On the Optimality of Forward Markets

By ROBERT M. TOWNSEND*

Kenneth Arrow's seminal article on the role of securities in the optimal allocation of risk bearing provided a convenient framework in which problems involving choice under uncertainty could be analyzed. By extending the commodity space to include random states of nature, classic results on the existence and optimality of a competitive equilibrium were made applicable to uncertain situations. Yet many authors have commented on the existence of the small number of markets in which claims contingent on the realization of a state are actively traded. In particular the existence of futures or forward markets in which unconditional rather than contingent claims are traded is regarded by some as a phenomenon in need of an explanation, and by others as prima facie evidence of some inefficiency.

The purpose of this paper is to show that in some cases any equilibrium allocation resulting from the operation of competitive prestate noncontingent forward markets and competitive poststate spot markets is Pareto optimal, and that any Pareto optimal allocation can be supported as a competitive equilibrium of these markets with appropriate redistribution of endowments. These propositions turn on the fact that if equilibrium spot prices satisfy certain conditions then a restriction to the trading of forward contracts will not be constraining in an equilibrium; that is, agents can achieve precisely the same allocation with forward and spot markets as they could with markets in which claims could be traded for any commodity contingent on any state.

In his article Arrow stressed that in actual markets risk bearing is not allocated by the sale of claims against specific commodities but rather by the sale of securities payable in money, and he argued that any optimal allocation could be achieved with an elementary set of such securities. These Arrow-Debreu securities, as they have become known, suffice because their returns span the space of all possible returns. That is, any security whatever can be regarded as a bundle of these elementary securities, and, as has been noted by many authors, if an arbitrary set of securities spans the space of all possible returns, then such a set of securities is essentially equivalent to the set of Arrow-Debreu securities. In particular Steinar Ekern and Robert Wilson, and Roy Radner have argued that equities or shares may have the spanning property, and Steven Ross has made a similar argument for options. This paper shows that a forward contract may be viewed as a security whose return is the amount of the numeraire good (i.e., the price) for which it can be exchanged in the spot market of each state. Thus if the rank of the matrix of spot prices is equal to the number of states,[1] forward contracts also have the spanning property, and the results of this paper may be viewed as an extension of Arrow's results.[2]

*Carnegie-Mellon University. An earlier version of this paper appeared in my doctoral dissertation presented at the University of Minnesota, July 1975. I am especially indebted to Neil Wallace for his advice and encouragement. I also acknowledge the assistance of Paul Anderson and helpful comments from John Chipman, John Danforth, Hayne Leland, Stephen Salant, Leonard Shapiro, and the referee, but claim full responsibility for any errors or ambiguities. Financial support from the Board of Governors of the Federal Reserve System and from the Federal Reserve Bank of Minneapolis is gratefully acknowledged. The views expressed herein are solely my own and do not necessarily represent the views of the Bank or of the Federal Reserve System.

[1]This is not possible if there are more states than commodities.

[2]The extension however is not quite as immediate as it may first appear to be. Both Arrow's model and his result have been given a variety of interpretations. Arrow modeled a distribution economy in which money as an actual commodity plays a role. In contrast in the exchange economy of this paper there is no money per se. Thus Arrow's proof may not be ap-

This paper proceeds as follows. Section I presents the assumptions and technology of a pure exchange economy and describes the operation of two exchange regimes—complete prestate markets for contingent claims with no poststate spot markets and non-contingent forward markets with poststate spot markets. Section II formalizes the two welfare propositions given above and outlines their proofs. These results are then interpreted by way of some examples which clarify the nature of the spanning property. Section III presents an example which emphasizes the general equilibrium hedging property of forward contracts and provides further insight into the workings and welfare implication of forward markets when market structure is incomplete. Section IV presents some concluding remarks. Formal proofs are shown in the Appendix.

