

Investor Sentiments

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Abstract

We consider a general class of games that have been used to model many economic problems where players' sentiments are believed to play an important role. Dropping the common-prior assumption, we identify the relevant notion of sentiments for strategic behavior in these games. This notion is tied to how likely a player thinks that some other player has a more optimistic outlook than himself when they obtain their private information. Under this notion, we show that sentiments have a profound effect on strategic outcomes—even with vanishing uncertainty.

1 Introduction

There is a common perception that investor sentiments play an important role in many economic problems, such as bank runs, development traps, and currency attacks, in the sense that the outcomes may differ widely in situations with similar economic fundamentals. For example, an economic program may succeed in one country and fail in another country with similar economic fundamentals. Economists have discussed the issue informally using multiple equilibria. Unfortunately, the arguments based on multiple equilibria are very fragile, in that multiple equilibria tend to disappear when one introduces small amounts of incomplete information, as demonstrated by Carlsson & van Damme (1993) and shown in greater generality by Frankel, Morris & Pauzner (2003) and Weinstein & Yildiz (2007).

In this paper, we offer a new methodology to model investor sentiments. We consider a large class of incomplete information games with a unique rationalizable strategy, where common prior assumption need not hold. The unique rationalizable action of a given type depends not only on his beliefs about the fundamentals but also on his beliefs about the

other players' beliefs about the fundamentals, etc. We identify the key properties of beliefs about the other players' beliefs that determine their unique rationalizable action. We call these key properties the relevant sentiments for strategic behavior. We demonstrate that the sentiments can be as important as fundamentals in determining players' actions. Note that, in order to distinguish the effect of sentiments from that of fundamentals, our notion of sentiments does not include the beliefs directly about the fundamentals, which are often identified with fundamentals in economic models.¹

Our theoretical contribution is the identification of the relevant sentiments for some important games in applied literature. In these games, each player has two actions. One of the actions corresponds to taking a particular action, such as investing in a project, short-selling a currency to attack it, or revolting against an oppressive government, etc. We call this action investment. The return to investment depends on how many other players invest and on the fundamentals. The other action simply corresponds to not investing and yields zero. We assume that some aspects of the payoff from investment are unknown but the players have private information about these aspects. Each individual's outlook about this payoff depends on his private information, and hence in general players have different outlooks, some having more optimistic outlooks than the others.

When there are only two players, in order to determine the relevant sentiments, one only needs to ask a player: what is the probability that the other player has a more optimistic outlook than yours? The answer, denoted by q , measures the relevant sentiments. Note that $q - 1/2$ is the more than normal likelihood of a player being more optimistic according to another player. If players had a common prior, then under symmetry q would be equal to $1/2$. In a way, there would be no room for individual sentiments about the payoffs. We do not assume a common prior and let q vary.

The relevant sentiment q captures the players' strategic behavior precisely as follows. According to the unique solution, a player invests if and only if the sum of q and his expectation about the payoff from the joint investment exceeds a threshold value. When the uncertainty is small, the above expectation is approximately the payoff parameter itself. Then, in the unique solution, q is a direct substitute to the payoff parameter, and hence it plays the same

¹There is no meaningful distinction between the beliefs directly about fundamentals and the fundamentals themselves in a purely game-theoretic model, as we cannot know whether a player's beliefs are consistent with the "true" probabilities, which are neither given nor relevant for the analysis. On the other hand, it is meaningful to ask a player i whether he thinks that another player j is more optimistic than he should be, or whether he thinks that j thinks that i is more optimistic than he should be, etc., because the answers to all of these questions are encoded in his beliefs about the other players' beliefs.

intuitive role the payoff parameter does. As q or the payoff from joint investment increases, it becomes more likely that players invest. In fact, by varying q , we can make either of the actions the unique rationalizable action whenever the complete-information game has multiple equilibria.

We also consider the games of regime change with a continuum of players. If enough players take a costly action towards a regime change, then regime changes. The required number of players can depend on the underlying state. In these games, in order to determine the relevant sentiments, one now needs to ask a player: what is the probability that there are enough people who have more optimistic outlooks than you have? We show once again that the unique rationalizable strategy is tied to the so determined sentiments, and players take actions towards changing the regime more often when they have higher sentiments.

Our results are consistent with the experimental evidence on global games (e.g., Heinemann, Nagel & Ockenfels (2004) and Shurchkov (2007)). In these experiments, subjects use cutoff values that are significantly lower than the cutoff predicted by the global games literature with common prior. In our model, this corresponds to q being significantly higher than $1/2$, i.e. subjects finding it likely that the fellow players are optimistic, which is also consistent with the empirical evidence on self-serving biases.

In a non-strategic environment, the relevant measure of an investor's sentiments would be his bias about the value of the project. If an investor had excessively optimistic views about the project, he would invest, and he would not invest if his views were excessively pessimistic. In a world with little uncertainty, such a bias would be small, having a negligible impact on the individual's behavior. One would have thought that in a strategic environment, the players' views about the other players' biases would be the relevant measure of sentiments. As it turns out, however, the relevant measure of investor sentiments, q , is independent of the presence of such biases or the level of uncertainty. Even when it is common knowledge that the players' signals are unbiased or when there is only a negligible amount of uncertainty, q may take any value in between 0 and 1, and hence the players may have substantial strategically-relevant sentiments that lead to substantially different outcomes. More interestingly, as the underlying uncertainty decreases, (small) variations in players' beliefs lead to larger variations in q , and their effect on the strategic behavior gets larger. This is also consistent with the experiments of Heinemann et al. (2004), where the deviation from the common-prior cutoff is much larger in the "complete-information" treatment.

