

GMM is Inadmissible Under Weak Identification

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Abstract

We consider estimation in moment condition models and show that under any bound on identification strength, asymptotically admissible (i.e. undominated) estimators in a wide class of estimation problems must be uniformly continuous in the sample moment function. GMM estimators are in general discontinuous in the sample moments, and are thus inadmissible. We show, by contrast, that bagged, or bootstrap aggregated, GMM estimators as well as quasi-Bayes posterior means have superior continuity properties, while results in the literature imply that they are equivalent to GMM when identification is strong. In simulations calibrated to published instrumental variables specifications, we find that these alternatives often outperform GMM.

Keywords: Limit Experiment, Weak Identification, Nonlinear GMM

JEL Codes: C11, C12, C20

1 Introduction

Generalized method of moments (GMM) estimators are ubiquitous in empirical economics, and many popular estimation methods including linear and nonlinear instrumental variables, moment-matching, and many examples of maximum likelihood can be cast as special cases. Appropriately constructed GMM estimators are known to be efficient in large samples, in the sense of minimizing mean squared error over a large class of estimators, provided model parameters are strongly identified (i.e. the data are sufficiently informative) and other regularity conditions hold (see Hansen 1982, Chamberlain 1987).

Unfortunately, in many contexts of economic interest the data provide only limited information about model parameters (Mavroeidis et al. 2014, Armstrong 2016, Andrews et al. 2019). In such cases asymptotic results assuming strong identification can be unreliable, and weak-identification approximations, which model the informativeness of the data as limited even

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in large samples, often provide a better description of finite-sample behavior (Staiger and Stock 1997, D. Andrews and Cheng 2012, Andrews and Mikusheva 2022). Standard arguments for the efficiency of GMM no longer apply under weak identification, raising the question of whether GMM estimators should be used in such settings and, if not, what alternatives we should prefer.

We study asymptotic optimality under weak identification using a limit experiment derived in Andrews and Mikusheva (2022). This limit experiment implies that there generally exists no single best estimator under weak identification, since optimizing performance over different parts of the parameter space leads to different estimators. A minimal requirement for reasonable estimators in this setting is that they be admissible, which means there exists no alternative estimator which performs at least as well for all parameter values and strictly better for some. Our main result shows that GMM estimators are inadmissible in decision problems where (i) the loss function is bounded and uniformly convex and (ii) identification strength is bounded. Since these results hold for a wide class of loss functions and any bound on identification strength, they establish a strong sense in which GMM performs poorly under weak identification.

Our proof for inadmissibility is non-constructive, in the sense that it does not deliver a dominating estimator, but nonetheless suggests directions for improvement. Specifically, we show that admissible estimators in the limit experiment must be uniformly continuous in the sample moments. To prove this result we first note that by a complete class theorem any admissible estimator under a convex loss function must be equal to the limit of Bayes decision rules for some sequence of priors. We prove, however, that under bounds on identification strength Bayes decision rules for bounded, uniformly convex loss functions are uniformly continuous in the sample moments, so small changes in the moments lead only to small changes in the estimator. Since uniform continuity is preserved under limits, admissible estimators inherit this property. GMM estimators, by contrast, change discontinuously in the sample moments when the minimizer of the sample GMM objective function is non-unique, and so fail to satisfy this necessary condition for admissibility.³

Motivated by the necessity of continuity for admissibility, we next explore more continuous

³The results of Guggenberger and Smith (2005) imply that under regularity conditions all Generalized Empirical Likelihood (GEL) estimators are first-order asymptotically equivalent to continuously updated GMM under weak identification. Consequently, all of our results for GMM also apply to GEL estimators, so the latter are likewise inadmissible under weak identification. Motivated by this equivalence, we do not separately discuss GEL approaches for the remainder of the paper.

alternatives to GMM. To derive positive results we focus on squared error loss, for which we show that admissible estimators must be Lipschitz in the sample moments. We discuss two estimators: first a bagged (or bootstrap aggregated) GMM estimator, and second a quasi-Bayes posterior mean. While we do not claim these estimators are admissible, we show that they have better continuity properties than GMM under weak identification, while existing results imply that both are asymptotically equivalent to GMM under strong identification and standard regularity conditions.

The first alternative estimator we discuss, bagged GMM, corresponds to the average of the GMM estimator across bootstrap realizations. Bagging smooths the discontinuities in the GMM estimator, and we show that bagged GMM is Lipschitz in many cases. Bagged GMM has a Bayesian interpretation, corresponding to the posterior mean of the GMM estimand under an uninformative prior that does not impose correct specification of the GMM model. Standard results on bootstrap bias correction (see e.g. Horowitz, 2001, Chen and Hall 2003) imply that bagged GMM is asymptotically equivalent to GMM in the strongly-identified case.⁴

Quasi-Bayes puts a prior on the structural parameters, treats the GMM objective as a negative log-likelihood, and combines the two to compute a quasi-posterior distribution. This approach was initially proposed by Chernozhukov and Hong (2003) for settings where minimization is computationally intractable, and they showed that quasi-Bayes is asymptotically equivalent to GMM under strong identification. More recently, Andrews and Mikusheva (2022) showed that quasi-Bayes arises as the limit of a sequence of Bayes decision rules under weak identification. In the present paper, we show that quasi-Bayes posterior means are Lipschitz in the GMM objective function. While quasi-Bayes is not in general Lipschitz in the moments, we show that it is Lipschitz in the special case where (i) the structural parameter takes only a finite number of possible values and (ii) the J -statistic for testing over-identifying restrictions is bounded.

We compare these estimators in simulation designs (from Andrews et al., 2019) calibrated to linear instrumental variables specifications published in the American Economic Review. These published specifications show substantial evidence of weak instruments, with 56 of the 121 specifications we consider having an expected first-stage F statistic smaller than 10. We find that bagged GMM estimators for bounded target parameters typically have smaller

⁴As Chen and Hall (2003) showed for estimating equation models, however, bagging is essentially the opposite of standard bias-correction, and so will increase higher-order bias in the well-identified case.

mean squared error than their conventional counterparts, consistent with poor performance for GMM under weak identification. We further find that the performance of quasi-Bayes depends strongly on the prior. Specifically, quasi-Bayes estimators with a flat prior perform the worst of all estimators considered, while quasi-Bayes estimators with a novel invariant prior (motivated by invariance in the spirit of Jeffreys, 1946) are much more competitive.

Section 2 describes the estimation problem we consider, the limit experiment (based on Andrews and Mikusheva 2022) in which we conduct our analysis, and defines a theoretical measure of identification strength. Section 3 shows that admissible estimators under bounds on identification strength must be uniformly continuous in the moments, and shows that GMM fails to satisfy this condition. Section 4 turns to alternative estimators, discussing bagged GMM in Section 4.1 and quasi-Bayes in Section 4.2. Section 5 compares the performance of these estimators in simulation.

2 Setting

Consider a researcher who observes a sample of independent and identically distributed observations $X^n = \{X_i, i = 1, \dots, n\}$ with $X_i \in \mathcal{X}$. The distribution of the data is related to a structural parameter $\theta^* \in \Theta$, which we assume satisfies a moment condition $\mathbb{E}[\phi(X_i, \theta^*)] = 0$ for $\phi(\cdot, \cdot)$ a known \mathbb{R}^k -valued function of the data and parameters.

This researcher wants to choose an action a from a compact and convex set of possible actions $\mathcal{A} \subseteq \mathbb{R}^p$, where the loss from taking action a when the true parameter value is θ^* is given by $L(a, \theta^*)$. We assume that L is bounded, as well as continuous and uniformly convex in a .

Definition 1 *Loss function $\{L(a, \theta) : \theta \in \Theta\}$ is uniformly convex in a if there exists a strictly increasing non-negative function $\kappa : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ such that for all $a, a' \in \mathcal{A}$, $\theta \in \Theta$, and $t \in [0, 1]$ the following inequality holds:*

$$L(t \cdot a + (1-t) \cdot a', \theta) \leq t \cdot L(a, \theta) + (1-t) \cdot L(a', \theta) - t(1-t)\kappa(\|a - a'\|).$$

A canonical example of a loss function is squared error loss for a transformation r of θ^* :

$$L(a, \theta^*) = (r(\theta^*) - a)'(r(\theta^*) - a). \tag{1}$$

The loss function (1) satisfies our assumptions so long as the function $r(\cdot):\Theta\rightarrow\mathbb{R}^p$ is bounded. More generally, any loss of the form $L(a,\theta^*)=\ell(r(\theta^*)-a)$ with $r(\cdot)$ bounded and $\ell(\cdot)$ strictly convex satisfies our conditions, though the class of uniformly convex loss functions, and hence the class of decision problems covered by our results, is much larger. The researcher’s goal is to select a decision rule $\delta_n:\mathcal{X}^n\rightarrow\mathcal{A}$ that yields low risk, or expected loss, $\mathbb{E}[L(\delta_n(X^n),\theta^*)]$, where θ^* and the distribution of X are both unknown. Since convex loss functions are commonly used in point-estimation problems we interpret δ_n as an estimator.

GMM estimators are popular in this setting. GMM estimates θ^* by minimizing the norm of the scaled sample moments $g_n(\theta)=\frac{1}{\sqrt{n}}\sum_{i=1}^n\phi(X_i,\theta)$,

$$\hat{\theta}_n=\operatorname{argmin}_{\theta\in\Theta}g_n(\theta)'\Xi_n(\theta)g_n(\theta),$$

for a potentially data- and parameter-dependent weighting matrix $\Xi_n(\theta)$. We will interpret the GMM decision rule as selecting an action a through the plug-in method, $\delta_n^{GMM}(X^n)\in\operatorname{argmin}_{a\in\mathcal{A}}L(a,\hat{\theta}_n)$.⁵ For instance, under squared-error loss (1), GMM uses the plug-in estimator $\delta_n^{GMM}(X^n)=r(\hat{\theta}_n)$. Well-known asymptotic arguments (see Hansen 1982) provide conditions under which $r(\hat{\theta}_n)$ is consistent for $r(\theta^*)$ and asymptotically normal as $n\rightarrow\infty$. These results further establish that if $\Xi_n(\theta^*)$ is proportional to the inverse of the variance of $g_n(\theta^*)$ then the GMM estimator is asymptotically efficient under squared error loss, in the sense that δ_n^{GMM} minimizes the asymptotic risk $\lim_{n\rightarrow\infty}n\cdot\mathbb{E}[L(\delta_n(X^n),\theta^*)]$ over a large class of estimators.⁶

Standard asymptotic results for GMM require, among other assumptions, that θ^* be point identified and strongly identified. Specifically, the moment condition $\mathbb{E}[\phi(X_i,\theta)]=0$ should be uniquely solved at θ^* , and the sample moment function $g_n(\theta)$ should be well-separated from zero, asymptotically, outside infinitesimal neighborhoods of θ^* . These point- and strong-identification assumptions are a poor fit for many economic applications, so Andrews and Mikusheva (2022) derived an alternative asymptotic efficiency theory for moment condition models with weak and partial identification.

The weak identification asymptotics of Andrews and Mikusheva (2022) models the sample of size n as generated by a distribution P_n , where $P_n\rightarrow P_0$ at a \sqrt{n} rate as $n\rightarrow\infty$, and identification fails completely at P_0 in the sense that $\{\theta:E_{P_0}[\phi(X_i,\theta)]=0\}=\Theta$. Hence, identification

⁵Manski (2021) termed decision rules of this type “as-if optimization.”

⁶Uniform integrability conditions are needed to ensure that $\lim_{n\rightarrow\infty}n\cdot\mathbb{E}[L(\delta_n(X^n),\theta^*)]$ is well-behaved. Absent such conditions, analogous results hold for trimmed losses.

fails in the limit.⁷ For finite n , by contrast, the model of Andrews and Mikusheva (2022) implies that $E_{P_n}[\phi(X_i, \theta)] = \frac{1}{\sqrt{n}}m(\theta) + o\left(\frac{1}{\sqrt{n}}\right)$, where m satisfies $m(\theta^*) = 0$. Consequently, under mild regularity conditions the scaled sample moments converge in distribution to a Gaussian process, $g_n(\cdot) \Rightarrow g(\cdot)$ where

$$g(\cdot) \sim \mathcal{GP}(m, \Sigma), \quad (2)$$

for an unknown mean function m satisfying $m(\theta^*) = 0$ and a consistently estimable covariance function Σ , with $\Sigma(\theta, \tilde{\theta}) = \text{Cov}(\phi(X_i, \theta), \phi(X_i, \tilde{\theta}))$. We treat the structural parameter θ^* as a well-defined economic quantity that may or may not be point-identified by the moment conditions. Hence, it is meaningful to discuss the “true” value of θ^* even when m has more than one zero so θ^* is set-identified.

For estimators, including GMM, that can be written as a function of the sample moments and the estimated covariance function $\hat{\Sigma}$, it follows that $\delta_n(X^n) = \delta(g_n, \hat{\Sigma}) \Rightarrow \delta(g, \Sigma)$ under mild continuity conditions. Hence, in order to understand the large-sample behavior of these estimators it suffices to study their behavior in the limit problem. This implies that, consistent with the results of e.g. Stock and Wright (2000), GMM estimators are inconsistent when identification is weak, with non-normal limit distributions. Consequently, standard efficiency arguments for GMM no longer apply.

To provide an alternative efficiency criterion, Andrews and Mikusheva (2022) further showed that (2) is the “limit experiment” in this setting. In the limit experiment, as in the finite-sample problem, the goal is to choose an estimator δ which now maps realizations of $g(\cdot)$ to estimates $\delta(g, \Sigma) \in \mathcal{A}$, in a way which yields a low risk $\mathbb{E}_m[L(\delta(g, \Sigma), \theta^*)]$, where $\mathbb{E}_m[\cdot]$ denotes the expectation taken under (2). Andrews and Mikusheva (2022) showed that the risk in the limit experiment lower-bounds the (appropriately scaled) asymptotic risk in the original problem. Thus, if we can find an optimal estimator $\delta(g, \Sigma)$ in the limit experiment, the plug-in estimator $\delta_n(X^n) = \delta(g_n, \hat{\Sigma})$ will be asymptotically optimal under weak identification.

Motivated by the results of Andrews and Mikusheva (2022), the following sections focus on properties for the limit experiment (2). First, however, we introduce two special cases and characterize the parameter space for the limit experiment.

