less, than they produce.

It is easy to accommodate agents who do not work into this class of firms. First, include in the set of actions a zero effort level and in the set of capital inputs a zero input level. Second, assume that when the zero input levels are combined they produce zero output with probability one. There are no incentive constraints for this effort level, so there is no consumption variation. Thus the price of the one position in this contract is simply the price of the employee's consumption. As we will see in the example, these "firms" are purchased by individuals with large endowments of capital, the "idle rich."

B. *Supervisor-Worker Firms*

A supervisor-worker firm has a worker who operates the technology and a supervisor who monitors him or her. We assume that the supervision makes the worker's effort public. We require that to monitor the worker, the supervisor must supply an equal amount of working time, as if working together. Most production within a firm requires multiple employees coordinating their efforts with supervision, and this assumption is a simple way of modeling this.

Let $j = 1$ index the worker position and let $j = 2$ index the supervisor position. We assume that utility is affected by the position, so the utility function is $U(c, a, j)$. Output is a function of only capital and the worker's effort; the production function is $f_{sw}(q|a_1, k)$. Efforts, capital, and output are public information. A *contract* for this class of firm is consumption sharing rules $c_1(q)$ and $c_2(q)$, effort levels a_1 and a_2, and capital input k. It is these joint consumption and production features that make these contracts club goods.

The monitoring assumption requires that the supervisor work an equal amount, so we require $a_1 = a_2$.

Definition 3. A supervisor-worker contract b_{lm} is a vector $(c_1(q), c_2(q), a_1, a_2, k)$ such that $a_1 = a_2$.

There are no incentive constraints in this type of firm.[4] Because there is no moral hazard problem, a supervisor-worker firm will be able to insure against its idiosyncratic shock. Since agents are risk averse, supervisor-worker firms will always make use of this option. Here, in equilibrium, $c_1(q)$ and $c_2(q)$ will be constant functions: consumption will not vary with output.

The employees' indirect utilities from contract $b_{lm} = (c_1(q), c_2(q), a_1, a_2, k)$ are

[4] Incentive constraints are needed if the supervisor observes only a signal correlated with the agent's effort, as in Holmström (1979). To model this, just let z be a publicly observed signal that is determined, along with output, by a combination of effort, the capital input, and a hidden shock, i.e., $f_{sw}(q, z|a_1, a_2, k)$. An incentive constraint on the agent's effort would be required, and consumption would depend on the output and the signal. In this generalization, it would make sense to drop the requirement that $a_1 = a_2$. In yet a further generalization, the supervisor's effort would also be private information, and he would have to be given an incentive to monitor.

$$u(b_{lm}, j) = \sum_q f_{sw}(q|a_1, k)U(c_j(q), a_j, j), \quad j = 1, 2.$$

The capital used by the contract is

$$r_k(b_{lm}) = k,$$

and the expected net consumption used is

$$r_c(b_{lm}) = \sum_q f_{sw}(q|a_1, k)[c_1(q) + c_2(q) - q].$$

C. *Relative-Performance Firms*

There are two agents who work separate projects, each with his own input and output. As in Holmström and Milgrom (1990), the projects are connected only through common shocks. There is no supervision.

Let $j = 1, 2$ index the two positions and the two projects. Utility does not depend inherently on the position, so we write it as $U(c, a)$. The capital used on project j is k_j, the effort applied to it is a_j, and the output of it is q_j. Because of the common shock, a joint conditional probability distribution over both outputs is needed. This distribution is

$$f_r(q_1, q_2|a_1, a_2, k_1, k_2).$$

Each effort is private information, whereas capital and outputs are public information. Because of the common shock, a project's output gives information about effort on both projects, so it is desirable to link each agent's consumption to output on both projects. This feature gives the problem its joint, or club, character. A *contract* is consumption sharing rules $c_1(q_1, q_2)$ and $c_2(q_1, q_2)$, effort levels a_1 and a_2, and capital levels k_1 and k_2.

There are incentive constraints for each person in a relative-performance firm. The standard assumption is for each agent to take the other agent's action as given.[5] The incentive constraints for worker 1 are

$$\sum_{q_1, q_2} f_r(q_1, q_2|a_1, a_2, k_1, k_2)U(c_1(q_1, q_2), a_1) \geq$$
$$\sum_{q_1, q_2} f_r(q_1, q_2|\hat{a}_1, a_2, k_1, k_2)U(c_1(q_1, q_2), \hat{a}_1) \quad \forall \hat{a}_1.$$

The incentive constraints for worker 2 are similar.

Definition 4. An incentive-compatible, relative-performance con-

[5] Sometimes it is valuable not to allow each worker to observe the other worker's recommended effort. This requires some randomization in the recommended efforts. We do not explicitly consider that possibility here.

tract b_{lm} is a vector $(c_1(q_1, q_2), c_2(q_1, q_2), a_1, a_2, k_1, k_2)$ that satisfies both agents' incentive constraints.

Indirect utilities from contract $b_{lm} = (c_1(q_1, q_2), c_2(q_1, q_2), a_1, a_2, k_1, k_2)$ are

$$u(b_{lm}, j) = \sum_{q_1, q_2} f_r(q_1, q_2 | a_1, a_2, k_1, k_2) U(c_j(q_1, q_2), a_j), \quad j = 1, 2.$$

The resources used by a b_{lm} contract are

$$r_k(b_{lm}) = k_1 + k_2,$$

$$r_c(b_{lm}) = \sum_{q_1, q_2} f_r(q_1, q_2 | a_1, a_2, k_1, k_2)[c_1(q_1, q_2) + c_2(q_1, q_2) - q_1 - q_2].$$

D. Team Production Firms

In the team production model of Holmström (1982), there are multiple agents who jointly work a single project. Let there be n_τ positions, indexed by $j = 1, \ldots, n_\tau$. Utility does not depend on position, so we write it as $U(c, a)$.

