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Learning Purified Mixed Equilibria[1]

Glenn Ellison

Department of Economics, Massachusetts Institute of Technology, 50 Memorial Drive, Cambridge, Massachusetts 02142
gellison@mit.edu

and

Drew Fudenberg

Department of Economics, Harvard University, Cambridge, Massachusetts 02138
dfudenberg@harvard.edu

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We examine the local stability of mixed equilibria in a smoothed fictitious play model. Our model is easy to analyze and yields the same conclusions as other models in 2×2 games. We focus on 3×3 games. Contrary to some previous suggestions, learning can sometimes provide a justification for complicated mixed equilibria. Whether an equilibrium is stable often depends on the distribution of payoff perturbations. The totally mixed equilibria of zero sum games are generically stable, and the totally mixed equilibria of symmetric games with symmetric perturbations are generically unstable. *Journal of Economic Literature* Classification Number: C72.

1. INTRODUCTION

In this paper we explore a potential justification for mixed-strategy equilibria based on the idea that an equilibrium distribution might arise in a large population as the result of a learning process in the style of fictitious play. The model is similar in spirit to that of Fudenberg and Kreps [4],

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who proposed a "smoothed" version of fictitious play in which small payoff perturbations in the style of Harsanyi's purification theorem make players' behavior random. That paper and several subsequent analyses have shown that in 2×2 games play in smoothed learning models converges to the mixed equilibrium in games like matching pennies, while the seemingly unreasonable mixed equilibria of coordination games are unstable.

While the 2×2 results are quite intuitive, when we should regard a mixed equilibrium as being reasonable or unreasonable is much less obvious in more complicated games. In fact, it is not clear from the existing literature whether we should ever expect players to learn to play mixed equilibria that give positive weight to more than two strategies. All 2×2 games with a unique mixed equilibrium are strategically equivalent to zero-sum games. In a zero-sum game with a unique mixed equilibrium, a player who is trying to maximize his own payoff will try to minimize his opponent's payoff. If his opponent is maximizing, this will involve equating the opponent's payoff across the available pure strategies. In more complicated games, however, maximizing one's own payoff and equating one's opponent's payoffs across pure strategies are not clearly linked, and thus the previous intuition does not provide a clear reason to think mixed equilibria should arise. Moreover, previous results for 3×3 games under other learning/evolutionary models have emphasized instability. For example, Krishna and Sjostrom [15] show that generically the continuous time version of the standard fictitious play model cannot converge cyclically to a mixed strategy equilibrium in which both players use more than two strategies; and Hofbauer and Sigmund [10] note that the totally mixed equilibria of 3×3 games are never asymptotically stable under the two-population replicator dynamics.

In this paper, we take a first step beyond the 2×2 case in exploring the behavior of a smooth learning model in 3×3 games. We regard our results as being favorable to the ideas that learning models can provide a justification for some mixed equilibria and can help us think about when mixed equilibria are or are not reasonable.

In our model, a continuum of player 1's and a continuum of player 2's are randomly matched in continuous time to play a fixed strategic-form game, with players observing the aggregate frequencies of each strategy in each population. There is a small degree of heterogeneity in the preferences of the players in each population, and hence it is possible, for example, for those player 1's with a slight preference for Heads in a matching pennies game to learn to play Heads while those player 1's with a slight preference for Tails come to play Tails. In this way, the aggregate distribution of the player 1's play can resemble the mixed strategy $(\frac{1}{2}H, \frac{1}{2}T)$ even though no individual player 1 uses a mixed strategy.

The model is designed to be easy to analyze and to share the appealing features of other forms of smoothed fictitious pay. If the time average of play converges then the instantaneous distribution of play converges as well, and the common limit is a "purified" mixed equilibrium of the game. In 2×2 "games of conflict" with a unique mixed equilibrium, the purified equilibrium is asymptotically stable; in a coordination game the (inferior) mixed equilibrium is a steady state of the learning process, but is unstable.

The largest part of the paper is devoted to an analysis of the local stability of equilibria in 3×3 games. Our first result provides a general answer, showing that whether a totally mixed equilibrium is locally stable when the payoff perturbations are sufficiently small depends on whether three necessary and jointly sufficient conditions on the payoffs of the game and the distribution of the heterogeneity are satisfied. The conditions include a generalization of the game of coordination/game of conflict dichotomy and conditions related to the possibility of cycling and other behaviors. We provide intuition for the conditions by examining a number of examples and restricted classes of games.

We regard our results as fairly supportive of the idea that populations may learn to play certain mixed strategy equilibria. While mixed equilibria can fail to be stable even when they are unique, the set of games in which the mixed equilibrium is stable is not a measure zero subset. More strongly, in simulations of a model with i.i.d. payoff perturbations we find that the totally mixed equilibria of games which do not have a Pareto superior equilibrium are stable about 70% of the time.

Another of our findings is that once one moves beyond the 2×2 case, whether an equilibrium is locally stable is no longer independent of the distribution of payoff heterogeneity. Nonetheless, one can obtain useful insights by looking at how the stability of a game is jointly determined by the nature of the game and the distribution of heterogeneity. In zero-sum games we find a generic stability result. When the distribution of heterogeneity is the same in the two populations, we find that the mixed equilibria of symmetric games are generically unstable.

Our formal model is similar to that of Fudenberg and Kreps [4], although we have a continuum of players on each side and we work from the outset in continuous time, rather than using stochastic approximation techniques to relate the asymptotic behavior of a discrete-time model and its continuous times analog. While the alternative model used here suggests different interpretations and applications, from a mathematical standpoint the choice between them is solely a matter of convenience if the goal is to characterize long-run behavior. Benaim and Hirsch [2], Kaniovski and Young [14], and Fudenberg and Levine [5, 7] consider more general forms of smoothed fictitious play. Our paper differs from these because we focus first on the sort of smoothing that arises from payoff perturbations and then on various combinations of restrictions on the form of the perturbations and the nature of the game, in order to derive sharper conclusions.[2]

Hopkins [12] (independently of this paper) explores the behavior of the smoothing of fictitious play proposed by Fudenberg and Levine [7]. He shows that the concavity of the smoothing function implies that the linearization of the dynamic is what he calls a positive definite adaptive (PDA) dynamic.[3] The behavior of a PDA dynamic is very easy to analyze when all eigenvalues of the game's payoff matrix have real parts which are of the same sign. In a one population model this allows him to obtain a general and elegant proof of the stability of learning in symmetric zero-sum games and in games where the payoff matrix is negative definite.

2. THE MODEL

Let G be a two-player game with strategy space $A_1 \times A_2$ and payoff functions $g_i: A_1 \times A_2 \to \mathfrak{R}$. Since our interest is in studying the stability of "totally mixed" Nash equilibria in which all strategies are played with positive probability, we restrict attention to games where such equilibria exist. We further specialize to the (generic) subclass of games where exactly one totally mixed equilibrium exists; denote this equilibrium σ^*. Note that we do not rule out the possibility of G having other equilibria that are not totally mixed.

We will be interested in *population games* derived from G, by which we mean games in which a continuum of player 1's and a continuum of player 2's are randomly matched in continuous time to play a game with the same strategy space as G, but where each player i's payoff is perturbed according to his/her "type" $\theta_i \in \Theta_i$, which is simply a function from A_i to $\mathfrak{R}$. We assume that the distribution of types of player i has a uniformly bounded, continuous, and strictly positive density on $\mathfrak{R}^{A_i}$ and denote the joint distribution over types by F. When a player i of type θ_i plays a_i and is matched with an opponent who plays a_{-i} his payoff in the ε-perturbation of the game, (G^ε, F), is

$$g_i^\varepsilon(a_i, a_{-i}; \theta_i) = g_i(a_i, a_{-i}) + \varepsilon \theta_i(a_i).$$

Although the perturbed game depends on the distribution of types F as well as ε, we will often just write G^ε for (G^ε, F) when this will not cause confusion.

[2] See Fudenberg and Kreps [4], Jordan [13], Aoyagi [1], Ellison [3] and chapter 2 of Fudenberg and Levine [6] for other discussions of fictitious play and its drawbacks.

[3] The concept is similar to Hofbauer and Sigmund's [11] "adaptive" dynamics.

An *equilibrium distribution* for G^ε is a strategy profile σ^ε with the property that for all i and a_i a fraction $\sigma_i^\varepsilon(a_i)$ of the player i's have a_i as a best response to σ_{-i}^ε. A *purifying sequence* for a mixed strategy equilibrium σ^* is a sequence of equilibrium distributions σ^ε of (G^ε, F) that converges to σ^* as $\varepsilon \to 0$.[4]

The population game is very much in the spirit of the perturbed games used in Harsanyi's [8] purification theorem, but as in Fudenberg and Kreps we assume that the perturbations to player i's payoff depend only on his action and not on the complete strategy profile and that the perturbations have unbounded rather than compact support.[5] We also do not require that the perturbations be independently distributed across a player's strategies.

Our learning model is a straightforward adaptation of fictitious play to the continuous-time, random-matching, large-population environment, with players assumed able to observe the distribution of strategy choices in the population as a whole. Formally, write $\sigma_{i\tau}(\theta_i)$ for the play of a player i of type θ_i at time τ. Write $x_{i\tau}$ for the distribution of play of the population of player i's at time τ,

$$x_{i\tau} = \int_{\Theta_i} \sigma_{i\tau}(\theta_i)\, f_i(\theta_i)\, d\theta_i,$$

and $s_{i\tau}$ for the time average of the play of the player i's between time 0 and τ,

$$s_{i\tau} = \frac{1}{\tau} \int_{r=0}^{\tau} x_{ir}\, dr.$$

At each time $\tau > 0$ each of the player i's is assumed to have the same beliefs over the strategies his opponent might use, with the expected distribution

[4] The name is justified by the fact that almost all types of player i have a strict preference for a single pure strategy given any distribution of opponents' play.