⁷The results of Andrews and Mikusheva (2022) allow that the parameter space in the finite-sample model may be a larger set $\tilde{\Theta} \supset \Theta$. Any $\theta \in \tilde{\Theta} \setminus \Theta$ is eventually ruled out by the data with probability one, however, so here we take $\tilde{\Theta} = \Theta$ to simplify the exposition.

Special Case: Linear IV For our first special case we consider the linear IV model. Suppose $X_i = (Y_i, D_i, Z_i')$ for $Y_i \in \mathbb{R}$ an outcome of interest, $D_i \in \mathbb{R}$ an endogenous regressor, and $Z_i \in \mathbb{R}^k$ a vector of instruments. The familiar linear IV estimators correspond to GMM with moment condition $\phi(X_i, \theta) = (Y_i - D_i\theta)Z_i$ and different choices of weighting matrix, for instance $\Xi_n = (\frac{1}{n} \sum Z_i Z_i')$ for two-stage least squares.

Weak-identification asymptotics in this case correspond to weak IV asymptotics as in Staiger and Stock (1997), and model the first-stage parameter as shrinking with the sample size to ensure that it cannot be distinguished from zero with certainty, with $\mathbb{E}_{P_n}[D_i Z_i] = \frac{1}{\sqrt{n}} \pi^*$ for a fixed vector π^* . The \mathbb{R}^k -valued Gaussian process $g(\cdot)$ is linear in θ , and so is fully characterized by its intercept $g(0) = \xi_0$ and slope $\frac{\partial}{\partial \theta} g(\theta) = -\xi_1$, where

$$(\xi_0', \xi_1')' \sim N((\pi^{*'} \theta^*, \pi^{*'})', \Omega_\xi), \quad \Omega_\xi = \text{Var}((Z_i' Y_i, Z_i' D_i)'). \quad (3)$$

Intuitively, ξ_0 corresponds (up to a linear transformation) to the reduced-form coefficient from regressing Y_i on Z_i , while ξ_1 corresponds to the first-stage regression of D_i on Z_i . The two-stage least squares estimator for θ is

$$\hat{\theta} = \underset{\theta \in \Theta}{\text{argmin}} (\xi_0 - \xi_1 \theta)' \Xi (\xi_0 - \xi_1 \theta) = \frac{\xi_1' \Xi \xi_0}{\xi_1' \Xi \xi_1}, \quad (4)$$

for $\Xi = \mathbb{E}[Z_i Z_i']^{-1}$, and the corresponding GMM estimator for $r(\theta)$ under squared error loss (1) is $\delta(g, \Sigma) = r(\hat{\theta})$. \square

Special Case: Finite Θ For our second special case we consider a potentially nonlinear moment condition $\phi(X_i, \theta)$ but restrict the structural parameter space to contain a finite, but potentially very large, number of points $\Theta = \{\theta_1, \dots, \theta_s\}$. While theoretical models in economics are typically written using continuous parameterizations, computational implementation is limited by machine precision, so the case with a finite parameter space Θ is arguably a better description of empirical practice.

Weak-identification asymptotics in this setting correspond to the weak-GMM asymptotics of Stock and Wright (2000), and imply that the mean of the moments is of the same order as sampling uncertainty, $\mathbb{E}_{P_n}[\phi(X_i, \theta)] = \frac{1}{\sqrt{n}} m(\theta) + o\left(\frac{1}{\sqrt{n}}\right)$, so $\mathbb{E}_{P_n}[g_n(\theta)] = m(\theta) + o(1)$. The limit experiment thus corresponds to observing the sk -dimensional normal vector $g = (g(\theta_1)', \dots, g(\theta_s)')' \sim N(m, \Sigma)$ for $m \in \mathbb{R}^{sk}$ and Σ an $(sk) \times (sk)$ matrix. We assume

for this example that Σ has full rank. The GMM estimator $\hat{\theta}$ for θ solves

$$g(\hat{\theta})'\Xi(\hat{\theta})g(\hat{\theta}) = \min\{g(\theta_1)'\Xi(\theta_1)g(\theta_1), \dots, g(\theta_s)'\Xi(\theta_s)g(\theta_s)\},$$

and the GMM estimator for $r(\theta)$ under squared error loss (1) is $\delta^{GMM}(g, \Sigma) = r(\hat{\theta})$. \square

2.1 Parameter Space for the Limit Experiment

To complete our description of the limit experiment (2) we need to specify the parameter space. As in the finite sample problem, we take the parameter space for the structural parameter θ^* to be Θ . We assume that Θ is compact under a topology that renders $\Sigma(\theta, \tilde{\theta})$ continuous on the product space $\Theta \times \Theta$, that $\Sigma(\theta, \theta)$ has full rank for all θ , and that $g(\cdot)$ is continuous almost surely.⁸ Andrews and Mikusheva (2022) shows that the parameter space for the functional parameter m in the limit experiment is related to the reproducing kernel Hilbert space (RKHS) associated with Σ , which we denote by \mathcal{H} .⁹ This is the largest set of mean functions such that we cannot tell with certainty whether a given draw g was generated by $\mathcal{GP}(m, \Sigma)$ or $\mathcal{GP}(0, \Sigma)$ (see e.g. Theorem 54 of Berlinet and Thomas-Agnan, 2004). Since $m=0$ corresponds to the case of complete non-identification of θ^* , \mathcal{H} is thus the largest parameter space for m such that the data never rule out complete identification failure.

For the purposes of the present paper, it is helpful to work with a particular representation of \mathcal{H} . Consider the mean-zero Gaussian process $G \sim \mathcal{GP}(0, \Sigma)$ corresponding to the noise in the moment process, $G = g - m$. Denote by $\mathcal{C}(\Theta, \mathbb{R}^k)$ the space of \mathbb{R}^k -valued continuous functions on Θ with norm $\|f\|_\infty = \max_{j=1, \dots, k} \sup_{\theta \in \Theta} |f_j(\theta)|$ for $f \in \mathcal{C}(\Theta, \mathbb{R}^k)$, and note that under our assumptions $G \in \mathcal{C}(\Theta, \mathbb{R}^k)$ with probability one. Let η_j denote the point-evaluation functional at point θ_j , $\eta_j(f) = f(\theta_j)$, and note that $\mathbb{E}[G(\cdot)\eta_j(G)'] = \Sigma(\cdot, \theta_j)$. If we consider linear combinations of such functionals $\eta(\cdot) = \sum_j \eta_j(\cdot)' a_j$, one can show that $\mathbb{E}[G(\cdot)\eta(G)] = \sum_j \Sigma(\cdot, \theta_j) a_j$ is a function

⁸Almost sure continuity is a mild regularity condition in our setting. As noted in Section 2.1.2 of Van der Vaart and Wellner (1996), almost-sure continuity with respect to a particular Σ -induced seminorm, and total boundedness of Θ in the same seminorm, is necessary for the sample moments to satisfy a uniform central limit theorem, $g_n(\cdot) \Rightarrow g(\cdot) \sim \mathcal{GP}(m, \Sigma)$, while Lemma 1.3.1 in Adler and Taylor (2007) implies that continuity under this seminorm is equivalent to continuity in θ .

⁹For finite sets of vectors $\{a_i\}_{i=1}^s \subset \mathbb{R}^k$ and $\{\theta_i\}_{i=1}^s \subset \Theta$, consider functions of the form $\sum_{i=1}^s \Sigma(\cdot, \theta_i) a_i$, with scalar product $\left\langle \sum_{i=1}^s \Sigma(\cdot, \theta_i) a_i, \sum_{j=1}^{s^*} \Sigma(\cdot, \theta_j^*) b_j \right\rangle_{\mathcal{H}} = \sum_{i=1}^s \sum_{j=1}^{s^*} a_i' \Sigma(\theta_i, \theta_j^*) b_j$. The RKHS \mathcal{H} is the completion of $\{\sum_{i=1}^s \Sigma(\cdot, \theta_i) a_i : a_i \in \mathbb{R}^k, \theta_i \in \Theta, s < \infty\}$ under $\langle \cdot, \cdot \rangle_{\mathcal{H}}$.

in \mathcal{H} and, moreover, that all functions in \mathcal{H} may be written as $\mathbb{E}[G(\cdot)\eta(G)]$ for some η .

Formally, let \mathbb{H} be the space of continuous linear functionals on $\mathcal{C}(\Theta, \mathbb{R}^k)$ with the norm $\|\eta\|^* = \sup_{f \in \mathcal{C}(\Theta, \mathbb{R}^k), \|f\|_\infty \leq 1} |\eta(f)|$. For each $\eta \in \mathbb{H}$ we define the Pettis integral of η as $m_\eta(\cdot) \equiv \mathbb{E}[G(\cdot)\eta(G)]$. The RKHS \mathcal{H} is the image of \mathbb{H} under the Pettis integral.

Lemma 1 *The image of \mathbb{H} under the Pettis integral transformation coincides with the RKHS: $\mathcal{H} = \{m_\eta : \eta \in \mathbb{H}\}$. Furthermore, the transformation is continuous with $\|m_\eta\|_\infty \leq \sigma^2(G)\|\eta\|^*$, where $\sigma^2(G) = \sup_{\|\eta\|^* \leq 1} \mathbb{E}[\eta(G)^2]$ is finite.*

Since the mean function must be zero at the true parameter value, $m(\theta^*) = 0$, we can write the parameter space in the limit experiment as

$$\Gamma = \{(\theta^*, m_\eta) : \theta^* \in \Theta, \eta \in \mathbb{H}, m_\eta(\theta^*) = 0\}.$$

Bounding Identification Strength Our main result concerns parameter spaces that bound the norm of η , which we interpret as a measure of identification strength. To understand this interpretation, consider a restricted parameter space with $\|\eta\|^*$ bounded by a positive constant W ,

$$\Gamma_W = \{(\theta^*, m_\eta) : \theta^* \in \Theta, \eta \in \mathbb{H}, \|\eta\|^* \leq W, m_\eta(\theta^*) = 0\}.$$

At one extreme, if $W = 0$, $\Gamma_0 = \Theta \times \{0\}$ implies that $m(\theta) = 0$ for all θ , so θ^* is completely unidentified. At the other extreme $\Gamma_\infty = \bigcup_W \Gamma_W = \Gamma$, so for unrestricted W we recover the original parameter space Γ . Between these two extremes, Lemma 1 shows that for any $(\theta^*, m) \in \Gamma_W$, $\|m\|_\infty \leq \sigma^2(G)W$. Since we observe only a noisy measure of m , $g(\cdot) \sim \mathcal{GP}(m, \Sigma)$, bounds on $\|m\|_\infty$ limit the ease with which we can distinguish $m(\theta)$ from 0 for any θ value and so limit how informative the data can be about θ^* . Thus, we can interpret Γ_W as a parameter space which imposes a uniform upper bound on the strength of identification.

In infinite-dimensional settings bounds on $\|\eta\|^*$ imply upper, but not in general lower, bounds on $\|m\|_\infty$. Consequently, bounding $\|\eta\|^*$ is more restrictive than bounding $\|m\|_\infty$, so weak identification neighborhoods defined using $\|\eta\|^*$ are “weaker” in that sense. The choice of the norm $\|\cdot\|^*$ as our measure of identification strength is natural, since we show in the proof of our main result that this is the norm that translates changes in the moments $\|g - g'\|_\infty$ to changes in the likelihood. In the finite dimensional case, however, the choice of norm is unimportant since the norms $\|\cdot\|^*$ and $\|\cdot\|_\infty$ are equivalent.

Finite-Dimensional Limit Experiments In many cases of empirical interest the limit experiment is finite-dimensional, in the sense that $g(\cdot)$ can be written as a function of a finite-dimensional normal random vector or, equivalently, that the covariance function Σ has a finite number of nonzero eigenvalues.

Definition 2 *The limit experiment is finite-dimensional if the covariance function Σ has finitely many nonzero eigenvalues.*

Most of our results apply to both finite- and infinite-dimensional limit experiments, but the interpretation of some conditions is simpler in the finite-dimensional case. Finite-dimensional limit experiments can arise in many ways, for instance because the support \mathcal{X} of the data is finite, because the moments are additively or multiplicatively separable in the data, $\phi(X_i, \theta) = \phi_1(X)\phi_2(\theta)$ for $\phi_1(\theta) \in \mathbb{R}^{k \times d}$ and $\phi_2(x) \in \mathbb{R}^d$, or because the parameter space is finite. Whatever the source of finite dimension, our bounds on identification strength are particularly easy to interpret in this case. Specifically, since all norms are equivalent on finite-dimensional spaces, there exists a (Σ -dependent) constant λ such that $\lambda^{-1}\|m_\eta\|_\infty \leq \|\eta\|^* \leq \lambda\|m_\eta\|_\infty$, so bounds on $\|\eta\|^*$ not only imply, but are also implied by, bounds on $\|m\|_\infty$.

Special Case: Linear IV (continued) Recall that in the linear IV model $g(\theta) = \xi_0 - \xi_1\theta$ for (ξ_0, ξ_1) a Gaussian vector in \mathbb{R}^{2k} . This is therefore a finite-dimensional setting. The mean function is $m(\theta) = \pi^*(\theta^* - \theta)$, so bounding $\|m\|_\infty$ is equivalent to bounding the first-stage π^* . Consequently, for π_η^* the first-stage implied by η and $\|\pi_\eta^*\|$ its Euclidean norm, there exists a constant λ^* such that $\lambda^{*-1}\|\pi_\eta^*\| \leq \|\eta\|^* \leq \lambda^*\|\pi_\eta^*\|$, and bounding $\|\eta\|^*$ is equivalent to bounding the first-stage coefficient π^* . \square

Special Case: Finite Θ (continued) In this example the process g reduces to a Gaussian vector in \mathbb{R}^{sk} , so this is again a finite-dimensional case. Thus, there exists a constant λ^* for which $\lambda^{*-1}\|m_\eta\|_\infty \leq \|\eta\|^* \leq \lambda^*\|m_\eta\|_\infty$, and bounding identification strength in terms of $\|\eta\|^*$ is equivalent to bounding $\|m\|_\infty$, the maximal deviation of the moments from zero. \square

3 Admissibility

Recall that the limit experiment corresponds to observing $g \sim \mathcal{GP}(m, \Sigma)$, where $(\theta^*, m) \in \Gamma$. The researcher aims to choose an estimator δ that yields a low risk $\mathbb{E}_m[L(\delta(g, \Sigma), \theta^*)]$ for the loss function L , where since Σ is known in the limit experiment we abbreviate $\delta(g, \Sigma) = \delta(g)$ going forward. Unfortunately there is not a uniformly best estimator in this setting, as minimizing risk at different parameter values $(\theta^*, m), (\theta'^*, m') \in \Gamma$ usually leads to distinct estimators δ and δ' . It is however without loss of performance to limit attention to the set of admissible estimators.