An employee in position j supplies effort a_j. There is a single capital input k and a single scalar output q. The conditional probability distribution of output is

$$f_\tau(q | a_1, \ldots, a_{n_\tau}, k).$$

Each agent's effort is private information, whereas capital and output are public information. A *contract* is consumption sharing rules $c_j(q)$, $j = 1, \ldots, n_\tau$; effort levels a_j, $j = 1, \ldots, n_\tau$; and capital input k.

The incentive constraints for the agent in position 1 are

$$\sum_q f_\tau(q | a_1, a_2, \ldots, a_{n_\tau}, k) U(c_1(q), a_1) \geq$$
$$\sum_q f_\tau(q | \hat{a}_1, a_2, \ldots, a_{n_\tau}, k) U(c_1(q), \hat{a}_1) \quad \forall \hat{a}_1.$$

The incentive constraints for the other agents are similar.

DEFINITION 5. An incentive-compatible, team production contract b_{lm} is a vector $(c_1(q), \ldots, c_{n_\tau}(q), a_1, \ldots, a_{n_\tau}, k)$ that satisfies the incentive constraints for each position.

Indirect utilities from contract $b_{lm} = (c_1(q), \ldots, c_{n_\tau}(q), a_1, \ldots, a_{n_\tau}, k)$ are

$$u(b_{lm}, j) = \sum_q f_\tau(q | a_1, \ldots, a_{n_\tau}, k) U(c_j(q), a_j), \quad j = 1, \ldots, n_\tau.$$

The resources used by contract b_{lm} are

$$r_k(b_{lm}) = k,$$

$$r_c(b_{lm}) = \sum_q f_\tau(q|a_1, \ldots, a_{n_\tau}, k)\left[\sum_{j=1}^{n_\tau} c_j(q) - q\right].$$

The consumption of a team's members need not equal their joint output in all states; surpluses and deficits are transferred to or from other firms. In the language of the team literature emanating from Holmström (1982), "budget balance" is not required to hold. In that literature, the role of the principal is to punish the agents by relaxing the constraint that team members' consumption sum to the team's output. When the principal consumes some of the output, he indirectly punishes the agents by lowering their consumption. In our model, trade with the market performs that role.

IV. Existence and Welfare Theorems

Despite the unusual commodities, our club economy is very similar to the classical general equilibrium model of Arrow, Debreu, and McKenzie. The commodity space is Euclidean, the consumption sets are compact and convex, the utility functions are linear, and the production set is a convex cone. Because the market-clearing conditions for positions within a firm hold at equality, there is not free disposal, but the classical model has been extended to include this case. The only difference is the assumption of a finite number of types of agents rather than a finite number of agents, but this is a minor difference.

All the standard general equilibrium results hold for this economy. Competitive equilibria exist, they are Pareto optimal, and Pareto optima can be supported as competitive equilibria.[6] The only caveats that apply are the usual ones, namely, that agents not be satiated and a cheaper point exists. Both of these assumptions are satisfied for reasonable specifications of our economy. For example, as long as the set of feasible incentive-compatible contracts includes contracts with low effort and a high level of consumption that is unattainable, then agents cannot be satiated. Cheaper points also exist if agents are endowed with positive amounts of capital. Proofs of the theorems are contained in the technical appendix (E. S. Prescott and Townsend 2005).

We assumed that agents can trade only a finite number of contracts

[6] One interesting feature of this environment is that there are Pareto optima that cannot be supported as competitive equilibria if the only difference in income between agents comes from their capital holdings. As we will see in the example, these Pareto optima can be supported if income transfers are also allowed.

for two reasons. First, if agents trade lotteries over a continuum of contracts, the commodity space is the space of signed measures. Welfare and existence theorems are available for these economies, but the exposition and proofs of these theorems are considerably simpler in finite-dimensional, Euclidean spaces.[7] Furthermore, as the grid of contracts gets finer, the economy will approach that of the continuum economy.

The second reason for the finiteness assumption is that it is helpful for computing solutions to examples. The Appendix describes an algorithm for solving the Pareto program. The second part of this algorithm describes a method that generates a rich, yet finite, set of incentive-compatible contracts for each type of firm.

V. Analysis of Prices

Existence of an optimum to the production sector's problem requires

$$\sum_{j=1}^{n_l} p(b_{lm}, j) \leq p_c r_c(b_{lm}) + p_k r_k(b_{lm}), \quad l = 1, \ldots, L, \ m = 1, \ldots, M_l. \tag{9}$$

If $\delta(b_{lm}) > 0$, then the corresponding equation in (9) holds with equality.

Equation (9) is the entry condition for firms. The only firms that form are those for which the price of positions equals the resource cost of creating them. The price of a position measures its utility value. For a type i agent, let v_i be the Lagrangian multiplier on his probability measure constraint, (1), and let γ_i be the multiplier on his budget constraint, (3). From the consumer's first-order conditions, for $i = 1, \ldots, I$, $l = 1, \ldots, L$, $m = 1, \ldots, M_l$, and $j = 1, \ldots, n_l$,

$$u(b_{lm}, j) - v_i - \gamma_i p(b_{lm}, j) \leq 0, \tag{10}$$

with (10) holding at equality if $x_i(b_{lm}, j) > 0$. Now consider a b_{lm} contract for which $\delta(b_{lm}) > 0$. Let $i(j)$ indicate that a type i agent is in position j. Substituting (10) into (9) gives

$$\sum_{j=1}^{n_l} \frac{u(b_{lm}, j)}{\gamma_{i(j)}} = p_c r_c(b_{lm}) + p_k r_k(b_{lm}) + \sum_{j=1}^{n_l} \frac{v_{i(j)}}{\gamma_{i(j)}}. \tag{11}$$

Equation (11) equates the marginal benefit to the marginal cost of creating an additional class l firm with contract b_{lm} and in which each position j is filled with agent $i(j)$. The left-hand side is the marginal benefit. It sums over employees' utilities, weighted by the inverse of marginal utilities of income. The right-hand side is the marginal cost.