[5] The issue of bounded versus unbounded support matters when considering equilibrium distributions that approximate a pure-strategy equilibrium of the original game, but will not matter here. The restriction to payoff perturbations that only depend on a player's own type seems a reasonable simplification and is unimportant in the 2×2 case, but we should point out that in our more general setting the restriction may tend to favor the stability of the steady state. Our intuition here comes from considering a game where all payoff functions are identically zero. Here, the adjustment dynamics are degenerate on the unperturbed game, while in the perturbed game each player type has a dominant strategy and so the equilibrium distribution is globally stable. Thus, stability always obtains when the payoff perturbations are large compared to the payoff differences. In contrast, Harsanyi's more general perturbations need not lead to stability even when the perturbations are large.

of the opponent's play being a weighted average of the time average of play to date and an initial fictitious history μ_{i0}, i.e.,

$$\mu_{i\tau}(a_{-i}) = \frac{\tau s_{-i\tau}(a_{-i}) + \tau_0 \mu_{i0}(a_{-i})}{\tau + \tau_0}.$$

Players are assumed to choose strategies myopically given these beliefs, so that

$$\sigma_{i\tau}(\theta_i) = \text{Argmax}_\sigma \, g_i^e(\sigma, \mu_{i\tau}; \theta_i).$$

We assume that the nature of the random matching is such that the instantaneous flow of matches exactly matches the distribution of play in the population. The evolution of beliefs is thus a deterministic process described by

$$\dot{\mu}_{i\tau} = \frac{x_{-i\tau} - \mu_{i\tau}}{\tau + \tau_0}.$$

If we renormalize the time dimension by writing t for $\log(\tau + \tau_0)$, the dynamics in t-space are stationary,

$$\dot{\mu}_{it} = x_{-it} - \mu_{it}.^6$$

Throughout this paper we will normalize the time dimension in this way and focus on the evolution of the players' beliefs over time.[7]

We will say that *the empirical averages converge to* σ if $\lim_{t \to \infty} s_{it} = \sigma_i$ for each player i. This is the sense in which the traditional fictitious play model sometimes converges to a mixed strategy equilibrium. Note that the empirical averages converge if and only if beliefs converge. We will say that *play converges to* σ if $\lim_{t \to \infty} x_{it} = \sigma_i$ for each i. This form of convergence is made possible by the smoothing caused by payoff heterogeneity and seems to better capture what we normally think of as mixing. The following proposition points out a couple of attractive features of the model.

PROPOSITION 1. (a) *The empirical averages converge to* σ *if and only if play converges to* σ.

(b) *If the empirical averages converge to* σ *then* σ *is an equilibrium distribution of* G^e.

[6] Note that for all i and a_i, $\partial\dot{\mu}_{it}(a_i)/\partial\mu_{it}(a_i) = -1$. This means that the system is volume contracting, a fact that has important implications in dimensions 1 and 2 but seems less useful more generally.

[7] Note that (up to a time normalization) the dynamics of our model coincide exactly with the dynamics which would be obtained from a variant of the model where the players beliefs reflect an exponentially weighted average of past play.

Any equilibrium distribution σ^ε of G^ε is a steady state of the learning dynamics.

Proof. (a) If s_t converges to σ then μ_{it} converges to σ_{-i} for each player i. Because x_{it} is a continuous function of μ_{it}, this implies that x_{it} converges. Given that s_{it} is converging to σ_i, σ_i is clearly the only possible limit for x_{it}. The reverse implication follows trivially from the definition of s_{it}.

(b) If x_t converges to σ, then for each player i we have $\sigma_i = \lim_{t \to \infty} x_{it}(\mu_{it}) = x_{it}(\sigma_{-i})$ which is exactly the condition for σ to be an equilibrium distribution. That beliefs corresponding to an equilibrium distribution are a steady state is obvious—when $\mu_{it} = \sigma^\varepsilon_{-i}$ for all i, $x_{-it}(\mu_{-it}) = \sigma^\varepsilon_{-i}$, and hence $\dot{\mu}_{it} = 0$. ▌

While any equilibrium distribution corresponds to a steady state of our model, the model allows us to draw distinctions between stable and unstable equilibria. When a mixed equilibrium is asymptotically stable we will regard the learning model as providing a reason to think that such an equilibrium might arise, and when a mixed equilibrium is unstable we will think of the model as suggesting that the equilibrium may be unreasonable. To avoid repeating the same phrases over and over, we will abuse terminology slightly and call a purifying sequence stable if there exists an $\bar{\varepsilon} > 0$ such that the elements of the purifying sequence σ^ε are asymptotically stable all $\varepsilon \in (0, \bar{\varepsilon})$. Similarly, we will call a purifying sequence unstable if there exists an $\bar{\varepsilon} > 0$ such that the σ^ε are asymptotically unstable all $\varepsilon \in (0, \bar{\varepsilon})$.[8]

3. 2×2 GAMES

This section analyzes the behavior of our model in 2×2 games. The results here are in large part a recapitulation of past work, but we feel they are worth presenting for a couple of reasons. First, the calculations are very straightforward, which may make them useful pedagogically. Second, laying out the details of this simple case will help set the stage for the analysis of 3×3 games in the next section.

Write a_{i1} and a_{i2} for the pure strategies available to player i. Let $\Delta_1 = g_1(a_{11}, a_{21}) + g_1(a_{12}, a_{22}) - (g_1(a_{12}, a_{21}) + g_1(a_{11}, a_{22}))$; this is positive or negative depending on whether player 1 is better off on the diagonal or off-diagonal boxes of the payoff matrix. Equivalently it can be thought of as measuring the degree to which player 1 views the actions as complements or substitutes. Similarly, let $\Delta_2 = g_2(a_{11}, a_{21}) + g_2(a_{12}, a_{22}) - (g_2(a_{12}, a_{21}) + g_2(a_{11}, a_{21}))$. We refer to a 2×2 game as a *game of coordination* if

[8] Note that in principle a purifying sequence might be neither stable nor unstable.

$\Delta_1 \Delta_2 > 0$. In the common-interest game shown below the definition is satisfied because each player has a preference for on-diagonal play. In the grab the dollar game the definition is satisfied because each player has a preference for off-diagonal play. We will refer to a 2×2 game as a *game of conflict* if $\Delta_1 \Delta_2 < 0$. The definition classifies a standard matching pennies game shown below as a game of conflict because the players have conflicting preferences as far as on/off diagonal play is concerned (the Δ_1 term is positive and the Δ_2 term is negative.)[9]

Matching pennies

	H	T
H	1, −1	−1, 1
T	−1, 1	1, −1

$\Delta_1 = 4$

$\Delta_2 = -4$

Conflict

Common interests

	A	B
A	2, 2	0, 0
B	0, 0	1, 1

$\Delta_1 = 3$

$\Delta_2 = 3$

Coordination

Grab the dollar

	In	Out
In	−1, −1	1, 0
Out	0, 1	0, 0

$\Delta_1 = -2$

$\Delta_2 = -2$

Coordination

PROPOSITION 2. (a) *if G is a game of conflict, then every purifying sequence for σ^* is stable.*

(b) *if G is a game of coordination, then every purifying sequence for σ^* is unstable.*

Proof. The space of beliefs (μ_1, μ_2) is nominally four dimensional, but the system is really only two dimensional because $u_{1t}(a_{22}) = 1 - \mu_{1t}(a_{21})$ and $\mu_{2t}(a_{12}) = 1 - \mu_{2t}(a_{11})$. Writing (z_1, z_2) for the $(\mu_{2t}(a_{11}), \mu_{1t}(a_{21}))$, the dynamics of the system (with the log-time normalization) are

$$\dot{z}_i = \text{Prob}\{\theta_i(a_{i1}) + g_i(a_{i1}, \mu_i)/\varepsilon > \theta_i(a_{i2}) + g_i(a_{i2}, \mu_i)/\varepsilon\} - z_i$$

$$= \tilde{F}_i((g_i(a_{i1}, \mu_i) - g_i(a_{i2}, \mu_i))/\varepsilon) - z_i,$$

where $\tilde{F}_i$ is the cumulative distribution function of $\theta_i(a_{i2}) - \theta_i(a_{i1})$.

[9] Note that the names "coordination" and "conflict" may seem odd when applied to games without a mixed equilibrium, as some games with a dominant strategy are classified as games of conflict.

Let $\{\sigma^\varepsilon\}$ be a purifying sequence for σ^*. As noted in Proposition 1, each σ^ε is a steady state of the dynamics. Write $\tilde{f}_i$ for the probability density function of $\theta_i(a_{i2}) - \theta_i(a_{i1})$, and let $h_i(\varepsilon) = \tilde{f}_i(1/\varepsilon(g_i(a_{i1}, \sigma^\varepsilon_{-i}) - g_i(a_{i2}, \sigma^\varepsilon_{-i})))$. Linearizing the differential equation around σ^ε, we find $\dot{z} \approx Az$ for

$$A = \begin{bmatrix} -1 & \frac{1}{\varepsilon} \Delta_1 h_1(\varepsilon) \\ \frac{1}{\varepsilon} \Delta_2 h_2(\varepsilon) & -1 \end{bmatrix},$$

It is a standard result in differential equations that σ^ε is asymptotically stable if all of the eigenvalues of A have negative real parts, and it is unstable if one of the eigenvalues has a positive real part.

The eigenvalues of A are the solutions to

$$(1+\lambda)^2 = (1/\varepsilon^2)\, \Delta_1 \Delta_2 h_1(\varepsilon)\, h_2(\varepsilon),$$

so that

$$\lambda = -1 \pm \frac{1}{\varepsilon} \sqrt{h_1(\varepsilon)\, h_2(\varepsilon)\, \Delta_1 \Delta_2}.$$

If G is a game of conflict, then the term in the square root sign is negative, so far for any ε both eigenvalues have a real part of -1, and the equilibrium is stable.

