Fully Sincere Voting*

Alexander Wolitzky
Massachusetts Institute of Technology
Cambridge, MA 02139
wolitzky@mit.edu

Abstract

In a general social choice framework where the requirement of strategy-proofness may not be sensible, we call a social choice rule *fully sincere* if it never gives any individual an incentive to vote for a less-preferred alternative over a more-preferred one and provides an incentive to vote for an alternative if and only if it is preferred to the default option that would result from abstaining. If the social choice rule can depend only on the number of votes that each alternative receives, those rules satisfying full sincerity are convex combinations of the rule that chooses each alternative with probability equal to the proportion of the vote it receives and an arbitrary rule that ignores voters' preferences. We note a sense in which the natural probabilistic analog of approval voting is the fully sincere rule that allows voters maximal flexibility in expressing their preferences and gives these preferences maximal weight.

JEL classification: D70
Keywords: Social choice theory, Voting theory, Strategy-proofness

Introduction

Much existing work on the incentive properties of economic mechanisms focusses on strategy-proofness, the property that players have dominant strategies. Strategy-proof mechanisms are desirable because they induce simple, dominance-solvable games. In the context of voting, strategy-proofness is also one possible explication of what it is for a mechanism to induce voters to express their preferences honestly; namely, it suggests that an honest vote is one that is not influenced by others' actions. For many reasonable-seeming social choice mechanisms, however, strategy-proofness is not well-motivated. Consider, for example, approval voting, the rule under which each individual is allowed to vote for any subset of the candidates and the candidate that receives the most votes

*I would like to thank Jerry Green for his invaluable advice and support throughout this project; Salvador Barberà, John Friedman, Drew Fudenberg, Jon Kolstad, Hervé Moulin, Andrea Wilson, Dan Wood, Muhamet Yildiz and two anonymous referees for extremely helpful comments; and the NSF for financial support.

is elected. Under approval voting, if there are three alternatives it is unclear whether one's dominant strategy should be to vote for one's first choice alone or for one's top two choices. Requiring strategy-proofness may also be unduly restrictive in principle if what is desired is never giving individuals incentives to misrepresent their preferences; to achieve this, one need require only that motivations to best-respond and to vote honestly coincide in every contingency, not that a single strategy always represents both a best response and an honest vote across all contingencies. Returning to the example of approval voting with three alternatives, one might think that voting for one's first choice alone would be the honest expression of one's preferences when everyone else is voting for one's second choice, while voting for one's top two choices would be the honest expression when everyone else is voting for one's third choice. This sense of voting for an alternative as expressing a preference for it relative to the outcome that would result from abstaining gives rise to our notion of fully sincere voting.

Recall that a mechanism is given by a strategy space and a mapping from strategy profiles to (lotteries over) outcomes, which we call a *vote-aggregation rule* or simply an *aggregation rule.* We call a vote-aggregation rule *fully sincere* if for any strategy space it gives each voter incentives to i) vote for the alternatives that she prefers to the default lottery that would result from abstaining (*strong sincerity*) and ii) never vote for a less-preferred alternative over a more-preferred one (*pairwise sincerity*). Correspondingly, we will call a mechanism *fully sincere* if its vote-aggregation rule is identical to a fully sincere vote-aggregation rule on the mechanism's strategy space. Strong sincerity, the more unusual component of our definition of full sincerity, is like a participation constraint; we require not only that an individual must be given an incentive to refrain from misrepresenting her preferences—that is, from voting for a less-preferred alternative over a more-preferred one—but also that she must be given an incentive to express her preference for any and all alternatives that she prefers to a default option. One could give this requirement a normative interpretation: voting for an alternative expresses a preference for that alternative and therefore an honest individual votes for those alternatives that she prefers and only those; or a positive one: voters may face some cost to casting each vote and therefore must be given a strict incentive to do so if the outcome is to depend on voters' preferences.

Our main result applies when the aggregation rule is constrained to be a function only of the total number of votes that each alternative receives (*vote-total dependence*), not an arbitrary function of individuals' strategies. This constraint is very restrictive in a technical sense but we believe it is quite natural and we provide a less explicit characterization for the general case in the appendix. We also note that this constraint may not be as restrictive as it first appears, since we allow an extremely broad class of strategy spaces. In particular, we will allow the space of possible actions (or *ballots*) for an individual to be an arbitrary subset of $\mathbf{Z}_+^M$, the set of nonnegative M-tuples of integers, where M is the number of possible alternatives, representing the combinations of votes that the individual is allowed to cast. Many common voting mechanisms use vote-total dependent aggregation rules, including approval voting and all scoring rules.

Under the assumption that the aggregation rule depends only on the number of votes each alternative receives, we present a complete characterization of all fully sincere mechanisms. Such rules are characterized by a single parameter and select each alternative with probability equal to a particular kind of convex combination of the proportion of the vote that was cast for that alternative and a noise term representing the outcome that would result if no votes were cast. The parameter has a natural interpretation as the responsiveness of the mechanism to individuals' preferences and, if we require that the mechanism be maximally responsive as well as fully sincere, we are left with the aggregation rule that selects each alternative with probability precisely equal to the proportion of the vote it receives. In particular, no deterministic aggregation rule is fully sincere, as deterministic aggregation rules cannot be sufficiently responsive for every voter to have a strict incentive to vote for those alternatives she prefers to the default outcome in every contingency. We also note that, when voters use pure strategies, the natural probabilistic analog of approval voting, which we call *random approval voting*, is in effect the fully sincere mechanism that allows voters the most flexibility in expressing their preferences and is maximally responsive to these preferences.

Finally, we use our characterization of fully sincere, vote-total dependent mechanisms to prove some simple facts about strategic behavior under fully sincere mechanisms with particular kinds of strategy spaces. We show that if a fully sincere mechanism is a *scoring rule*—that is, voters rank the alternatives, alternatives receive "scores" that depend only on the number of voters that rank them in each position, and the outcome depends only on these scores—then it is weakly dominant for each voter to rank the alternatives according to her preferences. Similarly, if strategy spaces are *triangular*, in that voters can always reallocate a vote from one alternative to another, then it is weakly dominant for each voter to cast all her votes for her most-preferred alternative under any fully sincere aggregation rule. In the diametrically opposed case where strategy space are *rectangular*, in that the number of votes that a voter casts for one alternative has no effect on the number of votes she can cast for any other alternative, fully sincere mechanisms are generally not dominance-solvable but they nonetheless always give each voter an incentive to vote for a most-preferred subset of the alternatives in any pure-strategy Nash equilibrium.

Related Literature

This paper follows in the tradition of using stochastic social choice rules as an "escape route" from the negative results of Arrow (1950, 1951) and Gibbard (1973) and Satterthwaite (1975). The use of randomization in collective choice is far more normatively appealing than in individual choice, where it often seems to suggest some kind of irrationality. For example, if I want to eat at restaurant a and you want to eat at restaurant b, flipping a coin might be a reasonable course of action. Gibbard (1978) made the first seminal contribution to this literature, showing that in a probabilistic framework the only strategy-proof mechanisms are those that are a mixture of a weighted random dictatorship and rules that restrict the ex post outcome to one of two fixed alternatives. Hylland (1980) shows that the only strategy-proof mechanisms that always select ex post Pareto optimal alternatives are weighted random dictatorships, and Barberà (1979) shows that the set of anonymous, neutral and strategy-proof probabilistic decision schemes—voting mechanisms with the strategy space of linear preference orders—are precisely those schemes that are a mixture of two basic kinds of rules, which can be interpreted as the probabilistic counterparts of Borda's and Condorcet's procedures.

More recently, Pattanaik and Peleg (1986) find that weighted random dictatorship is the only probabilistic decision scheme satisfying the probabilistic analogue of Arrow's independence of irrelevant alternatives condition, ex post Pareto-optimality, regularity—the condition that an alternative's likelihood of being selected cannot increase as the set of feasible alternatives expands—and a mild technical condition. Nandeibam (1995) notes that one can replace independence of irrelevant alternatives and this technical condition by strategy-proofness in Pattanaik and Peleg's result, and Clark (1992) reformulates their result in terms of social preferences rather than social choices, making the parallel with Arrow's theorem even clearer. Barberà et al. (1998) extend Gibbard's (1978) result in the context of expected utility maximizers and also study the smoothness properties of strategy-proof rules and, in papers that are somewhat related to each other, Barberà et al. (2001) and Benoit (2002) show Gibbard-Satterthwaite-like results in contexts where agents are allowed to express preferences over sets of alternatives and the social choice rule is then allowed to map these preferences to sets of alternatives. In more restrictive contexts, Ehlers et al. (2002) characterize strategy-proof probabilistic decision schemes under a minor technical condition when agents have one-dimensional single-peaked preferences; and Bogomolnaia et al. (2005) characterize efficient, strategy-proof, anonymous and neutral probabilistic mechanisms when agents have dichotomous preferences—that is, when agents rate each alternative as either "good" or "bad"—and these preferences are comparable across individuals.

Our research is also related to that of Brams and Fishburn (1978, 2002), who study voting mechanisms where individuals are permitted to vote for subsets of candidates and say that a voting rule is "sincere" if it never provides an incentive to include a less-preferred but not a more-preferred alternative in one's vote, the condition that we refer to as pairwise sincerity. Brams and Fishburn's research is similar to ours in that they study social choice mechanisms for which strategy-proofness is not an appropriate assumption and characterize those mechanisms satisfying an alternative criterion of sincerity. However, Brams and Fishburn rely on conditions on individuals' preferences and do not consider the case where individuals may be allowed to vote more than once for an alternative, stochastic social choice mechanisms, or the assumption of strong sincerity, so their techniques and results are very different from ours. It is interesting, nonetheless, that Brams and Fishburn find several senses in which approval voting is the "most sincere" deterministic voting mechanism while we find that random approval voting is in effect the "most sincere" stochastic mechanism.

The exposition of our results proceeds in three sections. Section I presents the formal framework for the paper, including the formal definition of full sincerity and some examples and preliminary results. We present our main result in Section II. Section III then analyzes the strategic structure of fully sincere mechanisms with certain strategy spaces. A theorem for the general, non-vote-total dependent case as well as most of the proofs are deferred to the appendix.

I Preliminaries

We suppose that N individuals must choose one alternative from a set X of cardinality M. Each individual i submits a ballot, s_i, from the space of her possible ballots, S_i. We write S for the product space of the S_i's and s for an element of S, and refer to S as a *strategy space* or *ballot space*. Throughout, we assume that S_i is a bounded subset of $\mathbf{Z}_+^M$ for all i, so S is a bounded subset of $(\mathbf{Z}_+^M)^N$.

Thus, a strategy s_i is an M-tuple of integers and we will refer to the jth coordinate of s_i, s_{i,x_j}, as the number of votes that individual i casts for the jth alternative. Note that we assume boundedness so that mechanisms will admit equilibria and our requirement that voters can only cast non-negative numbers of votes is a costless normalization.

A vote-aggregation rule f is a map from $(\mathbf{Z}_+^M)^N$ to $\Pi(X)$, the space of probability distributions over X, so if $s \in S$ then $f(s) \in \Pi(X)$. This represents the possibility that the social choice mechanism need not be deterministic; the special case of a deterministic aggregation rule is when the range of f contains only degenerate elements of $\Pi(X)$. We allow for stochastic aggregation rules because, as our main result shows, no deterministic aggregation rule satisfies our definition of full sincerity. We write $f(x, s)$ for the weight that f assigns to alternative $x \in X$ under strategy profile s. That is, $f(x, s)$ is the probability with which x is chosen under f if individual i selects strategy s_i for all i and $s = (s_1, \dots, s_N)$. We simply write 0 for the 0-vector in $(\mathbf{Z}_+^M)^N$, so $f(x, 0)$ is the weight assigned to x when no votes are cast. Note that a (finite) mechanism is simply a pair (S, f)[1].

