

Managing a Liquidity Trap: Monetary and Fiscal Policy*

Iván Werning, MIT

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Abstract

I study monetary and fiscal policy in liquidity trap scenarios, where the zero bound on the nominal interest rate is binding. I work with a continuous-time version of the standard New Keynesian model. Without commitment the economy suffers from deflation and depressed output. I show that, surprisingly, both are exacerbated with greater price flexibility. I find that the optimal interest rate is set to zero past the liquidity trap and jumps discretely up upon exit. Inflation may be positive throughout, so the absence of deflation is not evidence against a liquidity trap. Output, on the other hand, always starts below its efficient level and rises above it. Thus, monetary policy promotes inflation and an output boom. I show that the optimal prolongation of zero interest rates is related to the latter, not the former. I then study fiscal policy and show that, regardless of parameters that govern the value of “fiscal multipliers” during normal or liquidity trap times, at the start of a liquidity trap optimal spending is above its natural level. However, it declines over time and goes below its natural level. I propose a decomposition of spending according to “opportunistic” and “stimulus” motives. The former is defined as the level of government purchases that is optimal from a static, cost-benefit standpoint, taking into account that, due to slack resources, shadow costs may be lower during a slump; the latter measures deviations from the former. I show that stimulus spending may be zero throughout, or switch signs, depending on parameters. Finally, I consider the hybrid where monetary policy is discretionary, but fiscal policy has commitment. In this case, stimulus spending is positive and initially increasing throughout the trap.

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1 Introduction

The 2007-8 crisis in the U.S. led to a steep recession, followed by aggressive policy responses. Monetary policy went full tilt, cutting interest rates rapidly to zero, where they have remained since the end of 2008. With conventional monetary policy seemingly exhausted, fiscal stimulus worth \$787 billion was enacted by early 2009 as part of the American Recovery and Reinvestment Act. Unconventional monetary policies were also pursued, starting with “quantitative easing”, purchases of long-term bonds and other assets. In August 2011, the Federal Reserve’s FOMC statement signaled the intent to keep interest rates at zero until at least mid 2013. Similar policies have been followed, at least during the peak of the crisis, by many advanced economies. Fortunately, the kind of crises that result in such extreme policy measures have been relatively few and far between. Perhaps as a consequence, the debate over whether such policies are appropriate remains largely unsettled. The purpose of this paper is to make progress on these issues.

To this end, I reexamine monetary and fiscal policy in a liquidity trap, where the zero bound on nominal interest rate binds. I work with a standard New Keynesian model that builds on [Eggertsson and Woodford \(2003\)](#).¹ In these models a liquidity trap is defined as a situation where negative real interest rates are needed to obtain the first-best allocation. I adopt a deterministic continuous time formulation that turns out to have several advantages. It is well suited to focus on the dynamic questions of policy, such as the optimal exit strategy, whether spending should be front- or back-loaded, etc. It also allows for a simple graphical analysis and delivers several new results. The alternative most employed in the literature is a discrete-time Poisson model, where the economy starts in a trap and exits from it with a constant exogenous probability each period. This specification is especially convenient to study the effects of suboptimal and simple Markov policies—because the equilibrium calculations then reduce to finding a few numbers—but does not afford any comparable advantages for the optimal policy problem.

I consider the policy problem under commitment, under discretion and for some intermediate cases. I am interested in monetary policy, fiscal policy, as well as their interplay. What does optimal monetary policy look like? How does the commitment solution compare to the discretionary one? How does it depend on the degree of price stickiness? How can fiscal policy complement optimal monetary policy? Can fiscal policy mitigate the problem created by discretionary monetary policy? To what extent is spending gov-

¹[Eggertsson \(2001, 2006\)](#) study government spending during a liquidity trap in a New Keynesian model, with the main focus is on the case without commitment and implicit commitment to inflate afforded by rising debt. [Christiano et al. \(2011\)](#), [Woodford \(2011\)](#) and [Eggertsson \(2011\)](#) consider the effects of spending on output, computing “fiscal multipliers”, but do not focus on optimal policy.

erned by a concern to influence the private economy as captured by "fiscal multipliers", or by simple cost-benefit public finance considerations?

I first study monetary policy in the absence of fiscal policy. When monetary policy lacks commitment, deflation and depression ensue. Both are commonly associated with liquidity traps. Less familiar is that both outcomes are exacerbated by price flexibility. Thus, one does not need to argue for a large degree of price stickiness to worry about the problems created by a liquidity trap. In fact, quite the contrary. I show that the depression becomes unbounded as we converge to fully flexible prices. The intuition for this result is that the main problem in a liquidity trap is an elevated real interest rate. This leads to depressed output, which creates deflationary pressures. Price flexibility accelerates deflation, raising the real interest rate further and only making matters worse.

As first argued by [Krugman \(1998\)](#), optimal monetary policy can improve on this dire outcome by committing to future policy in a way that affects current expectations favorably. In particular, I show that, it is optimal to promote future inflation and stimulate a boom in output. I establish that optimal inflation may be positive throughout the episode, so that deflation is completely avoided. Thus, the absence of deflation, far from being at odds with a liquidity trap, actually may be evidence of an optimal response to such a situation. I show that output starts below its efficient level, but rises above it towards the end of the trap. Indeed, the boom in output is larger than that stimulated by the inflationary promise.

There are a number of ways monetary policy can promote inflation and stimulate output. Monetary easing does not necessarily imply a low equilibrium interest rate path. Indeed, as in most monetary models, the nominal interest rate path does not uniquely determine an equilibrium. Indeed, an interest rate of zero during the trap that becomes positive immediately after the trap is consistent with positive inflation and output after the trap.² I show, however, that the optimal policy with commitment involves keeping the interest rate down at zero longer. The continuous time formulation helps here because it avoids time aggregation issues that may otherwise obscure the result.

Some of my results echo findings from prior work based on simulations for a Poisson specification of the natural rate of interest. [Christiano et al. \(2011\)](#) reports that, when the central bank follows a Taylor rule, price stickiness increases the decline in output during a liquidity trap. [Eggertsson and Woodford \(2003\)](#), [Jung et al. \(2005\)](#) and [Adam and Billi](#)

²For example, a zero interest during the trap and an interest equal to the natural rate outside the trap. This is the same path for the interest rate that results with discretionary monetary policy. However, in that case, the outcome for inflation and output is pinned down by the requirement that they reach zero upon exiting the trap. With commitment, the same path for interest rates is consistent with higher inflation and output upon exit.

(2006) find that the optimal interest rate path may keep it at zero after the natural rate of interest becomes positive. To the best of my knowledge this paper provides the first formal results explaining these findings for inflation, output and interest rates.

An implication of my result is that the interest rate should jump discretely upon exiting the zero bound—a property that can only be appreciated in continuous time. Thus, even when fundamentals vary continuously, optimal policy calls for a discontinuous interest rate path.

Turning to fiscal policy, I show that, there is a role for government spending during a liquidity trap. Spending should be front-loaded. At the start of the liquidity trap, government spending should be higher than its natural level. However, during the trap spending should fall and reach a level below its natural level. Intuitively, optimal government spending is countercyclical, it leans against the wind. Private consumption starts out below its efficient level, but reaches levels above its efficient level near the end of the liquidity trap. The pattern for government spending is just the opposite.

The optimal pattern for total government spending masks two potential motives. Perhaps the most obvious, especially within the context of a New Keynesian model, is the macroeconomic, countercyclical one. Government spending affects private consumption and inflation through dynamic general equilibrium effects. In a liquidity trap this may be particularly useful, to mitigate the depression and deflation associated with these events.

However, a second, often ignored, motive is based on the idea that government spending should react to the cycle even based on static, cost-benefit calculations. In a slump, the wage, or shadow wage, of labor is low. This makes it is an opportune time to produce government goods. During the debates for the 2009 ARRA stimulus bill, variants of this argument were put forth.

Based on these notions, I propose a decomposition of spending into "stimulus" and "opportunistic" components. The latter is defined as the optimal static level of government spending, taking private consumption as given. The former is just the difference between actual spending and opportunistic spending.

I show that the optimum calls for zero stimulus at the beginning of a liquidity trap. Thus, my previous result, showing that spending starts out positive, can be attributed entirely to the opportunistic component of spending. More surprisingly, I then show that for some parameter values stimulus spending is everywhere exactly zero, so that, in these cases, opportunistic spending accounts for all of government spending policy during a liquidity trap. Of course, opportunistic spending does, incidentally, influence consumption and inflation. But the point is that these considerations need not figure into the calculation. In this sense, public finance trumps macroeconomic policy.

Another implication is that, in such cases, commitment to a path for government spending is superfluous. A naive, fiscal authority that acts with full discretion and performs the static cost-benefit calculation chooses the optimal path for spending.

These results assume that monetary policy is optimal. Things can be quite different when monetary policy is suboptimal due to lack of commitment. To address this I study a mixed case, where monetary policy is discretionary but fiscal policy has the power to commit to a government spending path. Positive stimulus spending emerges as a way to fight deflation. Indeed, the optimal intervention is to provide positive stimulus spending that rises over time during the liquidity trap. Back-loading stimulus spending provides a bigger bang for the buck, both in terms of inflation and output. Since price setting is forward looking, spending near the end promotes inflation both near the end and earlier. In addition, any improvement in the real rate of return near the end of the liquidity trap improves the output outcome level for earlier dates. Both reasons point towards increasing stimulus spending.

If the fiscal authority can commit past the trap, then it is optimal to promise lower spending immediately after the trap, and converge towards the natural rate of spending after that. Spending features a discrete downward jump upon exiting the trap. Intuitively, after the trap, once the flexible price equilibrium is attainable, lower government spending leads to a consumption boom. This is beneficial, for the same reasons that monetary policy with commitment promotes a boom, because it raises the consumption level during the trap. Thus, the commitment to lower spending after the trap attempts to mimic the expansionary effects that the missing monetary commitments would have provided.

The model is cast in continuous time and this is one of the distinguishing features of my analysis. Why is continuous time simpler and more powerful here? One answer is that continuous time is useful whenever one needs to solve for endogenous switching times, as in the balance of payment crisis model in [Krugman \(1979\)](#), the Baumol-Tobin model of inventory money demand, or in other *sS* menu-cost models. This is also the situation here because the solution has a bang-bang property, with the interest rate being kept at zero up to some endogenous exit time. In other words, the advantage has little to do with technical tools, such as the use of Pontryagin's maximum principle, and more to do with the fact that time is of the essence, that is, we are solving for an exit time and it is simpler and natural to allow that key choice variable to be continuous.

The rest of the paper is organized as follows. Section 2 introduces the model. Section 3 studies the equilibrium without fiscal policy when monetary policy is conducted with discretion. Section 4 studies optimal monetary policy with commitment. Section 5 adds fiscal policy and studies the optimal path for government spending alongside optimal

monetary policy. Section 6 considers mixed cases where monetary policy is discretionary, but fiscal policy enjoys commitment.

2 A Liquidity Trap Scenario

The model is a continuous-time version of the standard New Keynesian model. The environment features a representative agent, monopolistic competition and Calvo-style sticky prices; it abstracts from capital investment. I spare the reader another rendering of the details of this standard setting (see e.g. [Woodford, 2003](#), or [Galí, 2008](#)) and skip directly to the well-known log-linear approximation of the equilibrium conditions which I use in the remainder of the paper.

Euler Equation and Phillips Curve. The equilibrium conditions, log linearized around zero inflation, are

$$\dot{x}(t) = \sigma^{-1}(i(t) - r(t) - \pi(t)) \quad (1a)$$

$$\dot{\pi}(t) = \rho\pi(t) - \kappa x(t) \quad (1b)$$

$$i(t) \geq 0 \quad (1c)$$

where ρ, σ and κ are positive constants and the path $\{r(t)\}$ is exogenous and given. We also require a solution $(\pi(t), x(t))$ to remain bounded. The variable $x(t)$ represents the output gap: the log difference between actual output and the hypothetical output that would prevail at the efficient, flexible price, outcome. Inflation is denoted by $\pi(t)$ and the nominal interest rate by $i(t)$. Finally, $r(t)$ stands for the “natural rate of interest”, i.e. the real interest rate that would prevail in an efficient, flexible price, outcome with $x(t) = 0$ throughout.

Equation (1a) represents the consumer’s Euler equation. Output growth, equal to consumption growth, is an increasing function of the real rate of interest, $i(t) - \pi(t)$. The natural rate of interest enters this condition because output has been replaced with the output gap. Equation (1b) is the New-Keynesian, forward-looking Phillips curve. It can be restated as saying that inflation is proportional, with factor $\kappa > 0$, to the present value of future output gaps,

$$\pi(t) = \kappa \int_0^{\infty} e^{-\rho s} x(t+s) ds.$$

Thus, positive output gaps stimulate inflation, while negative output gaps produce deflation. Finally, inequality (1c) is the zero-lower bound on nominal interest rates (hereafter,

ZLB).

As for the constants, ρ is the discount rate, σ^{-1} is the intertemporal elasticity of substitution and κ controls the degree of price stickiness. Lower values of κ imply greater price stickiness. As $\kappa \rightarrow \infty$ we approach the benchmark with perfectly flexible prices, where high levels of inflation or deflation are compatible with minuscule output gaps.

A number of caveats are in order. The model I use is the very basic New Keynesian setting, without any bells and whistles. Basing my analysis on this simple model is convenient because it lies at the center of many richer models, so we may learn more general lessons. It also facilitates the normative analysis, which could quickly become intractable otherwise. On the other hand, the analysis abstracts from unemployment, and omits distortionary taxes, financial constraints and other frictions which may be relevant in these situations.

Quadratic Welfare Loss. I will evaluate outcomes using the quadratic loss function

$$L \equiv \frac{1}{2} \int_0^{\infty} e^{-\rho t} \left(x(t)^2 + \lambda \pi(t)^2 \right) dt. \quad (2)$$

According to this loss function it is desirable to minimize deviations from zero for both inflation and the output gap. The constant λ controls the relative weight placed on the inflationary objective. The quadratic nature of the objective is convenient and can be derived as a second order approximation to welfare around zero inflation when the flexible price equilibrium is efficient.³ Such an approximation also suggests that $\lambda = \bar{\lambda}/\kappa$ for some constant $\bar{\lambda}$, so that $\lambda \rightarrow 0$ as $\kappa \rightarrow \infty$, as prices become more flexible, price instability becomes less harmful.

The Natural Rate of Interest. The path for the natural rate $\{r(t)\}$ plays a crucial role in the analysis. Indeed, if the natural rate were always positive, so that $r(t) \geq 0$ for all $t \geq 0$, then the flexible price outcome with zero inflation and output gap, $\pi(t) = x(t) = 0$ for all $t \geq 0$, would be feasible and obtained by letting $i(t) = r(t)$ for all $t \geq 0$. This outcome is also optimal, since it is ideal according to the loss function (2).

The situation described in the previous paragraph amounts to the case where the ZLB constraint (1c) is always slack. The focus of this paper is on situations where the ZLB constraint binds. Thus, I am interested in cases where $r(t) < 0$ for some range of time.

³In order to be efficient, the equilibrium requires a constant subsidy to production to undo the monopolistic markup. An alternative quadratic objective that does not assume the flexible price equilibrium is efficient is $\frac{1}{2} \int_0^{\infty} e^{-\rho t} \left((x(t) - \bar{x})^2 + \lambda \pi(t)^2 \right) dt$ for $\bar{x} > 0$. Most of the analysis would carry through to this case.

For a few results it is useful to further assume that the the economy starts in a liquidity trap that it will eventually and permanently exit at some date $T > 0$:

$$\begin{aligned} r(t) &< 0 & t < T \\ r(t) &\geq 0 & t \geq T. \end{aligned}$$

I call such a case a *liquidity trap scenario*. A simple example is the step function

$$r(t) = \begin{cases} \underline{r} & t \in [0, T) \\ \bar{r} & t \in [T, \infty) \end{cases}$$

where $\bar{r} > 0 > \underline{r}$. I use the step function case in some figures and simulations, but it is not required for any of the results in the paper.

Finally, I also make a technical assumption: that $r(s)$ is bounded and that the integral $\int_0^t r(s)ds$ be well defined and finite for any $t \geq 0$.

3 Monetary Policy without Commitment

Before studying optimal policy with commitment, it is useful to consider the situation without commitment, where the central bank is benevolent but cannot credibly announce plans for the future. Instead, it acts opportunistically at each point in time, with absolute discretion. This provides a useful benchmark that illustrates some features commonly associated with liquidity traps, such as deflationary price dynamics and depressed output. I will also derive some less expected implications on the role of price stickiness. The outcome without commitment is later contrasted to the optimal solution with commitment.

3.1 Deflation and Depression

To isolate the problems created by a complete lack of commitment, I rule out explicit rules as well as reputational mechanisms that bind or affect the central bank's actions directly or indirectly. I construct the unique equilibrium as follows.⁴ For $t \geq T$ the natural rate is positive, $r(t) = \bar{r} > 0$, so that, as mentioned above, the ideal outcome $(\pi(t), x(t)) = (0, 0)$ is attainable. I assume that the central bank can guarantee this out-

⁴In this section, I proceed informally. With continuous time, a formal study of the no-commitment case requires a dynamic game with commitment over vanishingly small intervals.

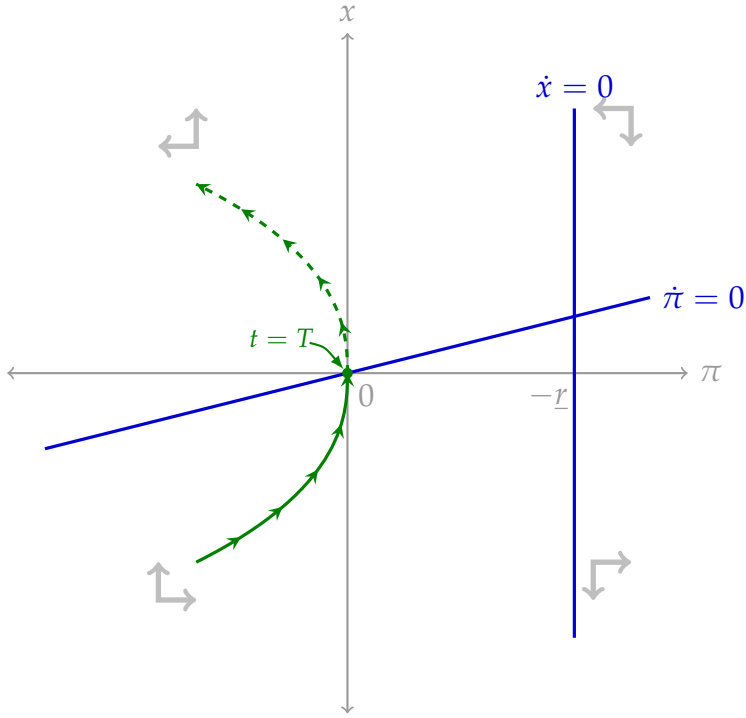


Figure 1: The equilibrium without commitment, featuring $i(t) = 0$ for $t \leq T$ and reaching $(0,0)$ at $t = T$.

come so that $(\pi(t), x(t)) = (0,0)$ for $t \geq T$.⁵ Taking this as given, at all earlier dates $t < T$ the central bank will find it optimal to set the nominal interest rate to zero. The resulting no-commitment outcome is then uniquely determined by the ODEs (1a)–(1b) with $i(t) = 0$ for $t \leq T$ and the boundary condition $(\pi(T), x(T)) = (0,0)$.