If G is a game of coordination, then the term in the square root sign is positive. The fact that σ^ε is a purified equilibrium implies that

$$\sigma_i^\varepsilon(a_{i1}) = \tilde{F}_i\left(\frac{1}{\varepsilon}(g_i(a_{i1}, \sigma^\varepsilon_{-i}) - g_i(a_{i2}, \sigma^\varepsilon_{-i}))\right).$$

Taking limits as ε goes to zero we find

$$\frac{1}{\varepsilon}(g_i(a_{i1}, \sigma^\varepsilon_{-i}) - g_i(a_{i2}, \sigma^\varepsilon_{-i})) \to \tilde{F}_i^{-1}(\sigma_i^*(a_{i1})),$$

so that the continuity of $\tilde{f}_i$ ensures that $h_i(\varepsilon)$ converges to a positive constant:

$$h_i(\varepsilon) \to \tilde{f}_i(\tilde{F}_i^{-1}(\sigma_i^*(a_{i1}))).$$

The term inside the square root sign in the expression for λ is thus converging to a positive constant as $\varepsilon \to 0$. For ε sufficiently small the system will then have one positive and one negative eigenvalue, and the purified equilibrium will be unstable. ∎

Remarks.

1. Note that classification of equilibria as being stable/unstable does not depend on the distribution of the payoff perturbations.

2. Increasing the degree of heterogeneity in the population (as parameterized by increasing ε) tends to contribute to the stability of mixed equilibria. In games with conflicting interests we have stability for all ε. In games with common interests, we also have stability whenever the $O(1/\varepsilon)$ term is not so large as to outweigh the -1 term in the expression for the eigenvalue.

3. Since the system is volume contracting and has dimension two, it cannot have a nontrivial limit cycle, nor can it have outwards spirals. Thus, in the case of conflicting interests, where the game has a unique Nash equilibrium, local and even global stability follows immediately without the need to compute the eigenvalues of the linearized system, as noted by Benaim and Hirsch [2].

4. While we focus in this paper on mixed strategy equilibria, our model has implications also for the stability of pure strategy equilibria in heterogeneous populations. We have shown (details available on request) that strict pure equilibria are locally stable in games of coordination —provided that additional technical assumption holds, essentially that the distribution of payoff perturbations puts vanishing weights on extreme values. Without this assumption, there may be multiple equilibrium distributions in the neighborhood of the strict equilibrium, and not all of these distributions need be stable.

5. The stability result could clearly be generalized well beyond the fictitious play model. The crucial features of the dynamics are that $\partial\dot{\mu}_{it}(a_{-i1})/\partial\mu_{it}(a_{-i1}) < 0$ and that $\partial\dot{\mu}_{it}(a_{-i1})/\partial\mu_{-it}(a_{i1})$ has the same sign as Δ_{-i}.

4. 3×3 GAMES

The stability conditions for mixed equilibria in 2×2 games had been informally known for a long time before the formal results of Fudenberg and Kreps [4], Benaim and Hirsch [2], and Kaniovski and Young [14]. It is less clear how intuitive notions of stability and instability extend to more general environments, and as we mentioned the existing results on learning mixed equilibria are predominantly negative. Krishna and Sjostrom [15] show that generically a continuous-time exact fictitious play model cannot converge to a cycle whose time-averages correspond to a mixed strategy equilibrium where both players mix over more than two pure strategies. In a two-population version of the replicator dynamic, mixed equilibria are never asymptotically stable, and Hofbauer [9] shows that in generic 3×3 games the replicator dynamic cannot have a neutrally stable steady state.

In this section, we explore the behavior of our learning model in 3×3 games and come to a much less negative conclusion. Our discussion begins with a general result providing a set of (almost) necessary and jointly sufficient conditions for stability. The theorem shows that whether a purification of a totally mixed equilibrium is stable for small ε depends on where the game fits into an extension of the coordination/conflict dichotomy and whether two additional conditions are satisfied. Subsequently, we present a number of examples and derive implications within particular classes of games to provide more intuition for the content of the general result.

4.1. *A General Analysis*

Write a_{11}, a_{12}, and a_{13} for player 1's pure strategies and a_{21}, a_{22}, and a_{23} for player 2's pure strategies. The dynamics of the learning process with the log-time normalization) are once again given by

$$\dot{\mu}_{it} = x_{-i}(\mu_{-it}) - \mu_{it}.$$

We analyze these dynamics as a four-dimensional system parameterized by

$$z_{1t} = \mu_{2t}(a_{11}) \qquad z_{2t} = \mu_{2t}(a_{12}) \qquad z_{3t} = \mu_{1t}(a_{21}) \qquad z_{4t} = \mu_{1t}(a_{22}).$$

Let $\{\sigma^\varepsilon\}$ be a purification of the totally mixed equilibrium σ^*. When the system is linearized about a given σ^ε we have $\dot{z} \approx Az$ with the matrix A being given by

$$A = \begin{bmatrix} -1 & 0 & m^1_{11}(\varepsilon) & m^1_{12}(\varepsilon) \\ 0 & -1 & m^1_{21}(\varepsilon) & m^1_{22}(\varepsilon) \\ m^2_{11}(\varepsilon) & m^2_{12}(\varepsilon) & -1 & 0 \\ m^2_{21}(\varepsilon) & m^2_{22}(\varepsilon) & 0 & -1 \end{bmatrix},$$

for $m^i_{jk}(\varepsilon) = (d/d\mu_{it}(a_{-ik}))\, x_i(\mu_{it})(a_{ij}) - (d/d\mu_{it}(a_{-i3}))\, x_i(\mu_{it})(a_{ij})$. We write M^ε_1 for the upper right 2×2 submatrix and M^ε_2 for the lower left 2×2 submatrix.

Computing $Det(A - \lambda I)$ we find that the eigenvalues of A are the roots of

$$(1+\lambda)^4 - Tr(M^\varepsilon_1 M^\varepsilon_2)(1+\lambda)^2 + Det(M^\varepsilon_1)\, Det(M^\varepsilon_2) = 0.$$

This quartic has no linear or cubic terms, so we can solve it to find

$$(1+\lambda)^2 = \tfrac{1}{2}\left(Tr(M^\varepsilon_1 M^\varepsilon_2) \pm \sqrt{Tr(M^\varepsilon_1 M^\varepsilon_2)^2 - 4Det(M^\varepsilon_1 M^\varepsilon_2)}\right),$$

which gives

$$\lambda = -1 \pm \frac{1}{\sqrt{2}} \sqrt{Tr(M_1^\varepsilon M_2^\varepsilon) \pm \sqrt{Tr(M_1^\varepsilon M_2^\varepsilon)^2 - 4Det(M_1^\varepsilon M_2^\varepsilon)}}.$$

PROPOSITION 3. *Let G be a 3×3 game, and let $\{\sigma^\varepsilon\}$ be a purifying sequence for its totally mixed equilibrium σ^*. Let M_1^ε and M_2^ε be the matrices defined above. Assume that for $i = 1, 2$ $\lim_{\varepsilon \to 0} \varepsilon M_i^\varepsilon$ exists and define $M_i \equiv \lim_{\varepsilon \to 0} \varepsilon M_i^\varepsilon$. Then, the purifying sequence is stable if all three of the conditions below are satisfied, and the sequence is unstable if the reverse of any of the inequalities holds strictly.*

(1) $Tr(M_1 M_2) < 0$

(2) $Tr(M_1 M_2)^2 > 4Det(M_1 M_2)$

(3) $Det(M_1)\, Det(M_2) > 0$

Proof. From the expression above we know that the eigenvalues are of the form

$$\lambda = -1 \pm \frac{1}{\sqrt{2}\,\varepsilon} \sqrt{x(\varepsilon) \pm \sqrt{y(\varepsilon)}},$$

for $x(\varepsilon) = Tr(\varepsilon M_1^\varepsilon \varepsilon M_2^\varepsilon)$ and $y(\varepsilon) = Tr(\varepsilon M_1^\varepsilon \varepsilon M_2^\varepsilon)^2 - 4Det(\varepsilon M_1^\varepsilon \varepsilon M_2^\varepsilon)$. Provided that $x(\varepsilon)$ and $y(\varepsilon)$ have well-defined limits x and y as ε approaches zero, the eigenvalues will have negative real parts for all small ε if $x + \sqrt{y}$ and $x - \sqrt{y}$ are both negative real numbers. This will be the case if $x < 0$, $y > 0$, and $x^2 > y$, which gives the three conditions above. Conversely, if any of the inequalities above is strictly violated we will have a root with a positive real part for sufficiently small ε. ∎

Remark. Note that the reason for the "sufficiently small ε" requirement varies between the stability and instability results. In deriving the stability result, "ε sufficiently small" is used only in a continuity argument letting us say that the conditions which hold for $M_1 M_2$ hold for $\varepsilon M_1^\varepsilon \varepsilon M_2^\varepsilon$ as well. For the instability result, the fact that $1/\varepsilon$ goes to infinity is needed so that small real components of the eigenvalues get blown up and overwhelm the minus one in the expression. We thus might expect that smaller ε's will be necessary to ensure that a system which is eventually unstable is in fact unstable.

4.2. *Interpretation of the General Result*

As a first step toward providing some intuition for the meaning of these conditions, we now provide formulas for the M_i matrices in terms of the payoffs of the game and the distribution of the types in the population.

Looking at a typical term, we see

$$m_{11}^1(\varepsilon)=\frac{d}{dz_3}\,\mathrm{Prob}\{a_{11}\in \operatorname*{Argmax}_a g_1^\varepsilon(a,\mu_1(z))\}\,|_{\mu_1=\sigma_2^\varepsilon},$$

where $\mu_1(z)$ is the distribution which places probability z_3 on a_{21}, probability z_4 on a_{22}, and probability $1-(z_3+z_4)$ on a_{23}. We will relate this to properties of the 2×2 "restricted games" obtained from G by deleting one strategy for each player. Let G_{mn} be the game obtained by deleting the mth row (strategy a_{1m}) and the nth column (strategy a_{2n}.) Analogously to the definition of Δ_i of Section 3, let $\Delta_i(G_{mn})$ measure whether player i is better off on the diagonal or off-diagonal of G_{mn}, e.g., $\Delta_1(G_{22})=g_1(a_{11},a_{11})+g_1(a_{13},a_{23})-(g_1(a_{13},a_{21})+g_1(a_{11},a_{23}))$.