We must emphasize three aspects of our methodology. First, the relevant measure of sentiments is determined by the payoff structure of the game, rather than being arbitrarily

chosen by the analyst. The relevant measure can be quite distinct for different payoff structures, i.e., there is no preconceived measure that fits all games. For example, in single-person decision problems and in games with linear best replies, the relevant sentiment is related to how much players over-estimate the fundamental. As we show, such a measure is irrelevant for the games we analyze in this paper. Second, we take the players' beliefs and thus their sentiments as exogenously given. Of course, one may want to understand how the beliefs and sentiments are formed endogenously and what factors affect them by analyzing a richer model, e.g. a dynamic model of belief formation. A crucial step in such an analysis is to determine the relevant sentiments for the game at hand, as we do in this paper. This is because focusing on a preconceived but irrelevant measure of sentiments may lead to wrong conclusions. For example, in a dynamic model, biases may asymptotically disappear, while the relevant sentiments, such as q , may remain large. Focusing on biases, one may falsely conclude that the sentiments become insignificant asymptotically. Third, strategic outcomes are determined by relevant sentiments (and the beliefs directly about the fundamentals), and these sentiments are given in the description of the game. If a researcher has an understanding of how certain factors affect the beliefs, then he may be able to learn about the effects of these factors on strategic outcomes by identifying the relevant sentiments. In this way, one may generate valuable insights. In contrast, the models based on multiple equilibria are silent about which equilibrium will be played in which situation (see Morris & Shin (1998) for a more detailed discussion).

Our paper is based on the analysis of global games, introduced by Carlsson & van Damme (1993), and popularized by Morris & Shin (1998), Morris & Shin (2002). Many authors have applied the global games approach to a wide range of economic problems. The existing literature assumes a common prior, assuming thereby that the only source of differences in beliefs is the differences in information (Aumann (1976)). They seem to leave no room for investors' sentiments to play a role. For example, in Morris & Shin (1998)'s model of currency attacks, economic fundamentals determine the unique equilibrium outcome in an intuitive way. Our contribution is to introduce sentiments to this analysis by dropping the common-prior assumption. Enriching the analysis in this way allows us to develop a notion of sentiments that captures the players' strategic behavior.

Weinstein & Yildiz (2007) have shown that any rationalizable action can be made uniquely rationalizable by perturbing the higher-order beliefs. In our framework, this means that when there are multiple rationalizable actions, in the nearby games all higher orders of beliefs are part of relevant sentiments and that sentiments play significant role. In such a general case,

the identification of the relevant sentiments is a daunting, impractical task without making specific assumptions on information structure. In this paper, we identify the relevant sentiments for the most canonical information structure in the applied literature. The relevant measure of sentiments for these games is simple and affects the strategies in an intuitive way. Moreover, the relevant sentiments can be solicited by a simple intuitive question about the second-order beliefs.

Independently, Morris & Shin (2007) aim to determine general conditions that lead to strategic behavior as described in the global games literature. Their approach is different and complementary to ours. Under similar supermodularity assumptions, they identify the properties of hierarchies of beliefs, specifically a common p -belief of a certain event, under which a given action is rationalizable. Clearly this is also a crucial step in our general methodology.

The common-prior assumption is a central tenet of traditional Economics. Nevertheless, empirical evidence suggests that economic agents do have heterogenous beliefs and systematic biases, such as self-serving biases, suggesting that this assumption is too restrictive to capture the behavior and reasoning of the actual agents. Consequently, a number of authors dropped this assumption to analyze the role of heterogenous beliefs in a variety of economic problems, such as financial markets (Harrison & Kreps (1978); Morris (1996)), information transmission (Banerjee & Somanathan (2001)), bargaining (Yildiz (2003)), and the theory of firm (Van den Steen (2005) and Gervais & Goldstein (2007)). Our analysis combines the methodology of this literature with incomplete information. Our notion of sentiments somewhat resembles the notion of optimism considered in Yildiz (2003). The two notions are different: our notion is ingrained in incomplete information, while his notion is based on complete information.

Equilibrium analysis without a common prior has weak foundations (Dekel, Fudenberg & Levine (2004)) and appears to be somewhat inconsistent as it does not allow biases about the strategies (Yildiz (2007)). Fortunately, the incomplete information games we consider are all dominance-solvable, and hence our analysis is not based on equilibrium. In particular, even if one introduces sentiments about strategies, these sentiments will not play any role, and our results will remain intact. Independently, Steiner & Stewart (2006) study long-term outcomes of learning through similarity and show that asymmetries in the similarity function have a similar effect as sentiments in our paper in determining which actions are rationalizable in two-player games.

In this paper, we allow for differing priors in order to analyze the players' sentiments directly. Under the common-prior assumption, one can take an indirect approach to investi-

gate the role of sentiments by analyzing the role of public information. Morris & Shin (2002) show that when public information is sufficiently precise relative to private information, public information also affects the unique rationalizable outcome; higher values of the public signals lead to lower cutoff values for investment, making investment more likely. There is a delicate balance here: if public information is too precise, then there will be multiple equilibria. Here, the public signal (or the prior beliefs) can substitute for sentiments and play a somewhat similar role to investor sentiments in our paper. Introduction of public information invites many other questions, such as whether a public signal can be sufficiently precise to affect the outcome but not too precise so that it does not lead to multiplicity (Angeletos & Werning (2006)). Our direct approach to the problem allows us to investigate the role of sentiments without dealing with such delicate issues.

In the next section, we formally introduce our notion of optimism and present our main result for two-player games. In Section 3, we extend our analysis to the games of regime change with a continuum of players. Section 4 concludes. Some of the proofs are relegated to the appendix.

2 Sentiments in a partnership game

In this section we consider a stylized two-person partnership game and identify the relevant notion of sentiments, which precisely captures players' strategic behavior.