Note that we have defined f over $(\mathbf{Z}_+^M)^N$ rather than over any particular $S \subseteq (\mathbf{Z}_+^M)^N$, so that f is defined even at infeasible strategy profiles for any mechanism (S, f). This approach helpfully allows us to focus attention on aggregation rules and strategy spaces separately, but the properties of a mechanism (S, f) that we consider depend only on the behavior of f at feasible strategy profiles $s \in S$.

Outside of Appendix A, we assume that f is *vote-total dependent*. Let $V_x(s)$ be the number of votes cast for alternative x under strategy profile s: $V_x(s) \equiv \sum_i s_{i,x}$ where $s_{i,x}$ is the x-coordinate of s_i; and let $V(s) = \sum_{x \in X} V_x(s)$, the total number of votes cast under strategy profile s.

[1]I thank an anonymous referee for calling my attention to the value of explicitly discussing mechanisms rather than only aggregation rules in this paper.

Definition 1 *f is* vote-total dependent *if $f(s) = f(s')$ for all s, s' such that $V_x(s) = V_x(s')$ for all $x \in X$*[2]*.*

To get some sense of the import of assuming that f is vote-total dependent, note that Condorcet's method as it is usually described—where S_i is the set of vectors containing each integer between 0 and $M-1$ exactly once and $f(x, s) = 1$ if $|i : s_{i,x} > s_{i,y}| > \frac{N}{2}$ for all $y \neq x$—does not use a vote-total dependent aggregation rule, as it is easy to construct a pair of strategy profiles with the same vote totals for each alternative but different Condorcet winners, while Borda's method—where S_i is as in Condorcet's method and $f(x, s) = \frac{1}{|W(s)|}$ if $x \in W(s)$, $f(x, s) = 0$ otherwise, where $W(s) = \{x : V_x(s) \in \arg\max_{y \in X} V_y(s)\}$—does. We discuss some other examples of vote-total dependent aggregation rules later in this section.

For future reference, let $s + x$ be the strategy profile identical to s except that $s_{1,x}$ is increased by one and note that, if f is vote-total dependent, $f(s+x)$ is equal to $f((s + x)')$ where $(s + x)'$ is the strategy profile identical to s except that $s_{i,x}$ is increased by one, for any i. We define $s - x$ analogously.

We also assume throughout that each individual i has a utility function $u_i : X \to \mathbf{R}$ and that individuals evaluate lotteries according to expected utility. That is, each individual i's utility from outcome $f(s)$ is

$$E\left[u_i(f(s))\right] \equiv \sum_{x \in X} f(x, s) u_i(x)$$

We may now formally define "fully sincere" aggregation rules and mechanisms. As indicated above, we first define two independent criteria of sincerity—*strong sincerity* and *pairwise sincerity*—and say that an aggregation rule is *fully sincere* if it satisfies both of them. We define strongly sincere and pairwise sincere aggregation rules in terms of the incentives they provide to voters and then characterize such rules in Lemmas 1 and 2.

Definition 2 *f satisfies* strong sincerity at s *if*

$$u_i(x) \geq \sum_{x' \in X} f(x', s) u_i(x')$$

if and only if

$$\sum_{x' \in X} f(x', s + x) u_i(x') \geq \sum_{x' \in X} f(x', s) u_i(x')$$

for all $u_i(\cdot)$ and $x \in X$. f is strongly sincere *if it satisfies strong sincerity at every $s \in (\mathbf{Z}_+^M)^N$.*

[2] An anonymous referee points out that a better term for this property might be "individual vote distribution independence," since the definition requires that f does not depend on anything other than the vote totals, not that it must vary with the vote totals. We agree with this point but persist in call such rules "vote-total dependent" to save space, with the understanding that "vote-total dependent" means "dependent *only* on vote totals."

That is, f is strongly sincere if under f a voter is willing to cast a vote for an alternative x if and only if she prefers x to the lottery that would result if she abstained from casting this vote, regardless of what other votes have been cast. Lemma 1 characterizes strongly sincere aggregation rules.

Lemma 1 *f is strongly sincere if and only if*

$$\frac{f(x,s+x)-f(x,s)}{1-f(x,s)}=\frac{f(y,s)-f(y,s+x)}{f(y,s)}>0$$

for all $x\neq y$ and s such that $f(x,s)<1$ and $f(y,s)>0$; $f(x,s)=1$ implies that $f(x,s+x)=1$; and $f(y,s)=0$ implies that $f(y,s+x)=0$ for all $x\neq y$.

Proof. See Appendix B. ■

Bracketing the second and third conditions, Lemma 1 says that an aggregation rule is strongly sincere if and only if the fraction of the weight not placed on an alternative x at s that is transferred to x when a vote for x is added at s equals the fraction of the weight placed on each alternative y at s that is transferred away from y when a vote for x is added at s. The intuition is that if adding a vote for x changed the relative weights on two other alternatives y and z, an individual might have an incentive to vote for x in order to manipulate these relative weights even if she does not actually like x, violating strong sincerity.

Next, we define pairwise sincerity.

Definition 3 *f satisfies* pairwise sincerity *at s if*

$$u_i(y)\geq u_i(z)$$

if and only if

$$\sum_{x'\in X} f(x',s+y)u_i(x')\geq\sum_{x'\in X} f(x',s+z)u_i(x')$$

for all $u_i(\cdot)$ and $y,z\in X$. f is pairwise sincere *if it satisfies pairwise sincerity at every $s\in(\mathbf{Z}_+^M)^N$.*

f is pairwise sincere if under f a voter prefers to cast a vote for y over z if and only if she prefers y to z. Lemma 2 characterizes pairwise sincere aggregation rules.

Lemma 2 *f is pairwise sincere if and only if $f(z,s+x)=f(z,s+y)$ and $f(x,s+x)>f(x,s+y)$ for all s, x, $y\neq x$, and $z\neq x$, y.*

Proof. See Appendix B. ■

Thus, an aggregation rule is pairwise sincere if and only if adding a vote for x and adding a vote for y have the same effect on the weight placed on any third alternative, z, and adding a vote for x increases the weight placed on x relative to adding a vote for y.

Neither strong sincerity nor pairwise sincerity imply the other. For example, it is easy to check that the aggregation rule

$$f(x,s)=\frac{V_x(s)^2}{\sum_{x'\in X}V_{x'}(s)^2}$$

if $s\neq 0$, $f(0)$ arbitrary, is strongly sincere but not pairwise sincere, since under this rule a vote for one's second-favorite alternative can be much more "effective" than a vote for one's first choice, more than offsetting the difference in utilities between the alternatives. Formally, $f(z,s+x)<f(z,s+y)$ if $V_x(s)>V_y(s)$, so failure of pairwise sincerity is immediate from Lemma 2. On the other hand, the aggregation rule

$$f(x,s)=\frac{V_x(s)}{2V(s)}+\frac{1}{2M}$$

is pairwise sincere but not strongly sincere when $M=3$. To see this, note that pairwise sincerity is clear from Lemma 2 but consider the following example: $X=\{a,b,c\}$, $u_i(a)=6$, $u_i(b)=5$, $u_i(c)=0$, $V_a(s)=1$ and $V_b(s)=V_c(s)=0$. Then $f(s)=\left(\frac{2}{3},\frac{1}{6},\frac{1}{6}\right)$ and $f(s+b)=\left(\frac{5}{12},\frac{5}{12},\frac{1}{6}\right)$, so i's expected utility from s is $\frac{29}{6}$, which is less than $u_i(b)=5$, while i's expected utility from $s+b$ is $\frac{55}{12}$, which is less than $\frac{29}{6}$, violating strong sincerity. The point is that under this aggregation rule adding a vote for b changes the relative weights on a and c, violating Lemma 1. In Section II, our main result shows that the only aggregation rules satisfying both pairwise sincerity and strong sincerity are convex combinations of the rule $f(x,s)=\frac{V_x(s)}{V(s)}$ and fixed rules that do not depend on s. This example shows that not every such convex combination satisfies strong sincerity.

Finally, we can formally define fully sincerity:

Definition 4 *f satisfies* full sincerity *at s if f satisfies strong sincerity and pairwise sincerity at s. f is* fully sincere *if f is strongly sincere and pairwise sincere.*

It is easy to see that the aggregation rule $f(x,s)=\frac{V_x(s)}{V(s)}$ is fully sincere[3]. Our main result shows that this is the fully sincere aggregation rule that is most responsive to voters' preferences.

Thus far all of our definitions have been for aggregation rules rather than mechanisms. The following is the natural definition of a fully (strongly, pairwise) sincere mechanism:

Definition 5 *A mechanism (S,f) is* fully *(*strongly*,* pairwise*)* sincere *if there exists a fully (strongly, pairwise) sincere aggregation rule $\widetilde{f}$ such that $f(s)=\widetilde{f}(s)$ for all $s\in S$.*

[3] We have not defined $f(0)$ here. Carefully speaking, any aggregation rule satisfying $f(x,s)=\frac{V_x(s)}{V(s)}$ for all $s\neq 0$ is fully sincere. We continue to refer to $f(x,s)=\frac{V_x(s)}{V(s)}$ as an aggregation rule for ease of reading, but this should be interpreted everywhere as $f(x,s)=\frac{V_x(s)}{V(s)}$ if $s\neq 0$ and $f(0)$ an arbitrary element of $\Pi(X)$.

It would not be appropriate to use the alternative definition "(S, f) is fully sincere if f is fully sincere" as then the sincerity of (S, f) would depend on the behavior of f outside of S which is irrelevant for the mechanism (S, f)[4]. Nevertheless, if f is fully sincere then (S, f) is fully sincere for any S. Our characterization of fully sincere aggregation rules in Section II leads to a simple characterization of fully sincere mechanisms.

It is worth observing that the statement "(S, f) is fully sincere" is very different from "f satisfies full sincerity at s for all $s \in S$." The former means that, as far as the mechanism is concerned, f is identical to some fully sincere aggregation rule; that is, if f is not fully sincere it is because of "problematic" behavior at some $s \notin S$. On the other hand, the latter means only that f is well-behaved in terms of local deviations from all $s \in S$, not that its behavior at all $s \in S$ is consistent with (global) full sincerity. For example, if $f(x, s) = \frac{V_x(s)}{V(s)}$ for all $s \in S$ but $f(x, s + x) = 0$ for some $s \in S$, $s + x \notin S$, then (S, f) is fully sincere but f does not satisfy full sincerity at all $s \in S$. Conversely, if $N = 1$, $X = \{a, b\}$, $S = S_1 = \{(0, 0), (1, 0), (4, 0)\}$, then it is easy to check that the aggregation rule $f(x, s) = \frac{V_x(s)}{V(s)}$ if $V_a(s) \leq 2$, $f(x, s) = \frac{\frac{1}{2}+V_x(s)}{1+V(s)}$ if $V_a(s) \geq 3$, is fully sincere at all $s \in S$ but cannot be extended to a fully sincere aggregation rule over all of $\mathbf{Z}^2$: while this aggregation rule is well-behaved in terms of local deviations from $(0, 0)$, $(1, 0)$ and $(4, 0)$, the change in the outcome from $(1, 0)$ to $(4, 0)$ is inconsistent with full sincerity, because $f(a, (1, 0)) > f(a, (4, 0))$ and Lemma 1 gives that, under any strongly sincere aggregation rule, the weight placed on an alternative must be weakly increasing in the number of votes it receives.

Let us present an example to illustrate the distinction between full sincerity and strategy-proofness and to simultaneously introduce *random approval voting*, our leading example of a fully sincere mechanism. Under random approval each voter can "approve" as many alternatives as she wishes and each alternative is selected with probability equal to the proportion of the "approvals" it receives. Formally:

Definition 6 Random approval voting *is the mechanism given by* $S_i = \left\{s_i \in \mathbf{Z}^M : s_{i,x} \in \{0, 1\} \text{ for all } x\right\}$ *and* $f(x, s) = \frac{V_x(s)}{V(s)}$.