[7] For their use in a private-information economy, see E. C. Prescott and Townsend (1984*a*).

The sum of the first two terms is the cost of physical resources. The last term is a measure of the scarcity value of the firm's employees. The multiplier v_i is the marginal increase in *total* utility accruing to type i agents from a marginal increase in the number of type i agents. *People* are scarce in this economy, so the opportunity cost of assigning an agent to a contract needs to be taken into account.

The relative prices of positions are also related to utilities. Consider the two different contract and position combinations (b_{lm}, j) and $(b_{l'm'}, j')$, with $x_i(b_{lm}, j) > 0$ and $x_i(b_{l'm'}, j') = 0$, for some i. Define

$$\Delta u \equiv u(b_{lm}, j) - u(b_{l'm'}, j'),$$

$$\Delta p \equiv p(b_{lm}, j) - p(b_{l'm'}, j').$$

From (10) we get

$$\frac{\Delta u}{\gamma_i} \geq \Delta p, \qquad (12)$$

so the price differential is bounded from above by the utility differential scaled by the inverse of the marginal utility of income. If an agent is assigned to both contract position combinations with positive probability, then (12) holds with equality.

Additional insight into prices comes from considering the self-employment firm defined in Section III*A*. For any such contract created in equilibrium, substituting for the r's gives

$$p(b_{lm}) = p_c \sum_q f(q|a, k)[c(q) - q] + p_k k.$$

The value of $p(b_{lm})$ is the entry fee. For a fixed level of capital, a higher entry fee usually corresponds to a higher level of expected consumption and a lower level of effort.

The value of $p(b_{lm})$ may be positive, negative, or zero. A contract includes a bundle of state-contingent consumption and output, and the difference between expected consumption and expected output can have any sign. For example, if a self-employment firm used no capital, faced no uncertainty, and had no incentive problem, so $c(q) = \bar{c}$, then the price of the firm would be $p_c(\bar{c} - q)$, which can have any sign. In a more standard commodity space, where consumption and output are not bundled together, $p_c\bar{c}$ would be an expense on the left-hand side of the consumer's budget constraint and $p_c q$ would be labor income on the right-hand side. If this type of firm were chosen with probability one, consumption satisfied $\bar{c} > q$, and the difference were paid exclusively with capital income, then the budget constraint would be

$$p_c\bar{c} = p_c q + p_k \kappa_i.$$

The entry fee would equal capital income and would simply correspond to consumption purchased from the output produced by other firms.

Alternatively, if $\bar{c} < q$, then purchasing this contract would generate income. In this case, the agent would also have to buy a position in an additional firm at a positive price for the budget constraint to hold.

VI. An Example

In this section, we report the results of a numerical example. We solve for a Pareto optimum and then find prices and capital endowments that support it as a competitive equilibrium. There are three possible classes of firms. The first class is the self-employment class described in Section III*A*. The second class is the supervisor-worker class described in Section III*B*. The final class of "firms" is an idle individual, who only consumes. Technically, this class can be viewed as a self-employment firm that uses no labor and no capital and whose employee faces no incentive constraints.

With the methods described in the Appendix, contracts are generated from the sets $c \in \{0.00, 0.02, 0.04, \ldots, 1.20\}$, $q \in \{0, 1\}$, $a \in \{0, 1, 2\}$, and $k \in \{0, 1, 2\}$. The set of consumptions is a finite grid, but it is meant to approximate a continuum of values.[8] Output can take on two values, capturing success or failure. Both effort and the capital input can take on one of three levels. The first two classes of firms require a minimum of one unit of capital to operate; anyone assigned zero units of capital is considered idle.

The first two classes of firms also require at least one unit of effort to operate. The highest effort level reflects working hard and the second effort level reflects slacking off. The minimal effort level can be viewed as the utility cost from just showing up to work. It also means that an agent assigned to a self-employment firm cannot deviate to the zero effort level. Individuals in the idle class must work zero units.

There is a utility function common to all three classes of firms. Idle individuals, the self-employed, and workers in the supervisor-worker firm face the utility function

$$U(c, a) = 2c^{0.5} - (a/4).$$

Supervisors face the utility function

$$U(c, a, s) = 2c^{0.5} - (a/40),$$

which we index by s to distinguish from the other utility function. The

[8] This method uses the consumption grid points to effectively generate a piecewise linear approximation to consumption. Consequently, solutions will sometimes include convex combinations of adjacent consumption grid points.

only difference between the utility functions is that supervisory effort is a lot less onerous.

The idle class of firms produce zero output with certainty. The other two classes face a common production technology $f(q|a, k)$, which is

$$f(q|a,\ k = 1) = \begin{cases} a & q = 0 & q = 1 \\ 1 & 0.8 & 0.2 \\ 2 & 0.5 & 0.5 \end{cases},$$

$$f(q|a,\ k = 2) = \begin{cases} a & q = 0 & q = 1 \\ 1 & 0.6 & 0.4 \\ 2 & 0.2 & 0.8 \end{cases}.$$

The aggregate capital endowment is $\kappa = 0.6$. The two types of agents are equal fractions of the population, so $\alpha_1 = \alpha_2 = 0.5$. Finally, the Pareto weights are $\lambda_1 = 0.16$ and $\lambda_2 = 0.84$, so type 2 agents are favored.

Pareto optimum.—The Pareto optimum is characterized by two types of supervisor-worker firms, one type of self-employment firm, and some type 2 agents who are idle. Each of the two supervisor-worker firms consists of a type 1 agent as the worker and a type 2 agent as the supervisor. For each, $k = 1$ and $a_1 = a_2 = 2$. Supervisors receive much higher consumption than workers, and consumption of all members does not vary over output. The only difference between the two contracts is the consumption level of the supervisor. For one of the firms it is 0.52 and for the other it is 0.54. This difference is an artifact of the consumption grid. With a continuum it would disappear and there would be only one type of supervisor-worker firm. Finally, there are 0.29 of the first type and 0.11 of the second type of firms. Table 1 presents a summary.