Consider the payoff matrix

	a	b
a	θ, θ	$\theta - 1, 0$
b	$0, \theta - 1$	$0, 0$

Player 1 chooses between rows, and player 2 chooses between columns. Here, action a corresponds to investing in a project, and b corresponds to not investing. If a player takes action b , not investing, then his payoff is zero. If he invests, his payoff depends on the other player's action. If the other player also invests, the project succeeds, and both players get θ . If the other player does not invest, the project fails, and the investing player incurs a cost, totalling a net benefit of $\theta - 1$.²

²This payoff structure corresponds to investing in a project that yields 1 if the project succeeds and 0 if it fails. A player has to incur a cost c when he invests in the project. Writing $\theta = 1 - c$ for the net return from the project, we obtain the payoff structure above.

When θ is in $(0, 1)$ and common knowledge, there are two equilibria in pure strategies and one equilibrium in mixed strategies. In the good equilibrium, anticipating that the other player invests, each player invests in the project, and each gets the positive payoff of θ . In the bad equilibrium, each player correctly anticipates that the other party will not invest, so that neither of them invest, yielding zero payoff for both players.

We assume that players do not know θ , but each of them gets an arbitrarily precise noisy signal about θ . In particular, each player i observes

$$x_i = \theta + \varepsilon\eta_i, \tag{1}$$

where η_i is a noise term that takes values in $[-1, 1]$ and $\varepsilon \in (0, 1)$ is a scalar that measures the level of uncertainty players face. In our model, the level of uncertainty will not play any role, and this will allow us to expose the differential role sentiments play in strategic and non-strategic environments. We assume that ex ante θ is distributed uniformly on a large interval $[-L, L]$ where ε/L is sufficiently small and $L \gg 1$. In order to develop a theory of sentiments, we allow each player to have his own subjective belief about (η_1, η_2) . We let Pr_i denote the probability distribution of player i and write E_i for the associated expectation operator. We assume that (θ, η_1, η_2) are independently distributed with a density according to each Pr_i . We assume that each player thinks that his own signal is unbiased:

$$E_i(\eta_i) = 0.$$

This is without loss of generality because a Bayesian player will always adjust his beliefs if he thinks that his signal is biased. For simplicity, we will also assume that the beliefs are symmetric. We assume that all of these are common knowledge.

Since players have private information, each player's beliefs about θ will depend on his signal x_i . In particular, for $x_i \in (0, 1)$, when player i observes x_i , his expected value for θ will be

$$E_i[\theta|x_i] = x_i.$$

When $\eta_j > \eta_i$, player j observes a signal x_j that is higher than player i 's signal x_i , and player j has a more optimistic outlook about the project than player i . In that case, we say that player j is *interim more optimistic than player i* .

In order to measure the *sentiment of player i about j* , we ask player i how likely it is that the other player j will end up being more optimistic than i . Formally, we measure the investor sentiments (about the other player) by

$$q \equiv \text{Pr}_i(\eta_j > \eta_i).$$

If players had a common prior (i.e. $\Pr_1 = \Pr_2$), then we would have

$$q = 1/2, \tag{CPA}$$

and each player would find either player equally likely to be more optimistic interim. Since we allow players to hold their own subjective beliefs, q may differ from $1/2$. Indeed, $q - 1/2$ is the amount of more than normal likelihood of player j being interim more optimistic according to player i . Hence, $q - 1/2$ is a measure of the level of optimism of player j according to player i . The next result establishes that q is indeed the relevant measure of sentiments for the partnership game, capturing players' strategic behavior precisely.

Proposition 1 *For any signal value $x_i \neq 1 - q$, the unique rationalizable action is*

$$s_i^*(x_i) = \begin{cases} a & \text{if } x_i + q > 1, \\ b & \text{if } x_i + q < 1. \end{cases}$$

Proof. Since the game is symmetric, monotone supermodular, the rationalizable strategies are bounded by extreme Bayesian Nash equilibria, which need to be symmetric and monotone (Milgrom & Roberts (1990) and Van Zandt & Vives (2007)).³ Also, b is strictly dominant at $x_i = -L$, and a is strictly dominant at $x_i = L$. Hence, each extreme equilibrium has a cutoff value $\hat{x} \in (-L, L)$, such that a player plays a if his signal is higher than \hat{x} , and he plays b if his signal is lower than \hat{x} . At the cutoff value, player i must be indifferent between a and b . But his payoff from playing a is

$$u_i(a|\hat{x}, s_j^*) = E_i[\theta|\hat{x}] - \Pr_i(x_j < \hat{x}|\hat{x}) = \hat{x} - (1 - q),$$

while his payoff from playing b is zero. Therefore,

$$\hat{x} = 1 - q.$$

That is, extreme equilibria are equal up to the specification of the action at the unique cutoff. Therefore, they are the only rationalizable strategies. ■

Excluding the degenerate signal value $x_i = 1 - q$, the first part of the proposition states that the resulting game is dominance-solvable. Hence, almost all known game-theoretic solution concepts agree on a unique solution. According to this unique solution, a player invests, playing action a , if and only if the sum of his expected value of the project, namely

³We have differing priors, but this is not important for the above results on supermodular games. For the sake of completeness, we present a direct proof in the appendix.

$x_i = E_i[\theta|x_i]$, and his sentiment, namely q , exceeds 1. When ε is close to zero, x_i is approximately equal to θ , and each player invests if and only if

$$\theta + q > 1.$$

Here, the economic fundamental, i.e., the value of the project, plays an intuitive role. Given any q , the players invest if the value of the project is sufficiently high, and they do not invest if that value is low. Somewhat more surprisingly, the investor sentiments play an equally important role. Given any value of the project, the players invest if they have sufficiently high sentiments, and they do not invest if their sentiment is low. In fact, the level of sentiments is a direct substitute for the value of the project.

In a non-strategic environment, a player's sentiments would have been measured by how inflated his expectation of the project is with respect to a reference expectation. Here, the reference expectation is the expectation the researcher finds reasonable, such as the average return from similar projects in the past. If the player were to incur a cost c for investing in a project with return θ , then he would have invested if $E_i[\theta|x_i] \geq c$. The difference between $E_i[\theta|x_i]$ and the reference expectation $E[\theta|x_i]$ would then measure the player's optimism. Such deviations would play a significant role in the player's behavior only in a highly uncertain environment. For example, with the noise structure in (1), $E_i[\theta|x_i]$ converges to θ as $\varepsilon \rightarrow 0$. In the limit, the player invests if and only if it is beneficial to do so.