Definition 3 *An estimator δ is dominated on $\tilde{\Gamma} \subseteq \Gamma$ if there exists another estimator δ' such that $\mathbb{E}_m[L(\delta'(g), \theta^*)] \leq \mathbb{E}_m[L(\delta(g), \theta^*)]$ for all $(\theta^*, m) \in \tilde{\Gamma}$, with a strict inequality for some $(\theta^*, m) \in \tilde{\Gamma}$. The estimator δ is admissible on $\tilde{\Gamma}$ if it is not dominated on $\tilde{\Gamma}$.*

An estimator is admissible if its performance, measured in terms of risk, cannot be uniformly improved. Since no admissible estimator dominates any other, selecting from among sets of admissible estimators requires taking a stand on how we value performance over different regions of the parameter space, for instance by specifying a prior and considering Bayes estimators as in Andrews and Mikusheva (2022). In the present paper we set a more modest goal, and aim to provide necessary conditions for admissibility under bounds on identification strength. Our main technical contribution is to establish a close connection between the set of admissible estimators under bounded identification strength and the set of estimators that are uniformly continuous in g .

Definition 4 *An estimator δ is almost-surely uniformly continuous with modulus of continuity $\tilde{\kappa}(\cdot)$, where $\tilde{\kappa}(\cdot)$ is continuous at zero and $\tilde{\kappa}(0) = 0$, if there exist another estimator δ^* such that $\delta(g) = \delta^*(g)$ for almost every g and $\|\delta^*(g) - \delta^*(g')\| \leq \tilde{\kappa}(\|g - g'\|_\infty)$ for all g, g' in the support of the process $\mathcal{GP}(0, \Sigma)$.*

Theorem 1 *Assume that the loss function $L(a, \theta)$ is bounded, continuous and uniformly convex in its first argument. Assume that an estimator δ is admissible on $\tilde{\Gamma}$, where $\tilde{\Gamma} \subseteq \Gamma_W$ for $W < \infty$. Then δ is almost-surely uniformly continuous with modulus of continuity $\tilde{\kappa}(x) = \kappa^{-1}(2\bar{L}W \cdot x)$ for $\bar{L} = \sup_{a, \theta} L(a, \theta)$.*

The proof of Theorem 1 builds on Theorem 2 of Andrews and Mikusheva (2022), which is itself a minor extension of a result from Brown (1986). That result, reproduced in the

appendix for completeness, shows that for convex loss functions admissible estimators must be the (almost everywhere) pointwise limit of Bayes decision rules for finitely-supported priors. Going beyond this result, we then prove that under bounded identification strength small changes in the moments g lead to only small changes in the posterior probability of different θ values.¹⁰ We further prove that for a bounded loss the posterior expected risk function is Lipschitz in g . Uniform convexity of the loss function allows us to translate this last property into uniform continuity of Bayes decision rules. Uniform continuity is preserved by pointwise limits, so the conclusion of the theorem follows.

If we consider the special case of squared error loss (1), the previous result can be strengthened to show that admissible decision rules must be Lipschitz in g .

Definition 5 *An estimator δ is almost-surely Lipschitz with Lipschitz constant K if there exists another estimator δ^* such that $\delta(g) = \delta^*(g)$ for almost every g and $\|\delta^*(g) - \delta^*(g')\| \leq K\|g - g'\|_\infty$ for all g, g' in the support of the process $\mathcal{GP}(0, \Sigma)$.*

Theorem 2 *Assume that an estimator δ is admissible on $\tilde{\Gamma}$, where $\tilde{\Gamma} \subseteq \Gamma_W$ for $W < \infty$ for squared error loss as defined in (1). Then δ is almost-surely Lipschitz with Lipschitz constant $K = \bar{r}\sqrt{p}W$, where $\bar{r} = \sup_\theta \|r(\theta)\|$.*

While Theorems 1 and 2 translate numerical bounds on identification strength to numerical bounds on the modulus of continuity or Lipschitz constant, selecting a value of W for a given application seems unappealing. Similarly, different researchers may prefer different loss functions. We next provide a necessary condition for admissibility under *any* bound on identification strength and *any* uniformly convex loss.

Corollary 1 *If δ is not almost-surely uniformly continuous, then it is inadmissible on $\tilde{\Gamma}$ for all $\tilde{\Gamma}$ with bounded identification strength (that is, all $\tilde{\Gamma}$ with $\tilde{\Gamma} \subseteq \Gamma_W$ for some $W < \infty$) and all bounded, continuous, and uniformly convex loss functions.*

Corollary 1 states that under any bound on the strength of identification, no matter how large, and any uniformly convex loss, admissible estimators are uniformly continuous in

¹⁰The choice of the norm $\|\cdot\|^*$ as our measure of identification strength is important for this step, since it allows us to bound the change in the likelihood between moment realizations g and g' in terms of $\|g - g'\|_\infty$.

the moment process g , so small changes in the realized sample moments (measured in the supremum norm $\|\cdot\|_\infty$) can induce only small changes in the estimate. Hence, if we observe that a given estimator fails to be uniformly continuous, it is unappealing in a very wide variety of decision problems. While uniform continuity may seem a minimal requirement, we show in the next section that GMM estimators do not have this property.

We show in Appendix B that bounded identification strength is crucial for the uniform continuity property. There, we provide an example with an unrestricted parameter space Γ where the limit of Bayes posterior means is a step function, and thus discontinuous.

3.1 Inadmissibility of GMM

While it is known that appropriately constructed GMM estimators are efficient in large samples when the model is strongly identified, this result breaks down under weak identification. We next show that GMM estimators are not generally uniformly continuous, and so are inadmissible under any bound on the strength of identification and for a wide variety of convex loss functions. GMM estimators in the limit experiment take the form

$$\delta^{GMM}(g, \Sigma) = \underset{a \in \mathcal{A}}{\operatorname{argmin}} L(a, \hat{\theta}), \quad \hat{\theta} \in \underset{\theta \in \Theta}{\operatorname{argmin}} g(\theta)' \Xi(\theta) g(\theta), \quad (5)$$

where $\Xi(\theta)$ is a deterministic weight function, corresponding to the probability limit of $\Xi_n(\theta)$. If there are multiple points where the second minimum is achieved, we assume that $\hat{\theta}$ applies some selection rule.

GMM estimators in the limit experiment are invariant to the scale of g .¹¹

Definition 6 *An estimator δ is scale-invariant if $\delta(c \cdot g, \Sigma) = \delta(g, \Sigma)$ for all g and all $c > 0$.*

This scale invariance is important for our purposes, since scale-invariant estimators in our setting are uniformly continuous if and only if they are constant.

Lemma 2 *Let δ be a scale-invariant estimator. If δ is almost-surely uniformly continuous with the modulus of continuity $\tilde{\kappa}(\cdot)$, where $\tilde{\kappa}(\cdot)$ is continuous at zero and $\tilde{\kappa}(0) = 0$, then there exists $a^* \in \mathcal{A}$ such that $\delta(g, \Sigma) = a^*$ almost surely.*

¹¹To be precise, GMM estimators are scale-invariant so long as the rule for selecting from a non-unique argmin is likewise invariant.

GMM estimators δ^{GMM} are scale-invariant and non-constant, so Lemma 2 implies that δ^{GMM} is not uniformly continuous. Hence, by Corollary 1, δ^{GMM} is inadmissible under bounded identification strength. The source of this inadmissibility is intuitive, namely that small changes in data can cause the GMM estimator to jump discontinuously. Our weakly identified setting is important for this result, and GMM estimators are known to be asymptotically admissible in many strongly-identified problems. Appendix C discusses why Lemma 2 does not apply in the strongly identified case.

Special Case: Linear IV (continued) In this example, δ is continuous in $g(\cdot)$ if and only if it is continuous in (ξ_0, ξ_1) , and the two-stage least squared estimator (4) is discontinuous when the first-stage estimate is zero, $\xi_1 = 0$. This is consistent with the intuition that instrumental variables estimation is badly behaved when the instrument is irrelevant.

Interestingly, if we use other instrumental variables estimators (with multiple instruments, $k > 1$) we may encounter additional points of discontinuity. For instance the limited information maximum likelihood estimator corresponds to GMM with weighting matrix $\Xi(\theta) = (\sigma_u^2 - 2\sigma_{uv}\theta + \sigma_v^2\theta^2)^{-1} \mathbb{E}[Z_i Z_i']^{-1}$ for σ_u^2 , σ_v^2 , and σ_{uv} the residual variances and covariance from regressing (Y, D) on Z . The resulting estimator $\hat{\theta}$ is discontinuous at $\xi_1 = 0$, but also at (ξ_0, ξ_1) where (i) the OLS and two-stage least squares estimates coincide and (ii) the reduced-form R^2 coefficient exceeds the first-stage R^2 , which may be interpreted as a sign of model misspecification – see Andrews (2019). \square

4 Alternative Estimators

In the last section we showed that GMM estimators are inadmissible under bounds on identification strength. Unfortunately our proof is non-constructive, and yields no characterization for a dominating estimator. The *reasons* for GMM’s inadmissibility are nonetheless instructive, and suggest a route to more reasonable estimators. The source of GMM’s inadmissibility is that it depends only on the minimizer of the GMM objective. This results in the scale-invariance discussed in the last section which, in turn, implies that the GMM estimate is not continuous in the moments g .

In this section we study two estimators which depend on the moments in a more continuous

way, the first based on bagging or bootstrap aggregation, and the second based on quasi-Bayes. In order to provide constructive results we focus on squared error loss (1), so Bayes estimators correspond to posterior means, and Theorem 2 implies that admissible estimators must be Lipschitz. Both bagged GMM and Quasi-Bayes are continuous in the moments, and both are Lipschitz under additional conditions.

While we show that bagged GMM and quasi-Bayes have better continuity properties than GMM when identification is weak, and existing theoretical results imply that both are efficient when identification is strong, their admissibility under weak identification is an open question. An alternative approach, which guarantees admissibility under weak identification, is to report Bayes posterior means based on full-support priors on Γ_W . To implement this approach in practice, however, one would have to take a stand on the identification strength bound W , which seems unlikely to be appealing. Moreover, in the infinite-dimensional case it is not obvious to us how to construct such priors or compute the resulting posteriors. Hence, while a fully Bayesian approach guarantees admissibility, we expect Quasi-Bayes and bagged GMM to be more attractive options in most applications.

4.1 Bagged GMM

One way to ensure continuity is to directly smooth the GMM estimator. For instance, we can average an estimator across bootstrap draws, yielding a bagged, or bootstrap aggregated, estimator. Bühlmann and Yu (2002) show that bagging can reduce both bias and variance when estimators are unstable, in the sense of being sensitive to small changes in the data. The instability of GMM under weak identification suggests that its performance might also be improved by bagging.

To formally introduce the bagged GMM estimator, again consider the limit experiment where we observe a single draw of the moment process $g \sim \mathcal{GP}(m, \Sigma)$. For an estimator $\delta(g)$ (which need not be GMM) let us draw independent Gaussian noise $\zeta \sim \mathcal{GP}(0, \Sigma)$ and define the bagged version of δ as the average of $\delta(g + \zeta)$ over noise realizations,

$$\delta^B(g) = \mathbb{E}[\delta(g + \zeta) | g].$$

We interpret $\delta^B(g)$ as a bagged estimator because the distribution of $g^* = g + \zeta$ given g is exactly the asymptotic distribution of the moments across bootstrap replications, conditional on the ini-

tial data delivering moments g (see e.g. Section 3.6 and Van der Vaart and Wellner 1996). Hence, $\delta^B(g)$ corresponds to the (asymptotic analog of the) average of $\delta(\cdot)$ across bootstrap draws.

We formalize the connection between bagging and smoothing in our setting by showing that in finite-dimensional limit experiments all bagged estimators are Lipschitz.

Proposition 1 *If the limit experiment is finite-dimensional, then for any estimator $\delta(\cdot)$ with range contained in \mathcal{A} the bagged estimator $\delta^B(g)$ is Lipschitz.*

Proposition 1 implies, in particular, that for $\delta^{GMM}(g)$ the GMM estimator as defined in (5), the bagged GMM estimator $\delta^{BGMM}(g) = \mathbb{E}[\delta^{GMM}(g+\zeta)|g]$ satisfies the global continuity property required by Corollary 1. On an intuitive level this estimator “averages out” the discontinuities of GMM, resulting in a Lipschitz estimator.¹² A practical limitation of the bagged GMM estimator is that it requires repeatedly minimizing the GMM objective function to compute $\delta^{GMM}(g+\zeta)$. In settings where minimization is difficult this can make computing the bagged estimator costly. If, on the other hand, a researcher is already using the bootstrap then the incremental cost of computing the bagged estimator is essentially zero.

To illustrate how bagging smooths the GMM estimator, we return to our examples.

Special Case: Finite Θ (continued) In this special case the bagged GMM estimator for $r(\theta)$ can be written as a weighted average across the possible values of $r(\theta)$. In particular, for $\hat{\theta}(g+\zeta)$ the GMM estimate based on moments $g+\zeta$, we can write the bagged GMM estimator as $\delta^{BGMM}(g) = \sum_{\theta \in \Theta} r(\theta) Pr\{\hat{\theta}(g+\zeta) = \theta | g\}$. Note, however, that the probability $Pr\{\hat{\theta}(g+\zeta) = \theta | g\}$ is simply the probability (conditional on g) that the collection of correlated random variables $\{(g(\theta) + \zeta(\theta))' \Xi(\theta)(g(\theta) + \zeta(\theta)) : \theta \in \Theta\}$ achieves its minimum at a particular θ . One can show that these probabilities are Lipschitz in g , from which it is immediate that $\delta^{BGMM}(g)$ is Lipschitz as well. \square

Special Case: Linear IV (continued) Recall that the limit experiment for linear IV reduces to observing the jointly normal random vector $\xi = (\xi'_0, \xi'_1)' \sim N((\pi^{*'}\theta^*, \pi^{*'})', \Omega_\xi)$, corresponding to the intercept and (negative) slope of g . We can correspondingly define

¹²The same is true for the whole family of estimators $\delta^\tau_{BGMM}(g) = \mathbb{E}[\delta(g+\tau\cdot\zeta)|g]$ for $\tau > 0$. However, values $\tau \neq 1$ complicate the bootstrap interpretation, as well as the Bayesian interpretation discussed below, so we focus on the case with $\tau = 1$.