Expanding the above expression for m_{11}^1 gives

$$\begin{aligned}
m_{11}^1(\varepsilon)&=\frac{d}{dz_3}\,\mathrm{Prob}\left\{\theta_1(a_{11})+\frac{1}{\varepsilon}\,g_1(a_{11},\mu_1(z))\geqslant\theta_1(a_{12})+\frac{1}{\varepsilon}\,g_1(a_{12},\mu_1(z))\right.\\
&\qquad\left.\text{and }\theta_1(a_{11})+\frac{1}{\varepsilon}\,g_1(a_{11},\mu_1(z))\geqslant\theta_1(a_{13})+\frac{1}{\varepsilon}\,g_1(a_{13},\mu_1(z))\right\}\Bigg|_{\mu_1=\sigma_2^\varepsilon}\\
&=\frac{d}{dz_3}\int_{\theta_1=-\infty}^{\infty}\int_{\theta_2=-\infty}^{\theta_1+1/\varepsilon\, g_1(a_{11}-a_{12},\mu_1(z))}\\
&\qquad\int_{\theta_3=-\infty}^{\theta_1+1/\varepsilon\, g_1(a_{11}-a_{13},\mu_1(z))} f(\theta_1,\theta_2,\theta_3)\,d\theta_3 d\theta_2 d\theta_1\Bigg|_{\mu_1=\sigma_2^\varepsilon}\\
&=\int_{\theta_1=-\infty}^{\infty}\int_{\theta_3=-\infty}^{\theta_1+k_{13}^{1\varepsilon}}\left(\frac{1}{\varepsilon}\frac{d}{dz_3}\,g_1(a_{11}-a_{12},\mu_1(z))\bigg|_{\mu_1=\sigma_2^\varepsilon}\right)\\
&\qquad\times f(\theta_1,\theta_1+k_{12}^{1\varepsilon},\theta_3)\,d\theta_3\,d\theta_1\\
&\quad+\int_{\theta_1=-\infty}^{\infty}\int_{\theta_2=-\infty}^{\theta_1+k_{12}^{1\varepsilon}}\left(\frac{1}{\varepsilon}\frac{d}{dz_3}\,g_1(a_{11}-a_{13},\mu_1(z))\bigg|_{\mu_1=\sigma_2^\varepsilon}\right)\\
&\qquad\times f(\theta_1,\theta_2,\theta_1+k_{13}^{1\varepsilon})\,d\theta_2\,d\theta_1\\
&=\int_{\theta=-\infty}^{\infty}\int_{\theta'=-\infty}^{\theta+k_{13}^{1\varepsilon}}\frac{1}{\varepsilon}\,\Delta_1(G_{32})\,f_i(\theta,\theta+k_{12}^{1\varepsilon},\theta')\,d\theta'\,d\theta\\
&\quad+\int_{\theta=-\infty}^{\infty}\int_{\theta'=-\infty}^{\theta+k_{12}^{1\varepsilon}}\frac{1}{\varepsilon}\,\Delta_1(G_{22})\,f_i(\theta,\theta',\theta'+k_{13}^{1\varepsilon})\,d\theta'\,d\theta,
\end{aligned}$$

where $k_{jk}^{i\varepsilon}=1/\varepsilon(g_i(a_{ij},\sigma^\varepsilon_{-i})-g_i(a_{ik},\sigma^\varepsilon_{-i}))$ and where we have written $g_i(a_{ij}-a_{ik},\sigma_{-i})$ for $g_i(a_{ij},\sigma_{-i})-g_i(a_{ik},\sigma_{-i})$.

As $\varepsilon \to 0$, $\varepsilon m^1_{11}(\varepsilon)$ will thus converge to

$$m^1_{11} = h^1_3 \Delta_1(G_{32}) + h^1_2 \Delta_1(G_{22}),$$

where the h^i_j are positive constants which depend on the density f_i of the type distribution. By analogy with Section 3, one would expect that $k^i_{jk} \equiv \lim_{\varepsilon \to 0} 1/\varepsilon\, k^{i\varepsilon}_{jk}$ exists and is independent of the particular purification; we verify this in the Appendix. The constants h^i_j thus also do not depend on the particular purifying sequence chosen and are given by

$$h^i_1 = \int_{\theta=-\infty}^{\infty} \int_{\theta'=-\infty}^{\theta - k^i_{12}} f_i(\theta', \theta, \theta + k^i_{23})\, d\theta'\, d\theta$$

$$h^i_2 = \int_{\theta=-\infty}^{\infty} \int_{\theta'=-\infty}^{\theta - k^i_{23}} f_i(\theta + k^i_{31}, \theta', \theta)\, d\theta'\, d\theta$$

$$h^i_3 = \int_{\theta=-\infty}^{\infty} \int_{\theta'=-\infty}^{\theta - k^i_{31}} f_i(\theta, \theta + k^i_{12}, \theta')\, d\theta'\, d\theta.$$

The constants h^i_j can be interpreted as the density of players who are indifferent between playing the two strategies other than a_{ij} against $\sigma^{\varepsilon}_{-i}$ and who prefer each of these strategies to a_{ij}. The h^i_j do clearly depend on the distribution of payoff perturbations.

More generally, a similar calculation shows that

$$m^i_{jk} = (-1)^{j-1}\, h^i_3\, \Delta_i(G_{3k'}) + h^i_{j'}\, \Delta_i(G_{j'k'}),$$

where j' is 2 when $j = 1$ and j' is 1 when $j = 2$, and likewise for k'.

We now return to the three conditions which are jointly sufficient for stability.

Condition (1) is fairly easy to interpret. Plugging the expression for m^i_{jk} given above into the condition $Tr(M_1 M_2) < 0$ we show in the Appendix that condition (1) is equivalent to

$$\sum_{j=1}^{3} \sum_{k=1}^{3} h^1_j h^2_k\, \Delta_1(G_{jk})\, \Delta_2(G_{jk}) < 0.$$

Each term in the sum reflects simply the extent to which one of the 2×2 submatrices of G is a game of conflict or coordination. Condition (1) is thus a requirement that, in terms of a weighted sum over the submatrices, the game is more one of conflict than one of coordination. For example, in a zero-sum game every submatrix is also a zero-sum game and hence is (at least weakly) a game of conflict. As a result, condition (1) will be satisfied (generically) regardless of the distribution of types (which just determines the weights in the average). Similarly in any supermodular game any submatrix is a game of coordination and condition (1) will fail to hold regardless of the distribution of types. Note, however, that in general whether the condition is satisfied will depend on the weights. For this reason, in contrast to the 2×2 case, we will not generally be able to classify an equilibrium of a game as being stable or unstable without making reference to a specific distribution of the heterogeneity in the population.

When condition (1) is strictly violated (and conditions (2) and (3) hold) we can see from the proof of Proposition 3 that the instability of the equilibrium results from the system having two positive real eigenvalues with corresponding eigenvectors reflecting distinct directions in which players can adjust their strategies and coordinate to improve their payoffs. Example 1 below provides one illustration of this. The mixed equilibrium which places probability one third on each of the three pure strategies is unstable. The first unstable eigenvector corresponds to both populations simultaneously putting more weight on A and less on B (or vice versa); the second corresponds to both populations putting less weight on C and adding weight equally on A and B.

EXAMPLE 1. Let G be the game below with σ^* the Nash equilibrium placing weight one third on each of the pure strategies, and suppose that the $\theta_i(a_i)$ are independently and identically distributed across strategies within each population. Then, condition (1) is strictly violated, while conditions (2) and (3) are satisfied (and thus any purifying sequence is unstable).

	A	B	C
A	3, 3	0, 0	0, 1
B	0, 0	3, 3	0, 1
C	1, 0	1, 0	1, 1

When the $\theta_i(a_i)$ are independently and identically distributed, there is a one third probability that each of player i's strategies has the largest θ, so there is an equilibrium distribution in which each action is played with probability one third in each population. The integrals defining the h_j^i are then also symmetric within each population, and so $h_1^1 = h_2^1 = h_3^1$ and $h_1^2 = h_2^2 = h_3^2$. The matrices M_i are proportional to whatever value is taken on by h_j^i, and the values of these constants simply scale up or down the expressions in each of the three conditions. Hence we may simplify the calculations without affecting whether the conditions hold by assuming that $h^i_j = 1$ for all i and j.

A direct calculation shows that the M_1 and M_2 matrices are then given by

$$M_1 = M_2 = \begin{bmatrix} 6 & -3 \\ -3 & 6 \end{bmatrix}$$

and that

$$M_1 M_2 = \begin{bmatrix} 45 & -36 \\ -36 & 45 \end{bmatrix}.$$

Condition (1) strictly fails because $Tr(M_1 M_2) = 90$ fails to be negative. The equilibrium is therefore unstable for small ε. One can also verify that conditions (2) and (3) are satisfied:

$Tr(M_1 M_2)^2 - 4Det(M_1 M_2) = 5184$ is positive

$Det(M_1 M_2) = 729$ is positive.

In general, whether condition (1) holds is not directly related to the existence of Pareto superior equilibria. Condition (1) may fail to be satisfied even in games with no pure or partially mixed equilibria.

We now discuss condition (3) which we think of as requiring that the players have common or opposite views of the relative gains from diagonal vs off-diagonal coordination. Recall that the condition is that $Det(M_1)$ $Det(M_2) > 0$. We show in the Appendix that

$$Det(M_i) = (h^i_1 h^i_2 + h^i_1 h^i_3 + h^i_2 h^i_3)(\Delta_i(G_{33})\, \Delta_i(G_{22}) - \Delta_i(G_{32})\, \Delta_i(G_{23})).$$

To help interpret this condition, recall that $\Delta_i(G_{33})$ measures the complementarity between a_{12} and a_{22} (relative to (a_{11}, a_{21})). Similarly $\Delta_i(G_{22})$ measures the complementarity between a_{13} and a_{23} and $\Delta_i(G_{32})$ measures the complementarity between a_{12} and a_{23}. If all actions are complements $Det(M_i)$ is thus positive if player i regards there as being greater complementarities between the actions on the diagonal than between the actions off the diagonal. Condition (3) requires that players 1 and 2 have common views as to where the complementarities are greater. The condition can also be thought of as requiring either that the payoff differences in the game for the two players both be log-supermodular or that they both be log-submodular.