In our games, the relevant measure of investor sentiments, q , is orthogonal to the non-strategic notion of sentiments above. Firstly, independent of the level ε of uncertainty, q determines whether the players will invest in a project, by augmenting their expected payoffs. In particular, suppose that ε is close to zero, so that there is only a negligible amount of uncertainty. Even in this extreme situation, q may be close to 1, in which case players will invest in the project so long as the project has some value ($\theta > 0$), or q may be close to 0, in which case players will not invest in the project unless it is a dominant strategy to do so. Therefore, the investor sentiments we uncover may affect the outcome even in a nearly certain environment, in which one might have thought that there is no room for optimism or pessimism.

Secondly, q affects the outcome regardless of whether players think that the other players are optimistic in the non-strategic sense, i.e., its effect is independent of $E_i[E_j[\theta|x_j]|x_i] - x_i$. To see this, suppose that

$$E_i(\eta_j) = 0,$$

i.e., it is common knowledge that players' signals are unbiased. In that case, we have $E_i[x_j|x_i] = x_i$, so that $E_i[E_j[\theta|x_j]|x_i] = x_i$, $E_i[E_j[E_i[\theta|x_i]|x_j]|x_i] = x_i$, \dots , up to arbitrarily high orders of expectation (when ε/L is small).⁴ Even in that case, q may take any value between 0 and 1, leading to either outcome.

We must emphasize that the relevant measure of sentiments that captures the strategic behavior varies with the payoff structure one considers. In the games of this section, the best response of a player depends only on his probability assessment about the other player's action. His strategic behavior is then determined by the probability of other player being interim more optimistic, independent of how optimistic the other player is expected to be. In games with highly nonlinear best response functions, the relevant measure of sentiments would be close to ours. On the other extreme, in games with linear best response functions, the relevant measure of optimism would be $E_i[\eta_j - \eta_i]$. (This is due to a known formula about the unique rationalizable strategy in such games. In these games, when the equilibrium is stable, the effect of sentiments, measured by $E_i[\eta_j - \eta_i]$, becomes negligible as the uncertainty vanishes.)

We have made several simplifying assumptions, which are maintained in most of the related applied papers. These assumptions do not play a significant role in the above result. We close this section by describing how these assumptions can be relaxed:

1. **Signal Structure:** We considered a signal structure with additive noise in (1). For our analysis it suffices that incomplete information game is monotone supermodular and the type space is totally ordered. In that case we simply define q as the probability that the other player's signal is higher than one's own.
2. **Uniformity of θ and Independence of (η_1, η_2) :** As in Carlsson and van Damme (1993), for small ε , it suffices that (i) θ has Lipschitz continuous density with a support containing $[-L, L]$ and (ii) (η_1, η_2) has a continuous density.
3. **Type-independent q :** In our model, the relevant sentiment q does not depend on the type, x_i . This stems from the standard additive noise structure *and* the simplifying assumption that θ is uniformly distributed. When the uniformity assumption is relaxed, the relevant sentiment is measured by $q(x_i) = \Pr_i(x_j \geq x_i|x_i)$, and it depends on the type. Nevertheless, our result remains true with this type-dependent senti-

⁴Precisely, given any K , there exists $\bar{\varepsilon} > 0$ such that for each $\varepsilon < \bar{\varepsilon}$ and $k < K$, the k th-order expectation of θ is x_i .

ment whenever the game is dominance solvable (as in general monotone supermodular games).

4. **Symmetry:** Without the symmetry assumption, we would define the sentiments by

$$q = \Pr_1(\eta_2 > \eta_1 - \Delta) = \Pr_2(\eta_1 > \eta_2 + \Delta), \quad (2)$$

where Δ is the unique solution to the equation $\Pr_1(\eta_2 > \eta_1 - \Delta) = \Pr_2(\eta_1 > \eta_2 + \Delta)$. The equilibrium strategies would be asymmetric, but q would play a similar role. As $\varepsilon \rightarrow 0$, the equilibrium strategies would converge to the strategy profile in Proposition 1 with q defined by (2).

3 Sentiments in games of regime change

In this section, we will consider a general class of games of regime change with a continuum of players. The games we consider will cover most games that have been used in the existing literature to model multiplicity and the role of incomplete information in many economic situations, such as currency attacks (Morris & Shin (1998)). For these games, we will determine the relevant measure of sentiments, which captures the players' strategic behavior.

Consider a continuum of players $[0, 1]$. Each player takes one of two actions, a and b . As in the previous section, the consequence of a depends on the regime that realizes and the payoff parameter θ . The other action, b , corresponds to remaining passive and always yields a constant payoff that is normalized to zero. There are two regimes: *status quo* and the *new regime*. Action a yields a payoff of $u(\theta) - c > 0$ under the new regime and $-c$ under the status quo for some $c > 0$. For any value of θ , if the measure of players who take action a exceeds $r(\theta)$, then the new regime is established; otherwise the status quo prevails. Here, θ measures the relative desirability of the new regime.⁵ We maintain the following regularity assumption throughout the section:

Assumption 1 *Function u is weakly increasing and continuous. Function r is continuous. There exist $\underline{\theta}$ and $\bar{\theta}$ such that $r(\theta) = 1$ when $\theta < \underline{\theta}$, $r(\theta) = 0$ when $\theta > \bar{\theta}$, and $r(\theta)$ is strictly decreasing on $(\underline{\theta}, \bar{\theta})$.*

⁵In the literature on regime changes, the fundamental usually represents the relative strength of the status quo, corresponding to putting the reverse order on θ . Our order makes the definitions of optimism and pessimism more straightforward.