This situation is depicted in Figure 1 which shows the dynamical system (1a)–(1b) with $i(t) = 0$ and depicts a path leading to $(0,0)$ precisely at $t = T$. Output and inflation are both negative for $t < T$ as they approach $(0,0)$. Note that the loci on which $(\pi(t), x(t))$ must travel towards $(0,0)$ is independent of T , but a larger T requires a starting point further away from the origin. Thus, initial inflation and output are both decreasing in T . Indeed, as $T \rightarrow \infty$ we have that $\pi(0), x(0) \rightarrow -\infty$.

Proposition 1. *Consider a liquidity trap scenario, with $r(t) < 0$ for $t < T$ and $r(t) \geq 0$ for $t \geq T$. Let $\pi^{nc}(t)$ and $x^{nc}(t)$ denote the equilibrium outcome without commitment. Then*

⁵Although this seems like a natural assumption, it presumes that the central bank somehow overcomes the indeterminacy of equilibria that plagues these models. A few ideas have been advanced to accomplish this, such as adhering to a Taylor rule with appropriate coefficients, or the fiscal theory of the price level. However, both assume commitment on and off the equilibrium path. Although this issue is interesting, it seem completely separate from the zero lower bound. Thus, the assumption that $(\pi(t), x(t)) = (0,0)$ can be guaranteed for $t \geq T$ allows us to focus on the interaction between no commitment and a liquidity trap scenario.

inflation and output are zero after $t = T$ and strictly negative before that:

$$\pi^{nc}(t) = x^{nc}(t) = 0 \quad t \geq T$$

$$\pi^{nc}(t) < 0 \quad x^{nc}(t) < 0 \quad t < T.$$

Moreover, $\pi(t)$ and $x(t)$ are strictly increasing in t for $t < T$. In the limit as $T \rightarrow \infty$, if the natural rate satisfies $\int_0^T r(t; T) ds \rightarrow -\infty$, then

$$\pi^{nc}(0, T), x^{nc}(0, T) \rightarrow -\infty.$$

The equilibrium features deflation and depression. The severity of both depend, among other things, on the duration T of the liquidity trap. Both become unbounded as $T \rightarrow \infty$. In this sense, discretionary policy making may have very adverse welfare implications.

This outcome coincides with the optimal solution with commitment if one constrains the problem by imposing $(\pi(T), x(T)) = (0, 0)$. In other words, the ability to commit to outcomes within the interval $t \in [0, T)$ is irrelevant; also, the ability to commit once $t = T$ is reached is also irrelevant. What is crucial is the ability to commit ex ante at $t < T$ to outcomes for $t = T$.

How can the outcome be so dire? The main distortion is that the real interest rate is set too high during the liquidity trap. This depresses consumption. Importantly, this effect accumulates over time. Even with zero inflation consumption becomes depressed by $\sigma^{-1} \int_t^T r(s) ds$. For example, with log utility $\sigma = 1$ if the natural rate is -4% and the trap lasts two years the loss in output is at least 8%. Moreover, matters are just made worse by deflation, which raises the real interest rate even more, further depressing output, leading to even more deflation, in a vicious cycle.

Note that it is the lack of commitment during the liquidity trap $t < T$ to policy actions and outcomes after the liquidity trap $t \geq T$ that is problematic. Policy commitment during the liquidity trap $t < T$ is not useful. Neither is the ability to announce a credible plan at $t = T$ for the entire future $t \geq T$. Indeed, if we add $(\pi(T), x(T)) = (0, 0)$ as a constraint, then the no commitment outcome is optimal, even when the central bank enjoys full commitment to any choice over $(\pi(t), x(t), i(t))_{t \neq T}$ satisfying (1a)–(1b) for $t < T$ and $t > T$. What is valuable is the ability to commit during the liquidity trap to policy actions and outcomes after the liquidity trap. In particular, to something other than $(\pi(T), x(T)) = (0, 0)$.

3.2 Elbow Room with a Higher Inflation Target: The Value of Commitment

Before studying optimal policy it is useful to consider the effects of commitment to simple non-optimal policies that avoid the depression and deflation outcomes obtained with full discretion.

Consider a plan that keeps inflation and output gap constant at

$$\pi(t) = -\underline{r} > 0 \quad x(t) = -\frac{1}{\kappa}\underline{r} > 0 \quad \text{for all } t \geq 0.$$

It follows that $i(t) = r(t) + \pi(t)$, so that $i(t) = 0$ for $t < T$ while $i(t) = \bar{r} + \bar{\pi} > \bar{r} > 0$ for $t \geq T$.

Although this policy is not optimal, it behaves well in the limit as prices become fully flexible. Indeed, in this limit as $\kappa \rightarrow \infty$ the output gap converges uniformly to zero while inflation remains constant. Thus, if we adopt the natural case where $\lambda = \bar{\lambda}/\kappa \rightarrow 0$, the loss function converges to its ideal value of zero, $L(\kappa) \rightarrow 0$. Compare this to the dire outcome without commitment in Proposition 2, where the output gap and losses converge to $-\infty$.

Just as in the case without commitment, this simple policy sets the nominal interest rate to zero during the liquidity trap, for $t < T$. Note that after the trap, for $t > T$, the nominal interest rate is actually set to a higher level than the case without commitment. Thus, the advantages of this simple policy do not hinge on lower nominal interest rates. Quite the contrary, higher inflation here coincides with higher nominal interest rates, due to the Fischer effect. One may still describe the outcome as resulting from looser monetary policy, but the point is that the kind of monetary easing needed to avoid the deflation and depression does not require lower equilibrium nominal interest rates. Obviously, these observations translates into long term interest rates at $t = 0$: a commitment to looser future monetary policy does not necessarily translate into lower yields on long term bonds. As we shall see in the next section, the optimal policy with commitment does feature lower, indeed zero, nominal interest rates.

This idea is more general. For any path for the natural interest rate $\{r(t)\}$, set a constant inflation rate given by

$$\pi(t) = \bar{\pi} = -\min_{t \geq 0} r(t)$$

and an output gap of $x(t) = \bar{x} = \kappa\bar{\pi}$. This plan is feasible with a non-negative nominal interest rates $i(t) \geq 0$. These simple policy capture the main idea behind calls to tolerate higher inflation targets that leave more “elbow room” for monetary policy during

liquidity traps (e.g. Summers, 1991; Blanchard et al., 2010). However, given the forward looking nature of inflation in this model, what is crucial is the commitment to higher inflation after the liquidity trap. This contrasts with the conventional argument, where a higher inflation rate before the trap serves as a precautionary sacrifice for future liquidity traps.

It is perhaps surprising that commitment to a simple policy can avoid deflation and depressed output altogether. Of course, they do so at the expense of inflation and overstimulated output. If the required inflation target $\bar{\pi}$ or output gap \bar{x} are large, or if the duration of the trap T is small, these plans may be quite far from optimal, since they require a permanent sacrifice for the loss function.⁶ This motivates the study of optimal monetary policy which I take up in the next section.

3.3 Harmful Effects from Price Flexibility

I now return to the case without without commitment. How is this bleak outcome affected by the degree of price stickiness? One might expect things to improve when prices are more flexible. After all, the main friction in New Keynesian models is price rigidities, suggesting that outcomes should improve as prices become more flexible. The next proposition shows, perhaps counterintuitively, that the reverse is actually the case.

Proposition 2. *Without commitment higher price flexibility leads to more deflation and lower output: if $\kappa < \kappa'$ then*

$$\pi^{nc}(t, \kappa') < \pi^{nc}(t, \kappa) < 0 \quad \text{and} \quad x^{nc}(t, \kappa') < x^{nc}(t, \kappa) < 0 \quad \text{for all } t < T.$$

Indeed, for given $T > 0$ and $t < T$ in the limit as $\kappa \rightarrow \infty$ then

$$\pi(t, \kappa), x(t, \kappa) \rightarrow -\infty \quad \text{and} \quad L(\kappa) \rightarrow \infty.$$

Sticky prices are beneficial because they dampen deflation, this in turn mitigates the depression. In fact, the most favorable outcome is obtained when prices are completely rigid, $\kappa = 0$. At the other end of the spectrum, in the limit of perfectly flexible prices, as $\kappa \rightarrow \infty$, the depression and deflation become unbounded.

⁶The reason the output gap \bar{x} is strictly positive is the New Keynesian model's non-vertical long-run Phillips curve. Some papers have explored modifications of the New Keynesian model that introduce indexation to past inflation. Some forms of full indexation imply that a constant level of inflation affects neither output nor welfare. Thus, with the right form of indexation very simple policies may be optimal or close to optimal. Of course, this is not the case in the present model without indexation.

To see this more clearly, note that the Phillips curve equation (1b) implies that, for a given negative output gap, a higher κ creates more deflation. More deflation, in turn, increases the real interest rate $i - \pi$. By the Euler equation (1a), this requires higher growth in the output gap, but since $x(T) = 0$ this translates into a lower level of $x(t)$ for earlier dates $t < T$. In words, flexible prices lead to more vigorous deflation, raising the real interest rate, increasing the desire for saving, lowering demand and depressing output. Lower output reinforces the deflationary pressures, creating a vicious cycle. The proof in the appendix echoes this intuition closely.

A similar result is reported in the analysis of fiscal multipliers by [Christiano et al. \(2011\)](#). They compute the equilibrium when monetary policy follows a Taylor rule and the natural rate of interest is a Poisson process, taking two values. They show numerically that output may be more depressed when prices are more flexible. They do not pursue a limiting result towards full flexibility.⁷ My result is somewhat distinct, since it applies to a situation with optimal discretionary monetary policy, instead of a given Taylor rule. Also, it holds for any deterministic path for the natural rate. Another difference is that with Poisson uncertainty an equilibrium fails to exist when prices are sufficiently flexible. Despite these differences, the logic for the effect is the same in both cases.⁸

The zero lower bound and the lack of commitment are not critical for this result. The same conclusions follow for any path of the natural rate $\{r(t)\}$ if we assume the central bank sets the nominal interest rate above the natural rate $i(t) = r(t) + \Delta$ with $\Delta > 0$ for some period of time $t \in [0, T]$ and then returns to implementing the first-best outcome $x(t) = \pi(t) = 0$ and $i(t) = r(t)$ for $t > T$.⁹ The zero lower bound and the lack of commitment serve to motivate such a scenario. However, another justification may be policy mistakes of this particular form, where interest rates are set too high (or too low) for a fixed amount of time.

As I discuss later, when the central bank commits to an optimal policy, price flexibility can be beneficial. Surprisingly, it may still be harmful, but it depends on parameters.

⁷Basically the same Poisson calculations in [Christiano et al. \(2011\)](#) appear also in [Woodford \(2011\)](#) and [Eggertsson \(2011\)](#), although the effects of price flexibility are not their focus and so they do not discuss its effects.

⁸[De Long and Summers \(1986\)](#) make the point that, for given monetary policy rules, price flexibility may be destabilizing, even away from a liquidity trap, in the sense of increasing the variance of output.

⁹Of course a symmetric result holds for $\Delta < 0$. There is a boom in output alongside inflation. The undesirable boom and inflation are amplified when prices are more flexible, in the sense of a higher κ .

Is there a Discontinuity at Full Flexibility? Not really.

The result that price flexibility makes matters worse may seem puzzling, especially in the limit, since it seems to contradict the notion that perfectly flexible prices lead to zero output gaps. That is, at $\kappa = \infty$ we expect $x(t) = 0$ for all $t \geq 0$, but, paradoxically, as $\kappa \rightarrow \infty$ we obtain $x(t) \rightarrow -\infty$ instead. Does this reveal an inherent discontinuity in the New Keynesian model?

No. The result is best seen as arising from a discontinuity in monetary policy at $\kappa = \infty$, not a discontinuity in the model itself. The equilibria obtained for finite κ described in Proposition 2 satisfy $\pi(t) \leq 0$ and $\pi(T) = 0$. If one takes these two features as a requirement then there is no equilibrium with $\kappa = \infty$. Economically, this suggests a form of continuity: an explosive outcome converges to a situation where an equilibrium seizes to exist. In any case, when $\kappa = \infty$, monetary policy must allow strictly positive inflation for an equilibrium to exist. In this sense, the discontinuity in outcomes is produced by a discontinuity in monetary policy regarding inflation at $\kappa = \infty$.

To see this more clearly, consider a liquidity trap scenario with $r(t) = \underline{r} < 0$ for $t < T$. Consider indexing monetary policy by a single parameter, $\bar{\pi}$, a target rate of inflation. Specifically, suppose $\pi(t) = \bar{\pi}$ for all $t \geq T$ and $i(t) = 0$ for $t < T$. For any finite κ this pins down an equilibrium uniquely. Note that if $\pi(\bar{T}) = 0$ then the equilibrium coincides with the discretionary case. Indeed, Proposition 2 still describes the outcome for $t \leq T$ in the limit $(1/\kappa_n, \bar{\pi}_n) \rightarrow (0, 0)$. However, suppose that as $1/\kappa_n \rightarrow 0$ we let $\{\pi_n\}$ to be an increasing sequence converging to $-\underline{r} > 0$. Provided this convergence is fast enough, the outcome converges to the one with flexible price so that $\lim_{n \rightarrow \infty} x_n(t) \rightarrow 0$ for all t .

Of course, the particular limit $(1/\kappa_n, \bar{\pi}_n) \rightarrow (0, 0)$ is motivated by the lack of commitment, but the point is that this can be seen as motivating a jump in monetary policy at $\kappa = \infty$, which creates the discontinuity in outcomes.

A more practical lesson from this discussion is that the benefits of flexible prices only accrue with monetary policies that accept higher inflation. In other words, price flexibility only does its magic if we use it.

4 Optimal Monetary Policy with Commitment

I now turn to optimal monetary policy with commitment. The central bank's problem is to minimize the objective (2) subject to (1a)–(1c) with both initial values of the states, $\pi(0)$ and $x(0)$, free. The problem seeks the most preferable outcome, across all those compatible with an equilibrium. In what follows I focus on characterizing the optimal

path for inflation, output and the nominal interest rate.¹⁰

4.1 Optimal Interest Rates, Inflation and Output

The problem can be analyzed as an optimal control problem with state $(\pi(t), x(t))$ and control $i(t) \geq 0$. The associated Hamiltonian is

$$H \equiv \frac{1}{2}x^2 + \frac{1}{2}\lambda\pi^2 + \mu_x\sigma^{-1}(i - r - \pi) + \mu_\pi(\rho\pi - \kappa x).$$

The maximum principle implies that the co-state for x must be non-negative throughout and zero whenever the nominal interest rate is strictly positive

$$\mu_x(t) \geq 0, \tag{3a}$$

$$i(t)\mu_x(t) = 0. \tag{3b}$$

The law of motion for the co-states are

$$\dot{\mu}_x(t) = -x(t) + \kappa\mu_\pi(t) + \rho\mu_x(t), \tag{3c}$$

$$\dot{\mu}_\pi(t) = -\lambda\pi(t) + \sigma^{-1}\mu_x(t). \tag{3d}$$

Finally, because both initial states are free, we have

$$\mu_x(0) = 0, \tag{3e}$$

$$\mu_\pi(0) = 0. \tag{3f}$$

Taken together, equations (1a)–(1c) and (3a)–(3f) constitute a system for $\{\pi(t), x(t), i(t), \mu_\pi(t), \mu_x(t)\}_{t \in [0, \infty)}$. Since the optimization problem is strictly convex, these conditions, together with appropriate transversality conditions, are both necessary and sufficient for an optimum. Indeed, the optimum coincides with the unique bounded solution to this system.

Suppose the zero-bound constraint is not binding over some interval $t \in [t_1, t_2]$. Then it must be the case that $\mu_x(t) = \dot{\mu}_x(t) = 0$ for $t \in [t_1, t_2]$, so that condition (3c) implies

¹⁰I do not dedicate much discussion to the question of implementation, in terms of a choice of (possibly time varying) policy functions that would make the optimum a unique equilibrium. It is well understood that, once the optimum is computed, a time varying interest rate rule of the form $i(t) = i^*(t) + \psi_\pi(\pi(t) - \pi^*(t)) + \psi_x(x(t) - x^*(t))$ ensures that this optimum is the unique local equilibrium for appropriately chosen coefficients ψ_π and ψ_x . [Eggertsson and Woodford \(2003\)](#) propose a different policy, described in terms of an adjusting target for a weighted average of output and the price level, that also implements the equilibrium uniquely.

$x(t) = \kappa\mu_\pi(t)$, while condition (3d) implies $\dot{\mu}_\pi(t) = -\lambda\pi(t)$. As a result,

$$\dot{x}(t) = \kappa\dot{\mu}_\pi(t) = -\kappa\lambda\pi(t) = \sigma^{-1}(i(t) - r(t) - \pi(t)).$$

Solving for $i(t)$ gives

$$i(t) = I(r(t), \pi(t)),$$

where

$$I(r, \pi) \equiv r(t) + (1 - \kappa\sigma\lambda)\pi,$$

is a function that gives the optimal nominal rate whenever the zero-bound is not binding. This is the interest rate condition derived in the traditional analysis that assumes the ZLB never binds (see e.g. [Clarida, Gali and Gertler, 1999](#), pg. 1683). Note that this rate equals the natural rate when inflation is zero, $I(r, 0) = r$. Thus, it encompasses the well-known price stability result from basic New-Keynesian models. Away from zero inflation, the interest rate generally departs from the natural rate, unless $\sigma\lambda\kappa = 1$.

Given this result, it follows that $I^*(r(t), \pi^*(t)) \geq 0$ is a necessary condition for the zero-bound not to bind. The converse, however, is not true.

Proposition 3. *Suppose $\{\pi^*(t), x^*(t), i^*(t)\}$ is optimal. Then at any point in time t either $i^*(t) = I(r(t), \pi^*(t))$ or $i^*(t) = 0$. Moreover if*

$$I(r(t), \pi^*(t)) < 0 \quad \forall t \in [t_0, t_1]$$

then

$$i^*(t) = 0 \quad \forall t \in [t_0, \hat{t}_1].$$

for $t_1 < \hat{t}_1$. Likewise, if $t_0 > 0$ then $i^*(t) = 0$ for $t \in [\hat{t}_0, t_0]$ for $\hat{t}_0 < t_0$.