As we have claimed, $f(x, s) = \frac{V_x(s)}{V(s)}$ is fully sincere, so random approval voting is a fully sincere mechanism. The specification of $f(s)$ when $V(s) = 0$ is arbitrary here.

Now suppose that $N = 2$, $X = \{a, b, c, d\}$, $u_1(a) = 5$, $u_1(b) = 2$, $u_1(c) = u_1(d) = 0$, $u_2(a) = u_2(b) = 0$, $u_2(c) = 2$, $u_2(d) = 5$ and consider random approval voting. We claim that there are two pure-strategy Nash equilibria of this mechanism—one where each player votes for her favorite alternative only and another where each player votes for her two favorite alternatives. To see that these are Nash equilibria, first note that $s_1 = \{a\}$, $s_2 = \{d\}$ yields payoff $\frac{5}{2}$ to both players while either players' best deviation—adding a vote

[4] I thank an anonymous referee for pointing this out.

for her second-favorite alternative—yields only $\frac{7}{3}$ for the deviator. Similarly, $s_1 = \{a, b\}$, $s_2 = \{c, d\}$ gives each player payoff $\frac{7}{4}$ but a players' best deviation—dropping her second-favorite alternative from her ballot—gives the deviator only $\frac{5}{3}$. It is easy to check that there are no other pure-strategy Nash equilibria. Note that neither player has a dominant strategy under this mechanism. We observe that the pure-strategy Nash equilibria in this example are strict, involve voting only for most-preferred sets of alternatives, and are Pareto-ranked. We investigate the generality of these observations in Section III.

II Characterization of Fully Sincere Aggregation Rules and Mechanisms

We are now ready to present our main result:

Theorem 1 *If $M = 2$, f is fully sincere if and only if $f(x, s + x) \geq f(x, s)$ for all s and x, with strict inequality if $f(x, s) < 1$. If $M \geq 3$, f is fully sincere if and only if there exists $g \in (0, 1]$ such that*

$$f(x, s) = \frac{(1 - g)f(x, 0) + gV_x(s)}{1 - g + gV(s)} \tag{1}$$

for all s and x.

Proof. See Appendix B. ■

When there are three or more alternatives, Theorem 1 says that an aggregation rule is fully sincere if and only if it is a special kind of convex combination of the rule $f(x, s) = \frac{V_x(s)}{V(s)}$ and the rule $f(x, s) = f(x, 0)$[5]. The weight given to each rule is parametrized by g; note that f is fully sincere only if g is strictly positive. g has a natural interpretation as the responsiveness of f to voters' preferences, as a higher g corresponds to less weight on the rule $f(x, s) = f(x, 0)$, which does not depend on s at all. An immediate consequence of Theorem 1 is that all fully sincere aggregation rules are stochastic. We also point out an easy corollary, which says that $g = 1$ defines the only fully sincere rule that can ever select an alternative deterministically at any strategy profile if it was not already chosen deterministically before any votes were cast:

Corollary 1 *Suppose that $M \geq 3$. If f is fully sincere and there exist an alternative x and strategy profile s such that $f(x, 0) < 1$ and $f(x, s) = 1$, then $f(x, s) = \frac{V_x(s)}{V(s)}$.*

[5] To see this, observe that

$$\frac{(1 - g)f(x, 0) + gV_x(s)}{1 - g + gV(s)} = \left(\frac{\frac{1-g}{V(s)}}{\frac{1-g}{V(s)} + g}\right) f(x, 0) + \left(\frac{g}{\frac{1-g}{V(s)} + g}\right) \frac{V_x(s)}{V(s)}$$

So, if $M \geq 3$, a fully sincere aggregation rule is a convex combination of $f(x, 0)$ and $\frac{V_x(s)}{V(s)}$ where the weight on $f(x, 0)$ is decreasing in $V(s)$.

Proof. Suppose $g < 1$ and $f(x,0) < 1$. Then $g > 0$ and $V(s) \geq V_x(s)$ imply that the numerator of equation (1) is strictly less than the denominator. So $f(x,s) < 1$. ■

Theorem 1 provides an easy method of checking whether or not a mechanism (S,f) is fully sincere; one must simply check whether or not there exist an $f(0) \in \Pi(X)$ and $g \in (0,1]$ such that $f(x,s)$ is given by equation (1) for all $s \in S$. If $0 \notin S$, this problem is equivalent to determining whether a system of $M \cdot |S| + 1$ equations, $f(x,s) = \frac{(1-g)f(x,0)+gV_x(s)}{1-g+gV(s)}$ for all $x \in X$ and $s \in S$ and $\sum_{x \in X} f(x,0) = 1$; and $M+1$ unknowns, $f(x,0)$ for all $x \in X$ and g, has a solution. If $0 \in S$, this is equivalent to the even simpler problem of determining whether a system of $M \cdot (|S| - 1)$ equations and a single unknown, g, has a solution, as in this case $f(0)$ is given.

Theorem 1 is nevertheless a result about aggregation rules only and does not refer to players' strategy spaces. In particular, Theorem 1 applies equally whether or not the mechanism is anonymous or neutral. For example, consider the election of corporate officers, where it is natural to give more votes to those individuals who own more stock in the company; or a referendum in which policymakers want to give an advantage to the status quo alternative. If issues like these are to be decided using a fully sincere rule, and if that rule is to be as responsive to voters' preferences as possible, they must be settled by the aggregation rule that chooses each alternative with probability equal to the proportion of the vote it receives.

III Strategic Analysis of Some Fully Sincere Mechanisms

While fully sincere mechanisms need not be strategy-proof, equilibria of fully sincere mechanisms nonetheless have several appealing properties if we impose additional assumptions on the strategy space. It should not be surprising that not all fully sincere mechanisms induce well-behaved games, as thus far we have assumed very little about the strategy spaces; if S_i consists of a small number of disparate points in $\mathbf{Z}^M$, player i's best response correspondence may jump unpredictably among these points as her opponents' strategies change. Therefore, we assume in this section that each player's strategy space is either *scoring*, in that her only decision is how to ordinally rank the alternatives; *triangular*, in that she is given a certain number of votes to allocate in any way she chooses among the alternatives; or *rectangular*, in that the number of votes she casts for one alternative does not affect how many votes she may cast for the others.

Definition 7 *S is* scoring *if for every i there exist positive integers $k_{i,1} \geq k_{i,2} \geq \ldots \geq k_{i,M}$ such that $S_i = \{s_i :$ there exists a permutation ρ on $\{1,\ldots,M\}$ such that $s_{i,x_j} = k_{i,\rho(j)}\}$.*

Note that a scoring rule is a vote-total dependent mechanism with a scoring strategy space.

Definition 8 *S is* triangular *if for every i there exists a positive integer k_i such that $S_i = \left\{s_i : \sum_{x \in X} s_{i,x} \leq k_i\right\}$.*

Pairwise sincerity immediately implies that voters have dominant strategies under fully sincere mechanisms with scoring or triangular strategy spaces:

Proposition 1 *Assume without loss of generality that $u_i(x_1) \geq u_i(x_2) \geq \ldots \geq u_i(x_M)$, and assume that (S, f) is pairwise sincere. If S is scoring, then the strategy s_i given by $s_{i,x_j} = k_{i,j}$ is weakly dominant for i under (S, f). If S is triangular, then the strategy s_i given by $s_{i,x_1} = k_i$, $s_{i,x_j} = 0$ for all $j \neq 1$ is weakly dominant for i under (S, f).*

Proof. See Appendix B. ■

Proposition 1 shows that pairwise sincerity (and therefore full sincerity) has little to offer relative to strategy-proofness when strategy spaces are scoring or triangular. The following restriction on strategy spaces is more interesting:

Definition 9 *S is* rectangular *if for every i and $x \in X$ there exist finite sets $S_{i,x} \subseteq \mathbf{Z}_+$ such that $S_i = \times_{x \in X} S_{i,x}$.*

The strategy space in random approval voting is rectangular, for example. We present three simple facts about strategic behavior under fully sincere mechanisms with rectangular strategy spaces. For the remainder of this section, continue to assume without loss of generality that $u_i(x_1) \geq u_i(x_2) \geq \ldots \geq u_i(x_M)$.

Definition 10 *A strategy s_i is m-*dichotomous *if $s_{i,x_j} = \max\left\{S_{i,x_j}\right\}$ for all $j \leq m$ and $s_{i,x_j} = \min\left\{S_{i,x_j}\right\}$ for all $j > m$. A strategy s_i is* dichotomous *if it is m-dichotomous for some m. A strategy s_i is* weakly m-dichotomous *if $s_{i,x} = \max\{S_{i,x}\}$ for all x such that $u_i(x) > u_i(x_m)$ and $s_{i,x} = \min\{S_{i,x}\}$ for all x such that $u_i(x) < u_i(x_m)$. A strategy s_i is* weakly dichotomous *if it is weakly m-dichotomous for some m.*

Proposition 2 *If (S, f) is fully sincere and S is rectangular then:*

1. *For any i and s_{-i}, every best response of i to s_{-i} is weakly dichotomous, and i has a dichotomous best response to s_{-i}.*
2. *For every s_{-i}, if there exist alternatives x' and x'' such that $V_{s_{-i}}(x') > 0$, $V_{s_{-i}}(x'') > 0$ and $u_i(x') > u_i(x'')$, then for any $\varepsilon > 0$ there exists a utility function $u_i'(\cdot)$ such that $|u_i'(x) - u_i(x)| < \varepsilon$ for all $x \in X$ and i has a unique best response to s_{-i} if she is given utility function $u_i'(\cdot)$.*

Proof. See Appendix B. ■

Proposition 2 shows that, generically, a voter under a fully sincere mechanism with a rectangular strategy space will have a unique best response to every pure strategy profile of her opponents. Furthermore, this best response will consist of her voting as much as possible for a "high" subset of alternatives and voting as little as possible for the remaining alternatives, where a subset of alternatives is *high* if every element it contains is preferred to every element it excludes, as in Brams and Fishburn (1978).

Next, note that with a rectangular strategy space a voter casts more votes when her opponents add votes for alternatives that she dislikes and casts fewer votes when her opponents add votes for alternatives that she likes:

Proposition 3 *If (S, f) is fully sincere, S is rectangular, i has a unique and m-dichotomous best response s_i to s_{-i} and a unique and m'-dichotomous best response s'_i to $s_{-i}+x'$[6], and $|S_{i,x_m}|$, $\left|S_{i,x'_m}\right| \geq 2$; then $m \geq m'$ if $u_i(x') \geq u_i(x_m)$ and $m \leq m'$ if $u_i(x') \leq u_i(x_m)$.*

Proof. See Appendix B. ■

Taken together, Propositions 2 and 3 present a fairly detailed picture of strategic behavior under fully sincere mechanisms with rectangular strategy spaces: voters vote as much as they can for some high subset of the alternatives, starting with their favorite alternative and adding further alternatives if and only if they prefer them to the lottery that would result if they did not add them. When one's opponents' votes are unfavorable, one casts more votes, as these votes take weight away from an unfavorable default lottery; conversely, one casts fewer votes when one's opponents' votes are favorable. Thus, voters do not have dominant strategies and in general they may require much information about others' strategies to compute their best responses.