One important observation follows immediately from the definition: whether condition (3) holds depends only on the game being played and not on the distribution of the payoff heterogeneity. The distribution of heterogeneity enters the expression only through the initial multiplicative term which is always positive.

A second immediate observation is that condition (3) is satisfied in generic symmetric games and in generic zero-sum games. In a symmetric game we have $\Delta_1(G_{33}) = \Delta_2(G_{33})$, $\Delta_1(G_{22}) = \Delta_2(G_{22})$, $\Delta_1(G_{32}) = \Delta_2(G_{23})$ and $\Delta_1(G_{23}) = \Delta_2(G_{32})$. In a zero-sum game we have $\Delta_1(G_{33}) = -\Delta_2(G_{33})$, $\Delta_1(G_{22}) = -\Delta_2(G_{22})$, $\Delta_1(G_{32}) = -\Delta_2(G_{32})$, $\Delta_1(G_{23}) = -\Delta_2(G_{23})$. Hence, in either case

$$Det(M_1)\,Det(M_2) = H_1 H_2 (\Delta_1(G_{33})\,\Delta_1(G_{22}) - \Delta_1(G_{32})\,\Delta_1(G_{23}))^2,$$

for $H_i = h_1^i h_2^i + h_1^i h_3^i + h_2^i h_3^i$, which is generically positive. Given that this term is usually strictly positive, we can think of the condition as holding whenever the game is nearly zero sum or nearly symmetric.

When condition (3) is strictly violated (and conditions (1) and (2) hold at least weakly) the matrix A has a single positive real eigenvalue which reflects a manner in which the strategy profiles in the two populations can be adjusted to improve the players' payoffs. Example 2 provides two illustrations where the improvement results from the populations shifting toward a Nash equilibrium which Pareto dominates the totally mixed equilibrium.

EXAMPLE 2. For each of the 3×3 games shown below, condition (3) is strictly violated for all distributions of the payoff heterogeneity. When the type distributions are independent and identical across types conditions (1) and (2) are satisfied or only weakly violated.

	a_{21}	a_{22}	a_{23}
a_{11}	5, 5	0, 3	0, 3
a_{12}	3, 0	2, 0	0, 2
a_{13}	3, 0	0, 2	2, 0

	a_{21}	a_{22}	a_{23}
a_{11}	9, 2	0, 0	0, 3
a_{12}	1, 0	7, 1	1, 13
a_{13}	0, 14	6, 15	3, 0

Perhaps the easiest way to verify that condition (3) fails is to compute the matrix of payoff differences, i.e., to write down lower right submatrix of the renormalized game where the players receive a payoff of zero whenever a_{11} or a_{21} is played. The renormalized games are:

	a_{22}-a_{21}	a_{23}-a_{21}
a_{12}-a_{11}	4, 2	2, 4
a_{13}-a_{11}	2, 4	4, 2

	a_{22}-a_{21}	a_{23}-a_{21}
a_{12}-a_{11}	15, 3	9, 12
a_{13}-a_{11}	15, 3	12, −15

In the game on the left we thus have $Det(M_1) = H_1(4 \cdot 4 - 2 \cdot 2) = 12H_1$ which is positive while $Det(M_2) = H_2(2 \cdot 2 - 4 \cdot 4) = -12H_2$ is negative. (We have again written H_i for $h_1^i h_2^i + h_1^i h_3^i + h_2^i h_3^i$.) In the game on the right $Det(M_1) = 45H_1$ and $Det(M_2) = -81H_2$.

An explicit calculation shows that conditions (1) and (2) are satisfied or weakly violated (in one case) when all of the h_j^i are set to one. With this assumption we have $Tr(M_1 M_2) = 0$ and $Tr(M_1 M_2)^2 - 4Det(M_1 M_2) = 5184$ for the game on the left and $Tr(M_1 M_2) = -144$ and $Tr(M_1 M_2)^2 - 4Det(M_1 M_2) = 151956$ for the game on the right.

Each of the above games has a single Nash equilibrium which Pareto dominates the totally mixed equilibrium. In the game on the left, coordinating on the first strategy gives each player a payoff of 5. While the game on the right has no pure strategy equilibria, it has two partially mixed equilibria: $(5/9a_{12} + 4/9a_{13}, 2/3a_{22} + 1/3a_{23})$ and $(14/15a_{11} + 1/15a_{13}, 1/4a_{21} + 3/4a_{23})$. The first of these provides both players with a higher payoff than the totally mixed equilibrium. In each case, the single positive real eigenvalue of the system reflects the possibility of the populations moving away from the totally mixed equilibrium in the direction of the superior equilibrium. In simulations we have found that all games violating condition (3) have pure or partially mixed equilibria, although these equilibria need not Pareto dominate the totally mixed equilibrium.

We think of condition (2) as a no-cycling condition. When condition (2) is strictly violated (and conditions (1) and (3) hold) the matrix A has a pair of complex eigenvalues with positive real parts when ε is sufficiently small, corresponding to an unstable subspace in which beliefs will spiral away from the purified equilibrium. See Fig. 1 for an illustration of the dynamics of such a system.

What condition (2) requires is that $Tr(M_1 M_2)^2 > 4Det(M_1 M_2)$. Recall from the discussion of condition (1) that $Tr(M_1 M_2)$ can be thought of as a measure of the degree to which the game is one of coordination or conflict. The requirement of condition (1) that $Tr(M_1 M_2)$ be negative is a requirement that the game be one of conflict. Condition (2) can be thought of as an additional requirement that the degree of conflict is sufficiently

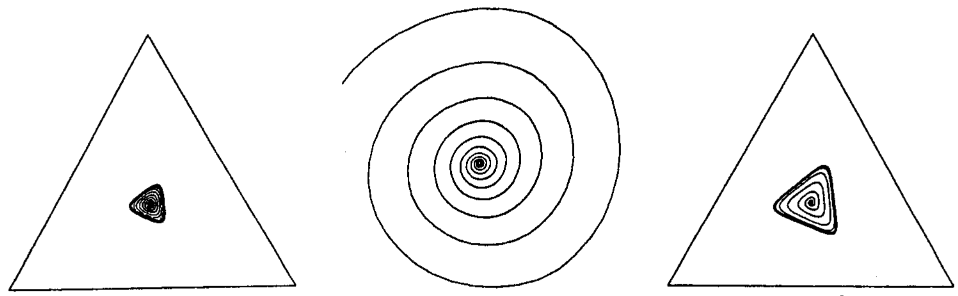

FIG. 1. Time path of player 1's play from symmetric initial condition when $\alpha > 0$.

large so as to overcome a tendency of the system to cycle (which apparently is captured by $Det(M_1 M_2)$).

The games in the example below contain illustrations of two extremes where the no-cycling condition will and will not hold. With i.i.d. type distributions the α-diagonal rock–paper–scissors game with $\alpha = \pm\sqrt{3}$ is just on the boundary between having common and conflicting interests on average. For values of α which are only slightly smaller than $\sqrt{3}$ in absolute value, the degree of conflict is thus insufficient to prevent cycling. The second game in the example is one where $Det(M_1 M_2) = 0$. Any degree of conflict is then sufficient to satisfy the no-cycling condition.

EXAMPLE 3. If G is the α-diagonal rock–paper–scissors game on the left below with $|\alpha| < \sqrt{3}$, $\alpha \neq 0$, and the distribution of payoff heterogeneity is independent and identical across strategies, the condition (2) is strictly violated and conditions (1) and (3) are satisfied. In the game on the right below condition (2) is satisfied regardless of the distribution of the payoff perturbations.

	R	P	S
R	α, α	$-1, 1$	$1, -1$
P	$1, -1$	α, α	$-1, 1$
S	$-1, 1$	$1, -1$	α, α

	a_{21}	a_{22}	a_{23}
a_{11}	$0, 0$	$1, -1$	$1, -1$
a_{12}	$-1, 1$	$3, -3$	$0, 0$
a_{13}	$1, -1$	$-1, 1$	$2, -2$

In the α-diagonal rock–paper–scissors game the symmetry of the perturbations again implies that it is a purified equilibrium to play each strategy with probability one third. We may thus again set all of the h_j^i equal to one to simplify the calculations. We then find

$$M_1 = M_2 = \begin{bmatrix} -3+3\alpha & -6 \\ 6 & 3+3\alpha \end{bmatrix}.$$

To show that condition (2) is strictly violated when $\alpha \neq 0$, we can then just use a direct calculation to show that $Tr(M_1 M_2)^2 - 4Det(M_1 M_2) = -3888\alpha^2$ is negative.

Condition (2) is the only condition that fails to hold in this example when $|\alpha| < \sqrt{3}$. $Tr(M_1 M_2) = -54 + 18\alpha^2$ is negative and hence condition (1) is satisfied. $Det(M_1 M_2) = 81(3+\alpha^2)^2$ is positive and hence condition (3) is satisfied.

In the game on the right $\Delta_i(G_{22}) = 0$ and $\Delta_i(G_{32}) = 0$, i.e., there are no complementarities between a_{13} and a_{23} or between a_{12} and a_{23} relative to (a_{11}, a_{21}). Hence, $Det(M_1 M_2) = 0$ for any distribution of the payoff perturbations. Because every 2×2 restricted game is at least weakly a game of conflict and several of them (e.g., G_{33}) have strictly conflicting interests $Tr(M_1 M_2)$ will be strictly negative for any distribution of the perturbations. Hence, condition (2) will always hold. (The discussion above has established also that condition (1) always holds and conditions (3) is weakly violated.)