The remaining portion of the population is assigned to single-agent firms. There is one kind of self-employment firm, and 0.10 of it. Its position is always filled by a type 1 agent. It uses $k = 2$ and $a = 2$. Because there is an incentive problem on its employee's effort, con-

TABLE 1
EQUILIBRIUM SUPERVISOR-WORKER CONTRACTS

Contract	$\delta(b_{lm})$	$x_1(b_{lm}, 1)$	$x_1(b_{lm}, 2)$	$x_2(b_{lm}, 1)$	$x_2(b_{lm}, 2)$
Supervisor-worker firm 1	.29	.58	. . .	. . .	.58
Supervisor-worker firm 2	.11	.22	. . .	. . .	.22

NOTE.—The contract for firm 1 is $c_1(q) = 0.02$ for all q, $c_2(q) = 0.52$ for all q, $a_1 = a_2 = 2$, and $k = 1$. The contract for firm 2 is $c_1(q) = 0.02$ for all q, $c_2(q) = 0.54$ for all q, $a_1 = a_2 = 2$, and $k = 1$.

TABLE 2
Equilibrium Single-Agent Contracts

Contract	$\delta(b_{lm})$	$x_1(b_{lm})$	$x_2(b_{lm})$
Self-employment firm	.10	.20	. . .
Idle "firm"	.10	. . .	.20

Note.—The contract for a self-employment firm is $c(q = 0) = 0.00$, $c(q = 1) = 0.098$, $a = 2$, and $k = 2$. The contract for idle individuals is $c(q) = 0.54$ for all q, $a = 0$, and $k = 0$.

TABLE 3
Prices for Positions in Each Firm

Contract	$j = 1$	$j = 2$ (Supervisor)
Supervisor-worker firm 1	−.439	1.569
Supervisor-worker firm 2	−.439	1.633
Self-employment firm	−.341	. . .
Idle "firm"	1.752	. . .

sumption varies with output. The agent receives $c(q = 1) = 0.0$ and $c(q = 2) = 0.098$.[9]

The last type of firm consists of type 2 agents who are idle. There are also 0.10 of these. These idle individuals do not work and receive a consumption level of 0.54. This level is the same as the one type 2's receive working in supervisor-worker firms (except for the differences resulting from the grid approximation). Type 2's are not assigned to any positions that face incentive constraints, so they are fully insured even across firm assignments. Table 2 reports the relevant statistics for both single-agent firms.

Supporting competitive equilibrium.—When we normalize the price of capital to be one, per capita wealth endowments needed to support this optimum as a competitive equilibrium are −0.419 for type 1 agents and 1.619 for type 2 agents. In this example, nonnegative assignments of capital are not enough to generate this wealth distribution. Instead, either negative assignments of capital or lump-sum taxes and transfers in the units of account are needed. (This is not true for all Pareto optima.) For example, one possible way to generate this wealth distribution is to give a per capita capital distribution of 0.0 to type 1 agents with a tax on them of −0.419 and a per capita capital distribution of 1.2 to type 2 agents with a transfer to them of 0.419.

Table 3 reports the prices of jobs in each of the four firms. Prices for the two supervisor-worker contracts are nearly identical. The price for the supervisor position in the second supervisor-worker firm is slightly

[9] Actually, if the agent produces the high output, he receives $c = 0.08$ with probability 0.11 and $c = 0.10$ with probability 0.89. Again, this is a lottery over consumption levels that would disappear with a continuum of consumption levels.

higher than the price in the first one because this supervisor receives slightly higher consumption. Prices for the worker position in these two firms are negative. Individuals are paid to be a worker in both firms. It is how type 1 agents earn “income” to overcome the tax in their budget constraint. Similarly, the price of the single position in the self-employment firm is negative. Here, because there is only one position in this firm, the price equals the firm’s cost of net expected consumption plus capital. In this firm net expected consumption is negative, so the firm supplies a surplus to the market. Conversely, the price of the idle “firm” is positive. The agent in this firm consumes a lot and produces zero. The price of this firm is just a pure purchase of consumption.

Each type 1 worker purchases a lottery between employment as a worker in a supervisor-worker firm and self-employment. Given the prices of the three contracts, -0.439, -0.439, and -0.341, and the chosen probabilities 0.58, 0.22, and 0.20, the worker exhausts his endowment of -0.419.

Endogenous industrial organization.—More generally, the particular optimum that will prevail depends on the Pareto weights and on the amount of economywide capital κ. Figure 1 describes parameter values for which supervisor-worker firms occur in equilibrium. For high aggregate capital levels and relatively equal Pareto weights (the upper right-hand corner of fig. 1), all firms are single-agent firms. As the aggregate capital level declines and the Pareto weights become more unequal, supervisor-worker firms begin to appear. The composition of these two-agent firms varies with the parameters. At low capital levels, 0.25 and below, the program assigns the low–Pareto weight (type 1) agents to supervise their fellow type 1 workers. As capital increases above 0.25, labor becomes more scarce. Some supervisor-worker firms are created in which a type 2 supervises a type 1. As capital increases above 0.5 and for low Pareto weights, all type 1’s are switched to workers and type 2’s supervise. Supervision is used to force type 1’s to work hard and transfer most of the consumption to type 2’s. For more equal Pareto weights, approximately 0.25 and above, and for capital above 0.5, there are still transfers, but less than at unequal Pareto weights. Fewer supervisor-worker firms are needed, so some type 1’s can be used as supervisors.