Recall that a mechanism is *anonymous* if it is invariant to permutations of the voters' names and *neutral* if it is invariant to permutations of the alternatives' names, so in particular an anonymous and neutral mechanism (S, f) with S rectangular has $S_{i,x} = S_{j,y}$ for all i, j and all x, $y \in X$. That is, an anonymous, neutral and rectangular ballot space is described by a single finite set $S_x \subseteq \mathbf{Z}_+$, the set of numbers of votes that each voter may case for each alternative. Proposition 2 gives that, generically, voters will vote as much as they can for a high subset of alternatives and as little as they can for all others so, if $0 \in S$, $|S| > 1$ and f is fully sincere with $g = 1$, then for generic payoffs the set of points in $\Pi(X)$ which are pure-strategy Nash equilibrium outcomes of (S, f) will not depend on S_x as long as S is anonymous, neutral and rectangular. The intuition is that $f(x, s) = \frac{V_x(s)}{V(s)}$ and changing S simply scales the equilibrium values of $V_x(s)$ and $V(s)$ by the same constant without changing the equilibrium values of $f(x, s)$. Formally, we have the following result:

Proposition 4 *Assume that $M \geq 3$, f is fully sincere with $g = 1$, and S and S' are anonymous, neutral and rectangular, with $0 \in S, S'$ and $|S|, |S'| > 1$. Then for any utility functions $\{u_i(\cdot)\}$ and any $\varepsilon > 0$ there exist utility functions $\{u'_i(\cdot)\}$ such that $|u'_i(x) - u_i(x)| < \varepsilon$ for all $x \in X$ and all i and, when payoffs are given by*

[6] Part 1 of Proposition 2 implies that if a voter has a unique best response, it must be m-dichotomous for some m.

$\{u_i'(\cdot)\}$, the set of pure-strategy Nash equilibrium outcomes of (S,f) is the same as the set of pure-strategy Nash equilibrium outcomes of (S',f). Furthermore, when payoffs are given by $\{u_i'(\cdot)\}$, if s is a pure-strategy Nash equilibrium of (S,f) with $f(s)$ nondegenerate, then $s\left(\frac{\overline{s'_{i,x}}}{\overline{s_{i,x}}}\right)$ is a pure-strategy Nash equilibrium of (S',f); and if s' is a pure-strategy Nash equilibrium of (S',f) with $f(s')$ nondegenerate, then $s'\left(\frac{\overline{s_x}}{\overline{s'_x}}\right)$ is a pure-strategy Nash equilibrium of (S,f); where $\overline{s_{i,x}} \equiv \max\{S_x\}$ and $\overline{s'_{i,x}} \equiv \max\{S'_x\}$.

Proof. See Appendix B. ■

Recalling that random approval voting is the mechanism given by $f(x,s) = \frac{V_x(s)}{V(s)}$ and S the unique anonymous, neutral, rectangular ballot space with $0 \in S$ and $\overline{s_{i,x}} = 1$, the following immediate corollary of Proposition 4 provides a sense in which random approval voting is equivalent to any other fully sincere mechanism with an anonymous, neutral and rectangular ballot space containing 0 and at least one other element:

Corollary 2 *Assume that $M \geq 3$, f is fully sincere with $g = 1$, and S' is anonymous, neutral and rectangular, with $0 \in S'$ and $|S'| > 1$. Then for any utility functions $\{u_i(\cdot)\}$ and any $\varepsilon > 0$ there exist utility functions $\{u_i'(\cdot)\}$ such that $|u_i'(x) - u_i(x)| < \varepsilon$ for all $x \in X$ and all i and, when payoffs are given by $\{u_i'(\cdot)\}$, the set of pure-strategy Nash equilibrium outcomes of (S',f) is the same as the set of pure-strategy Nash equilibrium outcomes of random approval voting. Furthermore, when payoffs are given by $\{u_i'(\cdot)\}$, if s is a pure-strategy Nash equilibrium of random approval voting with $f(s)$ nondegenerate, then $s\left(\overline{s'_{i,x}}\right)$ is a pure-strategy Nash equilibrium of (S',f); and if s' is a pure-strategy Nash equilibrium of (S',f) with $f(s')$ nondegenerate, then $s'\left(\frac{1}{\overline{s'_{i,x}}}\right)$ is a pure-strategy Nash equilibrium of random approval voting.*

Proof. Corollary 2 is a special case of Proposition 4. ■

We conclude this section with an example that shows that, unlike the properties that pure-strategy Nash equilibria are strict and that voters vote for high sets of alternatives, the property of the example at the end of Section I that pure-strategy Nash equilibria are Pareto-ranked is not very general: let $N = 2$, $X = \{a,b,c,d\}$, $u_1(a) = 9$, $u_1(b) = 8$, $u_1(c) = 6$, $u_1(d) = 0$, $u_2(a) = 6$, $u_2(b) = 0$, $u_2(c) = 9$ and $u_3(d) = 8$ and consider the game induced by random approval voting. It is easy to check, along the lines of the example at the end of Section I, that $s_1 = \{a,b,c\}$, $s_2 = \{c,d\}$ and $s_1 = \{a,b\}$, $s_2 = \{a,c,d\}$ are the only pure-strategy Nash equilibria of this game, and that player 1 prefers the second equilibrium while player 2 prefers the first. Furthermore, preferences in this example are both single-peaked and single-crossing with respect to the order (d,c,a,b). The equilibria of this game are not Pareto-ranked because, since each player's third-favorite alternative is the other player's favorite alternative, adding a vote for one's third-favorite alternative discourages one's opponent from voting for her third-favorite alternative by making her better off, so whichever player is fortunate enough to have her opponent vote for three alternatives does better in equilibrium. Finally, note that both pure-strategy Nash equilibria of this game are Pareto-dominated, for example by the lottery that puts weight $\frac{1}{2}$ on each of a and c. This shows that fully sincere mechanisms need not be Pareto-efficient.

Conclusion

In this paper we introduce a new concept of sincere voting, based on the idea that it is sincere for an individual to vote for an alternative if and only if she prefers that alternative to the default outcome that would result if she declined to cast that vote and to never vote for a less-preferred alternative over a more-preferred one. We suggest that a voting mechanism that induces sincere voting in this sense should not be said to be vulnerable to strategic misrepresentation of preferences, even though it is not strategy-proof. For while it is clearly a problem for voters to have conflicting impulses to vote strategically and sincerely, it is not clearly a problem for strategic and sincere concerns to point to different strategies in different contingencies, so long as in each contingency they point to the same one as each other.

We show that the only social choice functions that induce fully sincere voting in all contingencies are those that are a convex combination of the rule that selects each alternative with probability equal to the proportion of the vote it receives and a rule that selects an alternative arbitrarily. In particular, the fully sincere aggregation rule that is most responsive to voters' preferences is the rule under which an alternative that receives a certain proportion of the vote is chosen that proportion of the time. If this aggregation rule is adopted, the only remaining question in designing fully sincere mechanisms is that of what ballots to allow individuals to submit. The only finite, anonymous and neutral ballot spaces that do not require voters to trade off voting for one alternative against voting for another allow each individual to cast between 0 and k votes for each alternative, for k a positive integer, regardless of how many votes she casts for any other alternative. But under any fully sincere aggregation rule an individual then never has an incentive to cast any number other than 0 or k votes for an alternative when her opponents use pure strategies. Therefore, the pure-strategy equilibria of this rule are equivalent to those in the case where $k = 1$: random approval voting. So, when voters use pure strategies, random approval voting is in effect the voting mechanism with a rectangular ballot space that induces fully sincere voting and is maximally responsive to voters' preferences.

Appendix A: The Non-Vote-Total Dependent Case

In this appendix, we present an analog of Theorem 1 for the case where f need not be vote-total dependent. Our characterization for this more general case is much less explicit than Theorem 1, as in general the "influence" of an individual i's vote given s_{-i} can depend on the way in which the votes in s_{-i} are distributed across i's opponents and on i's identity without violating full sincerity.

If f is not vote-total dependent, $f(s_i + x, s_{-i}) \neq f(s_j + x, s_{-j})$ if $i \neq j$, in general, so we can no longer define full sincerity in terms of $f(s + x)$. Therefore, we must modify our definitions for the vote-total dependent case as follows: In this appendix, we say that f satisfies *strong sincerity at* s if $u_i(x) \geq \sum_{x' \in X} f(x', s) u_i(x')$ if and only if $\sum_{x' \in X} f(x', s_i + x, s_{-i}) u_i(x') \geq \sum_{x' \in X} f(x', s) u_i(x')$ for all $u_i(\cdot)$ and $x \in X$; and f satisfies *pairwise sincerity at* s if $u_i(y) \geq u_i(z)$ if and only if $\sum_{x' \in X} f(x', s_i + y, s_{-i}) u_i(x') \geq \sum_{x' \in X} f(x', s_i + z, s_{-i}) u_i(x')$ for all $u_i(\cdot)$ and $y \neq z \in X$. The definitions of strong sincerity, pairwise sincerity and full sincerity of aggregation rules and mechanisms for the non-vote-total dependent case build on these definitions exactly as in Section I. The following result is the analog of Theorem 1 in this setting:

Theorem 2 *If $M = 2$, f is fully sincere if and only if $f(x, s_i + x, s_{-i}) \geq f(x, s_i, s_{-i})$ for all i, x and s, with strict inequality if $f(x, s) < 1$. If $M \geq 3$, f is fully sincere if and only if*

$$\frac{f(x, s_i + x, s_{-i}) - f(x, s)}{1 - f(x, s)} = \frac{f(y, s) - f(y, s_i + x, s_{-i})}{f(y, s)} > 0 \tag{2}$$

for all i, $x \neq y$ and s such that $f(x, s) < 1$; $f(x, s) = 1$ implies that $f(x, s_i + x, s_{-i}) = 1$ for all i, x and s; and

$$\frac{f(x, s_i + x, s_{-i}) - f(x, s)}{1 - f(x, s)} = \frac{f(y, s_i + y, s_{-i}) - f(y, s)}{1 - f(y, s)} \tag{3}$$

for all i, $x \neq y$ and s such that $f(x, s) < 1$ and $f(y, s) < 1$.

Proof. See Appendix B. ■

The first two conditions in Theorem 2 simply restate Lemma 1. The third condition (equation (3)) says that the proportion of the weight not placed on an alternative x at s that is transferred to x when a vote for x is added at s by some individual i must equal the proportion of the weight not placed on another alternative y at s that is transferred to y when a vote for y is added at s by individual i. Equation (3) follows from combining strong sincerity and pairwise sincerity. Theorem 2 cannot be made more explicit because, in general, the proportion described above can depend on the distribution of the votes in s_{-i} as well as on the identity of voter i. These kinds of dependence are, of course, ruled out by vote-total dependence, which is what allows Theorem 1 to be so much cleaner than Theorem 2.

Appendix B: Proofs of Results

Proof of Lemma 1

First, note that the first two conditions in the Lemma—

$$\frac{f(x,s+x)-f(x,s)}{1-f(x,s)}=\frac{f(y,s)-f(y,s+x)}{f(y,s)}>0$$

for all $x \neq y$ and s such that $f(x,s)<1$ and $f(y,s)>0$, and $f(x,s+x)=1$ if $f(x,s)=1$—jointly imply the third condition, $f(y,s+x)=0$ if $f(y,s)=0$. This is obvious if $f(x,s)=1$, and follows if $f(x,s)<1$ because

$$\begin{aligned}
f(x,s+x)+\sum_{\substack{y\neq x:\\ f(y,s)>0}} f(y,s+x) &= f(x,s+x)+\sum_{\substack{y\neq x:\\ f(y,s)>0}} f(y,s)\left(1-\frac{f(x,s+x)-f(x,s)}{1-f(x,s)}\right)\\
&= f(x,s+x)+(1-f(x,s))\left(\frac{1-f(x,s+x)}{1-f(x,s)}\right)\\
&= 1
\end{aligned}$$

where the second equality uses the fact that $f(x,s)+\sum_{y\neq x: f(y,s)>0} f(y,s)=1$. Therefore, the "only if" direction is Step 1 of the proof of Theorem 1, which shows that strong sincerity implies the first two conditions in the Lemma.