I. Description of the Model and Exchange Regimes

The model is a pure exchange economy with random endowments. There are I consumers, S mutually exclusive states of the world, and C commodities. Let π_s denote the probability that state s will occur with $0 < \pi_s < 1$. In this context endowments and consumption should be indexed by the consumer i ($i = 1, 2, \ldots, I$), state s ($s = 1, 2, \ldots, S$), and commodity c ($c = 1, 2, \ldots, C$) to which they pertain. Hence let Z_{isc} and C_{isc} denote the endowment and consumption, respectively, of consumer i in state s of commodity c, and let Z_{is} and C_{is} denote the associated C dimensional vectors. Each consumer i maximizes expected utility:

$$\sum_{s=1}^{S} \pi_s U^i(C_{is})$$

Each is assumed to be risk averse in that function $U^i(\ )$ is strictly concave.[3]

There are various possibilities for trade in the model. In what follows two exchange structures will be imposed exogenously and then compared. In the first exchange regime there are complete prestate markets for contingent claims. Trading in the markets for such claims takes place before random endowments are known. Also in the first regime there is no trading in spot markets subsequent to the realization of the state. In the prestate markets each consumer can issue or purchase contingent claims, where each claim entitles the holder to one unit of a specified commodity if a particular state occurs, and zero otherwise. Let X_{isc} denote the number of such unit claims on commodity c in state s held by consumer i after trading in the market for claims (with associated C dimensional vector X_{is}). That is, $(X_{isc} - Z_{isc})$ is the demand for such claims by consumer i in the market for claims. Let r_{sc} denote the price of a unit claim on commodity c in state s in terms of some abstract unit of account. Then the budget constraint for consumer i in the markets for claims is of the form

$$\sum_{s=1}^{S} \sum_{c=1}^{C} r_{sc}(X_{isc} - Z_{isc}) = 0 \tag{1}$$

After endowments are realized and some state is known to pertain, claims are honored so that $C_{is} = X_{is}$. In summary, in the first exchange regime consumer i maximizes

$$\sum_{s=1}^{S} \pi_s U^i(X_{is}) \tag{2}$$

with respect to $\{X_{isc}\}$ subject to (1) with each $X_{is} \geq 0$.[4] An equilibrium in the first exchange regime is a set of claim prices $\{r^*_{sc}\}$ and an allocation $\{X^*_{isc}\} i = 1, 2, \ldots, I$ such that $\{X^*_{isc}\}$ is maximizing for each consumer i and there is equality of the number of claims bought and sold for each state s and each commodity c, that is,

---

plied directly. Though the principal results of this paper are known by some, there does appear to be a need for a clarifying exposition. (The analogue of Arrow's theorem for an exchange economy is presented in Appendix B.)

[3] It is also assumed that $U^i(\ )$ is continuously differentiable with $U^i_C(0) = \infty$.

[4] $\{X_{isc}\}$ denotes the SC dimensional vector with elements X_{is}, $s = 1, 2, \ldots, S$; $c = 1, 2, \ldots, C$. This shorthand notation is used below for this and other variables if no ambiguity results.

$$(3)\quad \sum_{i=1}^{I} (X^*_{isc} - Z_{isc}) = 0$$

$$s = 1,2,\ldots,S;\ c = 1,2,\ldots,C$$

In the first exchange regime there was a restriction that there be no trading in spot markets subsequent to the realization of a state. But that restriction cannot be constraining in a competitive equilibrium. For let P_{sc} denote the spot price of commodity c in state s (with C dimensional vector P_s) where the Cth commodity is chosen as the numeraire. Now suppose that in state s some auctioneer calls out the vector P^*_s where

$$(4)\qquad P^*_{sc} = r^*_{sc}/r^*_{sC}$$

If trade is permitted, each consumer i is then confronted in the spot market of state s with the following problem: maximize $U^i(C_{is})$ with respect to C_{is} subject to the budget constraint

$$\sum_{c=1}^{C} P^*_{sc}(C_{isc} - X^*_{isc}) = 0 \qquad \text{with } C_{is} \geq 0$$

It may be verified that $C_{is} = X^*_{is}$ solves this problem,[5] thus the prices $\{P^*_{sc}\}$ may be viewed as the *implicit* equilibrium spot prices of the first exchange regime.