The payoffs of any player i are presented in the following table, where A denotes the measure of players who take action a :

	$A \geq r(\theta)$	$A < r(\theta)$
a	$u(\theta) - c$	$-c$
b	0	0

For example, in the model of Morris & Shin (1998), we can interpret θ as the “weakness” of the currency so that the floating exchange rate f is decreasing in θ . Let c be the cost of short-selling the currency and e be the current exchange rate. The payoff of an attacker is $u(\theta) - c \equiv e - f(\theta) - c$ if the currency is floated and $-c$ if it remains pegged. For any θ , the central bank defends the currency if the measure A of attackers stays below $r(\theta)$ and floats it otherwise. Here, r is decreasing in θ because it is costlier to defend a weaker currency.

When $r(\theta) \in (0, 1)$ and θ is common knowledge, there are multiple equilibria: all players may play a or all players may play b . We will maintain the incomplete information structure of the previous section. Each player i observes a noisy signal

$$x_i = \theta + \varepsilon \eta_i,$$

where η_i is a noise term that takes values in $[-1, 1]$ and $\varepsilon \in (0, 1)$ is a scalar that measures the level of uncertainty players face. Recall that θ is distributed uniformly on $[-L, L]$. We will assume that $-L < \underline{\theta} - 1 < \bar{\theta} + 1 < L$. We will further assume that, under each players’ beliefs, the noise terms are all stochastically independent from each other and from θ . We will also assume that players’ beliefs are symmetric: according to each player $i \in [0, 1]$, the cumulative density function of η_i is F with density f , and the cumulative density of η_j is G with density g for each $j \neq i$. We will assume that both F and G are strictly increasing on $[-1, 1]$, i.e. we have full support.

Sentiments For this class of games, the relevant notion of sentiments is similar to the notion in the previous section. It is slightly more involved, as it depends on the quantiles of the other players:

$$q(p) = F(G^{-1}(p)). \quad (3)$$

The function q is illustrated in Figure 1. Here, $q(p)$ is the probability a player i assigns to the event that the measure of the players who have more optimistic outlooks than player i is at least $1 - p$. The relevant value of $1 - p$ will be the measure of (other) players needed for a regime change.

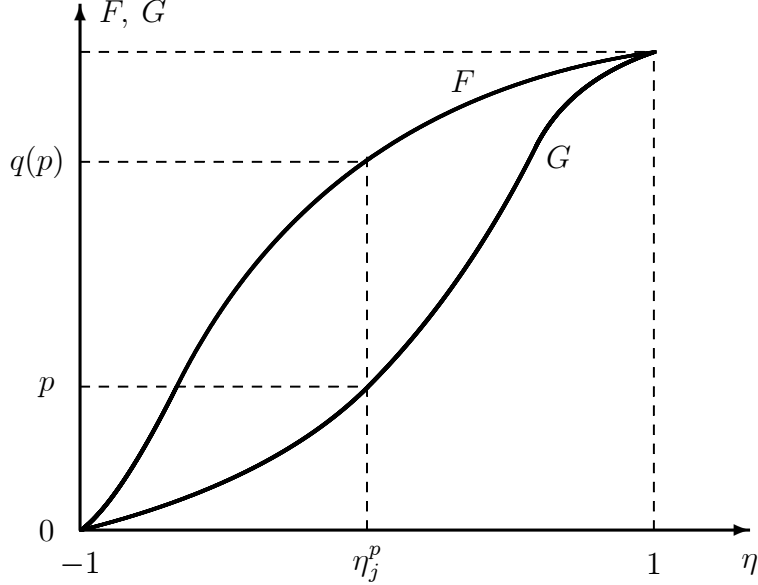


Figure 1: An illustration of $q(p)$ when there is optimism at all quantiles.

Notice that, under the common prior assumption, we would have $F = G$, and hence $q(p) = p$. Suppose now that player i thinks that the other players tend to have a more optimistic outlook in the sense that η_j first-order stochastically dominates η_i , i.e. $F \geq G$. Then, we have $q(p) \geq p$ for each p , and the inequality is strict whenever $F > G$ at the relevant value of the noise. On the other hand, if player i thinks that the other players tend to have a more pessimistic outlook, i.e., $F \leq G$, then we have $q(p) \leq p$ for each p . Hence,

$$q(p) - p = F(\eta_j^p) - G(\eta_j^p)$$

can be taken to be a measure of optimism (of the fellow players according to player i), where $\eta_j^p = G^{-1}(p)$ is the level of noise in the signal of the player at quantile p .

As the underlying uncertainty decreases, variations in beliefs typically lead to larger variations in q . For example, suppose that according to i , $\eta_i = \lambda \bar{\eta}_i$ and $\eta_j = \lambda \bar{\eta}_j + \beta$ for $j \neq i$ where $(\bar{\eta}_k)_{k \in N}$ are independently and identically distributed with cdf \bar{F} . Here, each player believes that the other players' signals have a bias β , and the level of uncertainty is scaled by λ . We have $F(\eta) = \bar{F}(\eta/\lambda)$ and $G(\eta) = \bar{F}((\eta - \beta)/\lambda)$. Consequently,

$$q(p) = F(G^{-1}(p)) = \bar{F}(\bar{F}^{-1}(p) + \beta/\lambda).$$

Hence, the effect of scaling up uncertainty by λ is the same as scaling $down$ the bias by λ .

The next result shows that q determines the players' strategies in an intuitive way when the noise parameter ε is small.