We also rely heavily on Step 1 of the proof of Theorem 1 for the "if" direction: Suppose that $f(x,s)<1$ and that $\frac{f(x,s+x)-f(x,s)}{1-f(x,s)}=\frac{f(y,s)-f(y,s+x)}{f(y,s)}$ for all $y\neq x$ such that $f(y,s)>0$. For any utility function u_i such that $\sum_{y\neq x} f(y,s)u_i(y)\neq 0$, this implies that

$$\begin{aligned}
\frac{\sum_{y\neq x}(f(y,s)-f(y,s+x))u_i(y)}{\sum_{y\neq x} f(y,s)u_i(y)} &= \frac{\sum_{y\neq x}\left(\frac{f(x,s+x)-f(x,s)}{1-f(x,s)}\right) f(y,s)u_i(y)}{\sum_{y\neq x} f(y,s)u_i(y)}\\
&= \frac{f(x,s+x)-f(x,s)}{1-f(x,s)}
\end{aligned}$$

for all x. This condition is the second-to-last displayed equation in Step 1 of the proof of Theorem 1, so the argument contained there gives that this condition implies that f satisfies strong sincerity at s with respect to x, in that

$$u_i(x)\geq \sum_{x'\in X} f(x',s)u_i(x')$$

if and only if

$$\sum_{x'\in X} f(x',s+x)u_i(x')\geq \sum_{x'\in X} f(x',s)u_i(x')$$

for all $u_i(\cdot)$.

Now suppose that $f(x,s)=1$ and that $f(x,s+x)=1$. Then f satisfies strong sincerity at s with respect to x trivially, because both of the above inequalities hold mechanically for any $u_i(\cdot)$.

Therefore, regardless of the value of $f(x,s)$, f satisfies strong sincerity at s with respect to x, so f satisfies strong sincerity at s with respect to all x; that is, f satisfies strong sincerity at s. This holds for all s, so f is strongly sincere.

Proof of Lemma 2

Suppose that f is pairwise sincere. First, let $u_i(x) = 1$, $u_i(y) = 0$ for all $y \neq x$. Then if there exists $y \neq x$ such that $f(x, s+x) \leq f(x, s+y)$ individual i strictly prefers x to y but is willing to add a vote for y rather than x at s, contradicting pairwise sincerity. Next, suppose that there exists $z \neq x, y$ such that $f(z, s+x) \neq f(z, s+y)$. Without loss of generality assume that $f(z, s+x) > f(z, s+y)$. Let $u_i(z) = 1$, $u_i(z') = 0$ for all $z' \neq z$. Then individual i weakly prefers y to x but is unwilling to vote for y rather than x at s, again contradicting pairwise sincerity.

Conversely, suppose that $f(z, s+x) = f(z, s+y)$ and $f(x, s+x) > f(x, s+y)$ for all s, x, y and $z \neq x$, y. First, suppose that $u_i(y) \geq u_i(z)$. Then $\sum_{x' \in X} f(x', s+y) u_i(x') \geq \sum_{x' \in X} f(x', s+z) u_i(x')$ if and only if $f(y, s+y)u_i(y) + f(z, s+y)u_i(z) \geq f(y, s+z)u_i(y) + f(z, s+z)u_i(z)$, which holds because $f(y, s+y) + f(z, s+y)$ must equal $f(y, s+z) + f(z, s+z)$ and $f(y, s+y) > f(y, s+z)$. Next, suppose that $\sum_{x' \in X} f(x', s+y) u_i(x') \geq \sum_{x' \in X} f(x', s+z) u_i(x')$. Then $f(y, s+y)u_i(y) + f(z, s+y)u_i(z) \geq f(y, s+z)u_i(y) + f(z, s+z)u_i(z)$ which implies $(f(y, s+y) - f(y, s+z))\, u_i(y) \geq (f(z, s+z) - f(z, s+y))u_i(z)$. This then implies that $u_i(y) \geq u_i(z)$ since $f(y, s+y) + f(z, s+y) = f(y, s+z) + f(z, s+z)$ and $f(y, s+y) - f(y, s+z) > 0$.

Proof of Theorem 1

$M = 2$ case:

Lemma 1 implies that f is strongly sincere if and only if $f(x, s+x) \geq f(x, s)$ with strict inequality if $f(x,s) < 1$. Lemma 2 shows that all such rules are also pairwise sincere, completing the proof.

"Only if" direction for the $M \geq 3$ case:

The proof of this direction proceeds in four steps. Suppose that $M \geq 3$ and let

$$g(x, s) \equiv \frac{f(x, s+x) - f(x, s)}{1 - f(x, s)}$$

so that $g(x,s)$ is the proportion of the weight not placed on x at s that is transferred to x when a vote for x is added at s, when this is well-defined. In Step 1 of the proof, we show that the difference between $f(s)$ and $f(s+x)$ is entirely determined by $g(x,s)$ for any strongly sincere aggregation rule. In Step 2, we show that if f is also pairwise sincere then $g(x,s)$ is independent of which alternative is added; that is, we show that $g(x,s) = g(y,s)$ for all $x, y \in X$, so that we can just write $g(s)$. In Step 3, we show that if $V(s) = V(s')$ then $g(s) = g(s')$, so that we can write $g(V)$. In Step 4, we show that $g(V)$ is completely determined by a single variable, $g(0) \equiv g$, and explicitly characterize all fully sincere aggregation rules in terms of g.

Step 1:

We first show that strong sincerity implies that

$$g(x,s) = \frac{f(y,s) - f(y,s+x)}{f(y,s)} > 0 \tag{4}$$

for all x, y and s such that $f(y,s) > 0$, and that $f(x,s) = 1$ implies that $f(x,s+x) = 1$ for all x and s.

To see this, first note that $\sum_{x' \in X} f(x', s+x) u_i(x') \geq \sum_{x' \in X} f(x', s) u_i(x')$ if and only if $u_i(x)(f(x,s+x) - f(x,s)) \geq \sum_{y \neq x} (f(y,s) - f(y,s+x)) u_i(y)$. We have two possible cases. If $f(x,s+x) - f(x,s) = 0$, then the second inequality in the definition of strong sincerity does not depend on $u_i(x)$. Since the equivalence in the definition must hold for all values of $u_i(x)$, this implies that the upper inequality in the definition must not depend on $u_i(x)$, either. And this occurs if and only if $f(x,s) = 1$. So we have that $f(x,s+x) - f(x,s) = 0$ if and only if $f(x,s) = 1$.

If $f(x,s+x) - f(x,s) \neq 0$, we can divide by this term, yielding $u_i(x) \geq \frac{1}{f(x,s+x) - f(x,s)} \sum_{y \neq x} (f(y,s) - f(y,s+x)) u_i(y)$.

Next, note that $u_i(x) \geq \sum_{x' \in X} f(x', s) u_i(x')$ if and only if $u_i(x)(1 - f(x,s)) \geq \sum_{y \neq x} f(y,s) u_i(y)$. So we have $u_i(x) \geq \frac{1}{1 - f(x,s)} \sum_{y' \neq x} f(y,s) u_i(y)$.

We now have that, as long as $f(x,s) \neq 1$, the assumption that f is strongly sincere at s with respect to x (in the sense defined in the proof of Lemma 1) is equivalent to the assumption that

$$u_i(x) \geq \frac{1}{f(x,s+x) - f(x,s)} \sum_{y \neq x} (f(y,s) - f(y,s+x)) u_i(y)$$

if and only if

$$u_i(x) \geq \frac{1}{1 - f(x,s)} \sum_{y \neq x} f(y,s) u_i(y)$$

for all $u_i(\cdot)$. Since this must hold for all values of $u_i(x)$, which appears only on the left hand side of both of these inequalities, this is equivalent to

$$\frac{\sum_{y \neq x} (f(y,s) - f(y,s+x)) u_i(y)}{f(x,s+x) - f(x,s)} = \frac{\sum_{y \neq x} f(y,s) u_i(y)}{1 - f(x,s)}$$

which in turn is equivalent the condition that

$$\frac{f(x,s+x) - f(x,s)}{1 - f(x,s)} = \frac{\sum_{y \neq x} (f(y,s) - f(y,s+x)) u_i(y)}{\sum_{y \neq x} f(y,s) u_i(y)}$$

whenever $\sum_{y \neq x} f(y,s) u_i(y) \neq 0$. Note that this last equivalence follows because if $\sum_{y \neq x} f(y,s) u_i(y) = 0$, then $\sum_{y \neq x} f(y,s+x) u_i(y) = 0$ as well, since the definition of $g(x,s)$ yields $\sum_{y \neq x} f(y,s+x) u_i(y) = \sum_{y \neq x} f(y,s) \left(1 - g(x,s)\right) u_i(y) =$

$(1 - g(x,s)) \sum_{y \neq x} f(y,s)u_i(y) = 0$, so the former displayed equation holds trivially.

Now suppose $u_i(y) = 1$ for any given y, $u_i(x') = 0$ for all $x' \neq y$. Then this equation reduces to

$$g(x,s) = \frac{f(y,s) - f(y,s+x)}{f(y,s)}$$

the first part of equation (4). Note that this must hold for all x, $y \neq x$, and s.

Furthermore, strong sincerity implies that $f(x,s+x) > f(x,s)$ if $f(x,s) < 1$. To see this, consider the utility function given by $u_i(x) = 0$, $u_i(y) = 1\ \forall y \neq x$ and suppose that there were some strategy profile s such that $f(x,s+x) \leq f(x,s)$ and $f(x,s) < 1$. Then $u_i(x) = 0 < 1 - f(x,s) = \sum_{x' \in X} f(x',s)u_i(x')$ but $\sum_{x' \in X} f(x',s+x)u_i(x') = 1 - f(x,s+x) \geq 1 - f(x,s) = \sum_{x' \in X} f(x',s)u_i(x)$, contradicting strong sincerity. Recalling the definition of $g(x,s)$, this implies that $g(x,s) > 0$ if $f(x,s) < 1$.

Step 2:

The next step in our proof is to show that, when $M \geq 3$, strong sincerity and pairwise sincerity imply that $g(y,s) = g(z,s)$ for all y, z and s such that $f(y,s)$, $f(z,s) < 1$.

To see this, assume towards a contradiction that there exist some s, y and z such that $g(y,s) - g(z,s) = \epsilon > 0$.

First, consider the case where $f(y,s) + f(z,s) < 1 - \delta$ for some $\delta > 0$. Let $u_i(z) = 1$, $u_i(y) = 1 - \epsilon\delta$, and $u_i(x) = 0\ \forall x \neq y, z$.

We claim that strong sincerity implies that voter i's expected utility when she adds a vote for y at s is $E\left[u_i(f(s))(1 - g(y,s))\right] + u_i(y)g(y,s)$. To derive this equation, first note that $\sum_{x' \in X} f(x',s+x)u_i(x') = \sum_{x' \in X} f(x',s)u_i(x') + \sum_{x' \in X}(f(x',s+x) - f(x',s))u_i(x')$. Now equation (4) gives $f(y,s+x) - f(y,s) = -f(y,s)g(x,s)$ for $y \neq x$, and $f(x,s+x) - f(x,s) = (1 - f(x,s))g(x,s)$ by definition of $g(x,s)$, so $\sum_{x' \in X} f(x',s)u_i(x') + \sum_{x' \in X}(f(x',s+x) - f(x',s))u_i(x') = E[u_i(f(s))] - \sum_{x' \in X} f(x',s)g(x,s)u_i(x') + g(x,s)u_i(x) = E[u_i(f(s))(1 - g(x,s))] + u_i(x)g(x,s)$. Similarly, i's expected utility when she adds a vote for z at s is $E[u_i(f(s))](1 - g(z,s)) + u_i(z)g(z,s)$. The difference between these is $-(g(y,s) - g(z,s))E[u_i(f(s))] + g(y,s)u_i(y) - g(z,s)u_i(z)$. Substituting $g(z,s) = g(y,s) - \epsilon$, $u_i(z) = 1$ and $u_i(y) = 1 - \epsilon\delta$ reduces this to $-\epsilon E[u_i(f(s))] + \epsilon - \epsilon\delta g(y,s)$. Now $f(y,s) + f(z,s) < 1 - \delta$ implies that $E[u_i(f(s))] < 1 - \delta$, as we have assumed that i gets utility weakly less than 1 from each alternative in X. So $-\epsilon E[u_i(f(s))] + \epsilon - \epsilon\delta g(y,s) > -\epsilon(1 - \delta) + \epsilon - \epsilon\delta g(y,s) = \epsilon\delta(1 - g(y,s)) \geq 0$, since $g(y,s) \leq 1$ by definition of g. Thus, we see that voter i would rather add a vote for y than for z at s, even though i strictly prefers z to y, contradicting pairwise sincerity.