However, if each consumer i knew prior to the realization of the state that he would have the opportunity to trade in spot markets at predetermined prices $\{P_{sc}\}$ as well as in markets for contingent claims at prices $\{r_{sc}\}$, each would solve the following recursive problem. First given income Y_{is} in state s in terms of the numeraire, commodity C, each consumer i would maximize $U^i(C_{is})$ with respect to C_{is} subject to the budget constraint $P_s \cdot C_{is} \leq Y_{is}$ with $C_{is} \geq 0$. Let $h_{is}(Y_{is}, P_s)$ denote the maximizing choice of C_{is}. Then define the indirect utility function $V^i(Y_{is}, P_s) = U^i[h_{is}(Y_{is}, P_s)]$. But Y_{is} is determined by the claims $\{X_{isc}\}$ acquired in the prestate markets for contingent claims. That is,

$$(5)\qquad Y_{is} = \sum_{c=1}^{C} P_{sc} X_{isc}$$

Hence in the market for contingent claims consumer i would maximize

$$(6)\qquad \sum_{s=1}^{S} \pi_s V^i\left(\sum_{c=1}^{C} P_{sc} X_{isc}, P_s\right)$$

with respect to $\{X_{isc}\}$ subject to the budget constraint (1) and income constraints $Y_{is} \geq 0$.[6] It should be clear that a maximizing choice $\{X^*_{isc}\}$ for this recursive problem at prices $\{P_{sc}\}$ and $\{r_{sc}\}$ cannot be unique. For if $\{X^*_{isc}\}$ were a maximizing choice, so also would be all bundles $\{X^{**}_{isc}\}$ such that

$$\sum_{c=1}^{C} P_{sc} X^{**}_{isc} = \sum_{c=1}^{C} P_{sc} X^*_{isc}$$

for each state s. Roughly speaking, given the opportunity to trade in spot markets at spot prices $\{P_{sc}\}$, consumer i cares only about the income he will have in the various states.

The indeterminacy of the recursive problem just described suggests that some further restrictions can be placed on trades without altering the ability of the consumer to acquire (ultimately) the maximizing consumption bundles. Indeed one such restriction was placed on the consumer in the first exchange regime—that there be no trading in spot markets.[7] This paper examines restrictions associated with forward contracts.

[5]For suppose $C_{is} = X^{**}_{is}$ solves this problem with $U^i(X^{**}_{is}) > U^i(X^*_{is})$ and $\sum_{c=1}^{C} (X^{**}_{isc} - X^*_{isc}) P^*_{sc} = 0$. Then from (4) one obtains $\sum_{c=1}^{C} (X^{**}_{isc} - X^*_{isc}) r^*_{sc} = 0$. This in conjunction with (1) establishes that $\{X^{**}_{isc}\}$ was obtainable in the first regime but not chosen, the desired contradiction.

[6]Here and below, these income constraints rule out bankruptcy; each consumer is assumed to honor all contracts into which he has entered, and with these constraints each has sufficient income to do so. However, it is *not* required that delivery be made in spot markets of commodities sold in the markets for claims; it is supposed that each consumer accepts delivery of all commodity bundles which when valued at spot prices yield incomes equivalent to the yield of the claim in question. It can also be established that under previous assumptions $V^i(\cdot, P_s)$ is strictly concave and continuously differentiable with $V^i_1(0, P_s) = \infty$. Hence in a maximizing position $Y_{is} > 0$ and the income constraints need not be made explicit.

[7]It can be established rigorously that such a restriction is not constraining.

For each consumer i and each commodity c these restrictions are of the form $(X_{isc} - Z_{isc}) = (X_{itc} - Z_{itc})$ for all states s and t. Thus for example if consumer i purchases a specified number of claims on commodity c contingent on state s, then he must also purchase the same number of claims on commodity c contingent on all other states. In effect only unconditional claims can be purchased or issued in such forward markets.

Thus in the second exchange regime of this paper each consumer can trade unconditional forward contracts in prestate markets and can also trade in poststate spot markets. The decision problem which confronts a consumer in such a regime is now formalized. Let Q_{ic} denote the number of unconditional claims on commodity c purchased forward by consumer i in forward markets. (Thus if Q_{ic} is negative, commodity c is sold forward.) Let f_c denote the forward price of an unconditional unit claim on commodity c in terms of some abstract unit of account. Then the budget constraint for consumer i in forward markets is

$$\sum_{c=1}^{C} f_c Q_{ic} = 0 \tag{7}$$

Having acquired forward contracts $\{Q_{ic}\}$, consumer i enters spot market s with income

$$Y_{is} = \sum_{c=1}^{C} P_{sc} Z_{isc} + \sum_{c=1}^{C} P_{sc} Q_{ic} \tag{8}$$

Thus, with trading permitted in spot markets, consumer i maximizes

$$\sum_{s=1}^{S} \pi_s V^i\left(\sum_{c=1}^{C} P_{sc}(Z_{isc} + Q_{ic}), P_s\right) \tag{9}$$

with respect to $\{Q_{ic}\}$ subject to the budget constraint (7).