$\nu = (\nu_0', \nu_1') \sim N(0, \Omega_\xi)$ as the intercept and (negative) slope of $\zeta \sim \mathcal{GP}(0, \Sigma)$. The two-stage least squares estimate of θ for moment realization $g + \zeta$ is thus

$$\hat{\theta}(\xi + \nu) = \frac{(\xi_1 + \nu_1)' \Xi (\xi_0 + \nu_0)}{(\xi_1 + \nu_1)' \Xi (\xi_1 + \nu_1)},$$

while the two-stage least squares estimate of $r(\theta)$ is $r(\hat{\theta}(\xi + \nu))$. The bagged two-stage least squares estimator is then

$$\delta^{BTSLS}(g) = E \left[r(\hat{\theta}(\xi + \nu)) | g \right] = \int r(\hat{\theta}(\xi + \nu)) f(\nu) d\nu = \int r(\hat{\theta}(\nu)) f(\nu - \xi) d\nu,$$

for $f(\cdot)$ the $N(0, \Omega_\xi)$ density. Note, however, that $f(\cdot)$ is Lipschitz, so since $r(\hat{\theta}(\cdot))$ is bounded, $\delta^{BTSLS}(g)$ is likewise Lipschitz. \square

The bagged GMM estimator also has a Bayesian interpretation. In the finite-dimensional case the mean function m is simply a finite-dimensional vector. For a flat (improper) prior on m , the posterior distribution on m after observing g corresponds to a $\mathcal{GP}(g, \Sigma)$ distribution, which is precisely the distribution of $g + \zeta$ conditional on g . Note, however, that the flat prior on m allows the possibility that $m(\theta) \neq 0$ for all θ and so does not impose correct specification of the GMM model. This raises the question of how to define the object of interest when GMM is misspecified. One approach is to focus on the GMM estimand or pseudo-true value $\theta^*(m) = \operatorname{argmin}_{\theta \in \Theta} m(\theta)' \Xi(\theta) m(\theta)$, which minimizes the population analog of the GMM objective. The bagged GMM estimator then corresponds to the posterior mean of $r(\theta^*(m))$ under the flat prior.

4.2 Quasi-Bayes

We could also take a more overtly Bayesian approach. For a prior π on Θ , the quasi-Bayes posterior mean of $r(\theta)$ in the limit experiment is

$$\delta_\pi^{QB}(g) = \int r(\theta) \frac{\exp(-\frac{1}{2}Q(\theta|g))}{\int \exp(-\frac{1}{2}Q(\theta|g)) d\pi(\theta)} d\pi(\theta), \quad (6)$$

where $Q(\theta|g) = g(\theta)' \Sigma(\theta, \theta)^{-1} g(\theta)$ is the continuously updated GMM objective function. This estimator corresponds to the posterior mean after updating $\pi(\theta)$ based on “log-likelihood” $-\frac{1}{2}Q(\theta|g)$, and was initially suggested by Chernozhukov and Hong (2003). Since $Q(\theta|g)$ is not in general the likelihood of the researcher’s model the interpretation of $\delta_\pi^{QB}(g)$ from a strict Bayesian perspective may not be obvious, but Andrews and Mikusheva (2022) showed

that this estimator arises as the limit of a sequence of Bayes posterior means for proper priors. Unlike the priors underlying bagged GMM, the priors that give rise to quasi-Bayes impose correct specification of the GMM model. See Chernozhukov and Hong (2003) and Andrews and Mikusheva (2022) for further discussion, as well as asymptotic results under both strong and weak identification.

A key feature of the quasi-Bayes approach for our purposes is that it takes a weighted average of $r(\theta)$ over the parameter space Θ , weighting by $\frac{\exp(-\frac{1}{2}Q(\theta|g))}{\int \exp(-\frac{1}{2}Q(\theta|g))d\pi(\theta)}$. It follows from this structure that quasi-Bayes is Lipschitz in the GMM objective function.

Lemma 3 *Quasi-Bayes is Lipschitz in the GMM objective function Q :*

$$\|\delta^{QB}(g) - \delta^{QB}(g')\| \leq K \|Q(\cdot|g) - Q(\cdot|g')\|_{\infty},$$

where $K = \frac{1}{2}\bar{r}\sqrt{p}$.

Unfortunately, $Q(\cdot|g)$ is continuous but not Lipschitz in the moments g . Consequently, the Lipschitz continuity required by Theorem 2 does not follow from Lemma 3. Indeed, while quasi-Bayes is continuous in g , it is not in general Lipschitz.¹³

Special Case: Finite Θ (continued) Suppose that the parameter space consists of just two points, $\Theta = \{0,1\}$, that we have a one-dimensional moment condition ($k=1$), and that $\Sigma = I_2$. Consider the quasi-Bayes estimator using a prior π that puts weight $\frac{1}{2}$ on each parameter value. The quasi-Bayes estimator of θ is

$$\delta_{\pi}^{QB}(g) = \frac{\exp(-\frac{1}{2}g(1)^2)}{\exp(-\frac{1}{2}g(0)^2) + \exp(-\frac{1}{2}g(1)^2)} = \frac{1}{1 + \exp(\frac{1}{2}g(1)^2 - \frac{1}{2}g(0)^2)}.$$

While this estimator is differentiable in $(g(0), g(1))$, it is not Lipschitz. Indeed,

$$\left. \frac{\partial \delta_{\pi}^{QB}(g)}{\partial g(0)} \right|_{g(0)=g(1)} = \frac{g(0)}{4},$$

which exceeds any finite constant for large values of both $g(0)$ and $g(1)$. Intuitively, when both $g(0)$ and $g(1)$ are large, $\delta_{\pi}^{QB}(g)$ behaves like $\delta(g) = \operatorname{argmin}_{\theta \in \{0,1\}} g(\theta)^2$. \square

¹³Since Andrews and Mikusheva (2022) showed that quasi-Bayes emerges as the limit of a sequence of Bayes posterior means, it may be surprising that it does not satisfy a necessary condition for admissibility under bounded identification strength. The sequences of priors underlying quasi-Bayes imply, however, that $\|m\|_{\infty} \rightarrow_p \infty$ and so correspond to the case of unbounded identification strength.

An interesting feature of this example is that the non-Lipschitz behavior of the quasi-Bayes estimator appears for realizations of g which suggest misspecification of the model. Specifically, the GMM model with parameter space $\Theta = \{0,1\}$ requires that either $m(0) = 0$ or $m(1) = 0$. Hence, under the model the distribution of $\min_{\theta \in \{0,1\}} g(\theta)^2$ is bounded by a χ_1^2 , and data realizations with both $g(0)$ and $g(1)$ large are highly unlikely. This suggests that if we limit attention to data realizations which appear consistent with the model the quasi-Bayes estimator may be Lipschitz. The following result shows that this is the case provided Θ is finite and π has full support.

Proposition 2 *Assume that the parameter space is finite, $|\Theta| < \infty$. For $C > 0$ define*

$$\mathcal{G}_C = \left\{ g : \inf_{\theta \in \Theta} Q(\theta|g) \leq C \right\}.$$

If π has support Θ , then the quasi-Bayes estimator $\delta_\pi^{QB}(g)$ is Lipschitz in g on \mathcal{G}_C .

The minimized GMM objective $Q(\theta|g)$ is often termed a J -statistic, and researchers commonly reject correct specification of the model when this statistic exceeds a threshold. Under the assumption of correct specification we have $\lim_{C \rightarrow \infty} \inf_{\gamma \in \Gamma} \mathbb{P}_\gamma \{g \in \mathcal{G}_C\} = 1$, so moment realizations $g \notin \mathcal{G}_C$ have low probability under all data generating processes consistent with the GMM model. Hence, for finite Θ , quasi-Bayes is Lipschitz over data realizations such that the GMM model is not rejected.

4.2.1 Default Priors

To apply the quasi-Bayes approach (unlike for bagged GMM) we must explicitly specify a prior $\pi(\theta)$. From a subjective Bayesian perspective the prior π on the GMM parameter θ should reflect the researcher’s beliefs about the structural parameters in a given application. In practice, however, subjective priors can be difficult to specify or controversial, and it may be helpful to have default options.

One common default is to use a flat prior, with π proportional to Lebesgue measure, where our assumption that Θ is compact ensures that this prior has finite mass. As has previously been observed in other contexts, however, “flatness” of a prior is parameterization-specific, and the use of flat priors can lead two researchers, estimating the same model on the same data but with different parameterizations, to different posteriors. This motivates us to seek parameterization-invariant default priors. Since the covariance function Σ , and its

domain $\Theta \times \Theta$, are known in the limit experiment they can be used to inform such a prior. Correspondingly, we define a default prior as a Σ -dependent probability measure on Θ , $\pi(\cdot; \Sigma)$.

To formally define reparameterization-invariance, let Ψ be a compact set and let $\vartheta: \Theta \rightarrow \Psi$ be a diffeomorphism between Θ and Ψ , corresponding to a reparameterization $\psi = \vartheta(\theta)$ of the model. This implies reparameterized moments $h(\cdot) = g(\vartheta^{-1}(\cdot))$ defined on Ψ , where by construction $h(\cdot) \sim \mathcal{GP}(m_h, \Sigma_h)$ for

$$m_h(\psi) = m(\vartheta^{-1}(\psi)), \Sigma_h(\psi_1, \psi_2) = \Sigma(\vartheta^{-1}(\psi_1), \vartheta^{-1}(\psi_2)).$$

We call a rule for constructing a prior invariant to re-parameterization if the pushforward of $\pi(\cdot; \Sigma)$ under $\vartheta(\cdot)$ is equal to $\pi(\cdot; \Sigma_h)$ for all re-parametrizations $\vartheta(\cdot)$, so the default delivers the same prior (and thus posterior) distribution regardless of the parameterization. The flat prior is invariant when ϑ is linear, but not generally otherwise.

The structure of the GMM model is also preserved under linear transformations of the moments. Specifically, let $B: \Theta \rightarrow \mathcal{B}$ be a differentiable function from Θ to the set \mathcal{B} of full-rank $k \times k$ matrices. We can define a new moment process $h(\theta) = B(\theta)g(\theta)$, where by construction $h(\cdot) \sim \mathcal{GP}(m_h, \Sigma_h)$ for

$$m_h(\theta) = B(\theta)m(\theta), \Sigma_h(\theta_1, \theta_2) = B(\theta_1)\Sigma(\theta_1, \theta_2)B(\theta_2)'$$

The moments h and g are one-to-one transformations of each other, and imply the same value for the continuously updated GMM objective. Hence, it is again natural to require that a default prior be invariant to such transformations, with $\pi(\cdot; \Sigma_h) = \pi(\cdot; \Sigma)$.

In parametric models the desire for a parameterization-invariant default prior has led to the use of the Jeffreys (1946) prior, which is usually defined to be proportional to $|i(\theta)|^{1/2}$, for $|i(\theta)|$ the determinant of the Fisher information. Our suggested default prior is based on an analogous idea. Specifically, recall that the Fisher information is equal to the variance of the score, that is, the gradient of the log likelihood. We might analogously try to form a default prior based on the variance of the gradient of the GMM objective, $\text{Var}(\frac{\partial}{\partial \theta} Q(\theta|g))$. Unfortunately, however, $\text{Var}(\frac{\partial}{\partial \theta} Q(\theta|g))$ depends on the mean function m , and so is unknown in general. To construct a feasible default prior, we instead consider the variance of the continuously updated GMM objective in the fully-unidentified case, $i(\theta; \Sigma) = \text{Var}(\frac{\partial}{\partial \theta} Q(\theta|G))$

for $G \sim \mathcal{GP}(0, \Sigma)$. The entries of $i(\theta; \Sigma)$ take the form

$$i_{jl}(\theta; \Sigma) = \text{tr} \left(\Sigma^{-1}(\theta, \theta) \left\{ \frac{\partial^2 \Sigma(\theta, \tilde{\theta})}{\partial \theta_j \partial \tilde{\theta}_l} - \frac{\partial \Sigma(\theta, \tilde{\theta})}{\partial \theta_j} \cdot \Sigma^{-1}(\theta, \theta) \frac{\partial \Sigma(\tilde{\theta}, \theta)}{\partial \tilde{\theta}_l} \right\} \Big|_{\tilde{\theta}=\theta} \right) \quad (7)$$

and so can be computed from Σ . Since the continuously updated GMM objective is unchanged by linear transformations of the moments, $i_{jl}(\theta; \Sigma)$ is also unchanged. Moreover, the same calculations which prove the invariance of Jeffreys prior for parametric models show that the default prior proportional to the square root of the determinant of $i(\theta; \Sigma)$,

$$\pi(\cdot; \Sigma) \propto |i(\cdot; \Sigma)|^{1/2} \quad (8)$$

is likewise invariant to reparameterization. Hence, the default prior (8) is invariant to both reparameterization and linear transformations of the moments, as desired.¹⁴

While we motivated our default prior (8) by analogy to Jeffreys prior, it also has a more direct interpretation. Since $G = g - m$ corresponds to the noise component of the GMM moments, the score $\frac{\partial}{\partial \theta} Q(\theta|G)$ measures the speed with which the noise in the moments changes at θ . One can show, however, that $E[\frac{\partial}{\partial \theta} Q(\cdot|G)] \equiv 0$, so $\text{Var}(\frac{\partial}{\partial \theta} Q(\theta|G)) = E[\frac{\partial}{\partial \theta} Q(\theta|G) \frac{\partial}{\partial \theta} Q(\theta|G)']$ measures the average (squared) magnitude of the score. Hence, the default prior (8) assigns more mass to regions of the parameter space where the noise component of the moments tends to change quickly in θ , and less to regions where the noise tends to change slowly.