VII. Extensions

A. Limited-Commitment Firms

Some forms of limited commitment can be incorporated into our methodology. Let $d \in D = \{0, 1\}$, where $d = 0$ means that the agent stays in the firm and does not run off with the capital. Conversely, $d = 1$ means

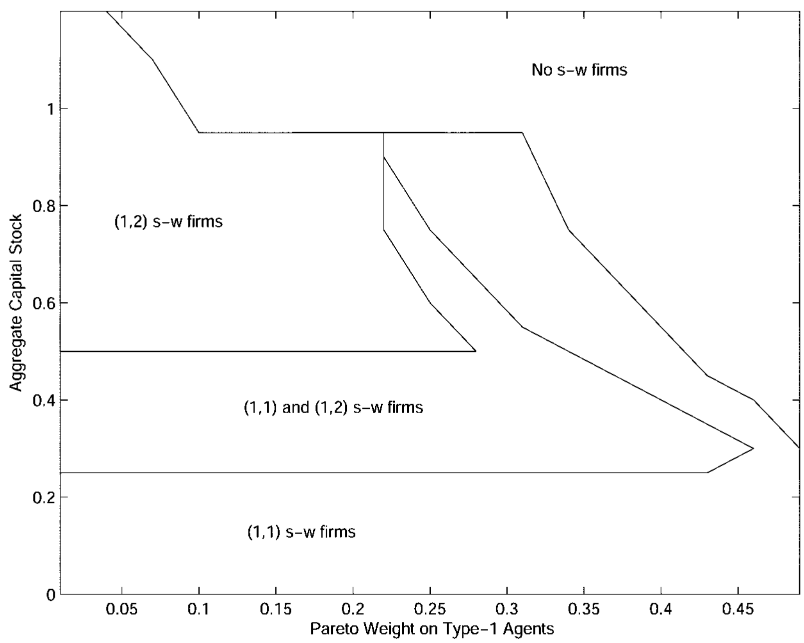

FIG. 1.—Occurrence of supervisor-worker firms as a function of the aggregate capital endowment and the distribution of Pareto weights. A (1, 1) supervisor-worker firm means that both members are type 1 agents. A (1, 2) supervisor-worker firm means that a type 1 agent is the worker and a type 2 agent is the supervisor. Note that in the figure the existence of a supervisor-worker firm does not preclude the existence of self-employment firms or idle individuals.

that the agent runs off with the capital. The decision to default and run off with the capital is made *after* the capital input is assigned but before the effort is taken. If an agent runs off with the capital, he converts it into consumption at some exogenous rate r with no effort supplied. In particular, we write $V(k, d = 1) \equiv U(rk, 0, w)$; utility from staying is unchanged from before.

Whether a default occurs is anticipated by the market as though it were part of the contract all along. Consequently, we treat this possibility as another restriction on the set of traded contracts, just like an incentive constraint. A self-employment contract now consists of a sharing rule $c(q)$, an effort level a, a capital input k, and a default decision d. If the contract specifies that $d = 0$, that is, the agent should not default, then the limited-commitment constraint for the self-employment firm is

$$\sum_q f(q|a, k)U(c(q), a, 1) \geq V(k, d = 1). \tag{13}$$

There is also an incentive constraint, which is almost identical to (8).

Only contracts that satisfy (13) and the incentive constraint are feasible.[10]

Similar extensions can be made to the supervisor-worker firms. One possibility is to assume that only the supervisor has the ability to run off with the capital. See our working paper (E. S. Prescott and Townsend 2000) for an example along these lines.

B. *Intrinsic Heterogeneity*

To incorporate heterogeneity in agents' abilities into our framework, we expand the commodity space by indexing firm classes by employees' intrinsic types. A type is *intrinsic* if it affects the set of incentive-compatible contracts. This can happen if agents' preferences or abilities differ. We assume that an agent's intrinsic type is public information.

As before, there are $i = 1, \ldots, I$ types of agents, but now each type is also identified with an intrinsic type $t = 1, \ldots, T$. Let $t(i)$ indicate type i agents' intrinsic type. Two types i_1 and i_2 may be of the same intrinsic type, that is, $t(i_1) = t(i_2)$. In this language, the earlier example has one intrinsic type, with the two agents' types differing only in their endowments.

There are $l = 1, \ldots, L$ classes of firms. Firm classes are now also distinguished by the assignment of intrinsic types to positions. For example, a two-person firm with one intrinsic type in position 1 and a different intrinsic type in position 2 is in a different class than the same firm in which both positions are filled by the first intrinsic type. Let $h_{ljt} = 0$ indicate that intrinsic type t cannot be in position j in firm l, and $h_{ljt} = 1$ means that the type can be.

Utility is $U_l(c, a, j, l)$, and indirect utility is $u_t(b_{lm}, j)$. If agent i is of intrinsic type t, then his consumption set is $X_t = x_i(b_{lm}, j) \geq 0$ such that

$$\sum_{l=1}^{L} \sum_{m=1}^{M_l} \sum_{j=1}^{n_l} x_i(b_{lm}, j) = 1$$

and

$$x_i(b_{lm}, j) = 0 \quad \text{if } h_{ljt} = 0,\ l = 1, \ldots, L,\ j = 1, \ldots, n_l \qquad (14)$$

hold. Equation (14) guarantees that an agent can purchase only a position in firm l with contract b_{lm} that takes his intrinsic type, t, in that position.

The rest of the problem is unchanged.

[10] If the contract specifies that the agent should leave, i.e., $d = 1$, then the agent receives $V(k, d = 1)$. For simplicity, we assume that if the agent stays when he is supposed to leave, then the contract sets consumption low enough to preclude this possibility. This avoids formally writing out the limited-commitment constraint along this branch.