An equilibrium of the second exchange regime is a set of forward prices $\{f_c^*\}$, a set of spot prices $\{P_{sc}^*\}$, a forward position $\{Q_{ic}^*\}$ $i = 1, 2, \ldots, I$ and a consumption allocation $\{X_{isc}^*\}$ $i = 1, 2, \ldots, I$, such that $\{Q_{ic}^*\}$ and $\{X_{isc}^*\}$ are maximizing for each consumer i in forward markets and spot markets, respectively. That is $\{Q_{ic}^*\}$ maximizes (9) subject to (7) under $\{f_c^*\}$ $\{P_{sc}^*\}$ with

$$X_{is}^* = h_{is}\left(\sum_{c=1}^{C} P_{sc}^*(Z_{isc} + Q_{ic}^*), P_s^*\right)$$

Forward markets clear for each commodity c,

$$\sum_{i=1}^{I} Q_{ic}^* = 0 \qquad c = 1, 2, \ldots, C \tag{10}$$

and spot markets clear for each commodity c in each state s,

$$\sum_{i=1}^{I} [X_{isc}^* - (Z_{isc} + Q_{ic}^*)] = 0 \tag{11}$$
$$s = 1, 2, \ldots, S \qquad c = 1, 2, \ldots, C$$

II. On the Equivalence of the Two Exchange Regimes

In this section it will be argued that, subject to some restrictions on the matrix of (implicit) spot prices, any Pareto optimal allocation can be supported as a competitive equilibrium of the second exchange regime with suitable redistribution of endowments and that competitive equilibria of the second regime are Pareto optimal. More formally we have

PROPOSITION 1: *Suppose that a Pareto optimal allocation* $\{X_{isc}^*\}$ $i = 1, 2, \ldots, I$ *can be supported as a competitive equilibrium of the first exchange regime with endowments* $\{Z_{isc}\}$ $i = 1, 2, \ldots, I$ *and claim prices* $\{r_{sc}^*\}$ *such that the* S x C *matrix* $p'' = [P_{sc}^*]$ *(where* $P_{sc}^* = r_{sc}^*/r_{s\bar{c}}^*$*) is of rank* S*. Then* $\{X_{isc}^*\}$ $i = 1, 2, \ldots I$ *can be supported with the same endowments as a competitive equilibrium of the second exchange regime with forward markets in* S *commodities.*

PROPOSITION 2: *Suppose there exists a competitive equilibrium of the second exchange regime with forward markets in* S *commodities with spot prices* $\{P_{sc}^*\}$ *and consumption allocation* $\{X_{isc}^*\}$ $i = 1, 2, \ldots, I$ *such that the corresponding* S x S *matrix* $P = [P_{sc}^*]$ *is of rank* S*. Then the consumption allocation* $\{X_{isc}^*\}$ $i = 1, 2, \ldots, I$ *is Pareto optimal.*

The formal proofs of these propositions are contained in Appendix A, but are now outlined with some motivating remarks. As for the first proposition it is clear from the classical theorems of welfare economics that any Pareto optimal allocation can be supported as a competitive equilibrium of the first exchange regime with suitable redistribution of endowments in the various states. Having specified this same distribution of endowments, the second exchange regime is imposed. Then it is argued that each consumer is endowed implicitly with forward contracts; each consumer can issue forward contracts up to his ability to honor such claims with his income in the various states. Also, each consumer must have sufficient income in the various states to purchase the optimal allocation assigned to him. This determines the forward contracts he must acquire in the forward markets. It is then shown that at appropriately selected spot and forward prices the resulting excess demands are consistent with the budget constraint of each consumer and that the acquired forward contracts are indeed maximizing. Finally it is shown that all markets clear.

Proposition 1 also has an important corollary:

If there exists a competitive equilibrium of the first exchange regime such that the matrix P″ is of rank S, then there exists a competitive equilibrium of the second exchange regime with a consumption allocation which is Pareto optimal. This follows from the fact that the equilibrium allocation of the first regime is Pareto optimal and by hypothesis can be supported in a competitive equilibrium without any redistribution of endowments.

The idea underlying the proof of the second proposition is that an equilibrium consumption allocation of the second regime can be supported as an equilibrium allocation of the first regime and hence is Pareto optimal.