Special Case: Linear IV (continued) To explore the implications of our default prior in this example, let us partition the $2k \times 2k$ variance matrix into four $k \times k$ sub-matrices Ω_{jl} for $j, l \in \{0, 1\}$, where $\Omega_{jl} = \text{Cov}(\xi_j, \xi_l)$. Under this notation $i(\theta; \Sigma)$ is equal to the trace of

$$\begin{aligned} & (\Omega_{00} - (\Omega_{10} + \Omega_{01})\theta + \Omega_{11}\theta^2)^{-1} \times \\ & \left(\Omega_{11} - (\Omega_{01} - \Omega_{11}\theta)(\Omega_{00} - (\Omega_{10} + \Omega_{01})\theta + \Omega_{11}\theta^2)^{-1}(\Omega_{10} - \Omega_{11}\theta) \right), \end{aligned}$$

which can also be written as the relative variance:

$$\text{Var}(g(\theta))^{-1} \text{Var}(\xi_1|g(\theta)).$$

Thus, the default prior (8) favors parameter values where (i) the GMM moments predict the first-stage poorly (so $\text{Var}(\xi_1|g(\theta))$ is large) and (ii) the GMM moments themselves are not

¹⁴While we focus on the default prior (8), other invariant priors exist. For instance, for scalar θ the prior proportional to $|\Sigma(\theta, \theta)|^{-\frac{1}{2}} \left| \frac{\partial^2}{\partial \theta \partial \tilde{\theta}} \Sigma(\theta, \tilde{\theta}) - \frac{\partial}{\partial \theta} \Sigma(\theta, \tilde{\theta}) \Sigma(\theta, \theta)^{-1} \frac{\partial}{\partial \tilde{\theta}} \Sigma(\theta, \tilde{\theta}) \right|_{\tilde{\theta}=\theta}^{\frac{1}{2}}$ is also invariant.

too noisy (so $\text{Var}(g(\theta))$ is small). Since $\text{Var}(\xi_1|g(\theta)) \rightarrow 0$ and $\text{Var}(g(\theta)) \rightarrow \infty$ as $|\theta| \rightarrow \infty$, we see that the default prior density (8) converges to zero for θ large.

We obtain further simplifications when the reduced-form and first-stage errors are homoskedastic. In this case the matrices Ω_{jl} are proportional to each other,

$$\Omega_\xi = \begin{pmatrix} \sigma_u^2 & \sigma_{uv} \\ \sigma_{uv} & \sigma_v^2 \end{pmatrix} \otimes \tilde{\Omega},$$

for $\tilde{\Omega}$ a $k \times k$ matrix, and $\sigma_u^2, \sigma_v^2, \sigma_{uv}$ again the variances and covariance of the reduced-form and first-stage errors. In this case some algebra shows that our default prior is equal to a Cauchy distribution centered at $\frac{\sigma_{uv}}{\sigma_v^2}$. One can show, however, that $\frac{\sigma_{uv}}{\sigma_v^2}$ corresponds to the probability limit of the OLS estimator under weak-instrument asymptotics. Hence in the homoskedastic case our default prior corresponds to a Cauchy distribution centered at OLS. \square

5 Linear IV Simulations

While our theoretical results show that GMM estimators are dominated under bounds on identification strength, they do not imply that GMM is dominated by either bagged GMM or quasi-Bayes. Relative performance of these estimators in applications is thus an open question. We explore this comparison in the context of linear IV, using simulation designs based on Andrews et al. (2019). Andrews et al. (2019) calibrated simulations based on all instrumental variables specifications published in the American Economic Review from 2014 to 2018 for which sufficient information is available to estimate the variance matrix Ω_ξ in (3), yielding 124 specifications. We follow their simulation designs, and draw data from the normal model (3) with π^* equal to the first-stage estimate in the Andrews et al. (2019) data and θ^* equal to the two-stage least squares estimate. We consider six different estimators. The first two are GMM, specifically two-stage least squares, which as noted above corresponds to GMM with weighting matrix $\Xi(\theta) = E[Z_i Z_i']^{-1}$, and continuously updated GMM, which corresponds to GMM with weighting matrix $\Xi(\theta) = \Sigma(\theta)^{-1}$. We next report bagged versions of each GMM estimator. Finally, we report two quasi-Bayes estimators, the first using a flat prior $\pi(\theta) \propto 1$ and the second using the invariant default prior (8).

Our theoretical results require the loss function to be bounded. In the context of the linear IV model, this means that our results do not apply to unconstrained estimation of θ

under squared error loss. We address this in two ways, first considering estimation of θ under bounds on the parameter space and then considering estimation of the correlation between the structural and first-stage errors, which is a naturally bounded quantity.

In most economic applications of IV methods there are context-specific bounds on the range of plausible parameter values, for instance bounds beyond which a researcher would conclude that there is a problem with their data or their instrument. Motivated by this observation, we take the parameter space in specification s to be $\Theta_s = \left[\pm 20 \left| \frac{\sigma_{uv,s}}{\sigma_{v,s}^2} \right| \right]$ for $\sigma_{uv,s}$ the covariance of the reduced-form and first-stage errors in specification s , and $\sigma_{v,s}^2$ the variance of the first-stage error. As noted above $\frac{\sigma_{uv}}{\sigma_v^2}$ corresponds to the probability limit of the least squares estimator under weak instrument asymptotics, so this parameter space restricts the size IV coefficient to be no more than 20 times the size of the least squares coefficient. Nine of the just-identified specifications in the Andrews et al. (2019) data do not report estimates for $(\sigma_{uv}, \sigma_v^2)$, because replication data were not publicly available but Ω_ξ could be estimated based on published results. In these cases we set σ_{uv} equal to covariance of the reduced-form and first-stage estimates, and σ_v^2 equal to the variance of the first-stage estimate. Consistent with our motivation for restricted the parameter space, Θ_s contains the true IV coefficient in all but three of the 124 Andrews et al. (2019) specifications, so we limit attention to these 121 specifications for our analysis. We approximate the bagged estimators by averages over 400 bootstrap draws, and report results based on 10,000 simulation draws.

The restriction to a bounded parameter space is substantively important for the results we report below. Absent a restriction on the parameter space, the two-stage least squares estimator for the IV coefficient only has finite moments up to the degree of over-identification. Correspondingly, for an unrestricted parameter space the bagged two-stage least squares estimate is only defined for $k \geq 2$. The situation is, if anything, worse for the other estimators we consider: for instance, in linear IV models with homoskedastic errors the continuously updating GMM estimator reduces to the limited information maximum likelihood estimator, which lacks even a first moment. Our restriction to a bounded parameter space ensures that all estimators and moments are well-defined and finite, but for those cases where the restriction is important for e.g. existence of a given moment, the results necessarily depend on the choice of bounds.

Motivated by this sensitivity, we report three additional sets of results. In the main text we report results estimating the correlation between the structural and first-stage errors, which can

be shown to equal $r(\theta) = \frac{\sigma_{uv} - \theta\sigma_v^2}{\sigma_v\sqrt{\sigma_u^2 - 2\theta\sigma_{uv} + \theta^2\sigma_v^2}}$. This correlation measures the degree of endogeneity, and so may be of interest in its own right. Moreover, as recently highlighted by Angrist and Kolesár (2022), conventional confidence intervals are reliable in just-identified IV settings, even with weak instruments, so long as this correlation coefficient is not too large. Hence, we might also want to know $r(\theta)$ for that reason. Importantly for our current purposes, $r(\theta)$ is bounded by construction.

In the appendix we report results where we vary the definition of the parameter space, considering $\Theta_s = \left[\pm 40 \left| \frac{\sigma_{uv,s}}{\sigma_{v,s}^2} \right| \right]$ and $\Theta_s = \left[\pm 60 \left| \frac{\sigma_{uv,s}}{\sigma_{v,s}^2} \right| \right]$. Widening the bounds increases the errors of the IV coefficient estimates from all approaches, but the ordering of estimators by average performance is unchanged. As expected, the change in the bounds has almost no effect on the correlation coefficient estimates. Finally we report results, for two-stage least squares and bagged two-stage least squares only, which drop the bounds on the parameter space entirely. To ensure that the mean squared error is well-defined these results restrict attention to specifications with at least three instruments, $k \geq 3$. The performance gaps between these estimators are small in our $k \geq 3$ specifications with bounded parameter spaces, and remain small when we drop the bounds.

Results for IV Coefficient Table 1 reports our findings for the IV coefficient. For each specification s we consider the root mean squared error for each estimator, normalized by the two-stage least squares standard error σ_s^* in the Andrews et al. (2019) data to account for differences in units, $\sqrt{\mathbb{E}_s[(\delta(g) - \theta_s^*)^2]} / \sigma_s^*$, where $\mathbb{E}_s[\cdot]$ denotes the expectation in specification s . We report the average of this ratio for each estimator across four different categories based on the effective first-stage F statistic of Montiel-Olea and Pflueger (2013). The effective F statistic, which we denote by F , is a measure of instrument strength and in the just-identified case is equal to the squared t-statistic for testing $\pi^* = 0$, $\xi_1^2 / \text{Var}(\xi_1)$. See Montiel-Olea and Pflueger (2013) for details and motivation for this statistic. To complement these results, Figure 1 plots the root mean squared error for each alternative estimator, relative to its GMM counterpart, against the average effective F statistic $\mathbb{E}_s[F]$, limiting attention to specifications where $\mathbb{E}_s[F] \leq 50$ for visibility. To show differences based on the number of instruments we plot just-identified ($k=1$) specifications in blue, and over-identified ($k \geq 2$) specifications in black. For a more detailed picture of how performance varies with the number of instruments, Table 3 in the appendix reports average results when we bin specifications by the number of instruments.

A number of patterns emerge in Table 1 and Figure 1. First, two-stage least squares outperforms continuously updating GMM everywhere except the strongest identification category. The bagged GMM estimators each outperform their standard GMM analogs in most cases. Specifically, these estimators show substantial improvements in the category where identification is weakest ($\mathbb{E}_s[F] \leq 10$), a smaller improvement in the next-weakest category ($10 < \mathbb{E}_s[F] \leq 20$), and either a minimal improvement or a small deterioration in the second-strongest category ($20 < \mathbb{E}_s[F] \leq 50$). Quasi-Bayes with a flat prior underperforms all the other estimators, while quasi-Bayes with the invariant prior outperforms both GMM estimators except in the second-strongest category ($20 < \mathbb{E}_s[F] \leq 50$). The performance gap between the two quasi-Bayes approaches demonstrates the influence of the prior, and highlights that the greater smoothness of quasi-Bayes as a function of the moments does not guarantee improved performance for all priors. Finally, all estimators show very similar performance in the strongest category ($50 < \mathbb{E}_s[F]$), though two-stage least squares and bagged two-stage least squares are known to be inefficient under strong identification. Nonetheless, the estimator with the best average performance overall is bagged two-stage least squares, followed by bagged continuously updating GMM.

Results for Correlation Coefficient Table 2 reports our findings for the correlation coefficient $r(\theta) = \frac{\sigma_{uv} - \theta\sigma_v^2}{\sigma_v\sqrt{\sigma_u^2 - 2\theta\sigma_{uv} + \theta^2\sigma_v^2}}$. We again report the root mean squared error normalized by the delta-method standard error $\sigma_{r,s}^*$ for $r(\theta)$ in the Andrews et al. (2019) data, $\sqrt{\mathbb{E}_s[(\delta(g) - r(\theta_s^*))^2]}/\sigma_{r,s}^*$, and bin specifications based on the average effective first-stage F statistic. Figure 2 plots the root mean squared error for each alternative estimator, relative to its GMM counterpart, against the average effective F statistic $\mathbb{E}_s[F]$. Table 4 in the appendix reports average results when we bin specifications by the number of instruments.

The results for estimating the correlation coefficient are broadly consistent with those for the IV coefficient. In particular, two-stage least squares largely outperforms continuously updating GMM, and each bagged estimator substantially outperforms its GMM counterpart in the weakest category, with smaller gain in the stronger categories. Quasi-Bayes with a flat prior again underperforms the other estimators considered. One difference with the IV coefficient results is that quasi-Bayes with our invariant prior now slightly underperforms relative to two-stage least squares. All estimators again behave very similarly in the strongest specifications ($50 < \mathbb{E}_s[F]$). The best-performing estimator on average is bagged continuously

	$\mathbb{E}_s[F] \leq 10$	$10 < \mathbb{E}_s[F] \leq 20$	$20 < \mathbb{E}_s[F] \leq 50$	$50 < \mathbb{E}_s[F]$
Two-Stage Least Squares	1.37	1.20	1.02	1.00
Continuously Updating GMM	1.55	1.24	1.03	0.99
Bagged Two-Stage Least Squares	1.03	1.09	1.01	0.99
Bagged Continuously Updating GMM	1.04	1.12	1.05	0.99
Quasi-Bayes, Flat Prior	1.51	1.61	1.10	1.00
Quasi-Bayes, Invariant Prior	1.18	1.10	1.03	0.99
Number of Specifications	56	28	19	18

Table 1: Performance of IV coefficient estimators in Andrews et al. (2019) specifications. Entries correspond to the root mean squared error normalized by the standard error in the Andrews et al. (2019) data, $\sqrt{\mathbb{E}_s[(\delta(g) - \theta_s^*)^2]} / \sigma_s^*$, averaged across specifications. Columns correspond to ranges of values for the average effective first-stage F statistic of Montiel-Olea and Pflueger (2013).

updating GMM, followed by bagged two-stage least squares.

Overall, these simulation results show that bagged GMM estimators often outperform their standard GMM counterparts in the settings we consider, through neither dominates the other. Quasi-Bayes with a flat prior performs quite poorly, while quasi-Bayes with the invariant prior (8) is more competitive.