Proposition 2 Let $\hat{x} \in (\underline{\theta}, \bar{\theta})$ be the (unique) solution to the equation

$$q(1 - r(x))u(x) = c. \quad (4)$$

Then, for any i and any $x_i \neq \hat{x}$, there exists $\bar{\varepsilon} > 0$ such that for each $\varepsilon \in (0, \bar{\varepsilon})$, the unique rationalizable action for signal value x_i is

$$s_i^*(x_i) = \begin{cases} a & \text{if } x_i > \hat{x}, \\ b & \text{if } x_i < \hat{x}. \end{cases}$$

Proof. See the Appendix. ■

Proposition 2 states that, when ε vanishes, the relevant quantile is $1 - r(\hat{x})$, and the unique rationalizable action is determined by $q(1 - r(\hat{x}))$. Indeed, observing $x_i = \hat{x}$ player i knows that the fundamental is close to \hat{x} , and that approximately $r(\hat{x})$ investing players are needed for the regime to change. In turn, player i believes that the regime will change if a player at $p = 1 - r(\hat{x})$ quantile has a more optimistic outlook than he does, and the probability of such an event is exactly $q(1 - r(\hat{x}))$. For player i to be indifferent between investing and staying passive, his expected gain from investment—benefit $u(\hat{x})$ multiplied by the probability of regime change $q(1 - r(\hat{x}))$ —has to be equal to the cost c .

When the sentiments are high in the sense that the probability $q(1 - r(\hat{x}))$ of having enough more optimistic players is high, the underlying threshold level \hat{x} of the unique rationalizable strategy is low. To see this, fix the payoffs u and c , and vary the beliefs F and G . Each (F, G) leads to a unique level of relevant sentiments $\hat{q}^{F,G} = q(1 - r(\hat{x}^{F,G}))$, where $\hat{x}^{F,G}$ is the cutoff value corresponding to (F, G) . Then, we have

$$\hat{x}^{F,G} = u^{-1}(c/\hat{q}^{F,G}).$$

Hence, the higher the level of sentiments $\hat{q}^{F,G}$ for the relevant quantile in equilibrium, the lower the cutoff value $\hat{x}^{F,G}$. When $\hat{q}^{F,G}$ gets higher, the set of signal values and θ at which players take action a enlarges, and the regime change becomes more likely. The next result formalizes this comparative statics observation, and shows that if the level of sentiments increases at all quantiles, then, under the unique rationalizable strategy, action a is taken more often. Hence, when the sentiments are higher at all quantiles, the regime change is more likely.

Proposition 3 Fix u and c , and consider the family of beliefs (F, G) . Let $q^{F,G}$ be the sentiment function under (F, G) , where $q^{F,G}(p) = F(G^{-1}(p))$. Consider any (F, G) and (F', G') with

$$q^{F,G}(p) > q^{F',G'}(p) \quad (\forall p \in (0, 1)).$$

The cutoff for the unique rationalizable strategy in the limit $\varepsilon \rightarrow 0$ is lower under (F, G) :

$$\hat{x}^{F,G} < \hat{x}^{F',G'}.$$

That is, for any x_i , there exists $\bar{\varepsilon} > 0$ such that for any $\varepsilon < \bar{\varepsilon}$, if a is rationalizable at x_i under (F', G') , then a is uniquely rationalizable at x_i under (F, G) ; moreover, for some signal values x_i with positive probability, the unique rationalizable action is a under (F, G) and b under (F', G') . Finally, when ε is small, the probability of regime change is higher under (F, G) .

Proof. For every $x \in (\underline{\theta}, \bar{\theta})$, we have

$$q^{F,G} (1 - r(x)) u(x) > q^{F',G'} (1 - r(x)) u(x). \quad (5)$$

Moreover, $q^{F,G}$ and $q^{F',G'}$ are strictly increasing, and r is strictly decreasing in this region. Hence, both $q^{F,G} (1 - r(x)) u(x)$ and $q^{F',G'} (1 - r(x)) u(x)$ are strictly increasing functions of x . Then, (5) implies that the solution $\hat{x}^{F,G}$ to $q^{F,G} (1 - r(x)) u(x) = c$ is strictly less than the solution $\hat{x}^{F',G'}$ to $q^{F',G'} (1 - r(x)) u(x) = c$. The second statement in the proposition then follows from Proposition 2. To see the last statement in the proposition, observe that, by Proposition 2, as $\varepsilon \rightarrow 0$, the probability of regime change approaches to $\Pr_i(\theta > \hat{x}) = (1 - \hat{x}/L)/2$. ■

We have made several simplifying assumptions, which are maintained in most of the related applied papers. These assumptions do not play a significant role in the above result. We close this section by describing how these assumptions can be relaxed (without repeating the comments at the end of the previous section):

1. **Independence:** We assumed that under each player's beliefs the noise terms of the other players are identically and independently distributed. This assumption allows us to easily compute the p th quantile μ^p among all of the noise terms of the other players. (Indeed, if the noise terms are *iid* according to cdf G under each player's beliefs, then $\mu^p = G^{-1}(p)$ almost surely.) Our measure of sentiments captures each player's *aggregate* perceptions— $q(p)$ is a probability that at least p other players have a more optimistic outlook than a player in question does. Therefore, whether individual noise terms are identically and/or independently distributed under each player's beliefs is not crucial; all of our results remain intact as long as there is no aggregate uncertainty about the quantiles $\{\mu^p\}_{p \in [0,1]}$ (maintaining the assumptions of symmetry among individual

beliefs \Pr_i and of stochastic independence between the fundamental and the noise terms).⁶