It remains to examine the hypothesis that the matrix of spot prices have rank equal to the number of states. Essentially this hypothesis ensures that the returns of forward contracts span the space of all possible returns. In order to clarify the role of this spanning property two examples are now described, one with the spanning property and one without.

For the first example there are three commodities and three states and the 3 x 3 matrix of spot prices is assumed to be of rank three. Suppose that a nonnegative vector of incomes $\{Y_{is}; s = 1, 2, 3\}$ is to be attained by a forward position $\{Q_{ic}; c = 1, 2, 3\}$. Then equations (7) and (8) are of the form

$$\sum_{c=1}^{3} f_c Q_{ic} = 0 \tag{12}$$

$$Y_{is} = \sum_{c=1}^{3} P_{sc} Z_{isc} + \sum_{c=1}^{3} P_{sc} Q_{ic} \qquad s = 1, 2, 3 \tag{13}$$

Setting $f_3 = 1$, solving for Q_{i3} in (12), and substituting into (13) yields

$$Y_{is} = \hat{Y}_{is} + \sum_{c=1}^{2} (P_{sc} - f_c) Q_{ic} \qquad s = 1, 2, 3 \tag{14}$$

$$Y_{is} = \sum_{c=1}^{3} P_{sc} Z_{isc} \qquad s = 1, 2, 3 \tag{15}$$

Equation (14) is a parametric representation of a plane in three space through the endowed state distribution income point $\{\hat{Y}_{is}\}$. With suitable specification of the spot and forward prices, each consumer i can exchange income in any one state for income in any other without altering income in the third as illustrated in Figure 1. Thus in effect in the second regime each consumer i maximizes

$$\sum_{s=1}^{3} \pi_s V^i(Y_{is}, P_s)$$

with respect to $\{Y_{is}\}$ as determined by the choice of $\{Q_{ic}\}$ subject to constraints (14). Moreover as each consumer i is confronted with a budgetplane with the same gradient (determined by the prices $\{f_c\}$ and $\{P_{sc}\}$) each will have the same rate of substitution

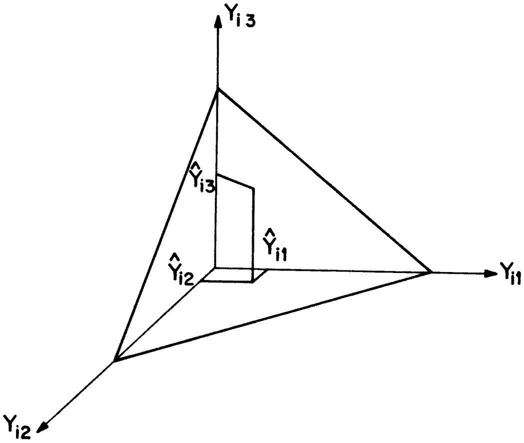

FIGURE 1

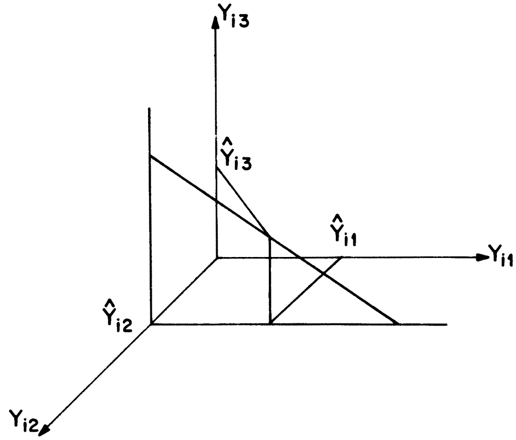

FIGURE 2

of income (the numeraire good) across states in an equilibrium, and the equilibrium allocation will be optimal.

For the second example there are three states but only two goods so that the matrix of spot prices cannot be of rank three. Then setting $f_2 = 1$, the analogue of (13) is of the form

$$(16) \quad Y_{is} = \hat{Y}_{is} + Q_{i1}(P_{s1} - f_1) \qquad s = 1,2,3$$

Equation (16) is a parametric representation of a line in three space. If for example $P_{11} < f_1 = P_{21} < P_{31}$ it is impossible to alter income in the second state. This is illustrated in Figure 2. If for example $P_{11} < f_1 < P_{21} < P_{31}$, it is impossible to alter income in one state without altering income in the other two. Though in an equilibrium of the second regime each consumer is confronted with a budget line of the same slope (determined by the prices $\{f_c\}$ and $\{P_{sc}\}$), in general each will not have the same rate of substitution of income across states. This example thus illustrates the potential for inefficiency when states outnumber commodities.