According to this result, the nominal interest rate should be held down at zero longer than what current inflation warrants. That is, the optimal path for the nominal interest rate is not the upper envelope

$$i^*(t) \neq \max\{0, I(r(t), \pi^*(t))\}.$$

Instead, the nominal interest rate should be set below this envelope for some time, at zero.

The notion that committing to future monetary easing is beneficial in a liquidity trap was first put forth by [Krugman \(1998\)](#). His analysis captures the benefits from future inflation only. It is based on a cash-in-advance model where prices cannot adjust within a period, but are fully flexible between periods. The first best is obtained by committing

to money growth and inducing higher future inflation. Thus, inflation is easily obtained and costless in the model. Eggertsson and Woodford (2003) work with the same New Keynesian model as I do here. They report numerical simulations where a prolonged period of zero interest rates are optimal. My result provides the first formal explanation for these patterns. It also clarifies that the relevant comparison for the nominal interest rate $i^*(t)$ is the unconstrained optimum $I(r(t), \pi^*(t))$, not the natural rate $r(t)$; the two are not equivalent, unless $\kappa\sigma\lambda = 1$. The continuous time framework employed here helps capture the bang-bang nature of the solution. A discrete-time setting can obscure things due to time aggregation.

One interesting implication of my result is that the optimal exit strategy features a discrete jump in the nominal interest rate. Whenever the zero-bound stops binding the nominal interest must equal $I(r(t), \pi^*(t))$, which given Proposition 3, will generally be strictly positive. Thus, optimal policy requires a discrete upward jump, from zero, in the nominal interest rate. Even when economic fundamentals vary smoothly, so that $I(r(t), \pi^*(t))$ is continuous, the best exit strategy calls for a discontinuous hike in the nominal interest rate.

Does commitment to the optimal policy, with prolonged zero interest rates, imply lower long term interest rates? Relative to the discretionary equilibrium are yields on long term bonds available at $t = 0$ lower? Not necessarily. Consider the liquidity trap scenario. Zero interest rates are generally prolonged past T , this lowers the yield rate on medium-term bonds at $t = 0$. However, upon exit from the zero lower bound at $\hat{T} > T$, the short-term interest rate is set to $I(r(\hat{T}), \pi(\hat{T})) \geq r(\hat{T})$ with strict inequality as long as $\kappa\sigma\lambda \neq 1$. Once again, as in Section 3.2, looser monetary policy, in this case optimal monetary policy, is not necessarily unambiguously associated with lower long-term interest rates. By implication, this may imply higher yield on very long term bonds.

The previous result characterizes nominal interest rates, but what can be said about the paths for inflation and output? This question is important for a number of reasons. First, output and inflation are of direct concern, since they determine welfare. In contrast, the nominal interest rate is merely an instrument to influence output and inflation. Second, as in most monetary models, the equilibrium outcome is not uniquely determined by the equilibrium path for the nominal interest rate. A central bank wishing to implement the optimum needs to know more than the path for the nominal interest rate. For example, the central bank may employ a Taylor rule centered around the target path for inflation $i(t) = i^*(t) + \psi(\pi(t) - \pi^*(t))$ with $\psi > 1$. Finally, understanding the outcome for inflation and output sheds light on the kind of policy commitment required.

The next proposition characterizes optimal inflation and output.

Proposition 4. *Suppose the first-best outcome is not attainable and that $\{\pi^*(t), x^*(t), i^*(t)\}$ is optimal:*

1. *Inflation must be strictly positive at some point in time: $\pi(t) > 0$ for some $t \geq 0$.*
2. *Output is initially negative $x(0) \leq 0$, but becomes strictly positive at some point, $x(t) > 0$ for some $t > 0$.*
3. *Furthermore, if (a) $\kappa\sigma\lambda = 1$ then inflation is initially zero and is nonnegative throughout, $\pi^*(0) = 0$ and $\pi^*(t) \geq 0$ for all $t \geq 0$; (b) $\kappa\sigma\lambda < 1$ then $\pi^*(t) > 0$ for all t ; (c) $\kappa\sigma\lambda > 1$ then $\pi^*(0) \leq 0$ with strict inequality if $x(0) < 0$.*

Inflation must be positive at some point. Depending on parameters, initial inflation may be positive or negative. In some cases, inflation may be positive throughout. Output, on the other hand, must switch signs. The initial recession is never completely avoided.

According to this proposition, there are two things optimal monetary policy accomplishes. First, it promotes inflation to mitigate or reverse the deflationary spiral during the liquidity trap. This lowers the real rate of interest, which lessens the recession. Second, it stimulates future output to create a boom after the trap. This percolates back in time, making consumers, who anticipate a boom, lower their desired savings. In other words, the root problem during a liquidity trap is that desired savings and the real interest rate are too high. Optimal policy addresses both.

In this model the two goals are related: inflation requires a boom in output. Thus, pursuing the first goal already leads, incidentally, to the second, and vice versa. However, the nominal interest rate path implied by Proposition 3 stimulates a larger boom than what is required by the inflation promise alone. To see this, suppose that along the optimal plan $I(r(t), \pi^*(t)) \geq 0$ for $t \geq t_1$, and $I(r(t), \pi^*(t)) < 0$ otherwise. The optimal plan then calls for $i^*(t) = 0$ over some interval $t \in [t_1, t_2]$. However, consider an alternative plan that has the same inflation at t_1 , so that $\pi(t_1) = \pi^*(t_1)$, but, in contradiction with Proposition 3, features $i(t) = I(r(t), \pi(t))$ for all $t \geq t_1$.¹¹ Suppose also that, for both plans, the long-run output gap is zero: $\lim_{t \rightarrow \infty} x(t) = \lim_{t \rightarrow \infty} x^*(t) = 0$. It then follows that $x(t_1) < x^*(t_1)$. In this sense, holding down the interest rate to zero stimulates a boom that is greater than the one implied by the inflation promise.

Proposition 4 singles out a case with $\kappa\sigma\lambda = 1$ where inflation starts and ends at zero and is positive throughout. This case occurs when the costate $\mu_\pi(t)$ on the Phillips curve

¹¹Note that, depending on the value of $\kappa\sigma\lambda$, the interest rate may even be greater than the natural rate $r(t)$. The fact that this policy is consistent with positive inflation and output after the trap even though it may have higher interest rates than the discretionary solution underscores, once again, that monetary easing does not necessarily manifest itself in lower equilibrium interest rates.

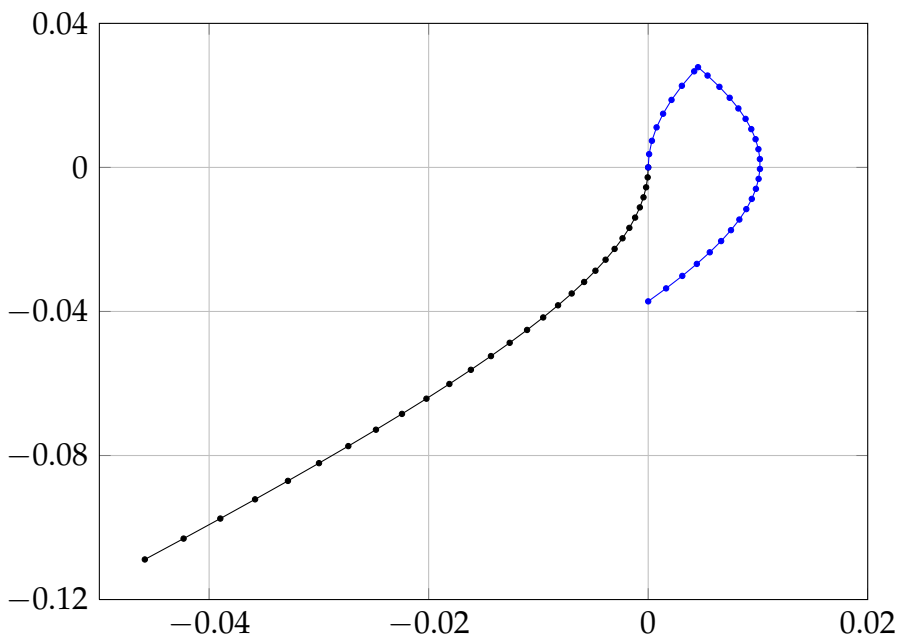


Figure 2: A numerical example showing the full discretion case (black) and optimal commitment case (blue).

is zero for all $t \geq 0$. This case turns out to be an interesting benchmark with other interesting implications for government spending.

Figure 2 plots the equilibrium paths for a numerical example. The parameters are set to $T = 2$, $\sigma = 1$, $\kappa = .5$ and $\lambda = 1/\kappa$. These choices are made for illustrative purposes and to ensure that $\kappa\sigma\lambda = 1$. They do not represent a calibration. The choices are tilted towards a flexible price situation. Relative to the New Keynesian literature, the degree of price stickiness is low (high κ) and the planner is quite tolerant of inflation (low λ). It is also common to set a lower value for σ , on the grounds that investment, which may be quite sensitive to the interest rate, has been omitted from the analysis.

The black line represents the equilibrium with discretion; the blue line, the optimum with commitment. With discretion output is initially depressed by about 11%, at the optimum this is reduced to just under 4%. The optimum features a boom which peaks at about 3% at $t = T$. The discretionary case features significant deflation. In contrast, because $\kappa\sigma\lambda = 1$ optimal inflation starts at zero and is always positive. Both paths end at origin, which represents the ideal first-best outcome. However, although the optimum reaches it later at $\hat{T} = 2.7$, it circles around it, managing to stay closer to it on average. This improves welfare.

One implication of Proposition 4 is that, whenever the first best is unattainable, optimal monetary policy requires commitment. Output is initially negative $x^*(0) \leq 0$, but

must turn strictly positive $x^*(t') > 0$ at some future date for $t' > 0$. This implies that, if the planner can reoptimize and make a new credible plan at time t' , then this new plan would involve initially negative output $x^*(t') \leq 0$. Hence, it cannot coincide with the original plan which called for positive output.

Note that the kind of commitment needed in this model involves more than a promise for future inflation, at time T , as in [Krugman \(1998\)](#). Indeed, my discussion here emphasizes commitment to an output boom. More generally, the planning problem features both π and x as state variables, so commitment to deliver promises for both inflation and output are generally required.

Liquidity traps are commonly associated with deflation, but these results suggest that the optimum completely avoids deflation in some cases. This is more likely to be the case if prices are less flexible (low κ), if the intertemporal elasticity of substitution is high (low σ), or if the central bank is not too concerned about inflation (low λ). Note that if we set $\lambda = \bar{\lambda}/\kappa$, then $\kappa\sigma\lambda = \lambda\sigma$, so the degree of price flexibility κ drops out of the condition determining the sign of initial inflation. In the other case, when $\kappa\sigma\lambda < 1$, the optimum does feature deflation initially, but transitions through a period of positive inflation as shown by [Proposition 4](#). Numerical simulations return to deflation and a negative output gap.

It is worth noting that prolonged zero nominal interest rates are not needed to promote positive inflation and stimulate output after the trap. Indeed, there are equilibria with both features and a nominal interest rate path given by $i(t) = \max\{0, I(r(t), \pi(t))\}$. In the liquidity trap scenario, the same is true for the interest rate path considered under pure discretion, $i(t) = 0$ for $t < T$ and $i(t) = r(t)$ for $t \geq T$. Without commitment, a unique equilibrium was obtained by adding the condition that the first best outcome $\pi(t) = x(t) = 0$ was implemented for $t \geq T$. However, positive inflation and output, $\pi(T), x(T) \geq 0$ are also compatible with this very same interest rate path. This is possible because equilibrium outcomes are not uniquely determined by equilibrium nominal interest rates. Policy may still be described as one of monetary easing, even if this is not necessarily reflected in equilibrium nominal interest rates.¹²

¹²To be specific, suppose policy is determined endogenously according to a simple Taylor rule, with a time varying intercept, $i(t) = \bar{i}(t) + \phi_\pi \pi(t)$ with $\phi_\pi > 1$. In the unique bounded equilibrium, a temporarily low value for $\bar{i}(t)$ typically leads to higher inflation $\pi(t)$, but not necessarily a lower equilibrium interest rate $i(t)$. The outcome for the nominal interest rate $i(t)$ depends on various parameters. Either way, the situation with temporarily low $\bar{i}(t)$ may be described as one of “monetary easing”.

4.2 A Simple Case: Fully Rigid Prices

To gain intuition it helps to consider the extreme case with fully rigid prices, where $\kappa = 0$ and $\pi(t) = 0$ for all $t \geq 0$.¹³ Consider the liquidity trap scenario, where $r(t) < 0$ for $t < T$ and $r(t) > 0$ for $t > T$, and suppose we keep the nominal interest rate at zero until some time $\hat{T} \geq T$, and implement $x(t) = \pi(t) = 0$ after \hat{T} . Output is then

$$x(t; \hat{T}) \equiv \sigma^{-1} \int_t^{\hat{T}} r(s) ds.$$

When $\hat{T} = T$, the integrand is always negative, so that output is negative: $x(t, T) < 0$ for $t < T$. In fact, the equilibrium coincides with the full discretion case isolated by Proposition 1. When $\hat{T} > T$ the integral includes strictly positive values for $r(t)$ for $t \in (T, \hat{T}]$. This increases the path for $x(t; \hat{T})$. For any date $t \leq T$ output increases by the constant $\sigma^{-1} \int_T^{\hat{T}} r(s) ds > 0$. Starting at $t = 0$, output rises and peaks at T , then falls until it reaches zero at \hat{T} . The boom induced at T percolates to earlier dates, increasing output in a parallel fashion.

Larger values of \hat{T} shrink the initially negative output gaps, but lead to larger positive gaps later. Starting from $\hat{T} = T$ an increase in \hat{T} improves welfare because the loss from positive output gaps are second order, while the gain from reducing existing negative output gaps is first order. Formally, minimizing the objective $V(\hat{T}) \equiv \frac{1}{2} \int_0^{\infty} e^{-\rho t} x(t; \hat{T})^2 dt$ yields

$$V'(\hat{T}^*) = r(\hat{T}^*) \sigma^{-1} \int_0^{\hat{T}^*} e^{-\rho t} x(t; \hat{T}^*) dt = 0,$$

and it follows that $T < \hat{T}^* < \bar{T}$ where $x(0, \bar{T}) = 0$. According to this optimality condition, the present value of output should be zero $\int_0^{\infty} e^{-\rho t} x(t) dt = 0$, implying that the current recession and subsequent boom should average out, in present value.

When prices are fully rigid inflation is zero regardless of monetary policy. Hence, creating inflation cannot be the purpose of monetary easing. Instead, committing to zero nominal interest rates is useful here because it creates an output boom after the trap. This boom helps mitigate the earlier recession. The logic here is completely different from the one in Krugman (1998), which isolated the inflationary motive for monetary easing. Next I turn to a graphical analysis of intermediate cases, where both motives are present.

¹³The same conditions we will obtain for $\kappa = 0$ here can be obtained if we consider the limit of the general optimality conditions derived above as $\kappa \rightarrow 0$. However, it is more revealing to derive the optimality condition from a separate perturbation argument.

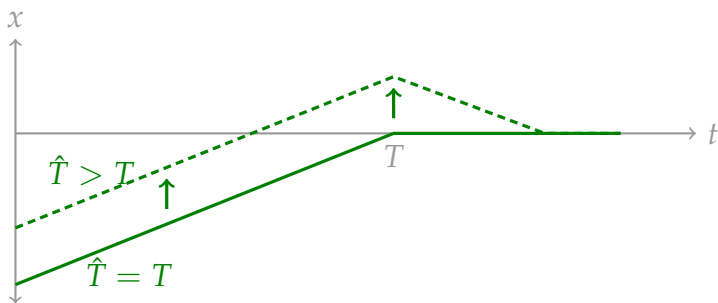


Figure 3: Fully rigid prices. The path for output with $\hat{T} = T$ and $\hat{T} > T$.

4.3 Stitching a Solution Together: A Graphical Representation

To see the solution graphically, consider the particular liquidity trap scenario with the step function path for the natural rate of interest: $r(t) = \underline{r} < 0$ for $t < T$ but $r(t) = \bar{r} \geq 0$ for $t \geq T$. It is useful to break up the solution into three separate phases, from back to front. I first consider the solution after some time $\hat{T} > T$ when the ZLB constraint is no longer binding (Phase III). I then consider the solution between time T and \hat{T} with the ZLB constraint (Phase II). Finally, I consider the solution during the trap $t \in [0, T]$ (Phase I).

After the Storm: Slack ZLB Constraint (Phase III). Consider the problem where the ZLB constraint is ignored, or no longer binding. If this were true for all time $t \geq 0$ then the solution would be the first best $\pi(t) = x(t) = 0$. However, here I am concerned with a situation where the ZLB constraint is slack only after some date $\hat{T} > T > 0$, at which point the state $(\pi(\hat{T}), x(\hat{T}))$ is given and no longer free, so the first best is generally not feasible.

The planning problem now ignores the ZLB constraint but takes the initial state (π_0, x_0) as given. Because the ZLB constraint is absent, the constraint representing the Euler equation is not binding. Thus, it is appropriate to ignore this constraint and drop the output gap $x(t)$ as a state variable, treating it as a control variable instead. The only remaining state is inflation $\pi(t)$.¹⁴ Also note that the path of the natural interest rate $\{r(t)\}$ is irrelevant when the ZLB constraint is ignored.

¹⁴One can pick any absolutely continuous path for $x(t)$ and solve for the required nominal interest rate as a residual: $i(t) = \sigma \dot{x}(t) + \pi(t) + r(t)$. Discontinuous paths for $x(t)$ can be approximated arbitrarily well by continuous ones. Intuitively, it is as if discontinuous paths for $\{x(t)\}$ are possible, since upward or downward jumps in $x(t)$ can be engineered by setting the interest rate to ∞ or $-\infty$ for an infinitesimal moment in time. Formally, the supremum for the problem that ignores the ZLB constraint, but carries both $\pi(t)$ and $x(t)$ as states, is independent of the current value of $x(t)$. Since the current value of $x(t)$ does not meaningfully constrain the planning problem, it can be ignored as a state variable.