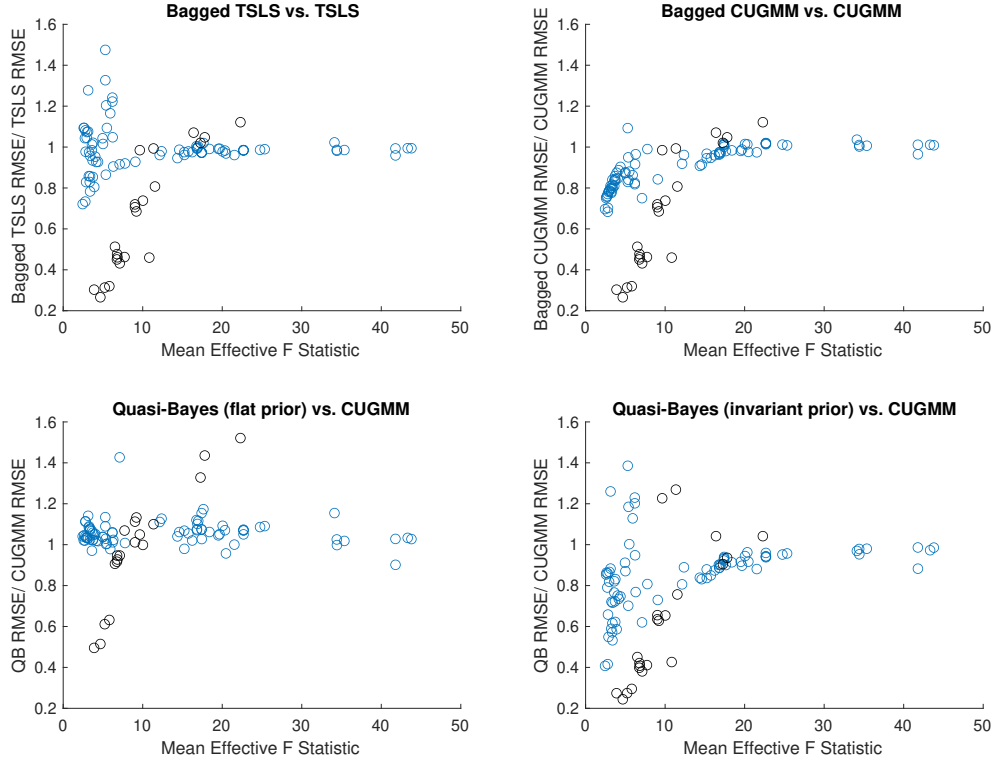


Figure 1: RMSE comparisons for IV coefficient estimators in Andrews et al. (2019) specifications. Each point corresponds to one of the Andrews et al. (2019) specifications. The vertical axis measures the ratio of root mean squared error for the alternative estimator compared to the GMM estimator for the IV coefficient, $\sqrt{\mathbb{E}_s[(\delta(g) - \theta_s^*)^2]} / \sqrt{\mathbb{E}_s[(\delta^{GMM}(g) - \theta_s^*)^2]}$. So, for instance, a value of 0.8 means the RMSE for the alternative estimator is 20% lower. The horizontal axis shows the average effective first-stage F statistic of Montiel-Olea and Pflueger (2013), $\mathbb{E}_s[F]$. We limit attention to specifications with $\mathbb{E}_s[F] \leq 50$ for visibility. Blue dots correspond to just-identified ($k=1$) specifications, while black dots correspond to over-identified ($k \geq 2$) specifications.

	$\mathbb{E}_s[F] \leq 10$	$10 < \mathbb{E}_s[F] \leq 20$	$20 < \mathbb{E}_s[F] \leq 50$	$50 < \mathbb{E}_s[F]$
Two-Stage Least Squares	1.09	1.10	1.01	0.99
Continuously Updating GMM	1.21	1.13	1.02	0.99
Bagged Two-Stage Least Squares	1.03	1.07	1.00	0.99
Bagged Continuously Updating GMM	0.99	1.09	1.02	0.99
Quasi-Bayes, Flat Prior	1.19	1.45	1.07	0.99
Quasi-Bayes, Invariant Prior	1.10	1.11	1.01	0.99
Number of Specifications	56	28	19	18

Table 2: Performance of correlation coefficient estimators in Andrews et al. (2019) specifications. Entries correspond to the root mean squared error normalized by the standard error in the Andrews et al. (2019) data, $\sqrt{\mathbb{E}_s[(\delta(g) - r(\theta_s^*))^2]}/\sigma_{r,s}^*$, averaged across specifications. Columns correspond to ranges of values for the average effective first-stage F statistic of Montiel-Olea and Pflueger (2013).

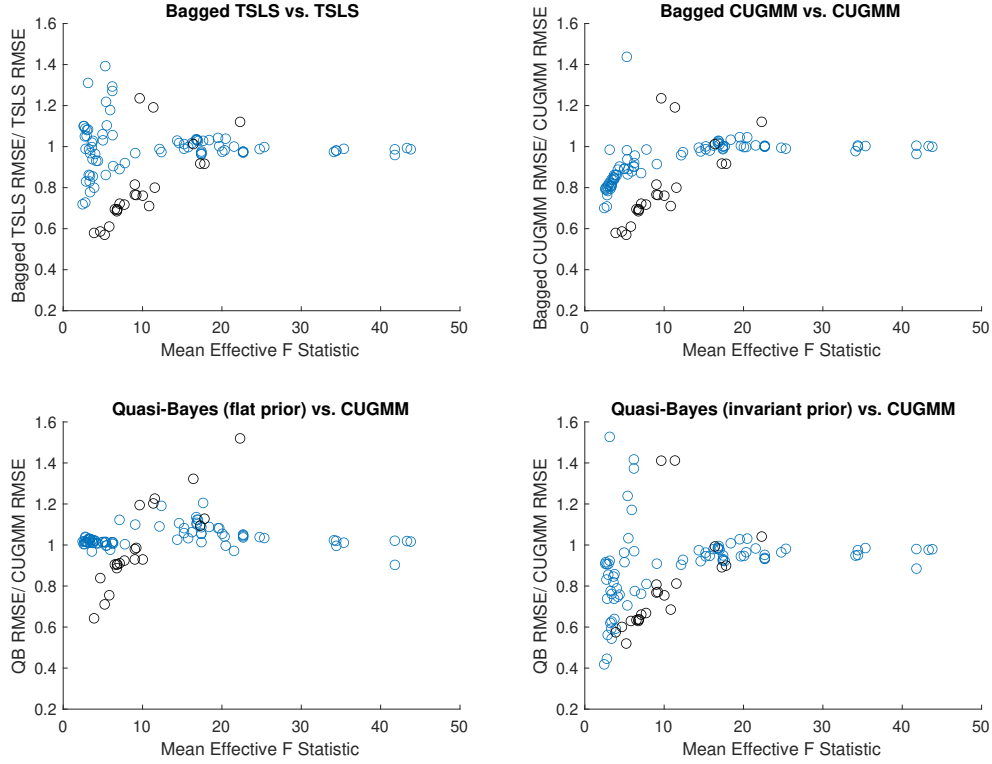


Figure 2: RMSE comparisons for correlation coefficient estimators in Andrews et al. (2019) specifications. Each point corresponds to one of the Andrews et al. (2019) specifications. The vertical axis measures the ratio of root mean squared error for the alternative estimator compared to the GMM estimator for the correlation coefficient, $\sqrt{\mathbb{E}_s[(\delta(g) - r(\theta_s^*))^2] / \mathbb{E}_s[(\delta^{GMM}(g) - r(\theta_s^*))^2]}$. So, for instance, a value of 0.8 means the RMSE for the alternative estimator is 20% lower. The horizontal axis shows the average effective first-stage F statistic of Montiel-Olea and Pflueger (2013), $\mathbb{E}_s[F]$. We limit attention to specifications with $\mathbb{E}_s[F] \leq 50$ for visibility. Blue dots correspond to just-identified ($k = 1$) specifications, while black dots correspond to over-identified ($k \geq 2$) specifications.

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A Proofs

Proof of Lemma 1 If $k=1$, the result is immediate from Theorem 2.1 of van der Vaart and van Zanten (2008). We are left to prove it for $k>1$. Define an augmented parameter space $\Theta^* = \Theta \times V$, where $V = \{v \in \mathbb{R}^k : \|v\|_1 = 1\}$ and consider a Gaussian process $g^*(\cdot)$ defined on Θ^* as $g^*(\theta, v) = v'g(\theta)$. Note that the process g^* , its mean $m^*(\theta, v)$ and its covariance function $\Sigma^*(\theta, v, \tilde{\theta}, \tilde{v})$ are one-to-one transformations of g , m , and Σ . For \mathcal{H}^* the RKHS associated with

Σ^* , \mathcal{H}^* is isometric to \mathcal{H} . Indeed, for $m^* \in \mathcal{H}^*$:

$$m^*(\theta, v) = \sum \alpha_i \Sigma^*(\theta_i, v_i, \theta, v) = \left(\sum \alpha_i v_i' \Sigma(\theta_i, \theta) \right) v = m(\theta)' v, \quad (9)$$

where $m \in \mathcal{H}$ and $\|m\|_{\mathcal{H}} = \|m^*\|_{\mathcal{H}^*}$.

We have assumed that $g(\cdot) \sim \mathcal{GP}(m, \Sigma)$ has almost-surely continuous sample paths, which implies that $G^* \sim \mathcal{GP}(0, \Sigma^*)$ can likewise be realized as a process with almost surely continuous sample paths. Let \mathcal{C}^* be the space of \mathbb{R} -valued continuous functions on Θ^* with the property that any $f^* \in \mathcal{C}^*$ can be represented as $f^*(\theta, v) = f(\theta)' v$ for $f \in \mathcal{C}(\Theta, \mathbb{R}^k)$ and $v \in V$. Due to the structure of Σ^* , realizations of the process G^* almost surely belong to \mathcal{C}^* and the process can be represented as $G^*(\theta, v) = v' G(\theta)$, where $G \sim \mathcal{GP}(0, \Sigma)$. Take any linear functional defined on the space of continuous functions with index set Θ^* and denote by η^* its restriction to \mathcal{C}^* . Since the relation between $f^* \in \mathcal{C}^*$ and $f \in \mathcal{C}(\Theta, \mathbb{R}^k)$ is one-to-one, we can define a linear functional on $\mathcal{C}(\Theta, \mathbb{R}^k)$ as $\eta(f) = \eta^*(f^*)$. This creates a one-to-one correspondence between linear functionals on \mathcal{C}^* and linear functionals on $\mathcal{C}(\Theta, \mathbb{R}^k)$. Note that the definition of the Pettis integral for process G^* depends on η^* only and all functionals that are the same once restricted to \mathcal{C}^* lead to the same Pettis integral:

$$m_{\eta^*}^*(\theta, v) = \mathbb{E}[G^*(\theta, v) \eta^*(G^*)] = \mathbb{E}[v' G(\theta) \eta^*(G^*)] = v' \mathbb{E}[G(\theta) \eta(G)] = v' m_{\eta}(\theta). \quad (10)$$

Due to Theorem 2.1 of van der Vaart and van Zanten (2008), \mathcal{H}^* coincides with the image of the space of linear functionals defined on \mathcal{C}^* under the Pettis integral transformation, while equation (2.4) in that paper together with the definition of $\sigma(G^*)$ establish that $\|m_{\eta^*}\|_{\infty} \leq \sigma^2(G^*) \|\eta^*\|^*$. Comparing equation (10) to (9), we see that the first statement of Lemma 1 holds. We further notice that all norms of starred objects coincide with the norms of the corresponding objects without stars. For example,

$$\|m\|_{\infty} = \sup_{j, \theta \in \Theta} |m_j(\theta)| = \sup_{v \in V, \theta \in \Theta} |v' m(\theta)| = \sup_{(\theta, v) \in \Theta^*} |m^*(\theta, v)| = \|m^*\|_{\infty}.$$

Note further that η and η^* have the same total variation norm.

$$\|\eta^*\|^* = \sup_{f^* \in \mathcal{C}^*, \|f^*\|_{\infty} \leq 1} \eta^*(f^*) = \sup_{f \in \mathcal{C}(\Theta, \mathbb{R}^k), \|f\|_{\infty} \leq 1} \eta(f) = \|\eta\|^*.$$

Finally,

$$\sigma^2(G^*) = \sup_{\|\eta^*\|^* \leq 1} \mathbb{E}[\eta^*(G^*)^2] = \sup_{\|\eta\|^* \leq 1} \mathbb{E}[\eta(G)^2] = \sigma^2(G).$$

This completes the proof. \square

Theorem 3 (Brown 1986, Andrews and Mikusheva 2022): For any parameter space $\tilde{\Gamma} \subseteq \Gamma$, any loss $L(a, \theta)$ which is convex in a for all θ , and any decision rule δ that is admissible on $\tilde{\Gamma}$, there exists a sequence of finitely supported priors π_r on $\tilde{\Gamma}$ and corresponding Bayes decision rules δ_{π_r} ,

$$\int \mathbb{E}_m[L(\delta_{\pi_r}(g), \theta^*)] d\pi_r(\theta^*, m) = \min_{\tilde{\delta}} \int \mathbb{E}_m[L(\tilde{\delta}(g), \theta^*)] d\pi_r(\theta^*, m),$$

such that $\delta_{\pi_r}(g) \rightarrow \delta(g)$ as $r \rightarrow \infty$ for almost every g .

Proof of Theorem 1 Consider the posterior risk function $R_\pi(a, g) = \mathbb{E}_\pi[L(a, \theta)|g]$. As the first step of the proof we show that for any finitely-supported prior π on $\tilde{\Gamma}$, the risk function $R_\pi(a, g)$ is Lipschitz in g :

$$|R_\pi(a, g) - R_\pi(a, g')| \leq \bar{L}W \|g - g'\|_\infty. \quad (11)$$

Let $\{(\theta_1, m_1), \dots, (\theta_J, m_J)\}$ be the support of π . For each m_j we know from Lemma 1 that there exists $\eta_{m,j} \in \mathbb{H}$ with $\|\eta\|^* \leq W$ and $m_j(\cdot) = \mathbb{E}[G(\cdot)\eta_{m,j}(G)]$. Further note that by e.g. Lemma 3.1 of van der Vaart and van Zanten (2008), for each $m \in \mathcal{H}$ the likelihood ratio for the measure Q_m corresponding to a $\mathcal{GP}(m, \Sigma)$ distribution, relative to $m' = 0$, takes the form $\frac{dQ_m}{dQ_0}(g) = \exp(\eta_m(g) - \frac{1}{2}\|m\|_{\mathcal{H}}^2)$.