2. **Symmetry:** The symmetry assumption is ingrained in our formulation in this section, but it is not crucial. We now illustrate how it can be relaxed on a specific example. As before, suppose that, under each player beliefs, the noise terms and θ are stochastically independent. Suppose that according to i , η_i is distributed according to $F(\eta)$, and, if $i \in [0, \frac{1}{2}]$, then each η_j is also distributed according to $F(\eta)$, while if $i \in (\frac{1}{2}, 1]$, then each η_j is distributed according to $G(\eta) = F(\eta - \beta)$. That is, half of the players believe that the other players' signals have a bias β . Then, for each ε , there is a pair of thresholds $\hat{x}^1(\varepsilon)$ and $\hat{x}^2(\varepsilon)$, for players $i \in [0, \frac{1}{2}]$ and $i \in (\frac{1}{2}, 1)$, respectively, which solve the following system

$$\begin{aligned} U^1(\hat{x}^1) &\equiv E_i \left[1_{\frac{1}{2}F(\eta_i) + \frac{1}{2}F(\eta_i - \Delta) + r(\theta) \leq 1} u(\theta) \mid \theta + \varepsilon\eta_i = \hat{x}^1 \right] = c, \text{ for } i \leq \frac{1}{2}, \\ U^2(\hat{x}^2) &\equiv E_i \left[1_{\frac{1}{2}G(\eta_i) + \frac{1}{2}G(\eta_i + \Delta) + r(\theta) \leq 1} u(\theta) \mid \theta + \varepsilon\eta_i = \hat{x}^2 \right] = c, \text{ for } i > \frac{1}{2}, \end{aligned}$$

where $\Delta = \frac{\hat{x}_1 - \hat{x}_2}{\varepsilon} \approx \beta$. When the noise vanishes, $\hat{x}^1(\varepsilon)$ and $\hat{x}^2(\varepsilon)$ converge to \hat{x} , and $\Delta(\varepsilon)$ converges to β . Letting $H(\eta) = \frac{1}{2}F(\eta) + \frac{1}{2}G(\eta) = \frac{1}{2}F(\eta) + \frac{1}{2}F(\eta - \beta)$, define

$$q(p) = F(H^{-1}(p)).$$

Then, as in Proposition 2, \hat{x} is the unique solution to $q(1 - r(x))u(x) = c$. Thus, the relevant measure of sentiments in this example corresponds to aggregation of sentiments over all players.⁷

4 Conclusion

In this paper, we considered a general class of games that have been used for modeling many economic problems where players' sentiments are believed to play an important role in the outcomes. Allowing players to have differing priors, we identified the relevant notion of sentiments for strategic behavior. We have shown that the relevant notion of sentiments asks: what is the probability that there are enough players who have more optimistic outlook

⁶This is so with a slight clarification that function G is to be defined as an inverse of the quantiles, $G(\mu^p) = p$, instead of being a cdf of a single individual noise term.

⁷Note that, the quantile function and the threshold \hat{x} , are exactly the same as if each player i has the same belief that half of η_j are distributed according to F and the other half according to G . Indeed, in this case, all the quantiles η^p are well-defined and satisfy $\eta^p = H^{-1}(p)$.

than you have—enough for the success of the project or the regime change? This notion of sentiments is significantly different from the relevant notion of sentiments in non-strategic environments. In the latter case, the relevant notion of sentiments would be the level of expectations for the player, and there would be little room for optimism or pessimism when there is little uncertainty. In strategic environments, the relevant measure of sentiments can vary arbitrarily and have large impact on the strategic behavior even when there is little uncertainty.

A A Direct Proof of Proposition 1

Given any belief, for player i , action a yields the expected payoff of

$$E_i[\theta|x_i] - \Pr_i(s_j(x_j) = b|x_i),$$

and action b yields 0. Moreover, for any $x_i \in (-L + \varepsilon, L - \varepsilon)$, $E_i[\theta|x_i] = x_i$. Hence, his unique best response is a if

$$x_i > \Pr_i(s_j(x_j) = b|x_i),$$

and b if $x_i < \Pr_i(s_j(x_j) = b|x_i)$. In particular, at round 0, we eliminate action a for each $x_i < 0 \equiv \underline{x}^0$ and eliminate action b for each $x_i > 1 \equiv \bar{x}^0$. For any $k > 0$, assume that at round $k - 1$ or before, action a has been eliminated for each $x_j < \underline{x}^{k-1}$ and action b has been eliminated for each $x_j > \bar{x}^{k-1}$ for some $\underline{x}^{k-1}, \bar{x}^{k-1} \in [0, 1]$. Assume also that player i assigns zero probability on strategies that have been eliminated at round $k - 1$ or before. Then,

$$\Pr_i(s_j(x_j) = b|x_i) \leq \Pr_i(x_j \leq \bar{x}^{k-1}|x_i) = \Pr_i(\eta_j \leq \eta_i + (\bar{x}^{k-1} - x_i)/\varepsilon),$$

for $s_j(x_j) = a$ whenever $x_j > \bar{x}^{k-1}$. Since the expression on the right-hand side is decreasing and continuous in x_i , there exists a (unique) \bar{x}^k such that

$$\Pr_i(\eta_j \leq \eta_i + (\bar{x}^{k-1} - \bar{x}^k)/\varepsilon) = \bar{x}^k, \tag{6}$$

and

$$\Pr_i(s_j(x_j) = b|x_i) \leq \Pr_i(\eta_j \leq \eta_i + (\bar{x}^{k-1} - x_i)/\varepsilon) < x_i$$

whenever $x_i > \bar{x}^k$. Hence, at round k , we eliminate action b for each $x_i > \bar{x}^k$. Notice that the sequence (\bar{x}^k) is non-increasing and positive, so that it converges to some \bar{x}^∞ . By (6),

$$\bar{x}^\infty = \Pr_i(\eta_j \leq \eta_i + (\bar{x}^\infty - \bar{x}^\infty)/\varepsilon) = \Pr_i(\eta_j \leq \eta_i) = 1 - q.$$

Similarly, one can construct a non-decreasing sequence (\underline{x}^k) , where action a is eliminated for each $x_i < \underline{x}^k$ at round k , and $\underline{x}^k \rightarrow \underline{x}^\infty = 1 - q$.