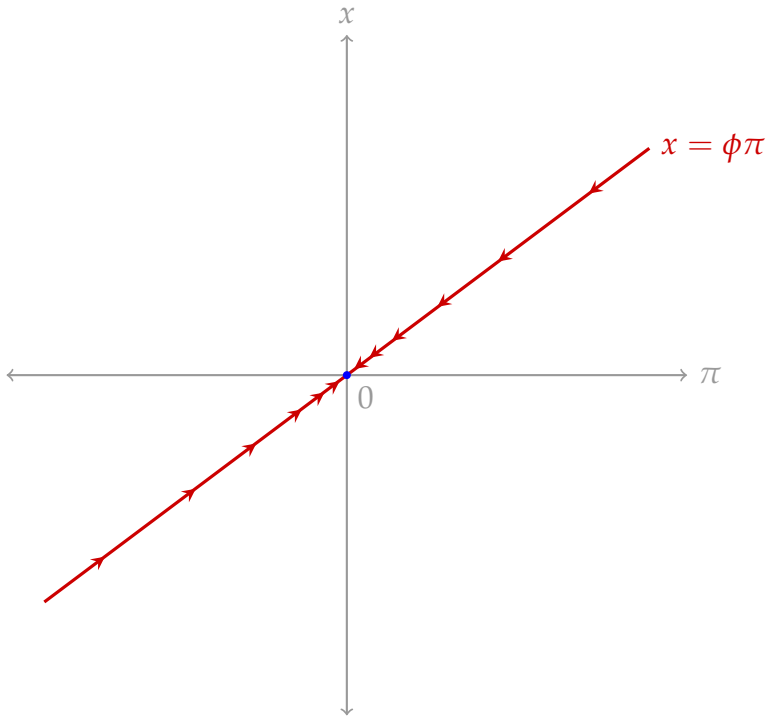


Figure 4: The solution without the ZLB constraint.

I seek a solution for output x as a function of inflation π . Using the optimality conditions with $\mu_x(t) = 0$ one can show that $i(t) = I(\pi(t), r(t))$ as discussed earlier, with output satisfying

$$x(t) = \phi\pi(t)$$

and costate $\mu_\pi(t) = \frac{\phi}{\kappa}\pi(t)$, where $\phi \equiv \frac{\rho + \sqrt{\rho^2 + 4\lambda\kappa^2}}{2\kappa}$ so that $\phi > \rho/\kappa$. The last inequality implies that the ray $x = \phi\pi$ is steeper than that for $\dot{\pi} = 0$. Thus, starting with any initial value of π the solution converges over time along the locus $x = \phi\pi$ to the origin $(\pi(t), x(t)) \rightarrow (0, 0)$. These dynamics are illustrated in Figure 4.

Just out of the Trap (Phase II). Consider next the problem for $t \geq T$ incorporating the ZLB constraint for any arbitrary starting point $(\pi(T), x(T))$. The problem is stationary since $r(t) = \bar{r} > 0$ for $t \geq T$.

If the initial state lies on the locus $x = \phi\pi$, then the solution coincides with the one above. This is essentially also the case when the initial state satisfies $x < \phi\pi$, since one can engineer an upward jump in x to reach the locus $x = \phi\pi$.¹⁵ After this jump, one proceeds with the solution that ignores the ZLB constraint. In contrast, the optimum features an

¹⁵For example, set $i(t) = \Delta/\varepsilon > 0$ for a short period of time $[0, \varepsilon]$ and choose Δ so that $x(\varepsilon) = \phi\pi(\varepsilon)$. As $\varepsilon \downarrow 0$ this approximates an upward jump up to the $x = \phi\pi$ locus at $t = 0$.

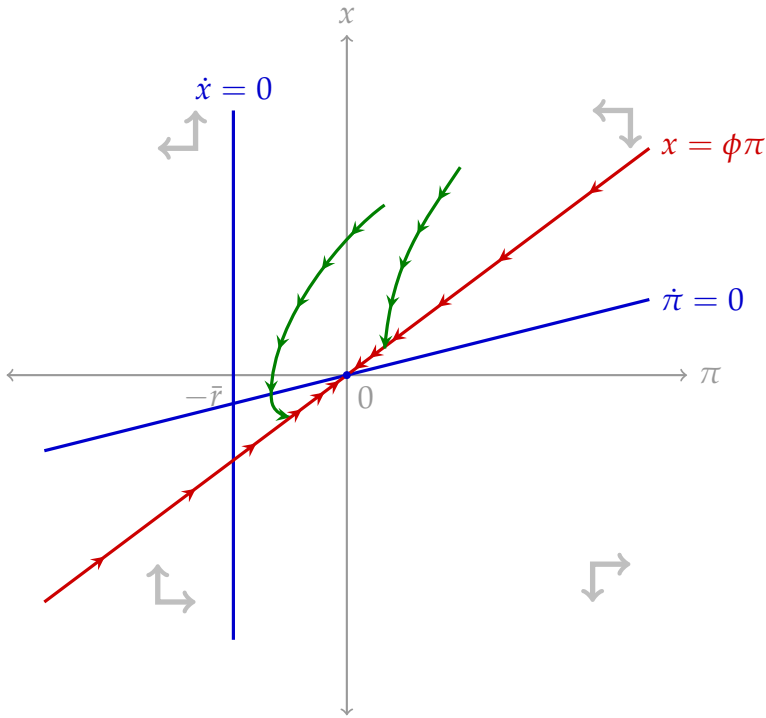


Figure 5: The solution for $t > T$ with the ZLB constraint.

initial state that satisfies $x > \phi\pi$. Intuitively, the optimum attempts to reach the red line as quickly as possible, by setting the nominal interest rate to zero until $x = \phi\pi$.

These dynamics are illustrated in Figure 4 using the phase diagram implied by the system (1a)–(1b) with $i(t) = 0$. The steady state with $\dot{x} = \dot{\pi} = 0$ involves deflation and a negative output gap: $\pi = -\bar{r} < 0$ and $x = -\frac{\rho}{\kappa}\bar{r} < 0$. As a result, for inflation rates near zero the output gap falls over time. As before, the red line denotes the locus $x = \phi\pi$, for the solution to the problem ignoring the ZLB constraint. For two initial values satisfying $x > \phi\pi$, the figure shows the trajectories in green implied by the system (1a)–(1b) with $i(t) = 0$. Along these paths $x(t)$ and $\pi(t)$ fall over time, eventually reaching the locus $x = \phi\pi$. After this point, the state follows the solution ignoring the ZLB constraint, staying on the $x = \phi\pi$ line and converges towards the origin.

During the Liquidity Trap (Phase I) During the liquidity trap $t \leq T$ the ZLB constraint binds and $i(t) = 0$. The dynamics are illustrated in Figure 6 using the phase diagram implied by equations (1a) and (1b) setting $i(t) = 0$. For reference, the red line denoting the optimum ignoring the ZLB constraint is also show.

Unlike the previous case, the steady state $\dot{x} = \dot{\pi} = 0$ for this system now has positive inflation and a positive output gap: $\pi = -\underline{r} > 0$ and $x = -\frac{\rho}{\kappa}\underline{r} > 0$. In contrast to the

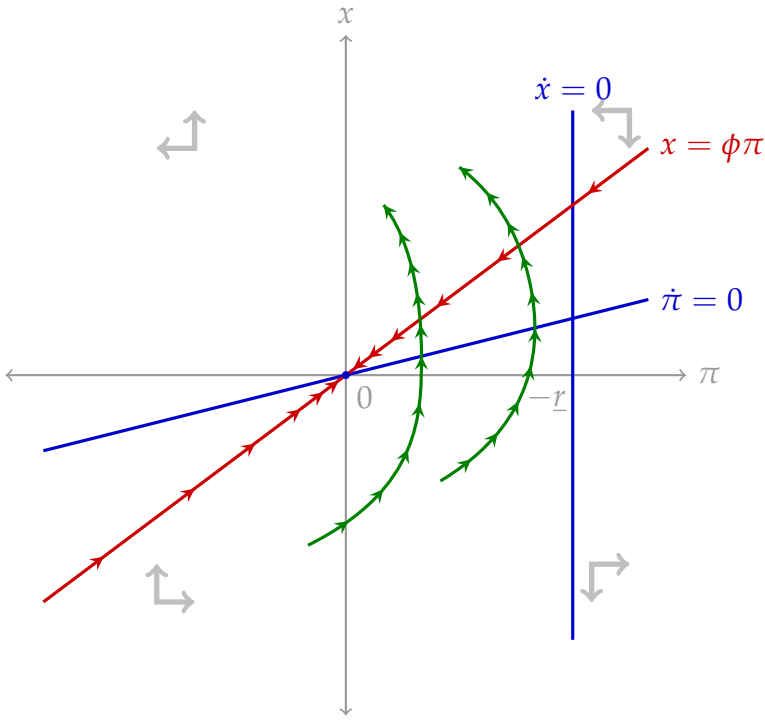


Figure 6: The solution for $t \leq T$ and $r(t) = -\bar{r} < 0$ with the ZLB constraint binding.

previous phase diagram, also featuring $i(t) = 0$, for inflation rates near zero the output gap rises over time. Two trajectories are shown in green. In one case, inflation is initially negative; in the other it is positive. Both cases have an the output gap initially negative and turning positive some time before $t = T$. Both trajectories start below the red line and end up above it at $t = T$.

Figure 7 puts the three phases together to display two possible optimal paths for all $t \geq 0$. The two trajectories illustrated in the figure are quite representative and illustrate the possibilities described in Proposition 4.

As these figures suggest one can prove that the nominal interest rate should be kept at zero past T . The following proposition follows from Proposition 3 and elements of the dynamics captured by the phase diagrams.

Proposition 5. *Consider the liquidity trap scenario with $r(t) = \underline{r} < 0$ for $t < T$ and $r(t) = \bar{r} > 0$ for $t \geq T$. Suppose the path $\{\pi^*(t), x^*(t), i^*(t)\}$ is optimal. Then there exists a $\hat{T} > T$ such that*

$$i(t) = 0 \quad \forall t \in [0, \hat{T}].$$

There are two ways of summarizing the optimal plan. In the first, the central bank commits to a zero nominal interest rate during the liquidity trap, for $t \in [0, T]$. It also makes a commitment to an inflation rate and output gap target $(\pi^*(T), x^*(T))$ after the

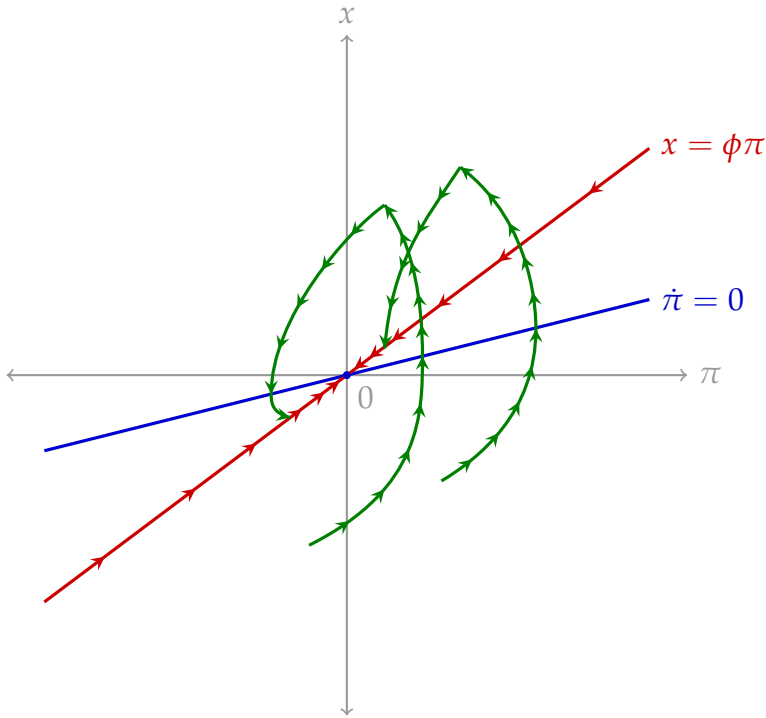


Figure 7: Two possible paths of the solution for $t \geq 0$.

trap. However, note that here

$$x^*(T) > \phi\pi^*(T)$$

so that the promised boom in output is higher than that implied by the inflation promise. Commitment to a target at time T is needed not just in terms of inflation, but also in terms of the output gap.

Another way of characterizing policy is as follows. The central bank commits to setting a zero interest rate at zero for longer than the liquidity trap, so that $i(t) = 0$ for $t \in [0, \hat{T}]$ with $\hat{T} > T$. It also commits to implementing an inflation rate $\pi(\hat{T})$ upon exit of the ZLB, at time \hat{T} . In this case, no further commitment regarding $x(\hat{T})$ is required, since $x(\hat{T}) = \phi\pi(\hat{T})$ is ex-post optimal given the promised $\pi(\hat{T})$. Note that the level of inflation promised in this case may be positive or negative, depending on the sign of $1 - \kappa\sigma\lambda$. A commitment to positive inflation once interest rates become positive is not necessarily a feature of all optimum.

5 Inflation or Boom?

It is widely believed that the main purpose of monetary easing in a liquidity trap is to promote inflation. The model confirms that an optimum has positive inflation and that it

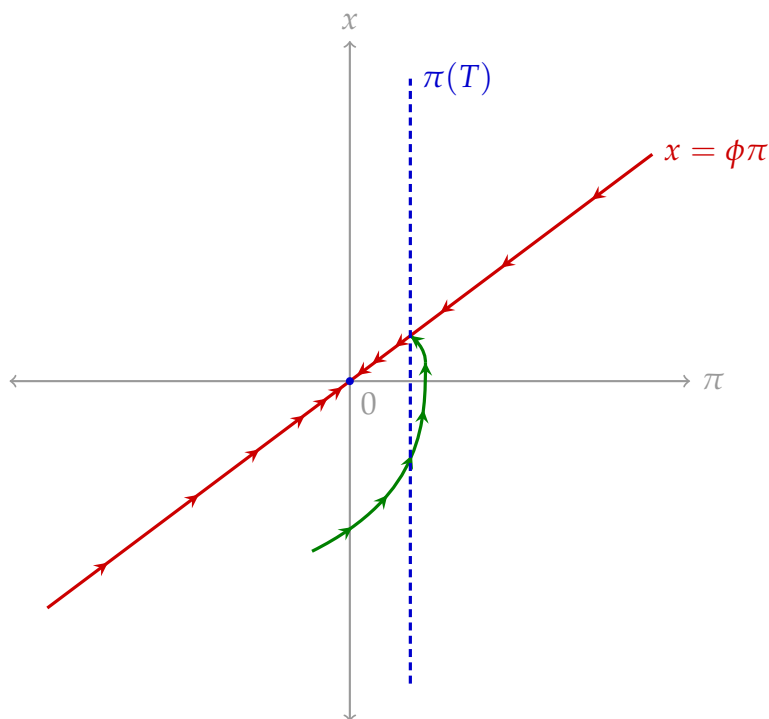


Figure 8: Commitment to inflation but not an additional boom in output.

commits to prolonging zero interest rates. Are the two connected?

I now argue that they are not. Recall that there is more to the it than inflation, since the optimum also calls for an output boom after the trap. I will argue that keeping interest rates at zero has everything to do with stimulating this boom and little, or nothing, to do with generating inflation. Three different special cases of the model will help cement this conclusion.

Fully Rigid Prices. When prices are fully rigid, so that $\kappa = 0$, inflation is just not in the cards, so only the output boom motive can be present. Yet I have shown in Section 4.2 that prolonging zero interest rates is still optimal in this case. Thus, promoting inflation is not necessary for a commitment to prolonging zero interest rates. The next example argues that it is also not sufficient.

Commitment to Inflation Promises Only. Consider a central bank in a liquidity trap scenario. Optimal policy can be summarized by a commitment to keeping the interest rate at zero up to time T together with a commitment to inflation and output upon exit, $(\pi(T), x(T))$. One can then imagine the central bank at $t = T$ re-optimizing and committing over the continuation plan $t \geq T$, subject to fulfilling its prior commitments to

inflation and output $\pi(T)$ and $x(T)$.¹⁶

Now consider stripping the central bank of a commitment to output $x(T)$. Suppose at $t = 0$ it can announce a commitment to keeping interest rates at zero up to time T and a promised exit inflation rate $\pi(T)$. In particular, it can make no binding promises regarding output $x(T)$. As before, at $t = T$ the central bank is allowed to optimize and commit over the continuation $t \geq T$, but it honors its prior commitment, in this case inflation $\pi(T)$ only.

Then, for any $\pi(T)$, as long $I(r(T), \pi(T)) > 0$ (which is guaranteed, for example, if $\kappa\sigma\lambda = 1$) the optimum will feature $i(t) = I(r(t), \pi(t)) > 0$ for all $t \geq T$. Index the resulting equilibrium by the choice of exit inflation $\pi(T)$ and note that setting $\pi(T) = 0$ leads to an outcome that is identical to that of the discretionary equilibrium. Things are only made worse by committing to negative exit inflation, $\pi(T) < 0$. Thus, some positive exit inflation, $\pi(T) > 0$, is desirable and strictly improves on the discretionary outcome, although it falls short of the full optimum.

This shows that a commitment to inflate does not lead to a commitment to prolong zero interest rates. Promising inflation is not sufficient for prolonged zero interest rates.¹⁷ Indeed, interest rates may be above or below the natural rate after $t = T$, depending on the sign of $\kappa\sigma\lambda - 1$. As pointed out in Section 3.2, once again a commitment to looser future monetary policy does not necessarily translate into lower long term interest rates or, equivalently, lower yields at $t = 0$ on long term bonds.

Of course, the discussion here is about equilibrium interest rates. One interpretation is that the equilibrium with $\pi(T) > 0$ is implemented by some form of *loose* monetary policy, even if it does not necessarily lead to lower interest rates *in equilibrium*. For example, one popular interpretation is that after time T policy is conducted according to a Taylor rule with a time varying intercept: $i(t) = \bar{i}(t) + \psi\pi(t)$ and $\psi > 1$. Then $\bar{i}(t) = -\psi\pi^*(t)$ which is indeed negative at $t = T$ and rises over time, converging to zero. In this sense, the central bank is committing to loose monetary policy. However, even in this case, it

¹⁶To see why this form of communication and commitment is enough, recall that the planning problem is recursive in the state variables $(\pi(t), x(t))$. Thus, given $(\pi(T), x(T))$ the continuation plan at $t = T$ coincides with the original plan at $t = 0$. Before T the commitment to set $i(t) = 0$ pins down a unique solution for the paths of $\pi(t)$ and $x(t)$.

¹⁷It is true, however, that the commitment to zero interest rates up to time T is binding. To see this suppose the central bank could only commit to exit inflation, but not to the interest rate path before T . As before, suppose at T it can bind itself to an optimal continuation plan given $\pi(T)$. The resulting equilibrium, for a given promise $\pi(T) > 0$, has $\lim_{t \rightarrow T} x(t) = 0$ and then jumps up discontinuously to $x(T) = \phi\pi(T)$. Intuitively, $x(t) > 0$ is not possible without commitment. The interest rate path is $i(t) = 0$ before T but includes an “instant” with an “infinite” interest rate at, or immediately before, $t = T$ that allows $x(t)$ to jump upward. Although peculiar, realistic values of ϕ are small, so this difference in the equilibrium is minor.