When α is small and nonzero, the A matrix for the α-diagonal rock–paper–scissors game has two complex roots with positive real parts, and thus the instability takes the form of beliefs spiraling away from the equilibrium. In the $\alpha > 0$ case if both populations start out playing their first strategy ("rock" in the rock–paper–scissors interpretation) with probability slightly more than 1/3, they will synchronously start to shift more weight to "paper," then more to "scissors," then more to "rock," etc., with the departures from $1/3 - 1/3 - 1/3$ mixing becoming more and more pronounced.

Because the player 1's and the player 2's always have the same strategy distribution along this spiral it is easy to graph. Figure 1 illustrates the time path of player 1's play from a symmetric initial condition near the equilibrium when the payoff shocks are independent normal random variables with standard deviation 0.1. The leftmost panel in the figure graphs the evolution of the time averages of play when $\alpha = 0.2$. Play slowly moves away from the equilibrium in a tight spiral and appears to settle down to a stable limit cycle. To provide more detail on behavior around the equilibrium, we present a 75-fold magnification of this figure in the center panel. the rightmost panel contains a similar graph for a game with $\alpha = 0.3$. As α increases the distribution of play in the populations seems to move away from the equilibrium much more quickly, and the limit cycle is farther from the equilibrium.

In the $\alpha < 0$ case the unstable spiral in virtually identical, although it is asymmetric with the two populations always having the opposite deviation from a $1/3 - 1/3 - 1/3$ mixture.

4.3. *Some Basic Facts*

In this section we give four simple propositions pointing out basic facts about the behavior of our model in 3×3 games: it is possible for totally mixed equilibria to be stable; stability cannot be defined independently of the distribution of the heterogeneity in the population; an equilibrium can be unstable even though it is the unique equilibrium of the game; and a totally mixed equilibrium can be stable even though it is Pareto dominated by another equilibrium. In each case, the results are established by means of examples.

First, in light of the fact that totally mixed equilibria in 3×3 games are never stable in the two-population replicator dynamics and that exact fictitious play also does not converge to such equilibria (Krishna and Sjostrom [15]) it is interesting to see that such equilibria can be locally stable in our model.

PROPOSITION 4. *There exist choices for the* 3×3 *game* G *and for i.i.d. type distributions such that there is a stable purifying sequence for* σ^*.

Proof. Consider the slight variant on rock–paper–scissors pictured below, where δ is a small positive number. Again suppose that the distributions of the types are independent and identical across strategies within each population. Once again, the game has been defined so that it (and each G^ϵ) has an equilibrium where each pure strategy is played with probability one third. The calculations required to apply the general result to determine whether this equilibrium is locally stable can again be simplified without loss of generality by assuming that $h^i_j = 1$ for all i and j.

	a_{21}	a_{22}	a_{23}
a_{11}	$\delta, 0$	$-1-2\delta, 1+2\delta$	$1+\delta, -1$
a_{12}	$1, -1$	$0, -\delta$	$-1, 1$
a_{13}	$-1, 1$	$1, -1-\delta$	$0, 0$

The M_1 and M_2 matrices are given by

$$M_1 = \begin{bmatrix} -3 & -6-6\delta \\ 6 & 3+3\delta \end{bmatrix}$$

$$M_2 = \begin{bmatrix} -3-3\delta & -6 \\ 6+6\delta & 3 \end{bmatrix}.$$

We can thus show that condition (1) is satisfied by noting that $Tr(M_1 M_2) = -54 - 54\delta - 36\delta^2$ is negative for all δ.[10]

Because the game is also nearly symmetric, one would expect condition (3) to hold for small δ. In fact, $Det(M_1) = Det(M_2) = 27 + 27\delta$, so condition (3) holds for any $\delta \neq -1$.

Finally, checking condition (2) we find

$$Tr(M_1 M_2)^2 - 4Det(M_1 M_2) = (54 + 54\delta + 36\delta^2)^2 - 4(27 + 27\delta)^2$$
$$= 1296\delta^2(\delta^2 + 3\delta + 3),$$

which is positive for all δ. ▮

It is also immediate from the proof that the set of parameters, i.e., the payoffs of the game and the six parameters describing the type distribution, for which totally mixed equilibria are stable is not of measure zero. For any $\delta > 0$ conditions (1), (2), and (3) hold strictly and are continuous in the parameters; hence the totally mixed equilibria of nearby models will also be stable.

Another example of a game with a stable totally mixed equilibrium can be obtained by slightly altering the payoffs in the game on the right in Example 2 so that condition (3) is strictly satisfied.

In 2×2 games, mixed equilibria are stable or unstable when ε is sufficiently small, independent of the distribution of the heterogeneity. In 3×3 games this is no longer true. We noted earlier that the form of condition (1) made it seem likely that the form of heterogeneity would matter; we now provide an example to verify this intuition.

PROPOSITION 5. *There exists a* 3×3 *game* G *with a unique totally mixed Nash equilibrium* σ^* *and two independent distributions* F *and* $\hat{F}$ *for the* $\theta_i(a_i)$ *such that* (G^ε, F) *has a stable purifying sequence for* σ^* *while every purifying sequence for* σ^* *in* $(G^\varepsilon, \hat{F})$ *is unstable.*

[10] The calculation was not really necessary, as each of the nine 2×2 submatrices has conflicting interests.

Proof. Let G be the α-diagonal rock–paper–scissors game of Example 3. We saw in Example 3 that when the distributions of the types were independent and identical across strategies, the symmetric purified equilibria were unstable for ε small.

Suppose now instead that the distribution of the types is such that $h^i_j = 1$ for all pairs (i, j) other than $(1, 2)$ with h^1_2 being different from one.[11] We saw in Example 3 that condition (3) was satisfied with i.i.d. heterogeneity. Condition (3) is independent of the distribution of heterogeneity, so it will continue to hold. It remains only to show that we can choose h^1_2 so that conditions (1) and (2) are satisfied.

The M_i matrices are then given by

$$M_1 = \begin{bmatrix} -3+3\alpha+2\alpha(h^1_2-1) & -6-(3-\alpha)(h^1_2-1) \\ 6 & 3+3\alpha \end{bmatrix}$$

$$M_2 = \begin{bmatrix} -3+3\alpha & -6 \\ 6 & 3+3\alpha \end{bmatrix}.$$

A direct calculation gives

$$Tr(M_1 M_2) = -54 + 18\alpha^2 - 18(h^1_2 - 1) + 6\alpha^2(h^1_2 - 1).$$

This is negative so condition (1) is satisfied for any $h^1_2 > 0$ if $\alpha \in [-1, 1]$.

Finally, condition (2) requires that

$$\begin{aligned} &Tr(M_1 M_2)^2 - 4Det(M_1 M_2) \\ &\quad = 36((9 - 6\alpha^2 + \alpha^4)(h^1_2 - 1)^2 - 72\alpha^2(h^1_2 - 1) - 108\alpha^2) > 0. \end{aligned}$$

For any fixed $h^1_2 \neq 1$ this inequality will be satisfied for α sufficiently close to zero. ∎

The α-diagonal rock–paper–scissors game can also be used to illustrate a couple of other properties of our model. First, we note that as in many other learning and evolutionary models a Nash equilibrium may be unstable even though it is unique.

PROPOSITION 6. *There exist* 3×3 *games G and distributions of the heterogeneity for which G has a unique Nash equilibrium* σ^* *that is totally mixed and for which any purifying sequence is unstable.*

[11] We can generate such a system with independent distributions for the types by choosing $\theta_1(a_{11})$, $\theta_1(a_{13})$, $\theta_2(a_{21})$, $\theta_2(a_{22})$, and $\theta_2(a_{23})$ to be independent standard normals, with the distribution of $\theta_1(a_{12})$ being normal with variance σ^2. We will then have that $h^1_1 = h^1_3 = c_1$, $h^1_2 = c_2$, and $h^2_1 = h^2_2 = h^2_3 = c_3$ with c_2/c_1 going smoothly from one to infinity as σ does the same. Dividing all of the h^1's and h^2's by c_1 and c_3, respectively, does not affect whether the conditions hold.

Proof. If one takes $\alpha < 0$ in the game of Example 3, then the mixed equilibrium placing weight one third on each pure strategy is unique. We saw in the example that whenever the type distributions were independent and identical across actions condition (2) always failed and hence any purifying sequence was unstable. ∎

Second, the reader may have noticed from our previous examples that in many cases in which an equilibrium is unstable it is Pareto dominated by a pure or partially mixed equilibrium. An implication of Proposition 6 is that the existence of such an equilibrium is not necessary for instability. It turns out that the existence of a Pareto superior equilibrium is also not sufficient to ensure that the totally mixed equilibrium is unstable.

PROPOSITION 7. *There exist* 3×3 *games* G *and distributions of the heterogeneity for which the unique totally mixed equilibrium of* G *has a stable purifying sequence despite* σ^* *being Pareto dominated by another Nash equilibrium of* G.

Proof. Again let G be the α-diagonal rock–paper–scissors game of Example 3 and suppose that the distribution of the types is such that $h^i_j = 1$ for all pairs (i, j) other than $(1, 2)$ with h^1_2 being different from one. For $\alpha > 1$, each of the diagonal boxes is a pure strategy equilibrium that Pareto dominates the totally mixed equilibrium. Let $\alpha = \sqrt{1.5}$. As in the proof of Proposition 5, condition (3) is satisfied for all h^1_2. Condition (1) becomes

$$Tr(M_1 M_2) = -27 - 9(h^1_2 - 1) > 0,$$

which is also satisfied for all $h^1_2 > 0$. Finally, condition (2) becomes

$$Tr(M_1 M_2)^2 - 4Det(M_1 M_2) = 36(2.25(h^1_2 - 1)^2 - 108(h^1_2 - 1) - 162) > 0.$$

This will be satisfied for h^1_2 sufficiently large. ∎

While the example above exploits a highly asymmetric type distribution, there are other examples which work with i.i.d. heterogeneity.

4.4. *How Common Is Stability? A Monte Carlo Experiment*

Looking at the various examples of mixed equilibria for which purifying sequences are unstable and the two examples we have given of models with stable equilibria, one might be led to wonder whether it is only in very special circumstances that our model supports the idea that learning can lead players to play mixed equilibria which are more complicated than those in 2×2 games. The message of this section is that this is not the case: it is quite common for purifications of totally mixed equilibria to be stable in our model.