VIII. Conclusion

Virtually any principal-agent problem fits into our framework if there is exclusiveness of contracting and the private information occurs *after* contracting. In addition to the models described in Sections III and VII, this class includes the general multiagent problems studied in Demski and Sappington (1984) and Mookherjee (1984); the collusion models studied in Holmström and Milgrom (1990), Ramakrishnan and Thakor (1991), Itoh (1993), and E. S. Prescott and Townsend (2002); and the task assignment models studied by Holmström and Milgrom (1991).

The commodity spaces we used are truly large ones. Every possible production process, broadly defined, is priced. For some applications, this means that the theory may have more normative than positive implications, as Arrow and Debreu state-contingent claims did when they were first introduced. For other applications, it may be desirable to examine alternative, but equivalent, decentralizations. For example, Cole and Prescott (1997) demonstrate that club economies with lotteries can be decentralized with actuarially fair income lotteries followed by deterministic purchases of club goods. Still, for yet other applications the richer commodity space has immediate useful descriptive implications. For example, to study labor markets for individuals who receive performance pay and work with other similarly compensated individuals, such as teams of executives, a commodity space that directly incorporates incentives and club effects would seem to be needed.

Appendix

Computing

The Pareto program is a linear program, but if there are many feasible contracts—as one typically wants in applications—it has a large number of variables and club constraints. Consequently, the constraint matrix can be too large to store in computer memory. This appendix describes an efficient algorithm for solving the program that exploits the special structure of the problem.

There are two parts to our algorithm. First, we eliminate the club constraints by developing an alternative representation of the Pareto program. Second, we avoid enumerating the variables by representing incentive-compatible contracts as extreme points of a small system of linear inequalities. Both steps are essentially applications of the Dantzig-Wolfe algorithm.[11]

For expositional purposes, we develop the algorithm for the economy described in the example, but without the third class of firms. Therefore, there are two types of agents, that is, $i = 1, 2$, and there are only two classes of firms. Let $l = 1$ index the self-employment class and let $l = 2$ index the supervisor-

[11] This algorithm was developed in Dantzig and Wolfe (1961) to solve linear programs with a particular structure. Descriptions of it can be found in advanced linear programming textbooks such as Bertsimas and Tsitsiklis (1997).

worker class. The algorithm is easily generalized to more types of agents and classes of firms.

Step 1: Eliminate the club constraints.—Consider the club constraint in (5) associated with a particular b_{1m} contract. The only variables with nonzero coefficients for this constraint are $x_1(b_{1m})$, $x_2(b_{1m})$, and $\delta(b_{1m})$. These variables and the club constraint form a block of nonzero coefficients in the constraint matrix that does not interact with other variables or club constraints. This block is

$$[\alpha_1 \quad \alpha_2 \quad -1]\begin{bmatrix} x_1(b_{1m}) \\ x_2(b_{1m}) \\ \delta(b_{1m}) \end{bmatrix} = [0]. \tag{A1}$$

This equation and the nonnegativity constraints on $x_1(b_{1m})$, $x_2(b_{1m})$, and $\delta(b_{1m})$ define a polyhedral cone with its vertex at the origin. As such, the system of equations can be represented as the set of all nonnegative linear combinations of its extreme rays. When we scale each ray to $\delta(b_{1m}) = 1$, the extreme rays of this cone are $(1/\alpha_1, 0, 1)$ and $(0, 1/\alpha_2, 1)$. Let $\delta^{(i)}(b_{1m})$, $i = 1, 2$, denote the quantity of these rays. Any $(x_1(b_{1m}), x_2(b_{1m}), \delta(b_{1m}))$ that satisfies (A1) also satisfies

$$(x_1(b_{1m}),\ x_2(b_{1m}),\ \delta(b_{1m})) = \delta^{(1)}(b_{1m})(1/\alpha_1,\ 0,\ 1) + \delta^{(2)}(b_{1m})(0,\ 1/\alpha_2,\ 1)$$

for some $\delta^{(i)}(b_{1m}) \geq 0$, $i = 1, 2$, and vice versa. Figure A1 illustrates.

The rays have a natural interpretation. One unit of the first ray corresponds to one self-employment firm that fills its position with a type 1 agent and uses contract b_{1m}. One unit of the second ray is interpreted similarly, but with its position filled by a type 2 agent. The quantities of the rays are related to the variables by

$$\begin{aligned} x_i(b_{1m}) &= \frac{\delta^{(i)}(b_{1m})}{\alpha_i}, \quad i = 1, 2, \\ \delta(b_{1m}) &= \sum_i \delta^{(i)}(b_{1m}). \end{aligned} \tag{A2}$$

Blocks for the supervisor-worker clubs are similar. For contract b_{2m}, the block is

$$\begin{bmatrix} \alpha_1 & 0 & \alpha_2 & 0 & -1 \\ 0 & \alpha_1 & 0 & \alpha_2 & -1 \end{bmatrix}\begin{bmatrix} x_1(b_{2m},\ 1) \\ x_1(b_{2m},\ 2) \\ x_2(b_{2m},\ 1) \\ x_2(b_{2m},\ 2) \\ \delta(b_{2m}) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

These equations and the nonnegativity constraints also define a polyhedral cone with its vertex at the origin. When we scale each ray to $\delta(b_{2m}) = 1$, the four extreme rays of this cone are $(1/\alpha_1, 1/\alpha_1, 0, 0, 1)$, $(1/\alpha_1, 0, 0, 1/\alpha_2, 1)$, $(0, 1/\alpha_1, 1/\alpha_2, 0, 1)$, and $(0, 0, 1/\alpha_2, 1/\alpha_2, 1)$. One unit of the first ray corresponds to a supervisor-worker firm that fills both of its positions with type 1 agents and uses contract b_{2m}. Let $\delta^{(i,i')}(b_{2m})$ denote the number of b_{2m} firms that consist of