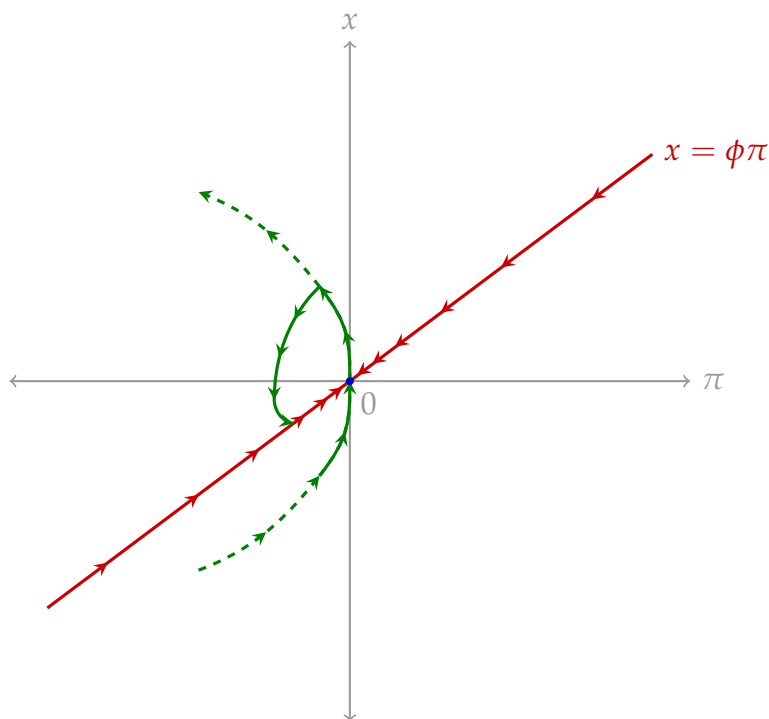


Figure 9: The optimum with the added constraint that $\pi(t) \leq 0$ for all $t \geq 0$.

does not involve a commitment to keeping interest rates at zero.

An Outside Constraint to Avoid Inflation. A third useful exercise is to consider imposing an arbitrary restriction to avoid positive inflation: $\pi(t) \leq 0$ for all $t \geq 0$. This restriction cannot be motivated within the basic New Keynesian model laid out here. The costs from inflation are already included in the loss function. However, one may still want to account for political or economic constraints outside the model that make an increase in inflation more costly. The extreme case is the one considered here, where inflation is just ruled out.

The optimum in this restricted case is illustrated in Figure 9. The optimal path goes along the same arc as the no-commitment solution shown in Figure 1. However, instead of reaching the origin at $t = T$ it now goes through the origin earlier and reaches a strictly positive output level at $t = T$. To minimize the quadratic objective it is best for output to take on both signs: the boom in output at later dates helps mitigate the recession early on. Positive inflation is avoided here by promising to approach, in the long run, the origin from the bottom-left quadrant, with deflation and negative output.

To sum up, if inflation is to be avoided because of some outside imposition, then the optimum still calls for a commitment to prolonged zero interest rates. A plan to inflate is

not needed to justify a commitment to keeping interest rates at zero longer.

Once again, this highlights the non-inflationary role monetary policy plays in a liquidity trap. Note that low interest rates are crucial in accomplishing this outcome. Indeed, if we considered the best equilibrium with the restriction that $\pi(t) \leq 0$ for all $t \geq 0$ and $\pi(t) = I(\pi(t), r(t))$ for $t \geq T$, then we isolate the no-commitment solution as shown in Figure 1.

6 Government Spending: Opportunistic and Stimulus

I now introduce government spending as an additional instrument. I first consider the full optimum, with commitment, over both fiscal and monetary policy.

As in [Woodford \(2011\)](#), the basic New Keynesian model is augmented with public goods provided by the government that enter the representative agent's utility function $U(c, g, n)$ and are produced by combining varieties in the exact same way as final consumption goods. As before, we focus on the linearized equilibrium conditions and a quadratic approximation to welfare. The planning problem becomes

$$\min_{c, \pi, i, g} \frac{1}{2} \int_0^{\infty} e^{-\rho t} \left((c(t) + (1 - \Gamma)g(t))^2 + \lambda \pi(t)^2 + \eta g(t)^2 \right) dt$$

subject to

$$\begin{aligned} \dot{c}(t) &= \sigma^{-1}(i(t) - r(t) - \pi(t)) \\ \dot{\pi}(t) &= \rho \pi(t) - \kappa (c(t) + (1 - \Gamma)g(t)) \\ i(t) &\geq 0 \end{aligned}$$

$$x(0), \pi(0) \text{ free.}$$

Here the constants satisfy $\Gamma \in (0, 1)$ and $\eta > (1 - \Gamma)\Gamma > 0$; the variable $c(t) = (C(t) - C^*(t))/C^*(t) \approx \log(C(t)) - \log(C^*(t))$ represents the private consumption gap, while $g(t) = (G(t) - G^*(t))/C^*(t)$ represents the government consumption gap, normalized by private consumption.

Note that this problem coincides with the previous one if one imposes $g(t) = 0$ for all $t \geq 0$. Government spending appears in the objective function here for two reasons: public goods are valued in the utility function and, through the resource constraint, they also affect the required amount of labor, for any given private consumption c . Spending does not affect the consumer's Euler equation, but does affect the Phillips curve. Intuitively, both private and public spending increase the wage, which creates inflationary pressure.

The coefficient $\Gamma \in (0, 1)$ represents the first best, or flexible-price equilibrium, government spending multiplier, i.e. for each unit increase in spending, output increases by Γ units, consumption is reduced by $1 - \Gamma$ units. The loss function captures this, because given spending g , the ideal consumption level is $c = -(1 - \Gamma)g$. The Phillips curve shows that $c = -(1 - \Gamma)g$ also corresponds to a situation with zero inflation, replicating the flexible-price equilibrium.

6.1 A Non-Optimal Policy of Filling in the Gap

The potential usefulness of the additional spending instrument g can be easily seen noting that spending can zero out the first two quadratic terms in the loss function, ensuring $c(t) + (1 - \Gamma)g(t) = \pi(t) = 0$ for all $t \geq 0$. This requires a particular path for spending satisfying

$$\dot{g}(t) = \frac{\sigma^{-1}}{1 - \Gamma}(r(t) - i(t)).$$

For simplicity, suppose we set $i(t) = 0$ for $t < T$ and $i(t) = r(t)$ for $t \geq T$. Then spending is declining for $t < T$ and given by

$$g(t) = \frac{\sigma^{-1}}{1 - \Gamma} \int_0^t r(s) ds + g(0).$$

After this, spending is flat $g(t) = g(T)$ for $t \geq T$. Since this policy ensures that the first two terms in the objective are zero, to minimize the quadratic loss from spending, the optimal initial value $g(0)$ is set to ensure that $g(t)$ takes on both signs: $g(0)$ is positive and $g(T)$ is negative. As a result, the same is true for consumption $c(t) = -(1 - \Gamma)g(t)$.

Although this plan is not optimal, it is suggestive that optimal spending may take on both positive and negative values during a liquidity trap. We prove this result in the next subsection.

6.2 The Optimal Pattern for Spending

It will be useful to transform the planning problem by a change variables. In fact, I will use two transformations. Each has its own advantages. For the first transformation, define the output gap $x(t) \equiv c(t) + (1 - \Gamma)g(t)$. The planning problem becomes

$$\min_{x, \pi, i, g} \frac{1}{2} \int_0^{\infty} e^{-\rho t} \left(x(t)^2 + \lambda \pi(t)^2 + \eta g(t)^2 \right) dt$$

subject to

$$\begin{aligned}\dot{x}(t) &= (1 - \Gamma)\dot{g}(t) + \sigma^{-1}(i(t) - r(t) - \pi(t)) \\ \dot{\pi}(t) &= \rho\pi(t) - \kappa x(t) \\ i(t) &\geq 0\end{aligned}$$

$x(0), \pi(0)$ free.

This is an optimal control problem with i and \dot{g} as controls and x , π and g as states. According to the objective, the ideal level of government spending, given current state variables $x(t)$ and $\pi(t)$ is always zero. However, spending also appears in the constraints, so it may help relax them. In particular, spending enters the constraint associated with the consumer's Euler equation. Indeed, the change in spending, \dot{g} , plays a role that is analogous to the nominal interest rate, but unlike the interest rate, the change in spending is not restricted by a zero lower bound.

Since government spending relaxes the Euler equation, it should be zero whenever the zero-bound constraint is not binding, which is the case when the zero lower bound is not binding. Conversely, if the zero-bound constraint binds and $i(t) = 0$ then government spending is generally non-zero.

Proposition 6. *The conclusions in Proposition 3, regarding the nominal interest rate $i(t)$, extend to the model with government spending. In addition, whenever the zero lower bound is not binding government spending is zero: $g(t) = 0$. Suppose the zero lower bound binds over the interval (t_0, t_1) and is slack in a neighborhood to the right of t_1 , then $g(t) < 0$ in a neighborhood to the left of t_1 . Similarly, if $t_0 > 0$ and the zero lower bound is slack to the left of t_0 then $g(t) > 0$ in a neighborhood to the right of t_0 . Government spending is always initially nonnegative $g(0) \geq 0$ and strictly so if $x(0) < 0$.*

The proposition suggests a typical pattern, confirmed by a number simulations, where spending is initially positive, then declines and becomes negative and finally returns to zero. In this sense, optimal government spending is front loaded.

It may seem surprising that spending takes on both positive and negative values. The intuition is as follows. Initially, higher spending helps compensate for the negative consumption gap at the start of a liquidity trap. However, recall that optimal monetary policy eventually engineers a consumption boom. If government spending leans against the wind, we should expect lower spending. The next subsection refines this intuition by decomposing spending into an opportunistic and a stimulus component.

Figure 10 provides a numerical example, following the same parametrization used for

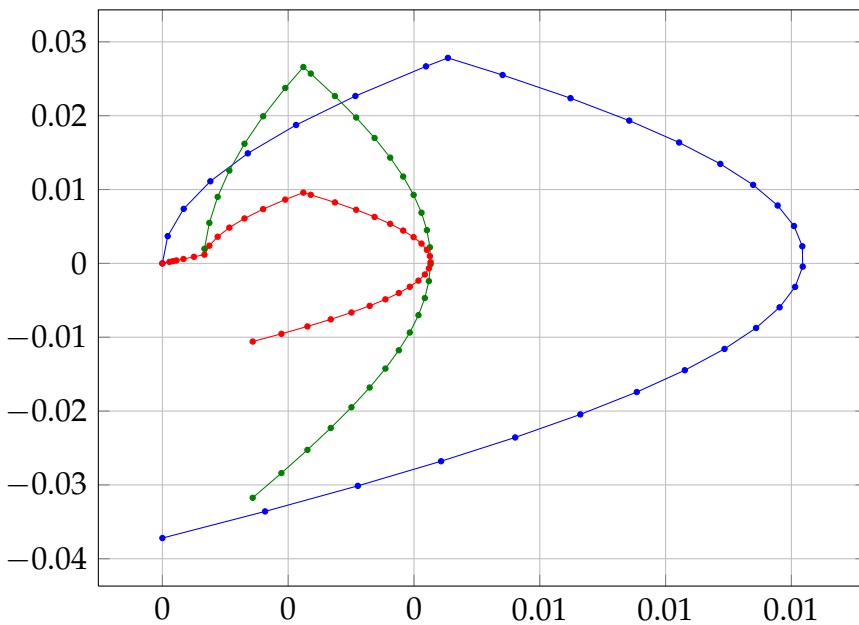


Figure 10: A numerical example. The optimum without spending (blue) versus the optimum with spending displaying output (red) and displaying consumption (green).

the example in Section 4, with the additional parameters $\Gamma = 0.5$ and $\eta = .5$. The figure shows both consumption and output. As we see from the figure consumption is not as affected as output is in this case.

6.3 Opportunistic vs. Stimulus Spending

Even a shortsighted government that ignores dynamic general equilibrium effects on the private sector, finds reasons to increase government spending during a slump. When the economy is depressed, the wage, or shadow wage, is lowered. This provides a cheap opportunity for government consumption.

To capture this idea, I define an opportunistic component of spending, the level that is optimal from a simple static, cost-benefit calculation. To see what this entails, take the utility function $U(C, G, N)$ and take some level of private consumption C as given. Then opportunistic spending is defined as the solution to $\max_G U(C, G, N)$ subject to the resource constraint relating C, G and N . For example, if the resource constraint were simply $N = C + G$, then one maximizes $U(C, G, C + G)$ taking private spending C as given. The first order condition $U_G = -U_N$ can be thought of as a Samuelsonian rule $U_G/U_C = -U_N/U_C$, equating the marginal utility of public goods relative to consumption goods to the marginal cost, equal here to the real wage. Private spending will generally affect the optimal level of public goods, by affecting the marginal benefit or marginal

cost. With separable utility, opportunistic spending is decreasing in private spending, because a fall in private consumption, holding the consumption of public goods constant, lowers the marginal disutility of labor.

I need to make this concept operational using the quadratic approximation to the welfare function and for variables expressed in terms of gaps. They translate to minimizing the loss function

$$g^*(c) \equiv \arg \max_g \left\{ (c + (1 - \Gamma)g)^2 + \eta g^2 \right\}.$$

Define stimulus spending as the difference between actual and opportunistic spending,

$$\hat{g}(t) \equiv g(t) - g^*(c(t)).$$

Note that

$$g^*(c) = -\frac{1 - \Gamma}{\eta} \psi c,$$

$$c + (1 - \Gamma)g^*(c) = \psi c,$$

with the constant $\psi \equiv \eta / (\eta + (1 - \Gamma)^2) \in (0, 1)$. Thus, opportunistic spending leans against the wind, $\psi < 1$, but does not close the gap, $\psi > 0$.

Using these transformations, I rewrite the planning problem as

$$\min_{\hat{x}, \pi, i, \hat{g}} \frac{1}{2} \int_0^\infty e^{-\rho t} \left(c(t)^2 + \hat{\lambda} \pi(t)^2 + \hat{\eta} \hat{g}(t)^2 \right) dt$$

subject to

$$\dot{c}(t) = \sigma^{-1}(i(t) - r(t) - \pi(t))$$

$$\dot{\pi}(t) = \rho \pi(t) - \kappa (\psi c(t) + (1 - \Gamma)\hat{g}(t))$$

$$i(t) \geq 0,$$

$$c(0), \pi(0),$$

where $\hat{\lambda} = \lambda / \psi$ and $\hat{\eta} = \eta / \psi^2$. According to the loss function, the ideal level of stimulus spending is zero. However, stimulus may help relax the Phillips curve constraint.

This problem is almost identical to the problem without spending. The only new optimality condition is

$$\hat{g}(t) = \frac{\kappa(1 - \Gamma)}{\hat{\eta}} \mu \pi(t). \quad (4)$$

This leads to the following result.

Proposition 7. *Stimulus spending is always initially zero: $\hat{g}(0) = 0$. (a) If $\kappa = 0$ or $\kappa\sigma\lambda = 1$ then stimulus spending is zero, $\hat{g}(t) = 0$ for all $t \geq 0$; (b) if $\kappa\sigma\lambda > 1$ then stimulus spending turns positive initially; (c) if $0 < \kappa\sigma\lambda < 1$ then stimulus spending turns negative initially.*

The proposition provides three instances where we should expect stimulus spending to be zero. First, unlike total spending, stimulus spending is always initially zero, so that total spending is entirely opportunistic.¹⁸ This result is especially relevant in a liquidity trap scenario, since then $t = 0$ coincides with the start of the liquidity trap and it implies that the entire initial increase in spending can be attributed to the opportunistic motive. Second, when prices are completely rigid, so that $\kappa = 0$, spending cannot affect inflation so that stimulus spending is zero and spending is always opportunistic. This highlights the somewhat indirect role that stimulus spending plays during a liquidity trap in this model as a promoter of inflation.¹⁹ Third, even for $\kappa > 0$ the costate for the Phillips curve, unlike the costate for the Euler equation, can take on both signs or remain at zero, depending on parameters. The third result exploits this fact to provide another benchmark, with $\kappa\sigma\lambda = 1$, where stimulus spending is always zero.

Whenever stimulus spending is zero it is as if spending were chosen according to a purely static, cost-benefit calculation that ignores any dynamic general equilibrium feedback effects on private spending. By implication, government spending can be chosen by a naive agency, lacking commitment, that performs such a static cost-benefit calculation. The fact that, in these special cases, government spending can be chosen without commitment contrasts sharply with the importance of commitment to monetary policy.

Figure 11 displays the optimal paths for total, opportunistic and stimulus spending for our numerical example (with the same parameters as those behind Figure 10). Spending starts at 2% of output above its efficient level. It then falls at a steady state reaching almost 2% below its efficient level of output. In this example, spending is virtually all opportunistic. Stimulus spending is virtually zero.

Away from this benchmark, numerical simulations show that stimulus starts at zero, it has a sinusoidal shape, switching signs once. When $\kappa\sigma\lambda > 1$ it first becomes positive, then turns negative, eventually converging to zero from below; when $\kappa\sigma\lambda < 1$ the reverse

¹⁸Another implication of equation (4) is that stimulus spending, unlike total spending, may be nonzero even when the zero lower bound constraint is not currently binding and will never bind in the future. This occurs whenever inflation is nonzero. Indeed, since total spending must be zero, stimulus spending must be canceling out opportunistic spending. This makes sense. If we have promised positive inflation, for example, then we require a positive gap. Opportunistic spending would call for lower spending, but doing so would frustrate stimulating the promised inflation.

¹⁹Although the model has Keynesian features and conclusions in the sense of price stickiness, private spending is forward looking and satisfies Ricardian equivalence, so it is not directly affected by government spending's effect on current income as in standard ISLM discussions.

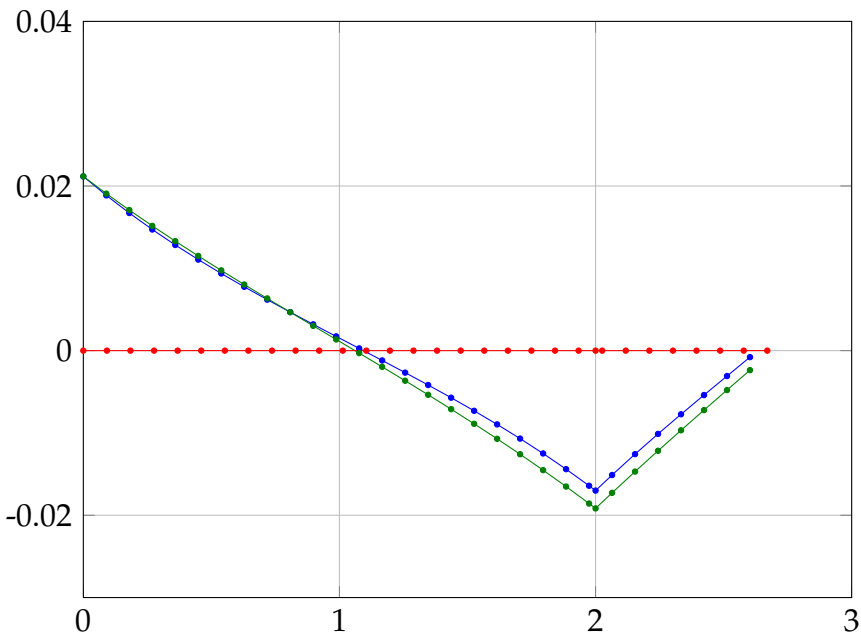


Figure 11: Total spending (blue), opportunistic spending (green) and stimulus spending (red) for a numerical example.

pattern obtains: first negative, then turns positive, eventually converging to zero from above. In most cases, stimulus spending is a small component of total spending.