B Proof of Proposition 2

We first present the unique rationalizable strategy for arbitrary ε :

Proposition 4 *For each ε , let $\hat{x}(\varepsilon)$ be the (unique) solution to the equation*

$$U(x) \equiv E_i [1_{r(\theta) \leq 1 - G(\eta_i)} u(\theta) | \theta + \varepsilon \eta_i = x] = c. \quad (7)$$

Then, for any $x_i \neq \hat{x}(\varepsilon)$, there exists a unique rationalizable action $s_i^(x_i; \varepsilon)$ for signal value x_i where*

$$s_i^*(x_i; \varepsilon) = \begin{cases} a & \text{if } x_i > \hat{x}(\varepsilon), \\ b & \text{if } x_i < \hat{x}(\varepsilon). \end{cases}$$

Here, $1_{r(\theta) \leq 1 - G(\eta_i)}$ is the characteristic function of the event $r(\theta) \leq 1 - G(\eta_i)$, taking value of 1 when $r(\theta) \leq 1 - G(\eta_i)$ and 0 otherwise. This is the event that the measure of people who are more optimistic than player i , namely $1 - G(\eta_i)$, is higher than what is needed for a regime change, namely $r(\theta)$. When the signal of player i is at the cutoff value, this event will correspond to a regime change because the people who take the costly action are exactly the same people who are more optimistic than player i .

Proof. As in the proof of Proposition 1, it suffices to show that there is a unique cutoff \hat{x} for a monotone symmetric equilibrium, and \hat{x} is the unique solution to (7). To this end, we will first show that the payoff from playing a is $U(\hat{x}) - c$ at any such cutoff \hat{x} . Since b yields 0, this shows that any cutoff \hat{x} must be a solution to (7). We will then show that (7) has a unique solution, showing the uniqueness of the cutoff.

Towards showing that the payoff from a is $U(\hat{x}) - c$ at the cutoff \hat{x} , note that if i takes action a , he will get $u(\theta) - c$ when the measure $A(\theta)$ of players who take action a exceeds $r(\theta)$ (and thus there is a regime change) and $-c$ otherwise. But a player j takes action a iff $x_j \geq \hat{x}$, i.e., iff $\eta_j \geq (\hat{x} - \theta) / \varepsilon$. Then, by the strong law of large numbers,

$$A(\theta) = \Pr_i(\eta_j \geq (\hat{x} - \theta) / \varepsilon) = 1 - G((\hat{x} - \theta) / \varepsilon) = 1 - G(\eta_i).$$

Hence, player i gets $u(\theta) - c$ if $1 - G(\eta_i) \geq r(\theta)$ and $-c$ if $1 - G(\eta_i) < r(\theta)$. Conditional on $x_i = \hat{x}$, the expected value of this is $U(\hat{x}) - c$.

In order to show that (7) has a unique solution, we will show that U is strictly increasing at any solution to (7). For this, it suffices to show: (i) $1_{G(\eta_i) + r(\theta) \leq 1} u(\theta)$ is increasing in $(\theta, -\eta_i)$, (ii) it is strictly increasing in θ with positive probability at the solution, (iii) conditional distribution of $(\theta, -\eta_i)$ on x is increasing in x_i in the sense of first-order stochastic dominance, and (iv) it is strictly increasing for θ . Now, (i) is true because r is decreasing and u and G are increasing; (ii) is true because by the assumption that $u(\theta) > c > 0$, at any solution to (7), the probability of $G(\eta_i) +$

$r(\theta) \leq 1$ and $G(\eta_i) + r(\theta) > 1$ are both positive. Finally, since θ is uniformly distributed over a large interval, (iii) and (iv) hold: the conditional distribution of $\eta_i = (x - \theta) / \varepsilon$ is independent of x (with $-L \ll x \ll L$), and the conditional distribution of θ is given by the cumulative distribution function $1 - F((x - \theta) / \varepsilon)$, strictly decreasing in x . ■

Proof of Proposition 2. We have

$$\begin{aligned}
\lim_{\varepsilon \rightarrow 0} U(x) &= \lim_{\varepsilon \rightarrow 0} E_i [1_{G(\eta_i) + r(\theta) \leq 1} u(\theta) | \theta + \varepsilon \eta_i = x] \\
&= \lim_{\varepsilon \rightarrow 0} E_i [1_{G(\eta_i) + r(x) \leq 1} u(x) | \theta + \varepsilon \eta_i = x] \\
&= \Pr_i (G(\eta_i) + r(x) \leq 1) u(x) \\
&= \Pr_i (\eta_i \leq G^{-1}(1 - r(x))) u(x) \\
&= F(G^{-1}(1 - r(x))) u(x) \\
&= q(1 - r(x)) u(x),
\end{aligned}$$

where the second equality is obtained using continuity of u and r and boundedness of η_i ; the third equality is by simple algebra, and the fourth equality is by monotonicity of G . Now, notice that in the limit $\varepsilon \rightarrow 0$, (7) has a unique solution, \hat{x} .⁸ Since the solution to (7) is upper-hemicontinuous in ε , this implies that $\hat{x}(\varepsilon) \rightarrow \hat{x}$ as $\varepsilon \rightarrow 0$. Therefore, for every $x_i < \hat{x}$ (resp., $x_i > \hat{x}$), there exists $\bar{\varepsilon} > 0$ such that for every $\varepsilon < \bar{\varepsilon}$, we have $\hat{x}(\varepsilon) > x_i$ (resp., $\hat{x}(\varepsilon) < x_i$), and thereby b (resp., a) is the unique rationalizable action at x_i . ■

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⁸Since $u(x) > c > 0$, at the solution, $q(1 - r(x)) \in (0, 1)$, and so that r is strictly decreasing. Moreover, since F and G are strictly decreasing, q is strictly increasing. Therefore, $q(1 - r(x)) u(x)$ is strictly increasing at the solution, showing that there can be at most one solution. Moreover, $q(1 - r(\underline{\theta})) = q(0) = 0$, so that $q(1 - r(\underline{\theta})) u(\underline{\theta}) = 0 < c$, and $q(1 - r(\bar{\theta})) = q(1) = 1$, so that $q(1 - r(\bar{\theta})) u(\bar{\theta}) = u(\bar{\theta}) > c$. Therefore, there exists a (unique) solution $\hat{x} \in (\underline{\theta}, \bar{\theta})$.

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