Define $\tilde{g} = g' - g$, and let $g_t = g + t \cdot \tilde{g}$. Note that

$$\mathbb{E}_\pi[L(a, \theta)|g_t] = \frac{\sum_j L(a, \theta_j) \exp\{\eta_{m,j}(g_t) - \frac{1}{2}\|m_j\|_{\mathcal{H}}^2\} \pi(\theta_j, m_j)}{\sum_j \exp\{\eta_{m,j}(g_t) - \frac{1}{2}\|m_j\|_{\mathcal{H}}^2\} \pi(\theta_j, m_j)}. \quad (12)$$

Linearity implies that $\eta_{m,j}(g_t) = \eta_{m,j}(g) + t\eta_{m,j}(\tilde{g})$, and thus

$$\frac{\partial}{\partial t} \exp\left\{\eta_{m,j}(g_t) - \frac{1}{2}\|m_j\|_{\mathcal{H}}^2\right\} = \eta_{m,j}(\tilde{g}) \exp\left\{\eta_{m,j}(g_t) - \frac{1}{2}\|m_j\|_{\mathcal{H}}^2\right\}.$$

By differentiating (12) we get

$$\frac{\partial}{\partial t} \mathbb{E}_\pi[L(a, \theta)|g_t] = \text{Cov}_\pi(L(a, \theta), \eta_m(\tilde{g})|g_t), \quad (13)$$

where the only posterior uncertainty about $\eta_m(\tilde{g})$ comes from the unknown parameter m , while \tilde{g} is fixed. The Cauchy-Schwarz inequality implies that

$$\|\text{Cov}_\pi(L(a, \theta), \eta_m(\tilde{g})|g_t)\| \leq \bar{L} \sqrt{\text{Var}(\eta_m(\tilde{g})|g_t)}.$$

For $(\theta_j, m_j) \in \Gamma_W$ we have $\|\eta_{m,j}\|^* \leq W$, thus $|\eta_{m,j}(\tilde{g})| \leq W\|\tilde{g}\|_\infty = W\|g-g'\|_\infty$, which implies that $\sqrt{\text{Var}(\eta_m(\tilde{g})|g_t)} \leq W\|g-g'\|_\infty$. Hence,

$$|R_\pi(a,g) - R_\pi(a,g')| = \left| \int_0^1 \frac{\partial}{\partial t} \mathbb{E}_\pi[L(a,\theta)|g_t] dt \right| \leq \bar{L}W\|g-g'\|_\infty.$$

We define the Bayes decision rule $\delta(g)$ for prior π as $R_\pi(\delta(g),g) = \min_{a \in \mathcal{D}} R_\pi(a,g)$. As the next step of the proof we show that for any finitely-supported prior π on $\tilde{\Gamma}$, the Bayes decision rule $\delta(g)$ is almost surely uniformly continuous in g and satisfies the inequality:

$$\|\delta(g') - \delta(g)\| \leq \kappa^{-1}(2\bar{L}W \cdot \|g-g'\|_\infty), \quad (14)$$

where $\kappa(\cdot)$ is the function from the definition of uniform convexity.

From the definition of the Bayes decision rule and equation (11) we have

$$R_\pi(\delta(g'),g) - R_\pi(\delta(g),g') \leq R_\pi(\delta(g'),g) - R_\pi(\delta(g'),g') \leq \bar{L}W\|g-g'\|_\infty.$$

Equation (11) also implies $R_\pi(\delta(g),g') - R_\pi(\delta(g),g) \leq \bar{L}W\|g-g'\|_\infty$. These inequalities result in the following statement:

$$R_\pi(\delta(g'),g) - R_\pi(\delta(g),g) \leq 2\bar{L}W \cdot \|g-g'\|_\infty. \quad (15)$$

The uniform convexity of the loss function $L(a,\theta)$ results in the uniform convexity of the posterior risk function $R_\pi(a,g)$ in its first argument with the same $\kappa(\cdot)$. So

$$R_\pi(ta + (1-t)a',g) \leq tR_\pi(a,g) + (1-t)R_\pi(a',g) - t(1-t)\kappa(\|a-a'\|).$$

Thus,

$$\begin{aligned} R_\pi(t\delta(g') + (1-t)\delta(g),g) - R_\pi(\delta(g),g) + t(1-t)\kappa(\|\delta(g') - \delta(g)\|) \\ \leq t(R_\pi(\delta(g'),g) - R_\pi(\delta(g),g)). \end{aligned}$$

The fact that $\delta(g)$ is a Bayes decision rule implies that

$$R_\pi(t\delta(g') + (1-t)\delta(g),g) - R_\pi(\delta(g),g) \geq 0.$$

Combining this with inequality (15) we obtain:

$$\kappa(\|\delta(g') - \delta(g)\|) \leq R_\pi(\delta(g'),g) - R_\pi(\delta(g),g) \leq 2\bar{L}W \cdot \|g-g'\|_\infty.$$

Since function κ is strictly increasing and thus invertible, we arrive at (14). The continuity of $\kappa(\cdot)$ at zero comes from continuity of the loss function $L(a,\theta)$.

As the final step in the proof of Theorem 1 we show that (14) holds for any admissible rule on $\tilde{\Gamma}$. Using an extension of a result from Brown (1986), Theorem 3 establishes that there exists a sequence of finitely supported priors π_s on $\tilde{\Gamma}$ and corresponding Bayes decision rules δ_{π_s} such that $\delta_{\pi_s}(g) \rightarrow \delta(g)$ as $s \rightarrow \infty$ for almost every g . Since $\delta_{\pi_s}(g)$ is uniformly continuous with modulus of continuity $\tilde{\kappa}$ for all s , however, it follows that δ_{π_s} converges pointwise to a limit δ^* which is also uniformly continuous with modulus of continuity $\tilde{\kappa}$. Hence, $\delta(g) = \delta^*(g)$ for almost every g . \square

Proof of Theorem 2 First consider $\tilde{\Gamma} \subseteq \Gamma_W$. Following exactly the same arguments as the ones used in the proof of statement (13), we can show that for any finitely-supported prior π on $\tilde{\Gamma}$, any moment realizations g, g' , and $g_t = g + t \cdot \tilde{g}$ for $\tilde{g} = g' - g$

$$\frac{\partial}{\partial t} \mathbb{E}_\pi[r(\theta)|g_t] = \text{Cov}_\pi(r(\theta), \eta_m(\tilde{g})|g_t),$$

where by the Cauchy-Schwarz inequality

$$\|\text{Cov}_\pi(r(\theta), \eta_m(\tilde{g})|g_t)\| \leq \bar{r} \sqrt{p} W \|g - g'\|_\infty,$$

from which it is immediate that $\mathbb{E}_\pi[r(\theta)|g]$ is Lipschitz in g :

$$\|\mathbb{E}_\pi[r(\theta)|g] - \mathbb{E}_\pi[r(\theta)|g']\| \leq K \|g - g'\|_\infty.$$

Given this Lipschitz property for Bayes posterior means, that δ is almost-everywhere Lipschitz follows by the same argument as in the proof of Theorem 1. \square

Proof of Corollary 1 Immediate from Theorem 1. \square

Proof of Lemma 2 Suppose that δ is both almost-surely uniformly continuous and scale-invariant. Consider two independent draws g and g' , and note that by scale-invariance we have

$$\delta(g, \Sigma) - \delta(g', \Sigma) = \delta(c \cdot g, \Sigma) - \delta(c \cdot g', \Sigma)$$

for all $c > 0$. However, the uniform continuity property implies for almost every (g, g') and any fixed c ,

$$\|\delta(g, \Sigma) - \delta(g', \Sigma)\| = \|\delta(c \cdot g, \Sigma) - \delta(c \cdot g', \Sigma)\| \leq \tilde{\kappa}(c \cdot \|g - g'\|_\infty)$$

with probability one. Sending c to zero leads us to the required statement. \square

Proof of Proposition 1 If the covariance function Σ has a finite number of nonzero eigenvalues, it follows that for $G \sim \mathcal{GP}(0, \Sigma)$ the process $G(\cdot)$ is a transformation of a finite-dimensional normal random vector, so we can write $G(\theta) = A(\theta)Y$ for $Y \in \mathbb{R}^q$ a standard normal random vector and $A(\cdot)$ a matrix-valued function that depends on Σ . Correspondingly, the RKHS \mathcal{H} can be written as $\{A(\cdot)x : x \in \mathbb{R}^q\}$.

Combining these observations, we can write $g(\cdot) = A(\cdot)y$ for $y \sim N(x, I)$, and any estimator $\delta(g)$ can be equivalently expressed as $\gamma(y) = \delta(A(\cdot)y)$. For v a standard normal random vector and ζ as defined in the main text, we likewise have the equality

$$\delta^B(g) \equiv \mathbb{E}[\delta(g + \zeta) | g] = \mathbb{E}[\gamma(y + v) | y] \equiv \gamma^B(y)$$

for the bagged estimators. Since $\Sigma(\theta, \tilde{\theta}) = A(\theta)A(\tilde{\theta})'$ while Σ is continuous and Θ is compact, the largest singular value of $A(\theta)$, $\sigma_{\max}(A(\theta))$, is uniformly bounded. For $\bar{\sigma} = \sup_{\theta \in \Theta} \sigma_{\max}(A(\theta))$,

$$\sup_{\theta \in \Theta} \|g(\theta) - \tilde{g}(\theta)\| \leq \sup_{\theta \in \Theta} \sigma_{\max}(A(\theta)) \|y - \tilde{y}\| \leq \bar{\sigma} \|y - \tilde{y}\|.$$

Hence, it suffices to show that $\gamma^B(y)$ is Lipschitz in y .

Note that for φ the standard (multivariate) normal density,

$$\gamma^B(y) = \int \gamma(y + v) \varphi(v) dv = \int \gamma(v) \varphi(v - y) dv.$$

Hence, if we let $y_t = y + t \cdot \tilde{y}$, we have

$$\begin{aligned} \frac{\partial}{\partial t} \gamma^B(y_t) \Big|_{t=0} &= \int \gamma(v) \frac{\partial}{\partial t} \varphi(v - y_t) \Big|_{t=0} dv \\ &= \int \gamma(v) (v - y)' \tilde{y} \varphi(v - y) dv = \text{Cov}_{v \sim N(y, I)}(\gamma(v), v' \tilde{y}), \end{aligned}$$

where $\text{Cov}_{v \sim N(y, I)}(\gamma(v), v')$ denotes the covariance of $\gamma(v)$ and v when $v \sim N(y, I)$. Note, however, that since the range of $\gamma(v)$ is contained in \mathcal{A} , Cauchy-Schwarz implies that for $\bar{a} = \sup_{a \in \mathcal{A}} \|a\|$,

$$\left\| \frac{\partial}{\partial t} \gamma^B(y_t) \Big|_{t=0} \right\| = \left\| \text{Cov}_{v \sim N(y, I)}(\gamma(v), v' \tilde{y}) \right\| \leq \bar{a} \sqrt{p} \|\tilde{y}\|,$$

which completes the proof. \square

Proof of Lemma 3 Let $Q(\theta) = Q(\theta | g)$ and $Q'(\theta) = Q(\theta | g')$, and consider

$$Q_t(\theta) = Q(\theta) + t(Q'(\theta) - Q(\theta)) = Q(\theta) + t \cdot \Delta(\theta)$$

for $\Delta(\theta) = Q'(\theta) - Q(\theta)$. Let us write $\mathbb{E}_\pi^{QB}[\cdot|Q]$ for the expectation under the quasi-Bayes posterior distribution, which draws θ from the distribution with density $\frac{\exp(-\frac{1}{2}Q(\theta))}{\int \exp(-\frac{1}{2}Q(\theta))d\pi(\theta)}$ relative to π , and define $\text{Cov}_\pi^{QB}(\cdot, \cdot|Q)$ analogously. Note that $\delta_\pi^{QB}(g) = \mathbb{E}_\pi^{QB}[r(\theta)|Q(\cdot|g)]$, and that

$$\begin{aligned} \frac{\partial}{\partial t} \mathbb{E}_\pi^{QB}[r(\theta)|Q_t] &= \frac{\partial}{\partial t} \left[\frac{\int r(\theta) \exp(-\frac{1}{2}Q_t(\theta)) d\pi(\theta)}{\int \exp(-\frac{1}{2}Q_t(\theta)) d\pi(\theta)} \right] \\ &= -\frac{1}{2} (\mathbb{E}_\pi^{QB}[r(\theta)\Delta(\theta)|Q_t] - \mathbb{E}_\pi^{QB}[r(\theta)|Q_t] \mathbb{E}_\pi^{QB}[\Delta(\theta)|Q_t]) \\ &= -\frac{1}{2} \text{Cov}_\pi^{QB}(r(\theta), \Delta(\theta)|Q_t) \end{aligned}$$

By the Cauchy-Schwarz inequality, however,

$$\|\text{Cov}_\pi^{QB}(r(\theta), \Delta(\theta)|Q_t)\| \leq \bar{r} \sqrt{p} \sup_\theta |\Delta(\theta)|,$$

which completes the proof. \square

Proof of Proposition 2 Similar to the proof of Lemma 3, let $\mathbb{E}_\pi^{QB}[\cdot|g]$ be the expectation under the quasi-Bayes posterior distribution, which draws θ from the distribution with density $\frac{\exp(-\frac{1}{2}Q(\theta|g))}{\int \exp(-\frac{1}{2}Q(\theta|g))d\pi(\theta)}$ relative to π , and define $\text{Cov}_\pi^{QB}(\cdot, \cdot|g)$, $\text{Var}_\pi^{QB}(\cdot|g)$ analogously. For $g_t = g + t \cdot \tilde{g}$, note that by Cauchy-Schwarz

$$\begin{aligned} \left\| \frac{\partial}{\partial t} \delta_\pi^{QB}(g_t) \right\| &= \left\| \frac{\partial}{\partial t} \mathbb{E}_\pi^{QB}[r(\theta)|g_t] \right\| = \frac{1}{2} \left\| \text{Cov}_\pi^{QB}(r(\theta), \tilde{g}(\theta)' \Sigma(\theta)^{-1} g_t(\theta)|g_t) \right\| \leq \\ &\quad \frac{1}{2} \bar{r} \sqrt{p} \sqrt{\mathbb{E}_\pi^{QB} \left[(\tilde{g}(\theta)' \Sigma(\theta)^{-1} g_t(\theta))^2 |g_t \right]}. \end{aligned}$$