To give a feel for how common stability is we conducted a Monte Carlo experiment. First, we randomly selected 100,000 3×3 payoff matrices. As noted earlier, the three conditions governing the stability of an equilibrium are unaffected by adding a constant to each player's payoff in one or more rows or columns (when the h^i_j are held constant). We therefore chose to construct the games by choosing randomly the eight parameters $\Delta_i(G_{jk})$, $i=1, 2,\ j, k=2, 3$, from independent standard normal distributions and then adding constants to each row and column so that the game has a mixed strategy equilibrium where each pure strategy is played with probability 1/3. In this experiment we found that 16.8% of the time a purifying sequence approaching this mixture in a model with independent and identically distributed types would be stable.

In a large number of the simulated games in which the purified equilibrium was unstable, it seemed that the instability was due to the game also having a pure strategy equilibrium which gave a higher payoff on which players would be led to coordinate. In the subsample of games that have no pure or partially mixed equilibria, we found the totally mixed equilibrium to be stable (for sufficiently small ε) 70.4% of the time.

4.5. *Zero-Sum Games*

More intuition for when games will be stable is provided by the following result, which shows that generically the purified equilibria of zero-sum games are stable. Because the stability conditions hold strictly on the generic set, this also tells us that, except in certain parts of the parameter space, games which are sufficiently close to being zero-sum (after adding separate constants to the rows and columns for each player) will also be stable.

PROPOSITION 8. *In the set of* 3×3 *zero-sum games which have a totally mixed Nash equilibrium, generically any purifying sequence for the totally mixed equilibrium is stable.*

Proof. Using the formula we derived in Section 4.2, condition (1) is equivalent to

$$\sum_{m,n} h^1_m h^2_n \Delta_1(G_{mn})\, \Delta_2(G_{mn}) < 0.$$

In a zero-sum game $\Delta_1(G_{mn}) = -\Delta_2(G_{mn})$ so each of the terms in this sum is less than or equal to zero and for a generic set of payoffs at least one is strictly negative. Hence condition (1) holds.

Next, we noted also in Section 4.2 that condition (3) is of the form

$$H_1 H_2(\Delta_1(G_{33})\,\Delta_1(G_{22}) - \Delta_1(G_{32})\,\Delta_1(G_{23}))(\Delta_2(G_{33})\,\Delta_2(G_{22}) - \Delta_2(G_{32})\,\Delta_2(G_{23})) > 0.$$

We noted there also that in a zero sum game

$$\Delta_1(G_{33})\,\Delta_1(G_{22}) - \Delta_1(G_{32})\,\Delta_1(G_{23}) = \Delta_2(G_{33})\,\Delta_2(G_{22}) - \Delta_2(G_{32})\,\Delta_2(G_{23}),$$

and thus condition (3) holds generically.

Finally when conditions (1) and (3) hold, condition (2) is equivalent to

$$-Tr(M_1 M_2) - 2\sqrt{Det(M_1 M_2)} > 0.$$

Each of the terms in the expression above can be written as a second degree polynomial (with no linear or constant term) in four parameters, $\Delta_1(G_{33})$, $\Delta_1(G_{32})$, $\Delta_1(G_{23})$, and $\Delta_1(G_{22})$, with the coefficients being functions of the h^i_j. An explicit calculation shows that

$$-Tr(M_1 M_2) - 2\sqrt{Det(M_1 M_2)} = v'Qv,$$

where v is the vector $(\Delta_1(G_{33}), \Delta_1(G_{32}), \Delta_1(G_{23}), \Delta_1(G_{22}))'$ and Q is the matrix

$$\begin{pmatrix} (h^1_1+h^1_3)(h^2_1+h^2_3) & -h^1_1h^2_1-h^1_3h^2_1 & -h^1_1h^2_1-h^1_1h^2_3 & h^1_1h^2_1-SH \\ -h^1_1h^2_1-h^1_3h^2_1 & (h^1_1+h^1_3)(h^2_1+h^2_2) & h^1_1h^2_1+SH & -h^1_1h^2_1-h^1_1h^2_2 \\ -h^1_1h^2_1-h^1_1h^2_3 & h^1_1h^2_1+SH & (h^1_1+h^1_2)(h^2_1+h^2_3) & -h^1_1h^2_1-h^1_2h^2_1 \\ h^1_1h^2_1-SH & -h^1_1h^2_1-h^1_1h^2_2 & -h^1_1h^2_1-h^1_2h^2_1 & (h^1_1+h^1_2)(h^2_1+h^2_2) \end{pmatrix}$$

In this expression we have written H for

$$\sqrt{(h^1_1h^1_2+h^1_1h^1_3+h^1_2h^1_3)(h^2_1h^2_2+h^2_1h^2_3+h^2_2h^2_3)}$$

and S for the sign of $Det(M_1)$.

To show that condition (2) holds generically, it will suffice to show that this quadratic form is always positive semidefinite and not identically zero. For $k \in \{1, 2, 3, 4\}$, let Q_k be the submatrix consisting of the first k rows and columns of Q. The matrix will be positive semidefinite if $Det(Q_k)$ is nonnegative for each k. Computing these determinants we find

$$Det(Q_1) = (h^1_1+h^1_3)(h^2_1+h^2_3) > 0$$

$$Det(Q_2) = (h^1_1+h^1_3)^2\,(h^2_1h^2_2+h^2_1h^2_3+h^2_2h^2_3) > 0$$

$$Det(Q_3) = 0$$

$$Det(Q_4) = 0.$$

Hence, the quadratic form is positive semidefinite and condition (2) holds generically, which completes the proof. ∎

4.6. *Symmetric Games with Symmetric Payoff Perturbations*

The fact that one cannot generally classify equilibria as stable or unstable without specifying distributions of heterogeneity is potentially troubling if one wants to use learning models to understand what kinds of mixed equilibria we might expect to observe. Given that we are assuming the amount of heterogeneity in preferences for each action is small, one could argue that it would be quite problematic to make specific assumptions about the relative degrees of heterogeneity in the various dimensions.

Nevertheless, in this section we will do exactly that and present an additional result that holds when he distribution of payoff perturbations is the same in the two populations.[12] In this case, we show that generically the totally mixed Nash equilibria of symmetric games are unstable.

PROPOSITION 9. *Let G be a symmetric 3×3 game with a totally mixed Nash equilibrium σ^*. Suppose also that the distributions of types in the two populations are identical and that*

$$h_1^1 \Delta_1(G_{11}) + h_2^1 \Delta_1(G_{22}) + h_3^1 \Delta_1(G_{33}) \neq 0.$$

Then, any purifying sequence for σ^* is unstable.

Proof. We show that conditions (1) and (2) are incompatible.

Given that the payoffs in the game are symmetric and the type distributions are identical we have $M_1 = M_2 = M$. Write m_{ij} for the ijth element of this matrix. Conditions (1) and (2) for stability require that

$$Tr(M^2) = (m_{11} + m_{22})^2 + 2(m_{12}m_{21} - m_{11}m_{22}) \leqslant 0$$

and

$$Tr(M^2) - 4Det(M^2)$$
$$= ((m_{11} + m_{22})^2 + 4(m_{12}m_{21} - m_{11}m_{22}))(m_{11} + m_{22})^2 \geqslant 0.$$

A simple calculation shows that

$$m_{11} + m_{22} = h_1^1 \Delta_1(G_{11}) + h_2^1 \Delta_1(G_{22}) + h_3^1 \Delta_1(G_{33}).$$

Hence, the condition in the proposition implies that $(m_{11} + m_{22})^2 > 0$. In this case, comparing the two conditions we see that conditions (1) and (2)

[12] Note that the payoff perturbations are not required to be symmetric with respect to the three pure strategies of each player.

can be simultaneously satisfied (or at least only weakly violated) only if $m_{12}m_{21} - m_{11}m_{22} \geqslant 0$. When this is true and $m_{11} + m_{22} \neq 0$, however, condition (1) is strictly violated. ∎

Remarks.

1. The basic insight from the proof above is that in symmetric games the conflicting interest and no-cycling conditions (1) and (2) are incompatible and cannot be satisfied if the perturbations are symmetric in the two populations.

2. The genericity restriction in the proposition

$$h_1^1 \Delta_1(G_{11}) + h_2^1 \Delta_1(G_{22}) + h_3^1 \Delta_1(G_{33}) \neq 0$$

fails for symmetric zero-sum games. Condition (2) is thus only weakly violated, and we cannot say whether the purified equilibria of these games are stable or unstable.

3. The assumption of symmetric perturbations is necessary to obtain this result. Recall that the symmetric game used in the proof of Proposition 5 has a totally mixed equilibrium which is stable for small ε with an asymmetric type distribution. When α is small in this game, only a small degree of imbalance in the type distributions is needed to make the equilibrium stable.

5. CONCLUSION

In this paper we have used a smoothed fictitious play model to try to better understand when purified mixed equilibria might be expected to emerge within populations of interacting agents. Our conclusions on stability in these games are more supportive of the idea that learning can lead to mixed equilibria than might have been expected given previous analyses of learning models in 3×3 games. One way to think intuitively of what our results are saying may be that the inherent stability of learning in a heterogeneous population (e.g., the model is globally convergent when the payoffs of the unperturbed game are all zero) is sufficient to overcome the instability caused by the fact that maximizing on's own payoff is different from minimizing one's opponent's payoff, provided that this difference is not too large.

In light of results on stochastic approximation, we would imagine that our results can be shown to carry over to other smoothings of fictitious play. One weakness of our analysis is that we assess the reasonableness of a mixed equilibrium using a purely local analysis, yet an unstable path away from an equilibrium may lead to a limit cycle that is close by, and a stable equilibrium may have a very small basin of attraction. It is conceivable that the basins of attraction of equilibria we identify as stable might also sometimes vanish in the limit as epsilon goes to zero. While we believe that a substantial degree of heterogeneity is an important feature of real world learning, a more complete analysis of the dynamics would clearly be valuable.