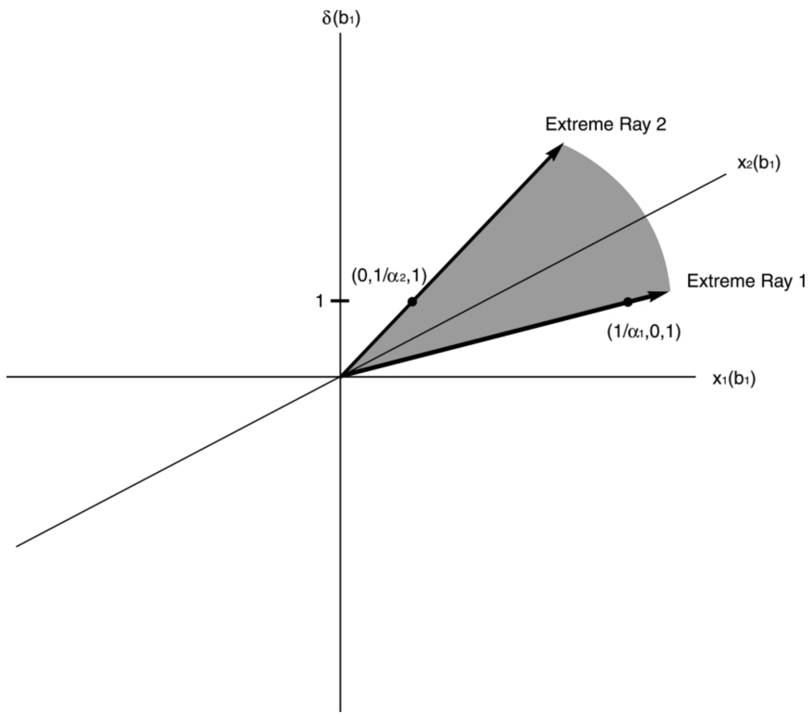

FIG. A1.—Description of the set $\{(x_1(b_{1m}),\ x_2(b_{1m}),\ \delta(b_{1m})) \in \Re^3_+ \mid \alpha_1 x_1(b_{1m}) + \alpha_2 x_2(b_{1m}) - \delta(b_{1m}) = 0\}$, which is illustrated by the shaded area. The two extreme rays are $(1/\alpha_1, 0, 1)$ and $(0, 1/\alpha_2, 1)$. Both are scaled to $\delta(b_1) = 1$. As is apparent in the figure, the above set can also be represented by the set of all nonnegative linear combinations of its extreme rays.

a type i as the worker and a type i' as the supervisor. The quantities of the rays are related to the variables by

$$x_i(b_{2m},1) = \sum_{i'} \frac{\delta^{(i,i')}(b_{2m})}{\alpha_i},$$

$$x_i(b_{2m},2) = \sum_{i'} \frac{\delta^{(i',i)}(b_{2m})}{\alpha_i},$$

$$\delta(b_{2m}) = \sum_{i,i'} \delta^{(i,i')}(b_{2m}). \tag{A3}$$

Now, define a new linear program in which the choice variables are nonnegative quantities of the rays. By definition, the club constraints are satisfied. Next, use (A2) and (A3) to put the objective function and the constraints that connect the blocks, that is, (1), (6), and (7), in terms of the new variables.

This alternative representation of the Pareto program (restricted to only b_{1m} and b_{2m} firms) is

$$\max_{\delta^{(i)}\geq 0,\delta^{(i,i')}\geq 0} \sum_i \lambda_i\Bigg[\sum_{m=1}^{M_1} \delta^{(i)}(b_{1m})u(b_{1m}) + \sum_{i'}\sum_{m=1}^{M_2} \delta^{(i,i')}(b_{2m})u(b_{2m},\ 1) + \sum_{i'}\sum_{m=1}^{M_2} \delta^{(i',i)}(b_{2m})u(b_{2m},\ 2)\Bigg]$$

subject to the probability measure constraints

$$\forall i,\quad \sum_{m=1}^{M_1} \delta^{(i)}(b_{1m}) + \sum_{i'}\sum_{m=1}^{M_2} \delta^{(i,i')}(b_{2m}) + \sum_{i'}\sum_{m=1}^{M_2} \delta^{(i',i)}(b_{2m}) = \alpha_i, \tag{A4}$$

the consumption constraint

$$\sum_i \sum_{m=1}^{M_1} \delta^{(i)}(b_{1m})r_c(b_{1m}) + \sum_{i,i'}\sum_{m=1}^{M_2} \delta^{(i,i')}(b_{2m})r_c(b_{2m}) \leq 0, \tag{A5}$$

and the capital constraint

$$\sum_i \sum_{m=1}^{M_1} \delta^{(i)}(b_{1m})r_k(b_{1m}) + \sum_{i,i'}\sum_{m=1}^{M_2} \delta^{(i,i')}(b_{2m})r_k(b_{2m}) \leq \kappa. \tag{A6}$$

There are no explicit club constraints, so with two types this program has only four constraints. However, it still has an enormous number of variables, even more than the original Pareto program. The next step develops a representation of these variables, the contracts, that is convenient for computing.

Step 2: A compact representation of the contracts.—Simplex-based algorithms, which are often used to solve linear programs, take a basic feasible solution, calculate the corresponding simplex multipliers, and then use these multipliers to check whether the solution can be improved on by introducing a nonbasic variable into the basis. For nonbasic variables $\delta^{(i)}(b_{1m})$, the condition to check is

$$\forall \delta^{(i)}(b_{1m}),\quad 0 \geq \lambda_i u(b_{1m}) - \mu_i - \mu_c r_c(b_{1m}) - \mu_k r_k(b_{1m}), \tag{A7}$$

where μ_i is the simplex multiplier on the type i constraint in (A4), μ_c is the simplex multiplier for (A5), and μ_k is the simplex multiplier for (A6).[12] If (A7) holds, then no $\delta^{(i)}(b_{1m})$ can be introduced into the basis to improve the value of the objective function. The condition for $\delta^{(i,i')}(b_{2m})$ is similar.