The case where $f(y,s) + f(z,s) = 1$ now follows easily. For so long as $f(y,s), f(z,s) \neq 1$ and $M \geq 3$, we can choose a third alternative x and repeat the above argument successively first for y and x and then for z and x, which shows that $g(y) = g(x) = g(z)$, as desired.

Step 3:

We now show that $f(x,s)$ depends on s only through $V_x(s)$ and $V(s)$. First, suppose that $f(x,s)=1$ for some x. We claim that this can occur only if $V_x(s)=V(s)$, that is only if x receives all votes cast. For suppose that s includes a vote for some alternative $y\neq x$. Towards a contradiction, we will show that $f(s-y)=f(s)$. First, note that $f(y,s)=0$ implies that $f(y,s-y)=0$, since $f(y,s)=f(y,s-y+y)\geq f(y,s-y)$ by strong sincerity. Now equation (4) gives $g(y,s)=\frac{f(z,s-y)-f(z,s)}{f(z,s-y)}=0$ for all $z\neq y$ such that $f(z,s-y)>0$. If $z\neq x$, then $f(z,s)=0$, so $\frac{f(z,s-y)-f(z,s)}{f(z,s-y)}=0$ cannot hold if $f(z,s-y)>0$. This implies that $f(z,s-y)=0$ for all $z\neq x,y$. Since the weights on all alternatives must sum to one, we obtain $f(x,s-y)=1$ and therefore $f(s)=f(s-y)$. So an individual i is always willing to add a vote for y at profile $s-y$, even if $u_i(y)<u_i(x)$, which contradicts strong sincerity. So $f(x,s)=1$ implies that $V_x(s)=V(s)$, so if $f(x,s)=1$ and strategy profile s' satisfies $V_x(s')=V_x(s)$ and $V(s')=V(s)$, then $s=s'$ so the result is trivial in this case.

Now suppose that $f(x,s)<1$, and consider two profiles s' and s'' such that $V_x(s')=V_x(s'')$ and $V(s')=V(s'')$. Note that we can construct a chain of profiles from s' to s'' changing one vote from some $y\neq x$ to some $z\neq x$ at each step. Now we have seen that strong sincerity gives that the effect on x of adding a vote for y to any s is completely determined by $g(y,s)$ (equation (4)), and we have from pairwise sincerity that $g(y,s)=g(z,s)$ for all y, z, s (Step 2). So the weight placed on x at two profiles that differ only in a switch of one vote from y to z must be identical, as this is the same as saying that the weight on x must be the same whether we add a vote for y or for z to the profile without either of the votes that we are switching. So we have $f(x,s')=f(x,s'')$, which yields our claim.

We now extend our definition of $g(x,s)$ to the case where $f(x,s)=1$, by stipulating that $g(x,s)=g(y,s)$ for any $y\neq x$ when $f(x,s)=1$. This is well-defined because $f(x,s)=1$ implies that $f(y,s)\neq 1$ for all $y\neq x$, and then Step 2 gives that $g(y,s)=g(z,s)$ for all $y,z\neq x$. We claim that with this convention, $g(x,s)$ actually depends only on $V(s)$. To see this, note that $g(x,s)=g(y,s)$ for all x, y and s, by Step 2 in the case where $g(x,s),g(y,s)<1$ and by this definition of $g(x,s)$ in the case where one of these equals one. Now, by the preceding argument, $f(x,s)$ and $f(x,s+x)$ can only depend on $V(s)$ and $V_x(s)$, so $g(x,s)$ can only depend on $V(s)$ and $V_x(s)$ as well, so we can write $g(x,s)=g(x,V_x(s),V(s))$. We must show that $g(x,V_x(s),V(s))=g(x,V_x(s'),V(s'))$ for any profiles s and s' such that $V(s)=V(s')$.

Without loss of generality, we can assume that s and s' differ only in that one vote switches between x and some $y\neq x$ as we move from s to s', for otherwise we can construct a chain of profiles from s to s' each differing by precisely one such switch and show the result for adjacent profiles in the chain. Since $M\geq 3$, there exists a $z\neq x$ such that $V_z(s)=V_z(s')$. Let $V=V(s)=V(s')$. Then $g(x,V_x(s),V)=g(z,V_z(s),V)$ by Step 2, $g(z,V_z(s),V)=g(z,V_z(s'),V)$ as $V_z(s)=V_z(s')$, and $g(z,V_z(s'),V)=g(x,V_x(s'),V)$ again by Step 2. So we have $g(x,V_x(s),V)=g(x,V_x(s'),V)$, as desired. So we can write $g(V(s))$ for $g(x,s)$.

Step 4:

Finally, we apply the fact that adding a vote for x and then a vote for some alternative $y \neq x$ must yield the same outcome as adding a vote for y and then a vote for x. In particular, the final weight placed on x must be the same under both of these scenarios. Note that $f(x, s+x) = f(x, s) + (1 - f(x, s))\, g(V(s))$ and $f(x, s+y) = f(x, s) - f(x, s)g(V(s))$, so adding a vote for x followed by a vote for y at s yields

$$f(x, s{+}x{+}y) = f(x, s){+}(1{-}f(x, s))g(V(s)){-}(f(x, s){+}(1{-}f(x, s))g(V(s)))g(V(s){+}1)$$

while adding a vote for y followed by a vote for x at s yields

$$f(x, s{+}x{+}y) = f(x, s){-}f(x, s)g(V(s)){+}(1{-}f(x, s){+}f(x, s)g(V(s)))g(V(s){+}1)$$

Subtracting the latter equation from the former and rearranging gives $g(V(s)) = g(V(s)+1)(1+g(V(s)))$. This equation holds for all $V(s)$, so we have

$$g(V(s)) = \frac{g(V(s)-1)}{1+g(V(s)-1)} = \frac{\frac{g(V(s)-2)}{1+g(V(s)-2)}}{1+\frac{g(V(s)-2)}{1+g(V(s)-2)}}$$

$$= \frac{g(V(s)-2)}{1+2g(V(s)-2)} = \ldots = \frac{g(0)}{1+V(s)g(0)}$$

So we see that f is entirely characterized by $g(0)$, the proportion of the weight previously shared among the other $M-1$ alternatives that is transferred to an alternative x under the profile where x receives one vote and no other votes are cast. We write $g \equiv g(0)$, and will now show that, for any fully sincere f,

$$f(x, s) = \frac{(1-g)f(x, 0) + gK}{1 + g(K + J - 1)}$$

where $f(x, 0)$ is the weight placed on x when no votes have been cast, $K \equiv V_x(s)$, and $J \equiv V(s) - V_x(s)$.

First, let Kx denote the profile with K votes for x and no votes for any other alternative. By the definition of $g(x, s)$,

$$f(x, Kx) = f(x, (K-1)x)(1 - g(K-1)) + g(K-1)$$

$$= f(x, (K-1)x)\left(1 - \frac{g}{1+(K-1)g}\right) + \frac{g}{1+(K-1)g}$$

$$= \left(f(x, (K-2)x)\left(1 - \frac{g}{1+(K-2)g}\right) + \frac{g}{1+(K-2)g}\right)\left(1 - \frac{g}{1+(K-1)g}\right)$$

$$+ \frac{g}{1+(K-1)g}$$

$$= \ldots$$

$$= f(x, 0)(1{-}g)\left(1 - \frac{g}{1+g}\right)\cdots\left(1 - \frac{g}{1+(K-1)g}\right){+}g\left(1 - \frac{g}{1+g}\right)\cdots\left(1 - \frac{g}{1+(K-1)g}\right)$$

$$+\frac{g}{1+g}\left(1-\frac{g}{1+2g}\right)\cdots\left(1-\frac{g}{1+(K-1)g}\right)+\frac{g}{1+2g}\left(1-\frac{g}{1+3g}\right)\cdots\left(1-\frac{g}{1+(K-1)g}\right)$$
$$+\ldots+\frac{g}{1+(K-1)g}$$

Now

$$\prod_{n=k}^{K-1}\left(1-\frac{g}{1+ng}\right)=\prod_{n=k}^{K-1}\left(\frac{1+(n-1)g}{1+ng}\right)=\frac{1+(k-1)g}{1+(K-1)g}$$

so

$$f(x,Kx)=f(x,0)\left(\frac{1-g}{1+(K-1)g}\right)+g\left(\frac{1}{1+(K-1)g}\right)+\frac{g}{1+g}\left(\frac{1+g}{1+(K-1)g}\right)$$
$$+\frac{g}{1+2g}\left(\frac{1+2g}{1+(K-1)g}\right)+\ldots+\frac{g}{1+(K-1)g}=\frac{f(x,0)(1-g)+Kg}{1+(K-1)g}$$

Next, let $Kx+Jy$ denote the profile with K votes for x and J votes for other alternatives. Recall that we can assume that these J votes are all cast for a single alternative $y\neq x$, as $f(x,s)$ depends only on $V_x(s)$ and $V(s)$. So

$$f(x,Kx+Jy)=\left(1-\frac{g}{1+(K+J-1)g}\right)f(x,Kx+(J-1)y)$$
$$=\left(1-\frac{g}{1+(K+J-1)g}\right)\left(1-\frac{g}{1+(K+J-2)g}\right)f(x,s+(J-2)y)=\ldots$$
$$=\prod_{n=K}^{K+J-1}\left(1-\frac{g}{1+ng}\right)f(x,Kx)=\frac{1+(K-1)g}{1+(K+J-1)g}f(x,Kx)$$
$$=\frac{(1-g)f(x,0)+gK}{1+g(K+J-1)}=\frac{(1-g)f(x,0)+gV_x(s)}{1-g+gV(s)}$$

"If" direction for the $M\geq 3$ case:

For the converse in the $M\geq 3$ case, showing pairwise sincerity is straightforward: If $f(x,s)=\frac{(1-g)f(x,0)+gV_x(s)}{1-g+gV(s)}$ with $g>0$, then $f(z,s+x)=f(z,s+y)$ and $f(x,s+x)>f(x,s+y)$ for all x, y, z and s, so Lemma 2 immediately yields pairwise sincerity. Strong sincerity is also direct, but requires some computation. To simplify notation in this part of the proof, we write V_x for $V_x(s)$ and V for $V(s)$ when doing so can cause no confusion. First, note that $f(x,s+x)-f(x,s)=\frac{(1-g)f(x,0)+g(V_x+1)}{1-g+g(V+1)}-\frac{(1-g)f(x,0)+gV_x}{1-g+gV}=\frac{g((1-g)(1-f(x,0))+g(V-V_x))}{(1+gV)(1+g(V-1))}$ and, for any $y\neq x$, $f(y,s+x)-f(y,s)=\frac{(1-g)f(y,0)+gV_y}{1+gV}-\frac{(1-g)f(y,0)+gV_y}{1+g(V-1)}=\frac{-g((1-g)f(y,0)+gV_y)}{(1+gV)(1+g(V-1))}$. Since the denominators in both of these equations are positive, we have $\sum_{x'\in X}f(x',s+x)u_i(x')\geq\sum_{x'\in X}f(x',s)u_i(x')$ if and only if $g((1-g)(1-f(x,0))+g(V-V_x))u_i(x)\geq\sum_{y\neq x}g((1-g)f(y,0)+gV_y)u_i(y)$.