Our restriction to 3×3 games is also a clear limitation of our work, and it would be nice to know that the basic lessons we have drawn carry over to larger classes of games. We would guess that zero-sum games will always be stable and that analogs of the conditions we have derived (in particular that the game be one of conflict) will still be necessary. However, the possibilities for cycling may be greater, and we do not know how these will affect the frequency with which mixed equilibria are stable.

APPENDIX

LEMMA 1. *$\lim_{\varepsilon \to 0} k^{i\varepsilon}_{jk}$ exists and is independent of the choice of purifying sequence.*

Proof. Fix a type distribution F and a purifying sequence $\{\sigma^{\varepsilon}\}$ for σ^*. Recall that $k^{i\varepsilon}_{jk} = 1/\varepsilon(g_i(a_{ij}, \sigma^{\varepsilon}_{-i}) - g_i(a_{ik}, \sigma^{\varepsilon}_{-i}))$, so player i prefers action j to action k if and only if $\theta_i(a_{ij}) - \theta_i(a_{ik}) > -k^{i\varepsilon}_{jk}$. Thus, the probability $\sigma^{\varepsilon}_i(a_{i1})$ that i plays a_{i1} is exactly the probability of the event $\{\theta_i(a_{i1}) - \theta_i(a_{i2}) > -k^{i\varepsilon}_{12}$ and $\theta_i(a_{i1}) - \theta_i(a_{i3}) > -k^{i\varepsilon}_{13}\}$. Hence $(k^{i\varepsilon}_{12}, k^{i\varepsilon}_{13})$ is a solution for (x, y) in the equation

$$\sigma^{\varepsilon}_i(a_{i1}) = \text{Prob}\{\theta_i(a_{i1}) - \theta_i(a_{i2}) > -x \text{ and } \theta_i(a_{i1}) - \theta_i(a_{i3}) > -y\}.$$

It is similarly also a solution to

$$\sigma^{\varepsilon}_i(a_{i2}) = \text{Prob}\{\theta_i(a_{i1}) - \theta_i(a_{i2}) < x \text{ and } \theta_i(a_{i3}) - \theta_i(a_{i2}) < x - y\}.$$

Since the type distribution F has a positive density everywhere, the first of these conditions holds on a downward-sloping curve in the space of pairs (x, y). The second condition holds on a curve with upward slope. Hence $(k^{i\varepsilon}_{12}, k^{i\varepsilon}_{13})$ is uniquely determined as the intersection of these two curves. Each of these curves shifts continuously in ε as $\varepsilon \to 0$, and hence $(k^i_{12}, k^i_{13}) \equiv \lim_{\varepsilon \to 0}(k^{i\varepsilon}_{12}, k^{i\varepsilon}_{13})$ is well defined.

By continuity (k^i_{12}, k^i_{13}) must satisfy both

$$\sigma^*_i(a_{i1}) = \text{Prob}\{\theta_i(a_{i1}) - \theta_i(a_{i2}) > -k^i_{12} \text{ and } \theta_i(a_{i1}) - \theta_i(a_{i3}) > -k^i_{13}\}$$

and

$$\sigma^*_i(a_{i2}) = \text{Prob}\{\theta_i(a_{i1}) - \theta_i(a_{i2}) < k^i_{12} \text{ and } \theta_i(a_{i3}) - \theta_i(a_{i2}) < k^i_{12} - k^i_{13}\}.$$

These equations again have a unique solution which shows that (k^i_{12}, k^i_{13}) does not depend on the particular purifying sequence for σ^* which was chosen. ▮

Derivation of Equivalent Form for Condition (1)

Expanding the terms in the matrices M_1 and M_2 we find

$$\begin{aligned}
Tr(M_1 M_2) &= m^1_{11} m^2_{11} + m^1_{12} m^2_{21} + m^1_{21} m^2_{12} + m^1_{22} m^2_{22} \\
&= (h^1_3\, \Delta_1(G_{32}) + h^1_2\, \Delta_1(G_{22}))(h^2_3\, \Delta_2(G_{23}) + h^2_2\, \Delta_2(G_{22})) \\
&\quad + (h^1_3\, \Delta_1(G_{31}) + h^1_2\, \Delta_1(G_{21}))(-h^2_3\, \Delta_2(G_{23}) + h^2_1\, \Delta_2(G_{21})) \\
&\quad + (-h^1_3\, \Delta_1(G_{32}) + h^1_1\, \Delta_1(G_{12}))(h^2_3\, \Delta_2(G_{13}) + h^2_2\, \Delta_2(G_{12})) \\
&\quad + (-h^1_3\, \Delta_1(G_{31}) + h^1_1\, \Delta_1(G_{11}))(-h^2_3\, \Delta_2(G_{13}) + h^2_1\, \Delta_2(G_{11}))
\end{aligned}$$

Grouping the sixteen terms in this product according to the $h^1_j h^2_k$ terms gives

$$\begin{aligned}
Tr(M_1 M_2) &= h^1_1 h^2_1\, \Delta_1(G_{11})\, \Delta_2(G_{11}) \\
&\quad + h^1_1 h^2_2\, \Delta_1(G_{12})\, \Delta_2(G_{12}) \\
&\quad + h^1_1 h^2_3\, (\Delta_1(G_{12})\, \Delta_2(G_{13}) - \Delta_1(G_{11})\, \Delta_2(G_{13})) \\
&\quad + h^1_2 h^2_1\, \Delta_1(G_{21})\, \Delta_2(G_{21}) \\
&\quad + h^1_2 h^2_2\, \Delta_1(G_{22})\, \Delta_2(G_{22}) \\
&\quad + h^1_2 h^2_3 (\Delta_1(G_{22})\, \Delta_2(G_{23}) - \Delta_1(G_{21})\, \Delta_2(G_{23})) \\
&\quad + h^1_3 h^2_1 (\Delta_1(G_{31})\, \Delta_2(G_{21}) - \Delta_1(G_{31})\, \Delta_2(G_{11})) \\
&\quad + h^1_3 h^2_2 (\Delta_1(G_{32})\, \Delta_2(G_{22}) - \Delta_1(G_{32})\, \Delta_2(G_{12})) \\
&\quad + h^1_3 h^2_3 (\Delta_1(G_{32})\, \Delta_2(G_{23}) - \Delta_1(G_{31})\, \Delta_2(G_{23}) \\
&\quad + \Delta_1(G_{31})\, \Delta_2(G_{13}) - \Delta_1(G_{32})\, \Delta_2(G_{13})).
\end{aligned}$$

It is then straightforward to show that each of the terms in this expression can be simplified to given the desired result, e.g.

$$\begin{aligned}
&\Delta_1(G_{32})\, \Delta_2(G_{22}) - \Delta_1(G_{32})\, \Delta_2(G_{12}) \\
&\quad = \Delta_1(G_{32})(g_2(a_{11} - a_{13}, a_{21} - a_{23}) - g_2(a_{12} - a_{13}, a_{21} - a_{23})) \\
&\quad = \Delta_1(G_{32})\, g_2(a_{11} - a_{12}, a_{21} - a_{23}) \\
&\quad = \Delta_1(G_{32})\, \Delta_2(G_{32}).
\end{aligned}$$

▮

Derivation of Equivalent Form for Condition (3)

Again, we start by expanding the terms in the matrix M_1 to find

$$\begin{aligned}Det(M_1) &= m^1_{11}m^1_{22} - m^1_{12}m^1_{21} \\ &= (h^1_3\,\Delta_1(G_{32}) + h^1_2\,\Delta_1(G_{22}))(-h^1_3\,\Delta_1(G_{31}) + h^1_1\,\Delta_1(G_{11})) \\ &\quad -(h^1_3\,\Delta_1(G_{31}) + h^1_2\,\Delta_1(G_{21}))(-h^1_3\,\Delta_1(G_{32}) + h^1_1\,\Delta_1(G_{12})).\end{aligned}$$

To simplify the notation in this proof we write x for $\Delta_1(G_{33})$, y for $\Delta_1(G_{32})$, z for $\Delta_1(G_{23})$, and w for $\Delta_1(G_{22})$. Each of the other Δ's in the expression above can be expressed in terms of these four parameters:

$$\begin{aligned}\Delta_1(G_{12}) &= w - y \\ \Delta_1(G_{21}) &= w - z \\ \Delta_1(G_{11}) &= x + w - y - z \\ \Delta_1(G_{31}) &= y - x.\end{aligned}$$

In terms of these parameters we have

$$\begin{aligned}Det(M_1) &= (h^1_3 y + h^1_2 w)(-h^1_3(y-x) + h^1_1(x+w-y-z)) \\ &\quad -(h^1_3(y-x) + h^1_2(w-z))(-h^1_3 y + h^1_1(w-y)).\end{aligned}$$

Multiplying out this expression gives a sum of 24 terms. Grouping these by the products of x, y, z, and w they contain gives

$$\begin{aligned}Det(M_1) = {} & (-h^1_3h^1_3 - h^1_3h^1_1 + h^1_3h^1_3 + h^1_3h^1_1)\,y^2 + (h^1_2h^1_1 - h^1_2h^1_1)\,w^2 \\ & + (h^1_3h^1_3 + h^1_3h^1_1 - h^1_3h^1_2 - h^1_3h^1_1)\,xy \\ & + (h^1_3h^1_1 - h^1_2h^1_3 - h^1_2h^1_1 - h^1_3h^1_1 + h^1_2h^1_3 + h^1_2h^1_1)\,yw \\ & + (h^1_3h^1_2 + h^1_2h^1_1 + h^1_3h^1_1)\,xw \\ & + (-h^1_3h^1_2 - h^1_2h^1_1 - h^1_3h^1_1)\,yz.\end{aligned}$$

Cancelling out a large number of terms gives

$$Det(M_1) = (h^1_1h^1_2 + h^1_1h^1_3 + h^1_2h^1_3)(xw - yz)$$

as desired. The expression for $Det(M_2)$ is symmetric. ∎

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