In our problem, there are too many b_{1m}'s to enumerate. For this reason, we use a compact representation of b_{1m}, $m = 1, \ldots, M_1$, that avoids directly checking (A7) for each m. We derive this representation for only the first class of firms. The representation of the second class can be similarly derived.

Let there be a grid of consumptions c, outputs q, actions a, and capital inputs k. Let $\pi(a, k)$ be the ex ante probability of being assigned an (a, k) pair, and let $\pi(c|q, a, k)$ be the conditional probability distribution of receiving consumption c given that output q was produced, action a was recommended, and capital input k was assigned. Together, these terms are analogous to the deterministic contract $(c(q), a, k)$ in definition 2. The only difference is the randomization,

[12] At an optimum the simplex multipliers are equal to the Lagrangian multipliers, and (A7) is the familiar first-order condition for optimality.

but that does not fundamentally change the problem. First, as we will see below, the relevant $\pi(a, k)$ will be degenerate, placing mass one or zero on each (a, k) combination. Second, under the usual assumptions on the utility function of concavity in consumption and separability, any randomization of consumption will be over adjacent points in the consumption grid and would go away with a sufficiently fine grid. In this case, $\pi(c|q, a, k)$ can be viewed as a piecewise linear approximation to richer consumption schedules.[13]

The next step in developing our compact representation is to describe the set of incentive-compatible contracts with a finite system of linear inequalities. We do this by embedding the contracts into a new variable. Let $\pi(c, q, a, k)$ be the joint probability distribution over consumption, output, effort, and the capital input. This distribution is related to the contractual terms, $\pi(a, k)$ and $\pi(c|q, a, k)$, and the exogenous technology, $f(q|a, k)$, by the identity

$$\pi(c, q, a, k) = \pi(c|q, a, k) f(q|a, k) \pi(a, k). \tag{A8}$$

First, we require that $\pi(c, q, a, k)$ be a probability measure by making it nonnegative and satisfy

$$\sum_{c,q,a,k} \pi(c, q, a, k) = 1. \tag{A9}$$

Second, we ensure that the joint distribution satisfies the identity (A8) by imposing the constraints

$$\forall \bar{q}, \bar{a}, \bar{k}, \quad \sum_{c} \pi(c, \bar{q}, \bar{a}, \bar{k}) = f(\bar{q}|\bar{a}, \bar{k}) \sum_{c,q} \pi(c, q, \bar{a}, \bar{k}). \tag{A10}$$

Finally, we make the allocation incentive compatible by requiring

$$\sum_{c,q} \pi(c, q, a, k) U(c, a) \geq \sum_{c,q} \pi(c, q, a, k) \frac{f(q|\hat{a}, k)}{f(q|a, k)} U(c, \hat{a}) \quad \forall k, a, \hat{a}. \tag{A11}$$

Let $\Pi_1 = \{\pi \in \Re^n_+ | \text{(A9), (A10), (A11)}\}$.[14] The set Π_1 contains all the incentive-compatible contracts; embedded in each $\pi(c, q, a, k) \in \Pi_1$ is an incentive-compatible contract $(\pi(a, k), \pi(c|q, a, k))$.

The set Π_1 is a finite system of linear inequalities with bounded solutions, so it can also be represented as the convex hull of its extreme points. Each one of these extreme points is a basic feasible solution to Π_1. Critically, for our purposes, there is a finite number of these extreme points.[15]

Let b_{1m}, $m = 1, \ldots, M_1$, be the set of extreme points, or basic feasible solutions, of Π_1. The utility and resource usages of a b_{1m} contract are

[13] For more general utility functions, such as nonseparability between consumption and effort, consumption lotteries may be beneficial, as in Gjesdal (1982). Our methods here automatically allow for this possibility.

[14] Detailed derivations of these constraints can be found in E. C. Prescott and Townsend (1984*b*) and Prescott (2004).

[15] Furthermore, to tie into our original definition of contracts, it easy to show that each one of these extreme points corresponds to a deterministic assignment of an (a, k) pair.

$$u(b_{1m}) = \sum_{c,q,a,k} \pi(c, q, a, k) U(c, a),$$

$$r_c(b_{1m}) = \sum_{c,q,a,k} \pi(c, q, a, k)(c - q),$$

$$r_k(b_{1m}) = \sum_{c,q,a,k} \pi(c, q, a, k) k. \qquad \text{(A12)}$$

With (A12), checking (A7) is equivalent to solving the linear program

$$\max_{\pi(c,q,a,k)} \sum_{c,q,a,k} \pi(c, q, a, k)[\lambda_i U(c, a, w) - \mu_c(c - q) - \mu_k k] - \mu_i \qquad \text{(A13)}$$

subject to

$$\pi(c, q, a, k) \in \Pi_1$$

and checking whether the value of the objective function is nonnegative. If the grids of c, q, a, and k are not too large, this linear program is relatively small.[16]

When solving (A13), the simplex algorithm searches over the extreme points of Π_1, that is, over b_{1m}, $m = 1, \ldots, M_1$. Thus it checks the optimality condition (A7). If at a solution the value of the objective function is nonpositive, then no $\delta^{(i)}(b_{1m})$ can be entered into the basis to improve the value of the objective function. If, instead, the value is positive, the solution is used to update the basis, new simplex multipliers to the alternative Pareto program are calculated, and the algorithm continues.

The advantage of this algorithm is that b_{1m}, $m = 1, \ldots, M_1$, does not need to be stored in computer memory. Instead, a b_{1m} is generated from Π_1 only as it is needed. The algorithm will converge because the simplex algorithm finds a solution in a finite number of iterations. Furthermore, speed is not usually an issue because simplex-based algorithms tend to quickly converge in practice.

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[16] Even if this linear program is large, it can be further decomposed (see Prescott 2004).

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