Next, note that $\frac{1}{1-f(x,s)} = \frac{1+g(V-1)}{(1-g)(1-f(x,0))+g(V-V_x)}$ and then $\frac{f(y,s)}{1-f(x,s)} = \frac{(1-g)f(y,0)+gV_y}{(1-g)(1-f(x,0))+g(V-V_x)}$. So we have $\sum_{x'\in X} f(x', s+x)u_i(x') \geq \sum_{x'\in X} f(x', s)u_i(x')$ if and only if $u_i(x) \geq \frac{1}{1-f(x,s)} \sum_{y\neq x} f(y,s)u_i(y)$. This is equivalent to the definition of strong sincerity if $f(x,s) < 1$, and strong sincerity is immediate if $f(x,s) = 1$.

Proof of Proposition 1

Suppose that S is scoring, and consider the strategy s_i given by $s_{i,x_j} = k_{i,j}$. Suppose that there exist s_i' and s_{-i} such that $E\left[u_i(f(s_i', s_{-i}))\right] > E\left[u_i(f(s))\right]$ and s_i' is a best response to s_{-i}. Since (S, f) is pairwise sincere, there exists a pairwise sincere aggregation rule $\widetilde{f}$ such that $f(s) = \widetilde{f}(s)$ for all $s \in S$, so in particular $E\left[u_i(\widetilde{f}(s_i', s_{-i}))\right] > E\left[u_i(\widetilde{f}(s))\right]$. Since $s_i' \neq s_i$, there exist x, $y \in X$ and $k > k'$ such that $s'_{i,x} = k'$ and $s'_{i,y} = k$ but $u_i(x) \geq u_i(y)$. If $u_i(x) > u_i(y)$ for any such x and y, then applying Lemma 2 $k - k'$ times at the strategy profile (s_i^*, s_{-i}), where s_i^* is given by $s^*_{i,z} = s'_{i,z}$ for all $z \neq x$, y, $s^*_{i,x} = s^*_{i,y} = k'$ implies that $E\left[u_i(\widetilde{f}(s_i'', s_{-i}))\right] > E\left[u_i(\widetilde{f}(s_i', s_{-i}))\right]$ where s_i'' is given by $s''_{i,z} = s'_{i,z}$ for all $z \neq x$, y, $s''_{i,x} = k$ and $s''_{i,y} = k'$, contradicting the hypothesis that s_i' is a best response to s_{-i}. If $u_i(x) = u_i(y)$ for all such x and y, then one can construct a chain of strategies leading from s_i' to s_i such that adjacent strategies in the chain differ only in that they switch the ranking of two alternatives x and y satisfying $u_i(x) = u_i(y)$. Lemma 2 implies that i must be indifferent between any two adjacent strategies in such a chain, which yields $E\left[u_i(\widetilde{f}(s_i', s_{-i}))\right] = E\left[u_i(\widetilde{f}(s))\right]$ by transitivity, another contradiction. So s_i must be a best response to any pure-strategy profile of i's opponents, s_{-i}. So s_i is weakly dominant.

Now suppose that S is triangular, and consider the strategy s_i given by $s_{i,x_1} = k_i$, $s_{i,x_j} = 0$ for all $j \neq 1$. Since (S, f) is pairwise sincere, there exists a pairwise sincere aggregation rule $\widetilde{f}$ such that $f(s) = \widetilde{f}(s)$ for all $s \in S$. Applying pairwise sincerity, $E\left[u_i(\widetilde{f}(s_i, s_{-i}))\right] \geq E\left[u_i(\widetilde{f}(s_i', s_{-i}))\right]$ for all s_i' such that $s'_{i,x_1} = k_i - 1$ and all $s_{-i} \in S_{-i}$. Applying pairwise sincerity a second time yields $E\left[u_i(\widetilde{f}(s_i, s_{-i}))\right] \geq E\left[u_i(\widetilde{f}(s_i'', s_{-i}))\right]$ for all s_i'' such that $s''_{i,x_1} = k_i - 2$ and all $s_{-i} \in S_{-i}$, and continuing in this manner shows that s_i is a best response to any opposing pure-strategy profile s_{-i}, so s_i is weakly dominant.

Proof of Proposition 2

By Lemma 1, i strictly prefers to add a vote for x at s if $u_i(x) > E[u_i(f(s))]$; strictly prefers to remove a vote for x if $u_i(x) < E[u_i(f(s))]$; and is indifferent between adding and removing a vote for x at s if $u_i(x) = E[u_i(f(s))]$. Since S is rectangular, i may add a vote for x at s if $s_{i,x} < \max\{S_{i,x}\}$ and may remove a vote for x at s if $s_{i,x} > \min\{S_{i,x}\}$. So if s_i is a best response to s_{-i} it must be the case that, for any x, either $u_i(x) > E[u_i(f(s))]$ and $s_{i,x} = \max\{S_{i,x}\}$; $u_i(x) < E[u_i(f(s))]$ and $s_{i,x} = \min\{S_{i,x}\}$; or $u_i(x) = E[u_i(f(s))]$. This shows that every best response of i to s_{-i} is weakly dichotomous. And setting $s_{i,x} = \max\{S_{i,x}\}$ whenever $u_i(x) = E[u_i(f(s))]$ yields a dichotomous best response for i to s_{-i}.

To prove the second part, note that Theorem 1 implies that $f(x', s_i', s_{-i}) > 0$ and $f(x'', s_i', s_{-i}) > 0$ for all $s_i' \in S_i$ whenever $V_{s_{-i}}(x') > 0$ and $V_{s_{-i}}(x'') > 0$, so $f(s_i', s_{-i})$ is nondegenerate whenever $V_{s_{-i}}(x') > 0$ and $V_{s_{-i}}(x'') > 0$. We claim that for any $\varepsilon > 0$ and for any utility function $u_i(\cdot)$, there exists a utility function $u_i'(\cdot)$ such that $|u_i'(x) - u_i(x)| < \varepsilon$ for all $x \in X$ and $u_i'(x) \neq E[u_i'(f(s))]$ for all x and all $s \in S$ for which $f(s)$ is nondegenerate. We proceed by induction on M. If $M = 2$, then any $u_i'(\cdot)$ such that $u_i'(x_1) \neq u_i'(x_2)$ satisfies the conditions of the claim. Now suppose that the claim holds for $M-1$. Then there exist $u_i'(x_1), \ldots, u_i'(x_{M-1})$ such that $|u_i'(x_j) - u_i(x_j)| < \varepsilon$ for all $j < M$ and $u_i'(x_j) \neq E[u_i'(f(s))]$ for all $j < M$ and all $s \in S$ for which $f(s)$ is nondegenerate and $f(x_M, s) = 0$. To prove the claim, it suffices to show that there exists a value $u_i'(x_M)$ such that $|u_i'(x_M) - u_i(x_M)| < \varepsilon$, $u_i'(x_M) \neq E[u_i'(f(s))]$ for all $s \in S$ for which $f(s)$ is nondegenerate, and $u_i'(x_j) \neq E[u_i'(f(s))]$ for all $j < M$ and all $s \in S$ for which $f(s)$ is nondegenerate and $f(x_M, s) > 0$. If $f(x_M, s) > 0$, then $u_i'(x_j) \neq E[u_i'(f(s))]$ for $j < M$ if and only if $u_i'(x_M) \neq \frac{1}{f(x_M,s)}\left(u_i'(x_j) - \sum_{k<M} f(x_k, s) u_i'(x_k)\right)$; and if $f(s)$ is nondegenerate, then $u_i'(x_M) \neq E[u_i'(f(s))]$ if and only if $u_i'(x_M) \neq \frac{1}{1-f(x_M,s)} \sum_{k<M} f(x_k, s) u_i'(x_k)$. Since S is finite, these inequalities hold for all but finitely many values for $u_i'(x_M)$, and there are also only finitely many values for $u_i'(x_M)$ such that $u_i'(x_j) = E[u_i'(f(s))]$ for some j and $s \in S$ with $f(s)$ nondegenerate. Therefore, there exists a value for $u_i'(x_M)$ such that $u_i'(x_j) \neq E[u_i'(f(s))]$ for all j and all $s \in S$ for which $f(s)$ is nondegenerate, and $|u_i'(x_M) - u_i(x_M)| < \varepsilon$, which proves the claim.

Let i have the utility function $u_i'(\cdot)$ constructed above. By the proof of the first part of the proposition, $s_{i,x_j} = \max\{S_{i,x_j}\}$ if $u_i'(x_j) > E[u_i'(f(s))]$ and $s_{i,x_j} = \min\{S_{i,x_j}\}$ if $u_i'(x_j) < E[u_i'(f(s))]$, if s_i is a best response to s_{-i}. When i has utility function $u_i'(\cdot)$, $u_i'(x_j) \neq E[u_i'(f(s))]$ for all x_j and all $s \in S$, so i has a unique best response to s_{-i}.

Proof of Proposition 3

Suppose towards a contradiction that $u_i(x') \geq u_i(x_m)$ and $m < m'$. Since $\left|S_{i,x_{m'}}\right| \geq 2$, $s'_{i,x_{m'}} = \max\{S_{i,x_{m'}}\} \neq \min\{S_{i,x_{m'}}\}$, so $u_i(x_{m'}) \geq E[u_i(f(s_i', s_{-i} + x'))]$ by strong sincerity. Since s_i' is i's unique best response to $s_{-i} + x'$ by assumption, $E[u_i(f(s_i', s_{-i} + x'))] > E[u_i(f(s_i, s_{-i} + x'))]$. Furthermore, $u_i(x_m) \geq E[u_i(f(s))]$, again by strong sincerity, which implies that $u_i(x') \geq E[u_i(f(s))]$. So by strong sincerity $E[u_i(f(s_i, s_{-i} + x'))] \geq E[u_i(f(s))]$ and therefore $u_i(x_{m'}) > E[u_i(f(s))]$ by a chain of inequalities. But strong sincerity then implies that $E[u_i(f(s_i + x_{m'}, s_{-i}))] \geq E[u_i(f(s))]$, which contradicts the assumption that s_i is i's unique best response to s_{-i}. A similar argument shows that if $u_i(x') \leq$

$u_i(x_m)$ then $m \leq m'$.

Proof of Proposition 4

Since $S \cup S^*$ is finite, the argument in the second paragraph of the proof of Proposition 2 with $S \cup S^*$ in place of S shows that for any $\varepsilon > 0$ and any utility function $u_i(\cdot)$, there exists a utility function $u_i'(\cdot)$ such that $|u_i'(x) - u_i(x)| < \varepsilon$ and $u_i'(x) \neq E\left[u_i'(f(s))\right]$ for all $x \in X$ and all $s \in S \cup S^*$ for which $f(s)$ is nondegenerate. Applying this fact to each voter, we have that for any $\varepsilon > 0$ and any $\{u_i(\cdot)\}$, there exist $\{u_i'(\cdot)\}$ such that, for all i, $|u_i'(x) - u_i(x)| < \varepsilon$ and $u_i'(x) \neq E\left[u_i'(f(s))\right]$ for all $x \in X$ and all $s \in S \cup S^*$ for which $f(s)$ is nondegenerate. Let voters' utility functions be given by $\{u_i'(\cdot)\}$. If f is fully sincere with $g = 1$, then $f(x,s) = \frac{V_x(s)}{V(s)}$ if $V(s) > 0$. Therefore, the second part of the proposition implies that the set of nondegenerate pure-strategy Nash equilibrium outcomes of (S,f) is the same as the set of nondegenerate pure-strategy Nash equilibrium outcomes of (S',f). We first show that the set of degenerate pure-strategy Nash equilibrium outcomes is the same for (S,f) and (S',f) and then prove the second part of the proposition.