By another application of Cauchy-Schwarz,

$$(\tilde{g}(\theta)' \Sigma(\theta)^{-1} g_t(\theta))^2 \leq \tilde{g}(\theta)' \Sigma(\theta)^{-1} \tilde{g}(\theta) \cdot Q(\theta|g_t),$$

so

$$\mathbb{E}_\pi^{QB} \left[(\tilde{g}(\theta)' \Sigma(\theta)^{-1} g_t(\theta))^2 |g_t \right] \leq \|\tilde{g}\|_{\Sigma, \infty}^2 \mathbb{E}_\pi^{QB}[Q(\theta|g_t)|g_t]$$

for

$$\|\tilde{g}\|_{\Sigma, \infty} = \sup_{\theta \in \Theta} \sqrt{\tilde{g}(\theta)' \Sigma(\theta)^{-1} \tilde{g}(\theta)} = \sup_{\theta \in \Theta} \sqrt{Q(\theta|\tilde{g})}.$$

Altogether, we obtain that

$$\left\| \frac{\partial}{\partial t} \mathbb{E}_\pi^{QB}[r(\theta)|g_t] \Big|_{t=0} \right\| \leq \frac{1}{2} \bar{r} \sqrt{p} \|\tilde{g}\|_{\Sigma, \infty} \sqrt{\mathbb{E}_\pi^{QB}[Q(\theta|g)|g]}.$$

Note, next, that

$$\mathbb{E}_\pi^{QB}[Q(\theta|g)|g] = \frac{\int Q(\theta|g)\exp(-\frac{1}{2}Q(\theta|g))d\pi(\theta)}{\int \exp(-\frac{1}{2}Q(\theta|g))d\pi(\theta)}.$$

Since the function $h(x) = x\exp(-\frac{1}{2}x)$ is maximized at $x=2$, if

$$\int \exp\left(-\frac{1}{2}Q(\theta|g)\right)d\pi(\theta) \geq \varepsilon,$$

then

$$E_\pi[Q(\theta|g)|g] \leq 2\exp(-1)\varepsilon^{-1}.$$

Note, however, that for $|\Theta|$ finite and $\underline{\pi} = \min_{\theta \in \Theta} \pi(\theta)$,

$$\int \exp\left(-\frac{1}{2}Q(\theta|g)\right)d\pi(\theta) = \sum_{\theta \in \Theta} \exp\left(-\frac{1}{2}Q(\theta|g)\right)\pi(\theta) \geq \exp\left(-\frac{1}{2}\min_{\theta \in \Theta} Q(\theta|g)\right)\underline{\pi}.$$

Hence, for $g \in \mathcal{G}_C$ as defined in the proposition,

$$\left\| \frac{\partial}{\partial t} \mathbb{E}_\pi^{QB}[r(\theta)|g_t] \Big|_{t=0} \right\| \leq \bar{r}\sqrt{p}\|\tilde{g}\|_{\Sigma, \infty} \exp\left(\frac{1}{2}C-1\right)\underline{\pi}^{-1},$$

which completes the proof. \square

B Example: Discontinuous Limit-of-Bayes

This appendix provides an example to demonstrate that without bounds on identification strength, the pointwise limit of Bayes posterior means can be discontinuous.

Let us continue the finite Θ special case discussed in main text, and further suppose that the parameter space consists of just two points, $\Theta = \{\theta_1, \theta_2\}$, that we have a one-dimensional moment condition ($k=1$), and that $\Sigma = I_2$. The function m is thus described by two numbers – the values at θ_1 and θ_2 . Consider prior π_C , supported on just two values of (θ, m) , which assigns probability $\frac{1}{2}$ to each of θ_1 and θ_2 and, conditional on $\theta = \theta_j$, implies that $(m(\theta_j), m(\theta_{-j})) = (0, C)$ with probability one, where θ_{-j} denotes the element of Θ other than θ_j . Suppose we are interested in estimating $r(\theta) = \theta$ under squared error loss, and note that the Bayes estimator corresponds to the posterior mean

$$\delta_{\pi_C}(g, \Sigma) = \mathbb{E}_{\pi_C}[\theta|g] = \frac{\sum_{j=1}^2 \theta_j \exp(-\frac{1}{2}g(\theta_j)^2 - \frac{1}{2}(g(\theta_{-j}) - C)^2)}{\sum_{j'=1}^2 \exp(-\frac{1}{2}g(\theta_{j'})^2 - \frac{1}{2}(g(\theta_{-j'}) - C)^2)}.$$

For a given g with $g(\theta_1) \neq g(\theta_2)$ $\mathbb{E}_{\pi_C}[\theta|g] \rightarrow \operatorname{argmin}_{\theta \in \{\theta_1, \theta_2\}} g(\theta)$ as $C \rightarrow \infty$. The limiting estimator $\delta(g) = \operatorname{argmin}_{\theta \in \{\theta_1, \theta_2\}} g(\theta)$ is a step function, and thus discontinuous.

C Strong-Identification Limit Experiment

This appendix outlines the limit experiment for strongly identified GMM and discusses why our inadmissibility argument for GMM estimators under weak identification does not translate to this setting.

In strongly identified GMM models, θ^* is consistently estimable at a \sqrt{n} rate. To obtain a non-trivial decision problem in the limit we must therefore focus attention on a shrinking neighborhood of the true parameter value, reparameterizing the model in terms of $\tau^* = \sqrt{n}(\theta_n^* - \theta_0)$, for θ_n^* the true parameter value in the sample of size n and θ_0 a fixed parameter value around which we localize the problem. Under mild conditions on the moments, the limit experiment for estimation of τ^* in this setting corresponds to observing

$$g(0) \sim N(-H\tau^*, \Omega) \quad (16)$$

for H and Ω known, non-stochastic matrices where H has full column rank, and $\tau^* \in \mathbb{R}^p$. See for instance Armstrong and Kolesar (2021).

Observing (16) is equivalent to observing the linear process $g(\cdot)$ where

$$g(\cdot) = \mathcal{GP}(m, \Sigma).$$

with $m(\tau) = H(\tau - \tau^*)$ and $\Sigma(\tau, \tilde{\tau}) = \Omega$, where both H and Σ are known. Hence, the strong-identification limit experiment (16) can be cast as a special case of our weak-identification limit experiment (2) with the additional restrictions that (i) m is linear with a known Jacobian and (ii) Σ is constant.

Our inadmissibility argument for GMM under weak identification relies on the fact that the GMM estimator is the same for moment realizations $g(\cdot)$ and $c \cdot g(\cdot)$ for all positive constants c . In the strongly identified case, however, $c \cdot g(\cdot)$ for $c \neq 1$ has a Jacobian different from H , and so lies outside the support of the process $g(\cdot)$. If we instead rescale the moments only at a given point τ , note that the GMM estimator for τ^* with weighting matrix Ξ can be written as $\hat{\tau} = \tau - (H'\Xi H)^{-1} H'\Xi g(\tau)$. Hence, in the strongly identified case GMM is not invariant to changing the scale of $g(\tau)$.

Indeed, a stronger statement is true: if we consider a flat prior on τ^* , the posterior mean is equal to the efficient GMM estimator with weighting matrix $\Xi = \Omega^{-1}$. Hence, in the strongly-

identified limit experiment the efficient GMM estimator can be written as the limit of a sequence of Bayes posterior means for proper priors, and our inadmissibility argument does not apply.

D Additional Simulation Results

This section reports additional simulation results to complement those reported in the main text. Tables 3 and 4 report average normalized mean squared error, for the IV and correlation coefficients respectively, when we bin specifications based on the number of instruments rather than the effective F statistic. We see that the gains for the alternative estimators are largest in the just-identified case, while the performance of the GMM and alternative estimators tends to be closer in the over-identified specifications.

Tables 5 and 6 report normalized root mean squared error (based on 1,000 simulation draws) averaged across all specifications when we set the parameter space Θ_s in specification s equal to $\left[\pm c \cdot \frac{\sigma_{uv}}{\sigma_v^2}\right]$ for $c \in \{20, 40, 60\}$. As these results show, the qualitative messages from our simulations are unchanged across these alternative parameter spaces. Specifically, Table 5 reports results for the IV coefficient, and we see that while the root mean squared error for all estimators grows as we widen the bounds on the parameter space, in every case bagged two-stage least squares performs the best, followed by bagged continuously updating GMM, then quasi-Bayes with an invariant prior, then the two GMM estimators, and finally quasi-Bayes with a flat prior. Table 6 reports results for the correlation coefficient and shows, as expected, that the bounds of the parameter space make almost no difference in this case.

Tables 7 and 8 drop boundedness of the parameter space altogether and report the mean squared error of two-stage least squares and bagged two-stage least squares estimators for the IV coefficient when we limit attention to specifications with $k \geq 3$. Table 7 bins the specifications based on the effective F-statistic, while Table 8 bins them based on the number of instruments. We see that in these specifications bagging makes very little difference: while the bagged estimator performs slightly better in some cases, and slightly worse in others, the overall performance difference between the two estimators is minimal. This is consistent with the results in Table 3, which show that the performance gains for bagged two stage least squares are coming from the specifications with $k=1$ and $k=2$.

	$k=1$	$k=2$	$k=3$	$k \geq 4$
Two-Stage Least Squares	1.84	1.06	1.01	0.98
Continuously Updating GMM	1.84	1.12	1.04	1.21
Bagged Two-Stage Least Squares	1.12	1.04	0.99	1.01
Bagged Continuously Updating GMM	1.12	1.08	1.03	1.00
Quasi-Bayes, Flat Prior	1.94	1.20	1.13	1.27
Quasi-Bayes, Invariant Prior	1.06	1.07	1.04	1.22
Number of Specifications	31	20	30	40

Table 3: Performance of IV coefficient estimators in Andrews et al. (2019) specifications. Entries correspond to the root mean squared error normalized by the standard error in the Andrews et al. (2019) data, $\sqrt{\mathbb{E}_s[(\delta(g) - \theta_s^*)^2]}/\sigma_s^*$, averaged across specifications. Columns correspond to varying degrees of over-identification.

	$k=1$	$k=2$	$k=3$	$k \geq 4$
Two-Stage Least Squares	1.20	1.06	1.00	1.01
Continuously Updating GMM	1.20	1.10	1.01	1.18
Bagged Two-Stage Least Squares	1.00	1.07	0.98	1.06
Bagged Continuously Updating GMM	1.00	1.10	1.00	1.01
Quasi-Bayes, Flat Prior	1.38	1.19	1.04	1.19
Quasi-Bayes, Invariant Prior	0.99	1.11	1.00	1.17
Number of Specifications	31	20	30	40

Table 4: Performance of correlation coefficient estimators in Andrews et al. (2019) specifications. Entries correspond to the root mean squared error normalized by the standard error in the Andrews et al. (2019) data, $\sqrt{\mathbb{E}_s[(\delta(g) - r(\theta_s^*))^2]} / \sigma_{r,s}^*$, averaged across specifications. Columns correspond to varying degrees of over-identification.

	$\Theta_s = \left[\pm 20 \frac{\sigma_{uv,s}}{\sigma_{v,s}^2} \right]$	$\Theta_s = \left[\pm 40 \frac{\sigma_{uv,s}}{\sigma_{v,s}^2} \right]$	$\Theta_s = \left[\pm 60 \frac{\sigma_{uv,s}}{\sigma_{v,s}^2} \right]$
Two-Stage Least Squares	1.22	1.34	1.42
Continuously Updating GMM	1.30	1.42	1.51
Bagged Two-Stage Least Squares	1.04	1.06	1.07
Bagged Continuously Updating GMM	1.05	1.07	1.09
Quasi-Bayes, Flat Prior	1.39	1.58	1.70
Quasi-Bayes, Invariant Prior	1.11	1.14	1.15
Number of Specifications	121	122	122

Table 5: Performance of IV coefficient estimators in Andrews et al. (2019) specifications based on 1000 simulation draws. Entries correspond the root mean squared error normalized by the standard error in the Andrews et al. (2019) data, $\sqrt{\mathbb{E}_s[(\delta(g) - \theta_s^*)^2]} / \sigma_s^*$, averaged across all specifications. Columns vary the size of the parameter space.

	$\Theta_s = \left[\pm 20 \frac{\sigma_{uv,s}}{\sigma_{v,s}^2} \right]$	$\Theta_s = \left[\pm 40 \frac{\sigma_{uv,s}}{\sigma_{v,s}^2} \right]$	$\Theta_s = \left[\pm 60 \frac{\sigma_{uv,s}}{\sigma_{v,s}^2} \right]$
Two-Stage Least Squares	1.07	1.07	1.07
Continuously Updating GMM	1.12	1.12	1.12
Bagged Two-Stage Least Squares	1.03	1.03	1.03
Bagged Continuously Updating GMM	1.02	1.03	1.03
Quasi-Bayes, Flat Prior	1.20	1.21	1.20
Quasi-Bayes, Invariant Prior	1.07	1.08	1.08
Number of Specifications	121	122	122

Table 6: Performance of correlation coefficient estimators in Andrews et al. (2019) specifications based on 1000 simulation draws. Entries correspond the root mean squared error normalized by the standard error in the Andrews et al. (2019) data, $\sqrt{\mathbb{E}_s[(\delta(g) - r(\theta_s^*))^2]}/\sigma_{r,s}^*$, averaged across all specifications. Columns vary the size of the parameter space.

	$\mathbb{E}_s[F] \leq 10$	$10 < \mathbb{E}_s[F] \leq 20$	$20 < \mathbb{E}_s[F] \leq 50$	$50 < \mathbb{E}_s[F]$
Two-Stage Least Squares	0.98	1.01	1.01	1.00
Bagged Two-Stage Least Squares	1.00	0.98	0.99	0.99
Number of Specifications	42	4	15	9

Table 7: Estimator performance in Andrews et al. (2019) specifications with $k \geq 3$ and an unbounded parameter space. Entries correspond to the root mean squared error normalized by the standard error in the Andrews et al. (2019) data, $\sqrt{\mathbb{E}_s[(\delta(g) - \theta_s^*)^2]}/\sigma_s^*$, averaged across specifications. Columns correspond to ranges of values for the average effective first-stage F statistic of Montiel-Olea and Pflueger (2013).

	$k=3$	$k \geq 4$
Two-Stage Least Squares	1.01	0.98
Bagged Two-Stage Least Squares	0.99	1.01
Number of Specifications	30	40

Table 8: Estimator performance in Andrews et al. (2019) specifications. Entries correspond to the root mean squared error normalized by the standard error in the Andrews et al. (2019) data, $\sqrt{\mathbb{E}_s[(\delta(g) - \theta_s^*)^2]}/\sigma_s^*$, averaged across specifications. Columns correspond to varying degrees of over-identification.