The results highlight that positive stimulus spending is just not a robust feature of the optimum for this model. Opportunistic spending does affect private consumption, by affecting the path for inflation. In particular, by leaning against the wind, it promotes price stability, mitigates both deflations and inflations. However, the effects are incidental, in that they would be obtained by a policy maker choosing spending that ignores these effects.

7 Spending with Discretionary Monetary Policy

I now relax the assumption of full commitment and consider the mixed case where monetary policy is assumed to be discretionary, as in Section 3, while fiscal policy can commit to a path for government spending, at least over some intermediate horizon.

One motivation for these assumptions is the fact that government spending decisions feature both legislative and technical lags in implementation over the medium run. Spending may be planned and legislated over a horizon of at least a year or two. Reversing these decisions may be politically difficult or simply impractical from a technical point of view, e.g. if infrastructure construction is under way. In sharp contrast, mone-

tary policy is determined almost instantaneously, at frequent meetings, and the nominal interest rate can easily react to the current state of the economy. This may tend to make monetary policy discretionary over the short and medium run—the relevant time frame for countercyclical policies, which is the present focus.

Of course, this comparison may be inverted when one considers the long run determinants of inflation and spending. At least for advanced economies, it may be more accurate to assume monetary policy can commit to avoid inflationary bias pressures. On the other hand, fiscal policy may not overcome time inconsistency or other political economy problems that lead to excess government spending and debt. The medium-run lags in fiscal policy are not helpful in resolving these biases in the level of spending. Likewise, the monetary authorities commitment to avoid inflationary pressures may be irrelevant (or even unhelpful) in solving the extraordinary kind of commitments needed during a liquidity trap. Thus, there is no contradiction in assuming that the relative commitment powers are different in the long run, than over the medium term required to respond to a liquidity trap.

7.1 Commitment to spending during the liquidity trap

I will consider a liquidity trap scenario, where $r(t) < 0$ for $t < T$ and $r(t) \geq 0$ for $t \geq T$. The central bank sets nominal interest rates with full discretion. In contrast, government spending can be credibly announced, at least until time T . I first assume that spending after T is chosen with discretion. This implies that $g(t) = 0$ for $t \geq T$. I then turn to the case where commitment is possible for the entire path of government spending.

Under these assumptions, at time $t = T$ the first best is attainable. I suppose then that the monetary and fiscal authorities jointly reoptimize and implement the first best: $c(t) = \pi(t) = g(t) = 0$ for all $t \geq T$. Next, I characterize the equilibrium for $t < T$.

For $t < T$, the central bank acts only takes into account the short run effects of its policy choices. At any point in time, it takes as given the behavior of future output and inflation. In continuous time, this implies that it only considers current output and inflation. However, the latter is essentially out of its control, since inflation is proportional to an integral of future output gaps. This makes the central bank's behavior easy to characterize. zero.

At any point in time t its choice for the nominal interest rate $i(t)$ depends entirely on the current output gap $x(t)$. Since there is no upper bound on the nominal interest rate $x(t) > 0$ is not possible: if $x(t) > 0$ occurred then the central bank would increase $i(t)$ without bound. In the limit, this would bring $x(t)$ immediately down to zero. If $x(t) < 0$ the unique optimum is to set $i(t) = 0$. When $x(t) = 0$ the central bank is content to keep

$x(t) = 0$ by setting $i(t)$ so that $\dot{x}(t) = 0$, unless this requires $i(t) < 0$ in which case it sets $i(t) = 0$.

This motivates the following problem

$$\int_0^T e^{-\rho t} ((c(t) + (1 - \Gamma)g(t))^2 + \lambda\pi(t)^2 + \eta g(t)^2) dt$$

$$\dot{c}(t) = \sigma^{-1}(i(t) - \pi(t) - r(t))$$

$$\dot{\pi}(t) = \rho\pi(t) - \kappa(c(t) + (1 - \Gamma)g(t))$$

$$c(t) + (1 - \Gamma)g(t) \leq 0$$

$$c(T) = \pi(T) = 0$$

$$i(t)(c(t) + (1 - \Gamma)g(t)) = 0$$

$$x(0), \pi(0) \text{ free}$$

This is the same problem as before, with the new constraints $x(t) \leq 0$ and $c(T) = \pi(T) = 0$ and $i(t)x(t) = 0$. The latter constraint implies that $i(t) = 0$ whenever $x(t) < 0$, since this is the best response from a monetary policy. As it turns out, this latter constraint is not binding anyway, because there is no conflict of interest when $x(t) < 0$ in setting the interest rate. Thus, my approach is to drop the constraint that $i(t)x(t) = 0$ and show that the solution to the relaxed problem satisfies it. In this formulation, c and π are state variables, while i and g are controls. Due to the presence of the inequality $c(t) + (1 - \Gamma)g(t) \leq 0$, one must invoke a generalized Maximum Principle for optimal control problems with mixed (state and control) constraints. Working with the implied necessary conditions yields the following sharp characterization.

Proposition 8. *Consider a liquidity trap scenario and assume that fiscal policy can commit to a path for spending $g(t)$ time $t \in [0, T]$ but monetary policy is discretionary. Then the equilibrium outcome satisfies the following properties: (a) consumption is negative and increasing; (b) total spending is positive; (c) opportunistic spending is positive and decreasing; (d) there exists a time $t^* \in (0, T)$ such that $x(t) < 0$ and $\pi(t) < 0$ for $t < t^*$ and $x(t) = \pi(t) = 0$ for $t \geq t^*$; (e) stimulus spending is initially zero, $\hat{g}(0) = 0$, but strictly increasing until $t = t^*$; (f) for $t \geq t^*$ total spending $g(t)$ is positive and decreasing, reaching zero at $t = T$.*

Since the output gap cannot be positive, the same is true for inflation, $\pi(t) \leq 0$ for all t . This then implies that consumption is negative and increasing, until it hits zero exactly

at $t = T$. As a result, opportunistic spending is positive and decreasing. As before, stimulus spending is initially zero. As long as $x(t) < 0$, so that the mixed constraint is slack, stimulus spending is positive and increasing over time because there is deflation and government spending can help relieve these deflationary pressures. However, due to these incentives, the output gap reaches zero at some intermediate date $t^* \in (0, T)$ and stays at zero thereafter. During this time, total spending is constrained and given by $g(t) = -c(t)/(1 - \Gamma) \geq 0$. Indeed, spending fills the gap, as discussed in Section 6.1, declining towards zero at $t = T$.

Why does the solution hit $x(t) = 0$ for some time? Note that with full commitment, the solution involved an output boom, with positive output gaps sometime before $t = T$. Since this is no longer possible, the constraint that $x(t) \leq 0$ binds over a latter interval of time.

7.2 Commitment to spending after the liquidity trap

I now relax the assumption that fiscal policy cannot commit past T and spending to be chosen for the entire future $\{g(t)\}_{t=0}^{\infty}$.

It is useful to split the problem into two segments, before and after T . The problem before T looks just as before but dropping the constraint that $\pi(T) = c(T) = 0$ and including the value function for the continuation in the objective

$$\int_0^T e^{-\rho t} ((c(t) + (1 - \Gamma)g(t))^2 + \lambda\pi(t)^2 + \eta g(t)^2) dt + e^{-rT} L_T(c(T), \pi(T))$$

where L_T is the value function for the continuation problem after T . It is defined as the solution to

$$L_T(c_T, \pi_T) = \int_0^{\infty} e^{-\rho s} ((c(T+s) + (1 - \Gamma)g(T+s))^2 + \lambda\pi(T+s)^2 + \eta g(T+s)^2) dt$$

subject to all of the constraints outlined in the planning problem from the previous subsection except with initial conditions $c(0) = c_T$ and $\pi(0) = \pi_T$. Note that because of the restriction that $x(T+s) \leq 0$, this loss function is only defined for $\pi_T \leq 0$.

Since the first best is attainable after T when $c(T) = \pi(T) = 0$, the only reason for deviating from this is to improve the solution before T . Deflation is not helpful, thus the solution still entails $\pi(T) = 0$. However, a consumption boom $c(T) > 0$ is very helpful because it raises consumption at all earlier dates where consumption is negative.

I now outline the solution to $L_T(c_T, 0)$ for positive consumption c_T . The basic idea is

that positive consumption is consistent with a zero output gap if government spending is negative. Thus, one feasible plan is to hold spending constant at a negative level consistent with c_T . A better solution, to minimize the losses coming from the ηg^2 term, is to start with this same level of spending, but have it increase over time to eventually reach zero. During this transition, consumption rises, which requires a lower interest rate. We are constrained then by the zero lower bound. Thus, the optimum calls for $i(T + s) = 0$ and

$$\tilde{g}(T + s) = -(1 - \Gamma)c_T + \frac{\sigma^{-1}}{1 - \Gamma} \int_0^s r(z) dz$$

for $t \in [T, T + \Delta]$, where Δ is defined so that $\tilde{g}(T + \Delta) = 0$, and $g(t) = 0$ for $t > T + \Delta$. Since $x(t) = \pi(t) = 0$, such a plan implies a tail cost

$$L_T(c_T, 0) \equiv \eta \int_0^\Delta e^{-\rho s} \eta \tilde{g}(T + s)^2 ds.$$

This function attains its minimum at $c_T = 0$ so that $\frac{\partial}{\partial c_T} L_T(0, 0) = 0$.

Turning to the problem before T , the new optimality condition is the transversality condition

$$\mu_c(T) = \frac{\partial}{\partial c_T} L_T(c(T), 0).$$

Since the solution entails $\mu_c(T) > 0$ it follows that $c(T) > 0$, which implies that $g(T) < 0$.

Proposition 9. *Consider a liquidity trap scenario and assume that fiscal policy can commit to a path for $g(t)$ for all time $t \geq 0$ but monetary policy is discretionary. Then the equilibrium outcome is as in Proposition 8 except that spending is negative at the end of the trap and consumption is positive, $g(T) < 0$ and $c(T) > 0$, and both converge to zero at some finite time $\hat{T} > T$. The nominal interest rate is zero up to \hat{T} , $i(t) = 0$ for $t \leq \hat{T}$.*

Optimal fiscal policy entails a commitment to austerity towards the end and after the liquidity trap, $g(t) < 0$ in the neighborhood of $t = T$. Recall that the full optimum, with commitment to both monetary and fiscal policy, also entailed $g(t) < 0$ towards the end of the trap, but not after the trap; optimal spending is zero for $t \geq T$ (see Proposition according to Proposition 6). In contrast, here spending serves the purpose of stimulating a boom in consumption, which helps mitigate the earlier consumption slump, much in the same way as optimal monetary policy would. Indeed, it is interesting to note that the optimal fiscal commitment induces the discretionary monetary authority to prolong zero interest rates. Although monetary policy cannot commit, when fiscal policy can, it creates a temporary drop in spending that lowers the interest rate that keeps the output gap at zero.

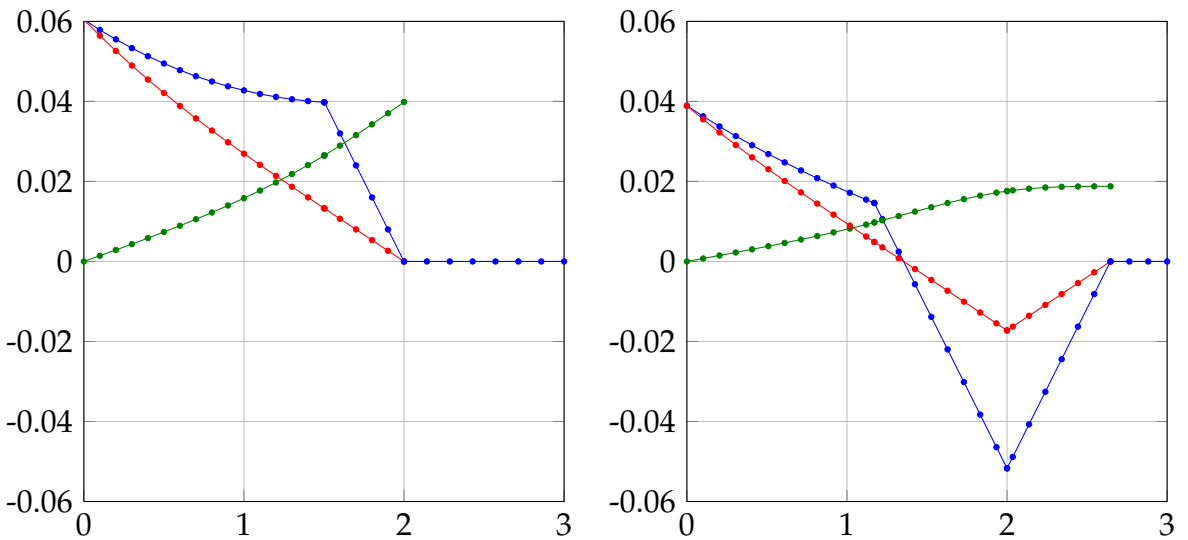


Figure 12: An example with mixed commitment. Left panel: $\{g(t)\}$ commitment up to T . Right panel: $\{g(t)\}$ commitment for all $t \geq 0$. Total spending (blue), opportunistic spending (green) and stimulus spending (red).

7.3 A Numerical Example

Figure 12 shows the equilibrium outcome for government spending as a fraction of output when monetary policy is discretionary but fiscal policy can commit. The example uses the same parameters and is thus directly comparable to the example displayed in Figure 11, which shows the full optimum, with commitment for both monetary and fiscal policy.

When fiscal policy can only commit during the trap optimal spending is higher than in the full commitment solution. Indeed, both opportunistic and stimulus spending are higher, since the recession and deflation are made worse by discretion in monetary policy. At some point (around $t = 1.5$) the solution hits the constraint that the output gap cannot be positive and spending is constrained from then onwards falling to zero at $t = T$.

When fiscal policy can commit past the trap, optimal spending is still higher than the full commitment solution, but lower than when spending can only commit during the trap. The solution now features a significant negative spending at $t = T$. This serves to create a temporary boom in consumption (of size $c(T) = (1 - \Gamma)g(T)$) as well as reduce the quadratic cost from spending. The austerity measure is temporary: some time before $t = 3$ the solution is back to $c(t) = \pi(t) = g(t) = 0$.

Overall, these examples illustrate that the solution for government spending may be quite affected by the lack of commitment in monetary policy and the assumed horizon of commitment for fiscal policy.

8 Conclusion

This paper has revisited monetary policy during a liquidity trap. The continuous time setup offers some distinct advantages in terms of the analysis and results that are obtained. Some of my results support the findings from prior work based on simulations. Optimal monetary policy in the model is engineered to promote inflation and an output boom. It does so, in part, by committing to holding the nominal interest rate at zero for an extended period of time.

To the best of my knowledge, my results on government spending have no clear parallel in the literature. In particular, the decomposition between opportunistic and stimulus spending is novel and leads to unexpected results.

When both fiscal and monetary policy are coordinated, I find that optimal government spending starts at a positive level, but declines and become negative. However, I show that most of these dynamics are explained by a cost-benefit motive for spending, which, by definition, ignores the effects this spending has on private consumption and inflation. At the model's optimum, stimulus spending is always initially zero. Moreover, depending on parameters, stimulus may be identically zero throughout or deviate from zero changing signs. However, simulations show stimulus spending playing a modest role.

This situation can be very different when monetary policy is suboptimal due to the lack of commitment. In this case, the model's optimal policy calls for positive and increasing stimulus spending during the trap and lower spending after the trap.

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A Proof of Proposition 1

Recall that $\pi(t) = x(t) = 0$ for $t \geq T$. In integral form the equilibrium conditions for $t < T$ are

$$\begin{aligned} x(t) &= \int_t^T \sigma^{-1}(r(s) + \pi(s))ds, \\ \pi(t) &= \kappa \int_t^T e^{-\rho(s-t)} x(s)ds. \end{aligned}$$

Substituting inflation $\pi(t)$ we can write this as a single condition for the output path $\{x(t)\}$

$$x(t) = \sigma^{-1} \int_t^T \left(r(s) + \kappa \int_s^T e^{-\rho(z-s)} x(z)dz \right) ds.$$

Define the operator associated with the right hand side of this expression:

$$T[x](t) = \sigma^{-1} \int_t^T \left(r(s) + \kappa \int_s^T e^{-\rho(z-s)} x(z)dz \right) ds = a(t) + \kappa \int_t^T m(z-t)x(z)dz$$

with $a(t) \equiv \sigma^{-1} \int_t^T r(z)dz$ and $m(s) \equiv (\sigma\rho)^{-1}(1 - e^{-\rho s})$, note that m is nonnegative, strictly increasing, with $m(0) = 0$ and $\lim_{s \rightarrow \infty} m(s) = M \equiv (\sigma\rho)^{-1} > 0$.

The operator T maps the space of continuous functions on $(-\infty, T]$ onto itself. An equilibrium x^* is a fixed point $T[x^*] = x^*$. Since an equilibrium represents a solution to an initial value problem for a linear ordinary differential equation, there is a unique fixed point x^* . The T operator is linear and monotone (since $m \geq 0$), so that if $x^a \geq x^b$ then $T[x^a] \geq T[x^b]$.

Fix an interval $[\hat{t}, T]$. Although T is not a contraction, starting from any continuous function x_0 that is bounded on $[\hat{t}, T]$ and defining $x_n \equiv T^n[x_0]$ we obtain a sequence that converges uniformly on $[\hat{t}, T]$ to the unique fixed point $x_n \rightarrow x^*$. To prove this claim, note that since $|x_0(t)| \leq B$ and $|r(t)| \leq R$ then

$$|x_1(t) - x_0(t)| \leq |a(t)| + \kappa \int_t^T m(z-t) |x_0(t)| \leq \left(\frac{R}{\kappa M} + B \right) \kappa M |T - t|.$$

In turn

$$\begin{aligned} |x_2(t) - x_1(t)| &= |T[x_1](t) - T[x_0](t)| \leq \kappa \int_t^T m(z-t) |x_1(z) - x_0(z)| dz \\ &\leq \kappa M \int_t^T |x_1(z) - x_0(z)| dz \leq \left(\frac{R}{\kappa M} + B \right) (\kappa M)^2 \frac{|T-t|^2}{2}. \end{aligned}$$

By induction it follows that

$$|x_n(t) - x_{n-1}(t)| \leq \left(\frac{R}{\kappa M} + B \right) (\kappa M)^n \frac{|T-t|^n}{n!}.$$

It follows that

$$\sum_{n=1}^{\infty} |x_n - x_{n-1}(t)| \leq \left(\frac{R}{\kappa M} + B \right) e^{\kappa M |T-t|}.$$

As a consequence for any $m \geq n$

$$|x_n - x_m(t)| \leq \sum_{j=n+1}^m |x_j - x_{j-1}(t)| \leq \sum_{j=n+1}^{\infty} |x_j - x_{j-1}(t)|$$

Since the right hand converges to zero as $n \rightarrow \infty$, for any $\varepsilon > 0$ and $t' < T$ there exists an N such that for all $n, m \geq N$ we have

$$|x_n(t) - x_m(t)| \leq \varepsilon$$

for all $t \geq t'$. Thus, $\{x_n\}$ is a Cauchy sequence on a complete metric space, implying that $x_n \rightarrow x^*$ where x^* is a bounded function. Since the operator T is continuous it follows that x^* is a fixed point $x^* = T[x^*]$.