An attempt is now made to relate Proposition 1 to the results of Arrow. His principal conclusion is that an optimal allocation risk bearing can be achieved in a distribution economy (with money) by competitive markets in elementary securities. He emphasizes that a security is a claim payable in money in contrast to claims against specific commodities. But of course in the context of an exchange economy money can be no more than a numeraire. Thus, for example, if the Cth good is selected as the numeraire of each spot market, an elementary security yielding one monetary unit in state s and zero otherwise can be nothing other than a claim on commodity C in state s. It is shown in Appendix B of this paper that in an exchange economy any optimal allocation can be achieved with a set of S securities with linearly independent returns, where these returns are in terms of the amount of the numeraire good which a bearer can purchase in the spot market of each state. This then is the generalized analogue of Arrow's theorem for an exchange economy. Arrow-Debreu securities (claims on the numeraire good only) can be viewed as a particularly simple set of such securities. And subject to a rank condition on a matrix of spot prices P, forward contracts also constitute a spanning set. A forward purchase on commodity c for example has state dependent return represented by the cth column of the matrix P. The condition that P be of rank S is equivalent to the condition that the column vectors of returns of the S forward "securities" be linearly independent.

III. Forward Trading as General Equilibrium Hedging

This section is intended to give some further insight into the workings of forward markets in a general equilibrium setting. A simple example is presented which illustrates that with active spot markets, forward contracts serve as a hedge against exogenously random endowments *and* endogenously random spot prices. This general equilibrium hedging model of forward markets may be contrasted with the classic partial equilibrium approach of John M. Keynes and John Hicks which emphasizes a distinction between hedgers and speculators. In particular in the model of this paper maximizing behavior on the part of risk-averse agents does not necessarily involve the elimination of risk by purchasing the consumption bundle forward. The example also allows some inferences concerning the existence and optimality of a competitive equilibrium of the second exchange regime when market structure is incomplete.

For the example there are two representative consumers, S states of the world ($S \geq 2$), and two commodities. The first consumer is endowed with the first commodity only, and the second consumer is endowed with the second commodity only. That is, $Z_{isc} = 0$ if $i \neq c$. Without loss of generality it is supposed that Z_{1s1} is strictly increasing in s. It is also assumed that Z_{2s2} is equal to some constant Z_2 for all states.

Preferences are identical for both consumers. Each has a utility function of the form ($U(\ ,\ ) = g[W(\ ,\ )]$ where $W(\ ,\ )$ displays constant elasticity of substitution and $g(\ )$ is a monotone increasing function. Hence $W(\ ,\ )$ is of the form

$$W(C_{is1}, C_{is2}) = [(\alpha)C_{is1}^{-\rho} + (1-\alpha)C_{is2}^{-\rho}]^{-1/\rho} \quad \text{if } \sigma \neq 1$$

$$W(C_{is1}, C_{is2}) = C_{is1}^{\alpha}C_{is2}^{1-\alpha} \quad \text{if } \sigma = 1$$

where σ, the elasticity of substitution, equals $1/(1+\rho)$ and $0 < \alpha < 1$. It is further assumed that $g(W) = W^{\mu}$ where $0 < \mu < 1$, or $g(W) = ln\ W$.

Let the second commodity be chosen as the numeraire in the forward markets. Then each consumer i maximizes

$$(17) \quad \sum_{s=1}^{S} \pi_s V\left(\sum_{c=1}^{2} Z_{isc}P_{sc} + Q_{i1}(P_{s1} - f_1), P_s\right)$$

with respect to Q_{i1}, yielding necessary and sufficient first-order conditions

$$(18) \quad \sum_{s=1}^{S} \pi_s(P_{s1} - f_1)V_1\left\{\sum_{c=1}^{2} Z_{isc}P_{sc} + \psi^i(f_1)[P_{s1} - f_1], P_s\right\} = 0$$

where $\psi^i(f_1)$ denotes the maximizing choice of Q_{i1} as a function of f_1. It can be shown that $\psi_1^i(f_1) < 0$.[8]