Suppose that s is a degenerate pure-strategy Nash equilibrium of (S,f) and denote by x the element satisfying $f(s,x) = 1$. Because S is anonymous, neutral and rectangular and satisfies $0 \in S$ and $|S| > 1$, each voter may add a vote for any alternative other than x at s. Since $f(x,s) = \frac{V_x(s)}{V(s)}$ if $V(s) > 0$, at s a voter will add a vote for any alternative y that she prefers to x, so since s is a Nash equilibrium it must be that x is each voter's (weakly) most-preferred alternative. Consider the strategy profile s' given by $s_{i,x} = 1$, $s_{i,y} = 0$ for all $y \neq x$ and all i. $s' \in S'$ by our assumptions on S', and, since x is each voter's most-preferred alternative, s' is a Nash equilibrium of S'. $f(s',x) = 1$, so any degenerate pure-strategy Nash equilibrium outcome of (S,f) is also a degenerate pure-strategy Nash equilibrium outcome of (S',f). The same argument goes through with S and S' switched, so the set of degenerate pure-strategy Nash equilibrium outcomes is the same for (S,f) and (S',f).

Now suppose that s is a pure-strategy Nash equilibrium of S with $f(s)$ nondegenerate. Since S is rectangular, f is strongly sincere, and $u_i'(x) \neq E\left[u_i'(f(s))\right]$ for all x, the proof of part 1 of Proposition 2 yields that i has a unique best response to s_{-i} which is m_i-dichotomous with m_i uniquely determined by the inequalities $u_i(x_{m_i}) > E\left[u_i'(f(s))\right]$ and $u_i(x_{m_i+1}) < E\left[u_i'(f(s))\right]$. Since $0 \in S$ and S is anonymous and neutral, this implies that under s each voter casts either 0 or $\overline{s_{i,x}}$ votes for every alternative. Let $s' = s\left(\frac{\overline{s'_{i,x}}}{\overline{s_{i,x}}}\right)$, which is well-defined because $|S| > 1$ implies that $\overline{s_{i,x}} \geq 1$. Under s', each voter casts either 0 or $\overline{s'_{i,x}}$ votes for every alternative, so $s' \in S'$ by our assumptions on S'. Since $f(x,s) = \frac{V_x(s)}{V(s)}$ if $V(s) > 0$, $f(x,s) = f(x,s')$. Now the proof of part 1 of Proposition 2 implies that each voter i has a unique best response to s'_{-i} which is m_i'-dichotomous with m_i' uniquely determined by the inequalities $u_i(x_{m_i'}) > E\left[u_i'(f(s'))\right]$ and $u_i(x_{m_i'+1}) < E\left[u_i'(f(s'))\right]$, so the fact that

$f(x,s) = f(x,s')$ implies that $m'_i = m_i$. And s'_i is precisely the m_i-dichotomous strategy in S', so s' is a pure-strategy Nash equilibrium of (S', f). The same argument with S and S' switched shows that if s' is a pure-strategy Nash equilibrium of (S', f) with $f(s')$ nondegenerate, then $s'\left(\frac{\overline{s_x}}{s'_x}\right)$ is a pure-strategy Nash equilibrium of (S, f).

Proof of Theorem 2

The "only if" direction of this theorem is a straightforward corollary to Theorem 1: The proof of $M = 2$ case is immediate from the proof of Theorem 1 and the proof of the $M \geq 3$ case follows steps 1 and 2 in the proof of Theorem 1 directly.

For the "if" direction, the $M = 2$ case is trivial but the $M \geq 3$ case is not quite as straightforward as in the more restrictive context of Theorem 1. For $M \geq 3$, we first show strong sincerity. Strong sincerity with respect to alternative x at s follows immediately if $f(x,s) = 1$. So suppose that $f(x,s) < 1$ and $u_i(x) \geq \sum_{x' \in X} f(x', s) u_i(x')$, which implies that $u_i(x) \geq \frac{1}{1-f(x,s)} \sum_{y \neq x} f(y,s) u_i(y)$. Then

$$\sum_{x' \in X} f(x', s_i + x, s_{-i}) u_i(x') - \sum_{x' \in X} f(x', s) u_i(x')$$

$$= ((f(x, s_i + x, s_{-i}) - f(x,s)) u_i(x) - \sum_{y \neq x} (f(y,s) - f(y, s_i + x, s_{-i})) u_i(y)$$

$$\geq \sum_{y \neq x} \left(\frac{f(x, s_i + x, s_{-i}) - f(x,s)}{1 - f(x,s)} f(y,s) - (f(y,s) - f(y, s_i + x, s_{-i})) \right) u_i(y)$$

Now for all $y \neq x$, equation (2) yields

$$\left(\frac{f(x, s_i + x, s_{-i}) - f(x,s)}{1 - f(x,s)} \right) f(y,s) - (f(y,s) - f(y, s_i + x, s_{-i}))$$

$$= (f(y,s) - f(y, s_i + x, s_{-i})) \left(\frac{f(y,s)}{f(y,s)} - 1 \right) = 0$$

So the sum over y of these terms equals 0 as well, so we have $\sum_{x' \in X} f(x', s_i + x, s_{-i}) u_i(x') \geq \sum_{x' \in X} f(x', s) u_i(x')$ as desired.

Conversely, suppose that $\sum_{x' \in X} f(x', s_i + x, s_{-i}) u_i(x') \geq \sum_{x' \in X} f(x', s) u_i(x')$. Then $((f(x, s_i + x, s_{-i}) - f(x,s)) u_i(x) \geq \sum_{y \neq x} (f(y,s) - f(y, s_i + x, s_{-i})) u_i(y)$, so by equation (2) either $f(y,s) = 0$ or$(1 - f(x,s)) \left(\frac{f(y,s)) - f(y, s_i + x, s_{-i})}{f(y,s)} \right) u_i(x) \geq \sum_{y \neq x} (f(y,s) - f(y, s_i + x, s_{-i})) u_i(y)$, where we recall that $\frac{f(y,s)) - f(y, s_i + x, s_{-i})}{f(y,s)}$ does not depend on y, by equation (2). Note that if $f(y,s) - f(y, s_i + x, s_{-i}) = 0$, then equation (3) yields $f(s) = f(s_i + x, s_{-i})$ which by equation (2) is possible if and only if $f(x,s) = 1$, whence equation (2) gives $f(x, s_i + x, s_{-i}) = 1$, so the result is immediate in this case. If $f(y,s) - f(y, s_i + x, s_{-i}) \neq 0$, we have $(1 -$

$f(x,s))u_i(x) \geq \sum_{y \neq x}(\frac{f(y,s)}{f(y,s)-f(y,s_i+x,s_{-i})})(f(y_i,s) - f(y,s_i+x,s_{-i}))u_i(y)$, so $u_i(x) \geq \frac{1}{1-f(x,s)} \sum_{y \neq x} f(y,s)u_i(y)$ and finally $u_i(x) \geq \sum_{x' \in X} f(x', s_i, s_i)u_i(x)$.

It remains only to show pairwise sincerity. First, suppose that $u_i(y) \geq u_i(z)$ and that $f(y,s)$, $f(z,s) < 1$. Then for all $x \neq y, z$, equations (2) and (3) give $\frac{f(x,s)-f(x,s_i+y,s_{-i})}{f(x,s)} = \frac{f(x,s)-f(y,s_i+z,s_{-i})}{f(x,s)}$. So for all $x \neq y, z$ we have $f(x, s_i + y, s_{-i}) = f(x, s_i + z, s_{-i})$ or $f(x,s) = 0$. Now note that if $f(x,s) = 0$ then $f(x, s_i + y, s_{-i}) = f(x, s_i + z, s_{-i}) = 0$ as well by strong sincerity, which we have already shown. This follows from considering, for example, $u_i(x) = 1$, $u_i(x') = 0$ for all $x' \neq x$. So we have $f(x, s_i + y, s_{-i}) = f(x, s_i + z, s_{-i})$ for all $x \neq y, z$, and therefore $\sum_{x' \in X} f(x', s_i + y, s_{-i})u_i(x') - \sum_{x' \in X} f(x', s_i + z, s_{-i})u_i(x') = (f(y, s_i+y, s_{-i}) - f(y, s_i+z, s_{-i}))u_i(y) - (f(z, s_i+z, s_{-i}) - f(z, s_i+y, s_{-i}))u_i(z)$. Now $f(y, s_i+y, s_{-i}) - f(y, s_i+z, s_{-i}) = f(z, s_i+z, s_{-i}) - f(z, s_i+y, s_{-i})$, since the sum of the weights on all alternatives must equal one at every profile. So $u_i(y) \geq u_i(z)$ implies that $\sum_{x' \in X} f(x', s_i + y, s_{-i})u_i(x') - \sum_{x' \in X} f(x', s_i + z, s_{-i})u_i(x') \geq 0$.

Finally, we must consider the cases where $f(y,s) = 1$ or $f(z,s) = 1$. Strong sincerity, which we have shown is implied by equations (2) and (3), gives that $f(x,s) = 0$ implies $f(x, s_i + z, s_{-i}) = f(x, s_i + y, s_{-i}) = 0$, so in these cases voting for y as opposed to z can only weakly transfer weight from z to y, so pairwise sincerity is immediate.

References

Arrow, K.J., 1950. A difficulty in the concept of social welfare. J. Polit. Economy 58, 328-346.

Arrow, K.J., 1951. Social Choice and Individual Values. Yale University Press, New Haven, CT; 2nd Edition 1963. Wiley, New York.

Barberà, S., 1979. Majority and positional voting in a probabilistic framework. Rev. Econ. Stud. 46, 389-397.

Barberà, S., Bogomolnaia A., van der Stel H., 1998. Strategy-proof probabilistic rules for expected utility maximizers. Math. Soc. Sci. 35, 89-103.

Barberà, S., Dutta B., Sen A., 2001. Strategy-proof social choice correspondences. J. Econ. Theory 101, 374-394.

Benoit, J.-P., 2002. Strategic manipulation in voting games when lotteries and ties are permitted. J. Econ. Theory 102, 421-436.

Bogomolnaia, A., Moulin H., Strong R., 2005. Collective choice under dichotomous preferences. J. Econ. Theory 122, 165-184.

Brams, S., Fishburn P.C., 1978. Approval Voting. Amer. Polit. Sci. Rev. 72, 831-847.

Brams, S., Fishburn P.C., 2002. Voting procedures. In: Arrow, K.J., Sen, A.K., Suzumura, K. (Eds.), Handbook of Social Choice and Welfare, Volume 1. Elsevier Science, B.V. 2002, pp. 173-236.

Clark, S., 1992. The representative agent model of probabilistic social choice. Math. Soc. Sci. 23, 45-66.

Ehlers, L., Peters H., Storcken T., 2002. Strategy-proof probabilistic decision schemes for one-dimensional single-peaked preferences. J. Econ. Theory 105, 408-434.

Gibbard, A.F., 1973. Manipulation of voting schemes: a general result. Econometrica 41, 587-601.

Gibbard A.F., 1978. Straightforwardness of game forms with lotteries as outcomes. Econometrica 46, 595-614.

Hylland, A., 1980. Strategyproofness of voting procedures with lotteries as outcomes and infinite sets of strategies. mimeo, University of Oslo.

Nandeibam, S., 1995. A note on randomized social choice and random dictatorships. J. Econ. Theory 66, 581-589.

Pattanaik, P., Peleg B., 1986. Distribution of power under stochastic social choice rules. Econometrica 54, 909-922.

Satterthwaite, M.A., 1975. Strategy-proofness and Arrow's conditions: existence and correspondence theorems for voting procedures and social welfare functions. J. Econ. Theory 10, 187-218.