Note that starting from the zero function $x_0(t) = 0$ for all t we obtain $x_1(t) = a(t) < 0$ for $t < T$. Since the operator T is monotone, the sequence $\{x_n\}$ is decreasing $x_0 \geq x_1 \geq \dots \geq x_n \geq \dots$. Thus, $x^*(t) < x_1(t) < 0$ for all $t < T$. This implies $\pi^*(t) < 0$ for $t < T$.

Next I establish that both inflation and output $x^*(t)$ and $\pi^*(t)$ are monotone. Note that $\frac{\partial}{\partial t} T[x](t) = a'(t) - \int_t^T m'(z-t)x(z)dz$. Since $a'(t) = -\sigma^{-1}r(t) > 0$ and $x^*(t) < 0$ for $t < T$ we have that $T[x^*](t) = x^*(t)$ is strictly increasing in t for $t < T$ and $\dot{x}^*(t) > 0$. Write

$$\pi^*(t) = \kappa \int_0^{\infty} e^{-\rho z} x^*(t+z) dz$$

Hence, differentiating we find $\dot{\pi}^*(t) = \kappa \int_0^{T-t} e^{-\rho z} \dot{x}^*(t+z) dz$, which implies $\dot{\pi}^*(t) > 0$ for $t < T$ given that $\dot{x}^*(t) > 0$ for $t < T$.

Finally note that

$$x^*(t; T) = \sigma^{-1} \int_t^T (r(s; T) + \pi^*(s; T)) ds \leq \sigma^{-1} \int_t^T r(s; T) ds$$

and since $\int_0^T r(s; T) ds \rightarrow -\infty$ as $T \rightarrow \infty$ it follows that $\int_t^T r(s; T) ds \rightarrow -\infty$, implying $x^*(t; T) \rightarrow -\infty$ for $t < T$. Using that $x^*(t, T) \leq 0$, this implies that

$$\begin{aligned} \pi^*(t, T) &= \kappa \int_t^\infty e^{-\rho(s-t)} x^*(s; T) ds \leq \kappa \int_t^{t+1} e^{-\rho(s-t)} x^*(s; T) ds \\ &\leq \kappa x^*(t+1; T) \int_t^{t+1} e^{-\rho(s-t)} ds \end{aligned}$$

As $T \rightarrow \infty$ we have that $x^*(t+1; T) \rightarrow -\infty$, so it follows that $\pi^*(t; T) \rightarrow -\infty$ for any t .

B Proof of Proposition 2

Consider two values $\kappa_0 < \kappa_1$ with associated equilibria x_0^* and x_1^* . Let $T[\cdot; \kappa]$ be the operator defined in the proof of Proposition 1 associated with κ . Define the sequence $x_n = T^n[x_0^*; \kappa_1]$. Since $x_0^*(t) < 0$ for $t < T$, it follows that $x_1(t) = T[x_0^*; \kappa_1] < T[x_0^*; \kappa_0] = x_0(t)$ for $t < T$. Since the operator $T[\cdot; \kappa_1]$ is monotone, this implies that $\{x_n\}$ is a declining sequence. Since the sequence converges to x_1^* . This proves that $x_0^*(t) > x_1(t) > \dots > x_n(t) > \dots > x_1^*(t)$ for $t < T$. This implies that $\pi_1^*(t) < \pi_0^*(t)$ for $t < T$.

To prove the limit result as $\kappa_1 \rightarrow \infty$ it suffices to show that $T[x_0^*; \kappa_1](t) \rightarrow -\infty$ for all $t < T$, since $x_1^*(t) < T[x_0^*; \kappa_1](t)$. That this is the case follows from

$$T[x_0^*; \kappa_1](t) = a(t) + \kappa_1 \int_t^T m(z-t) x_0^*(z) dz \leq \kappa_1 \int_t^T m(z-t) a(z) dz$$

where I am using that $x_0^*(t) < a(t)$ for $t < T$ as established in the proof of Proposition 1. The result follows from the fact that $\int_t^T m(z-t) a(z) dz < 0$ for all $t < T$. The same implication for $\pi^*(t; \kappa)$ then follows.

The following estimate for the loss function

$$L \geq \int_0^\infty e^{-\rho t} x(t)^2 dt \geq \int_0^{T/2} e^{-\rho t} x(t; \kappa)^2 dt \geq x(T/2; \kappa)^2 \int_0^{T/2} e^{-\rho t} dt$$

where I have used that x is nonpositive and increasing. The result then follows since $x(T/2; \kappa) \rightarrow -\infty$.

C Proof of Proposition 3

Differentiating the law of motion for μ_x we obtain a linear non homogeneous second-order differential equation

$$\begin{aligned}\ddot{\mu}_x(t) &= -\dot{x}(t) + \kappa\dot{\mu}_\pi(t) + \rho\dot{\mu}_x(t) \\ &= -\sigma^{-1}(i(t) - r(t) - \pi(t)) - \kappa\lambda\pi(t) + \kappa\sigma^{-1}\mu_x(t) + \rho\dot{\mu}_x(t)\end{aligned}$$

Equivalently, since $I^*(\pi, t) = r(t) + (1 - \sigma\kappa\lambda)\pi$, we can write this as

$$\ddot{\mu}_x(t) = \sigma^{-1} (I^*(\pi(t), t) - i(t)) + \kappa\sigma^{-1}\mu_x(t) + \rho\dot{\mu}_x(t).$$

Now suppose at the optimum we have $i(t) > 0$ for $t \in (t_1, t_1 + \varepsilon)$ with $\varepsilon > 0$. Then it must be that $\mu_x(t_1) = \dot{\mu}_x(t_1) = 0$, which provides a convenient initial condition for the differential equation derived above. The solution for μ_x is then

$$\mu_x(t) = \sigma^{-1} \int_t^{t_1} \omega(s-t) (I^*(\pi(s), s) - i(s)) ds \quad (5)$$

where

$$\omega(x) \equiv \frac{e^{-r_2x}}{r_1} (1 - e^{-r_1x}) \quad (6)$$

where $r_1 > 0$ and $r_2 < 0$ are the two distinct real roots to the characteristic equation $r^2 - \kappa\sigma^{-1}r - \rho = 0$. Note that $\omega(x) > 0$ for $x > 0$. It follows that if $i(t) = 0$ and $I^*(\pi(t), t) < 0$ for all $t \in (t_0, t_1)$, then $\mu_x(t) < 0$ for all $t \in [t_0, t_1)$. A contradiction with the optimality condition that $\mu_x(t) \geq 0$ for all $t \geq 0$. A symmetric argument holds for t_0 as long as $t_0 > 0$. This completes the proof.

Proof that solution to ODE takes stated form. Consider a 2nd order non-homogeneous ODE of the form

$$\alpha_1\mu(t) + \alpha_2\dot{\mu}(t) + \ddot{\mu}(t) = a(t)$$

where $a(t) \equiv \sigma^{-1} (I^*(\pi(t), t) - i(t))$, the coefficients are $\alpha_1 = -\kappa\sigma^{-1}$ and $\alpha_2 = -\rho$; the initial conditions are given by $\mu(t_1) = \dot{\mu}(t_1) = 0$. To solve this equation postulate that

$$-r_1\mu(t) + \dot{\mu}(t) = e^{r_2t} \int_{t_1}^t e^{-r_2s} a(s) ds \quad (7)$$

for some constants r_1 and r_2 , that will turn out to be the roots of the characteristic equa-

tion. This guess works because it implies, by differentiating both sides,

$$\begin{aligned} -r_1\dot{\mu}(t) + \ddot{\mu}(t) &= r_2e^{r_2t} \int_{t_1}^t e^{-r_2s} a(s) ds + a(t) = r_2(-r_1\mu(t) + \dot{\mu}(t)) + a(t) \\ &\Rightarrow r_1r_2\mu(t) - (r_1 + r_2)\dot{\mu}(t) + \ddot{\mu}(t) = a(t) \end{aligned}$$

Matching coefficients gives

$$\begin{aligned} r_1 + r_2 &= -\alpha_2 & r_1r_2 &= \alpha_1 \\ \Rightarrow r_1 + \frac{\alpha_1}{r_1} + \alpha_2 &= 0 \Rightarrow \alpha_1 + r_1\alpha_2 + r_1^2 &= 0 \end{aligned}$$

which is just the characteristic equation. In our case

$$r = \frac{-\alpha_2 \pm \sqrt{\alpha_2^2 - 4\alpha_1}}{2} = \frac{\rho \pm \sqrt{\rho^2 + 4\kappa\sigma^{-1}}}{2}$$

so that the two roots are real, distinct and one root is positive and the other negative. We can take $r_1 > 0$ and $r_2 < 0$.

Now write equation (7) as

$$\mu(t) = e^{r_1t} \int_{t_1}^t e^{-r_1s} A(s) ds$$

where $A(t) = \int_{t_1}^t e^{-r_2(s-t)} a(s) ds$. Or simply

$$\begin{aligned} \mu(t) &= \int_{t_1}^t \int_{t_1}^s e^{-r_2(z-t) - r_1(s-t)} a(z) dz ds \\ &= \int_{t_1}^t a(z) e^{r_2(t-z)} \left(\int_z^t e^{-r_1(s-t)} ds \right) dz \\ &= \int_{t_1}^t a(z) \Gamma(t-z) dz \end{aligned}$$

(it is useful to think of the change in the order of integration done in the second equality graphically: we are integrating over a triangle below the 45 degree line), where we have computed

$$e^{r_2(t-z)} \int_z^t e^{-r_1(s-t)} ds = \frac{-e^{r_2(t-z)}}{r_1} e^{-r_1(s-t)} \Big|_z^t = \frac{e^{r_2(t-z)}}{r_1} (e^{r_1(t-z)} - 1) \equiv \Gamma(t-z).$$

Note that $\Gamma(0) = 0$, $\Gamma(x) > 0$ for x and $\Gamma(x) < 0$ for $x < 0$. Finally when $t < t_1$ it is more convenient to define

$$\omega(x) \equiv -\Gamma(-x)$$

and write

$$\mu(t) = \int_{t_1}^t a(z)\Gamma(t-z)dz = - \int_t^{t_1} a(z)\Gamma(-(z-t))dz = \int_t^{t_1} a(z)\omega(z-t)dz$$

justifying the stated solution above.

D Proof of Proposition 4

The initial conditions require $\mu_x(0) = \mu_\pi(0) = 0$. The non-negativity requirement for μ_x then requires $\dot{\mu}_x(0) = -x(0) \geq 0$, implying $x(0) \leq 0$.

To establish that inflation must be positive at some point, consider the perturbation $x(t, \varepsilon) = x(t) + \varepsilon$ and

$$\pi(t, \varepsilon) = \kappa \int_0^\infty e^{-\rho s} x(t+s, \varepsilon) ds = \kappa \int_0^\infty e^{-\rho s} x(t+s) ds + \kappa \int_0^\infty e^{-\rho s} \varepsilon ds = \pi(t) + \frac{\kappa}{\rho} \varepsilon.$$

Note that the perturbation is feasible for all $\varepsilon > 0$ since higher inflation relaxes the ZLB constraint $\sigma \dot{x}(t) \geq -r(t) - \pi(t)$. In terms of the loss function

$$L(\varepsilon) = \frac{1}{2} \int_0^\infty e^{-\rho t} \left(x(t, \varepsilon)^2 + \lambda \pi(t, \varepsilon)^2 \right) dt$$

we have

$$L'(0) = \int_0^\infty e^{-\rho t} \left(x(t) + \lambda \frac{\kappa}{\rho} \pi(t) \right) dt = \frac{1}{\kappa} \pi(0) + \lambda \frac{\kappa}{\rho} \int_0^\infty e^{-\rho t} \pi(t) dt \leq 0.$$

Hence, negative inflation $\pi(t) \leq 0$ for all $t \geq 0$ with strict inequality over some range, implies $L'(0) < 0$, a contradiction with optimality.

To establish that $x(t)$ must be positive for some $t \geq 0$, proceed by contradiction. Suppose $x(t) \leq 0$ for all $t \geq 0$. This implies that $\pi(t) \leq 0$ for all $t \geq 0$, a contradiction.

Case with $\kappa\sigma\lambda = 1$. Now suppose $\kappa\sigma\lambda = 1$. Let us guess and verify that $\mu_\pi(t) = 0$ for all $t \geq 0$ and $\mu_x(t) = \sigma\lambda\pi(t)$. Note that

$$\dot{\mu}_\pi(t) = -\lambda\pi(t) + \sigma^{-1}\mu_x(t) = 0$$

and

$$\dot{\mu}_x(t) = -x(t) + \rho\sigma\lambda\pi(t) = \sigma\lambda(\rho\pi(t) - \kappa x(t)) = \sigma\lambda\dot{\pi}(t)$$

are both satisfied. Set $i(t) = 0$ whenever $\pi(t) > 0$ and $i(t) = r(t)$ otherwise. Set $\pi(0) = 0$. The unique bounded solution to the ODEs for x and π is optimal.

Since $\mu_x(0) = 0$ this implies that we must have $\pi(0) = 0$. Since $\mu_x(t) \geq 0$ it follows that $\pi(t) \geq 0$. Indeed, $\pi(t) = 0$ whenever the zero lower bound is not binding so that $\mu_x(t) = 0$ and $i(t) = I(r(t), \pi(t)) = I(r(t), 0) = r(t)$.

Case with $\kappa\sigma\lambda < 1$. I first show that $\pi(t) \geq 0$ for all $t \geq 0$. Define

$$f(t) \equiv e^{\rho t} \int_t^{\infty} e^{-\rho s} \mu_{\pi}(s) ds.$$

Suppose that there exists a date τ such that $\pi(\tau) < 0$. Then since

$$\mu_x(\tau) = \kappa^{-1}\pi(\tau) - \kappa f(\tau) \geq 0$$

it follows that $f(\tau) < 0$, which implies that $\mu_{\pi}(t) < 0$ for some $t \geq \tau$. Now define a date τ' as follows. If $\mu_{\pi}(\tau) \geq 0$, then let τ' be the earliest date after τ when μ_{π} turns strictly negative: $\tau' = \inf_t \{t : t \geq \tau \wedge \mu_{\pi}(t) < 0\}$. If $\mu_{\pi}(\tau) < 0$ then let τ' be the latest date before τ such that μ_{π} turns strictly negative: $\tau' = \sup_t \{t : t \leq \tau \wedge \mu_{\pi}(t) > 0\}$. Note that, in both cases, $f(\tau) = e^{-\rho(\tau'-\tau)}f(\tau') + e^{\rho\tau} \int_{\tau}^{\tau'} e^{-\rho s} \mu_{\pi}(s) ds < 0$ and by definition $\int_{\tau}^{\tau'} e^{-\rho s} \mu_{\pi}(s) ds > 0$. This implies $f(\tau') < 0$. By construction we also have $\mu_{\pi}(\tau') = 0$ and $\dot{\mu}_{\pi}(\tau') \leq 0$.

Next I argue that $f(\tau') < 0$, $\mu_{\pi}(\tau') = 0$ and $\dot{\mu}_{\pi}(\tau') \leq 0$ are incompatible. If $\pi(\tau') < 0$ then

$$\dot{\mu}_{\pi}(\tau') = -\lambda\pi(\tau') + \sigma^{-1}\mu_x(\tau') \geq -\lambda\pi(\tau') > 0.$$

While if $\pi(\tau') \geq 0$ then

$$\dot{\mu}_{\pi}(\tau') = (\sigma^{-1}\kappa^{-1} - \lambda)\pi(\tau') - \sigma^{-1}\kappa f(\tau') > 0.$$

Either way we obtain a contradiction.

This establishes that $\pi(t) \geq 0$ for all $t \geq 0$. Next I argue that $\pi(t) > 0$ for all $t \geq 0$. Towards a contradiction, suppose $\pi(t) = 0$ for some $t \geq 0$. If $\pi(t) = 0$ for all t , then $x(t) = 0$ for all $t \geq 0$ and this is the first best allocation, which is not possible if $r(t) < 0$ for some positive measure of time. Thus, consider the case where $\pi(t) \neq 0$ for some positive measure. Let τ denote the earliest date where $\pi(\tau) = 0$ and either $\int_{\tau-\varepsilon}^{\tau} \pi(t) > 0$

or $\int_{\tau}^{\tau+\varepsilon} \pi(t) > 0$.

By the arguments above it follows that $f(\tau) \leq 0$. If this inequality is strict then we can proceed as before and obtain a contradiction. Thus, suppose $f(\tau) = 0$. Note also that unless $\mu_{\pi}(\tau) = 0$ then $f(t) < 0$ for some t in the neighborhood of τ ; this also would lead to a contradiction. Thus, assume $\mu_{\pi}(\tau) = 0$. Using that $\int_t^{\infty} e^{-\rho(z-t)} \mu_{\pi}(z) dz = \int_t^{\tau} e^{-\rho(z-t)} \mu_{\pi}(z) dz$ since $f(\tau) = 0$, we have

$$\dot{\mu}_{\pi}(t) = (\sigma^{-1}\kappa^{-1} - \lambda)\pi(t) - \sigma^{-1}\kappa \int_t^{\tau} e^{-\rho(z-t)} \mu_{\pi}(z) dz$$

integrating yields

$$\mu_{\pi}(\tau) - \mu_{\pi}(t) = (\sigma^{-1}\kappa^{-1} - \lambda) \int_t^{\tau} \pi(s) ds - \sigma^{-1}\kappa \int_t^{\tau} \int_s^{\tau} e^{-\rho(z-s)} \mu_{\pi}(z) dz ds$$

Define the Piccard operator

$$T[\mu_{\pi}](t) = -(\sigma^{-1}\kappa^{-1} - \lambda) \int_t^{\tau} \pi(s) ds + \sigma^{-1}\kappa \int_t^{\tau} m(z-t) \mu_{\pi}(z) dz$$

where $m(s) = (\sigma\rho)^{-1}(1 - e^{-\rho s})$. This is similar to the operator defined in the proof of Proposition 1. As explained in more detail there, starting from the zero function, $\mu_{\pi}^0(t) = 0$ for all $t \geq 0$, and using that $\pi(t) \geq 0$ we construct a declining sequence $T^n[\mu_{\pi}^0] = \mu_{\pi}^n(t)$ that converges to the fixed point. Indeed, if $\int_{\tau-\varepsilon}^{\tau} \pi(s) ds > 0$, then $\mu_{\pi}^n(\tau - \varepsilon) < 0$, and the fixed point has $\mu_{\pi}(t) < 0$ for $t \in (\tau - \varepsilon, \tau)$. Likewise, if $\int_{\tau}^{\tau+\varepsilon} \pi(s) ds > 0$, then $\mu_{\pi}^n(\tau + \varepsilon) > 0$, so the fixed point has $\mu_{\pi}(t) > 0$ for $t \in (\tau, \tau + \varepsilon)$. Either way, it follows that we have a point τ' , with either $\tau' = \tau - \varepsilon$ or $\tau' = \tau + \varepsilon$, such that $f(\tau') < 0$. This leads to a contradiction as before. We conclude that $\pi(t) > 0$ for all $t \geq 0$.

Case with $\kappa\sigma\lambda > 1$. Assume that instead $\pi(0) > 0$, then

$$\mu_x(0) = \kappa^{-1}\pi(0) - \kappa f(0) = 0$$

implies that $f(0) > 0$, which implies that $\mu_{\pi}(t) > 0$ for some $t \geq 0$. Now define a date τ' as the earliest date when μ_{π} turns strictly positive: $\tau' = \inf_t \{t : t \geq 0 \wedge \mu_{\pi}(t) > 0\}$. Note that we have $f(0) = e^{-\rho\tau'} f(\tau') + \int_0^{\tau'} e^{-\rho s} \mu_{\pi}(s) ds > 0$ and by definition $\int_0^{\tau'} e^{-\rho s} \mu_{\pi}(s) ds \leq 0$. This implies $f(\tau') > 0$. By construction we also have $\mu_{\pi}(\tau') = 0$ and $\dot{\mu}_{\pi}(\tau') \geq 0$.

Next I argue that $f(\tau') > 0$, $\mu_{\pi}(\tau') = 0$ and $\dot{\mu}_{\pi}(\tau') \geq 0$ are incompatible. If $\pi(\tau') \leq 0$

then

$$\mu_x(\tau') = \kappa^{-1}\pi(\tau') - \kappa f(\tau') < 0$$

which is a contradiction with optimality. If $\pi(\tau') > 0$ then

$$\dot{\mu}_\pi(\tau') = (\sigma^{-1}\kappa^{-1} - \lambda)\pi(\tau') - \sigma^{-1}\kappa f(\tau') < 0.$$

A contradiction since by construction $\dot{\mu}_\pi(\tau') \geq 0$. This establishes that $\pi(0) \leq 0$.

Next I argue that $\pi(0) < 0$ if $x(0) < 0$. If $\pi(0) = 0$ then $f(0) = 0$. This implies that $\mu_\pi(0) = \dot{\mu}_\pi(0) = 0$. However, we also have $\dot{\pi}(0) < 0$ and thus

$$\dot{\mu}_\pi(0) = (\sigma^{-1}\kappa^{-1} - \lambda)\dot{\pi}(0) < 0.$$

This implies that $\mu_\pi(t) < 0$ in a neighborhood of $t = 0$. This implies that we can find a point $\tau' > 0$ such that $f(\tau') > 0$, $\mu_\pi(\tau') = 0$ and $\dot{\mu}_\pi(\tau') \geq 0$, which once again leads to a contradiction.

E Proof of Proposition 5

If $\kappa\sigma\lambda \geq 1$ then Proposition 4 implies that $\pi(t) \geq 0$ for all t . This implies that $I(r, \pi) = r + (1 - \kappa\sigma\lambda)\pi < 0$ for $t < T$ and the result follows from Proposition 3. Similarly, if $\kappa = 0$ the result is immediate since $\pi(t) = 0$ for all $t \geq 0$.

If instead $\kappa\sigma\lambda < 1$ and $\kappa \neq 0$ then we must rule out that $\pi(t) \geq -\underline{r}/(1 - \kappa\sigma\lambda) > -\underline{r} > 0$ for $t \leq T$. Towards a contradiction, suppose that $\pi(t') > -\bar{r}$ for some $t' < T$. Then since from $x(0) < 0$ it follows that $\rho\pi(t') - \kappa x(t') > 0$ (see the phase diagram). Thus, we have that $\dot{\pi}(t') > 0$ and $\dot{x}(t') < 0$. This situation is absorbing: for any $t \geq t'$ with $\rho\pi(t) - \kappa x(t) > 0$ and $\pi(t) > -\bar{r} \geq -r(t)$ we have $\dot{\pi}(t) > 0$ and $\dot{x}(t) < 0$. This is the case because at any point in time either $i(t) = 0$ which implies $\dot{x}(t) = \sigma^{-1}(-r(t) - \pi(t)) < 0$ or else $i(t) = I(r(t), \pi(t))$ in which case $\dot{x}(t) = -\kappa\lambda\pi(t) < 0$. In either case, $\dot{x}(t) \leq -\kappa\lambda\pi(t) < \kappa\lambda\underline{r} < 0$. It follows that $x(t) \rightarrow -\infty$ and $\pi(t) \rightarrow \infty$. This violates the present value condition $\pi(t) = \kappa \int_0^\infty e^{-\rho s} x(t+s) ds$ and is therefore not feasible.

Moreover, since $x(t) < 0$ for all $t \geq t''$ for some $t'' > t'$ it follows that $\dot{\pi}(t) > \rho\pi(t) - \kappa x(t) > \rho\pi(t)$ for all t above some t'' . This implies $e^{-\rho t}\pi(t) \rightarrow \infty$. As a result $\int e^{-\rho t}\pi(t)^2 dt = \infty$. Similarly, for all t large enough we have $\dot{x}(t) \leq -\kappa\lambda\pi(t) \leq -\kappa\lambda e^{\rho t}\pi(t'')$. This implies that $x(t) \leq \frac{-\kappa\lambda}{\rho}(e^{\rho t} - 1)\pi(t'')$ (recall that $x(t'') < 0$). This

implies that

$$e^{-\rho t} x(t) \leq \frac{-\kappa\lambda}{\rho} \pi(t'') + e^{-\rho t} A \leq \frac{-\kappa\lambda}{\rho} \pi(t'') + \max\{A, 0\}$$

implying that $\int e^{-\rho t} x(t)^2 dt = \infty$. As a result the loss is infinite, $L = \infty$, and this plan cannot be optimal even if we were to ignore the requirement that $\pi(t) = \kappa \int_0^\infty e^{-\rho s} x(t+s) ds$.

The argument above assumed no jumps in x . One may allow for upward jumps in the path for x , but they are not optimal. To see this, suppose an upward jump in x occurs at some point t' . Then $\lim_{t \downarrow t'} \dot{\mu}_x(t) > \lim_{t \uparrow t'} \dot{\mu}_x(t)$ and since $\mu_x(t) = 0$ for $t \in [t', t' + \varepsilon)$ then $\lim_{t \uparrow t'} \dot{\mu}_x(t) < 0$. Hence, $\mu_x(t) < 0$ in a neighborhood to the left of t' , a contradiction with the optimality conditions.

F Proof of Proposition 6

Write the Hamiltonian as

$$H = \frac{1}{2} (x^2 + \lambda \pi^2 + \eta g^2) + \mu_x((1 - \Gamma)\Delta + \sigma^{-1}(i - \pi - r)) + \mu_\pi(\rho\pi - \kappa x) + \mu_g \Delta$$

where $\dot{g}(t) = \Delta(t)$ is the law of motion for government spending and $\Delta(t) \in \mathbb{R}$ is a new, unrestricted, control variable. The initial state $g(0)$ is free. The optimality conditions are just as before with the addition of

$$(1 - \Gamma)\mu_x(t) + \mu_g(t) = 0$$

$$\dot{\mu}_g(t) = \rho\mu_g(t) - \eta g(t)$$

Solving for spending gives

$$g(t) = \frac{1 - \Gamma}{\eta} (-\rho\mu_x(t) + \dot{\mu}_x(t)) = \frac{1 - \Gamma}{\eta} (-x(t) + \kappa\mu_\pi(t)).$$

It follows that in regions where the zero lower bound is not binding, where $\mu_x(t) = \dot{\mu}_x(t) = 0$, spending is zero: $g(t) = 0$. It also follows that $g(0) \geq 0$ since $\mu_\pi(0) = 0$ and $x(0) \leq 0$ by Proposition 4. Indeed, $g(0) > 0$ if and only if $x(0) < 0$.

I now argue that the conclusions of Proposition 3 extend to the case with government spending, so that $I^*(r(t), t) > 0$ in a neighborhood to the left of t_1 . The same is true in a neighborhood to the right of t_0 as long as $t_0 > 0$. Differentiating the law of motion for the

costate μ_x and using the expression for $g(t)$ from above gives

$$\begin{aligned}\ddot{\mu}_x &= -\dot{x} + \kappa\dot{\mu}_\pi + \rho\dot{\mu}_x \\ &= -\sigma^{-1}(i - \pi - r) - (1 - \Gamma)\dot{g} - \kappa\lambda\pi + \kappa\sigma^{-1}\mu_x + \rho\dot{\mu}_x \\ &= -\sigma^{-1}(i - I(\pi(t), t)) - (1 - \Gamma)\frac{1 - \Gamma}{\eta}(-\rho\dot{\mu}_x + \ddot{\mu}_x) + \kappa\sigma^{-1}\mu_x + \rho\dot{\mu}_x.\end{aligned}$$

Rearranging we arrive at the second order ordinary differential equation

$$\left(1 + \frac{(1 - \Gamma)^2}{\eta}\right)\ddot{\mu}_x - \left(1 + \frac{(1 - \Gamma)^2}{\eta}\right)\rho\dot{\mu}_x - \kappa\sigma^{-1}\mu_x = a(t).$$

The associated characteristic equation has real positive and negative roots

$$r_1 = \frac{\rho + \sqrt{\rho^2 + \frac{4\kappa\sigma^{-1}}{1 + \frac{(1-\Gamma)^2}{\eta}}}}{2} > \rho \quad r_2 = \frac{\rho - \sqrt{\rho^2 + \frac{4\kappa\sigma^{-1}}{1 + \frac{(1-\Gamma)^2}{\eta}}}}{2} < 0.$$

This then implies that the solution for $\mu_x(t)$ is given by equations (5)–(6). The rest then follows as in the proof of Proposition 3.

When $t_0 > 0$, applying equation (7) we can write

$$\begin{aligned}\frac{\eta}{1 - \Gamma}g(t) &= -r_1\mu_x(t) + \mu_x(t) + (r_1 - \rho)\mu_x(t) \\ &= \sigma^{-1}e^{r_2t} \int_{t_0}^t e^{-r_2s}(I^*(r(s), s) - i(s))ds + (r_1 - \rho)\mu_x(t)\end{aligned}$$

In a neighborhood to the right of t_0 we have established that $(I^*(r(t), s) - i(t)) > 0$ and $\mu_x(t) > 0$ implying that $g(t) > 0$. A symmetric argument, but interchanging the two roots r_1 and r_2 in this expression, establishes that $g(t) < 0$ in a neighborhood to the left of t_0 .

G Proof of Proposition 7

The results immediately using equation (4) combined with the proof behind Proposition 4 part 3, which characterizes the sign of the multiplier $\mu_\pi(t)$.

H Proof of Proposition 8

An extended Maximum principle is required to attack this problem (see [Seierstad and Sydsæter \(1987\)](#) or [Vinter \(2000\)](#)). One forms the Hamiltonian

$$H \equiv \frac{1}{2}(c + (1 - \Gamma)g)^2 + \frac{1}{2}\lambda\pi^2 + \eta g^2 + \mu_c \left(\sigma^{-1}(i - \pi - r(t)) \right) + \mu_\pi (\rho\pi - \kappa(c + (1 - \Gamma)g))$$

and associated Lagrangian

$$L \equiv H + \phi(c + (1 - \Gamma)g).$$

The necessary conditions are then

$$\mu_c(t) \geq 0 \quad = 0 \text{ if } i(t) > 0$$

$$g(t) = \frac{(1 - \Gamma)}{(1 - \Gamma)^2 + \eta} (-c + \kappa\mu_\pi(t) - \phi(t))$$

where $\phi(t) \geq 0$ and

$$\phi(t)(c(t) + (1 - \Gamma)g(t)) = 0$$

The costates evolve according to

$$\begin{aligned} \dot{\mu}_c &= -\frac{\partial L}{\partial c} = -(c + (1 - \Gamma)g) + \rho\mu_c + \kappa\mu_\pi - \phi \\ \dot{\mu}_\pi &= -\frac{\partial L}{\partial \pi} = -\lambda\pi + \sigma^{-1}\mu_c \end{aligned}$$

and satisfy the initial conditions

$$\mu_c(0) = \mu_\pi(0) = 0.$$

Since $x(t) = c(t) + (1 - \Gamma)g(t) \leq 0$ it follows that $\pi(t) \leq 0$ for all $t \geq 0$. Also, $\mu_c(t) \geq 0$. Thus, $\mu_\pi(t)$ is non-negative and monotone increasing, strictly so unless $\pi(t) = x(t) = 0$ for all t .

Since

$$\dot{c}(t) = \sigma^{-1}(i(t) - \pi(t) - r(t)) \geq \sigma^{-1} - \underline{r} > 0$$

and $c(T) = 0$ it follows that $c(t)$ is strictly negative and strictly increasing over $t \in [0, T)$.

Next we argue that $\mu_c(t) > 0$ for all $t \in (0, T)$. Whenever $x(t) < 0$ we have

$$\dot{\mu}_c(t) = -x(t) + \rho\mu_c(t) + \kappa\mu_\pi(t)$$

and when $x(t) = 0$ we have

$$\dot{\mu}_c(t) = \rho\mu_c(t) - \frac{\eta}{(1-\Gamma)^2}c(t)$$

since $c(t) < 0$ for $t < T$ it follows that $\dot{\mu}_c(t) > 0$ and $\mu_c(t)$ is strictly increasing. Given that $\mu_c(0) = 0$, it follows that $\mu_c(t) > 0$ for $t \in (0, T)$.

Since $\mu_c(t) > 0$ it follows that $i(t) = 0$ almost everywhere for $t \in [0, T)$.

Next I argue that there is a date $t^* \in (0, T)$ such that for $t < t^*$ we have $x(t) < 0$ and for $t \geq t^*$ we have $x(t) = 0$. The optimality conditions imply that if $\phi(t) = 0$ then

$$c(t) + (1-\Gamma)g(t) = \frac{\eta}{(1-\Gamma)^2 + \eta}c(t) + \frac{(1-\Gamma)^2}{(1-\Gamma)^2 + \eta}\kappa\mu_\pi(t)$$

the right hand side is strictly increasing, thus, there is at most one time $t^* \in [0, T]$ where it equals zero. Note that the right hand side is strictly negative at $t = 0$ since $c(0) < 0$ and $\mu_\pi(0) = 0$. This implies that $\mu_\pi(t) > 0$ for $t > 0$. The right hand side is strictly positive at $t = T$ since $c(T) = 0$ and we have just established that $\mu_\pi(T) > 0$. Thus, t^* exists and $t^* \in (0, T)$.

It follows that for $t \in [0, t^*]$ we have

$$g(t) = \frac{(1-\Gamma)}{(1-\Gamma)^2 + \eta}(-c + \kappa\mu_\pi(t))$$

and hence $g(t) > 0$. Note that $g(t)$ is the sum of the opportunistic component, which is positive and decreasing, and the stimulus component, which is positive and increasing. Total spending may be non monotone.

I've shown that once the output gap hits zero it stays there $c(t) + (1-\Gamma)g(t) = 0$ and since $x(t) = \pi(t) = 0$ for $t \geq t^*$ and $c(T) = 0$, this implies that total spending equals

$$g(t) = -\frac{1}{1-\Gamma}c(t) = -\frac{1}{1-\Gamma} \int_t^T r(s)ds$$

for $t \in [t^*, T]$ which is positive and decreasing.

I Proof of Proposition 9

As discussed in the text, the optimality conditions are the same as in the proof of Proposition 8, except for the terminal conditions. The full set of terminal condition actually are

as follows:

$$\begin{aligned}\mu_c(T) &= \frac{\partial}{\partial c_T} L_T(c_T, \pi_T) \\ \mu_\pi(T) &\geq \frac{\partial}{\partial \pi_T} L_T(c_T, \pi_T)\end{aligned}$$

where the inequality must hold with equality if $\pi_T < 0$.

The function L_T is convex and attains its global minimum at $(c_T, \pi_T) = (0, 0)$. For given c_T the minimum for $L_T(c_t, \cdot)$ is attained at $\pi_T = 0$. Following the arguments in the proof of Proposition 8, the costates are nonnegative, $\mu_c(t), \mu_\pi(t) \geq 0$ for all $t \geq 0$. This implies that $\pi_T = 0$ and $c_T > 0$.

Since $i(t) \geq 0, r(t) \leq 0$ and $\pi(t) \leq 0$, consumption $c(t)$ is monotone increasing. I now argue that $c(0) \geq 0$ is never optimal. This would imply that if $\phi(t) = 0$ then

$$c(0) + (1 - \Gamma)g(0) = \frac{\eta}{(1 - \Gamma)^2 + \eta} c(0) + \frac{(1 - \Gamma)^2}{(1 - \Gamma)^2 + \eta} \kappa \mu_\pi(t)$$

since $\mu_\pi(0) = 0$ and $\mu_\pi(t) \geq 0$ this rules out $c(0) > 0$, since otherwise $x(0) > 0$ which is not feasible. Also, if $c(0) = 0$ then it must be that $x(t) = 0$ for all $t \in [0, T]$ since otherwise $\phi(t) = 0$ and $x(t) > 0$ for some $t \in (0, T)$. Also, given the characterization for $t \geq T$ we have $x(t) = \pi(t) = 0$ and $c(t) \geq 0$ for all $t \geq 0$. This implies that $g(t) \leq 0$ for all $t \geq 0$. But then the following plan improves welfare

$$\begin{aligned}\tilde{c}(t) &= c(t) - \varepsilon \\ \tilde{g}(t) &= g(t) + (1 - \Gamma)\varepsilon\end{aligned}$$

for small $\varepsilon > 0$. This is true because $\int_0^\infty e^{-\rho t} (x(t)^2 + \lambda \pi(t)^2) dt = 0$ and

$$\eta \frac{\partial}{\partial \varepsilon} \frac{1}{2} \int e^{-\rho t} (g(t) + (1 - \Gamma)\varepsilon)^2 = \eta \int e^{-\rho t} (g(t) + (1 - \Gamma)\varepsilon) > 0$$

for small enough $\varepsilon > 0$.

The rest of the arguments now follow the proof of Proposition 8.