It also can be shown that for this example equilibrium spot prices are independent of the existence and direction of forward trading as

$$(19) \quad P_{s1} = \left(\frac{\alpha}{1-\alpha}\right)\left(\frac{Z_2}{Z_{1s1}}\right)^{1/\sigma}$$

Consequently P_{s1} is strictly decreasing in s. It also follows that

$$(20) \quad P_{s1}Z_{1s1} = \left(\frac{\alpha}{1-\alpha}\right) Z_2^{1/\sigma}Z_{1s1}^{(\sigma-1)/\sigma}$$

Equation (20), which displays the value of the exogenous endowment of the first consumer as a function of s and σ, will be useful in what follows.

There remains the task of establishing the existence and direction of equilibrium forward trading. We have

PROPOSITION 3: *Under the assumptions of the example there exists a competitive*

[8]The objective function is strictly concave and continuously differentiable in Q_{i1}. Moreover the income constraints $Y_{is} \geq 0$ restrict the choice of Q_{i1} to a compact set. Hence there exists a unique maximizing choice of Q_{i1}. Also with $V_1^i(0, P_s) = \infty$, this choice must be an interior solution and the implicit function theorem applies. With decreasing absolute risk aversion, the derivative of $\psi^i(\ )$ can be signed.

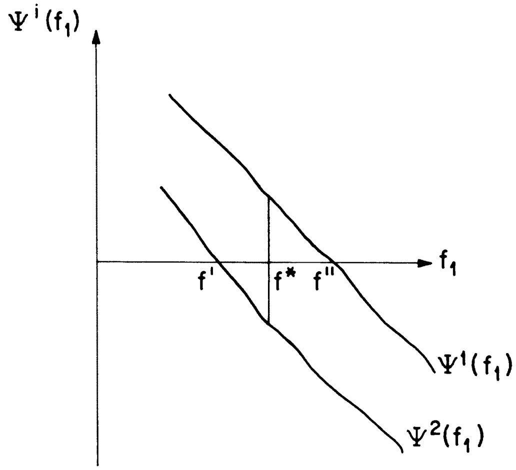

FIGURE 3

equilibrium of the second regime. Moreover

(i) *if* $\sigma > 1$, *then* $Q_{11} > 0$

(ii) *if* $0 < \sigma < 1$, *then* $Q_{11} < 0$

(iii) *if* $\sigma = 1$, *then* $Q_{11} = 0$

The idea underlying the proof of the proposition is illustrated in Figure 3. The object is to find forward prices f'' and f' at which the first and second consumer, respectively, would not wish to trade, and to show these prices differ in an appropriate way. The equilibrium price f^* can then be found and the properties of the proposition verified. A formal proof of the proposition is contained in Appendix C.

The results of the proposition are not as counterintuitive as they may first seem. Consider the case $\sigma > 1$. From (20), $P_{s1}Z_{1s1}$ is strictly increasing in s. Hence the first consumer is relatively more anxious to engage in a venture which is strictly decreasing in s than is the second consumer. Forward purchases of the first commodity with per unit return $(P_{s1} - f_1)$ represents such a venture. Thus each consumer purchases forward the single commodity with which he is endowed. Maximizing behavior in this general equilibrium hedging model need not entail purchasing the consumption bundle forward.

What can be said of the optimality of a competitive equilibrium allocation of the second exchange regime when market structure may be incomplete (as in the example of this section with $S > 2$)? If tastes are identical and homothetic (as in the example), it is possible to make some welfare comparisons. For if tastes are identical and homothetic, spot market prices are independent of the existence and direction of prestate forward trading. If there are forward markets and if a consumer chooses not to participate in such markets, then his consumption possibility set is precisely what it would have been had there been no forward markets at all. Hence the possibility of forward trading can only make him better off. This yields:

PROPOSITION 4: *If tastes are identical and homothetic, then a competitive equilibrium allocation of the second exchange regime is Pareto noninferior and possibly Pareto superior to the competitive equilibrium allocation with all markets for claims prohibited.*

IV. Concluding Remarks

Jacques Drèze has stressed the need for research into the functions and shortcomings of existing institutions and for the application of standard welfare economics based on Pareto optimality to limited exchange opportunities for risk bearing. The objective of this paper was to examine the workings and welfare implications of forward markets and to place those markets in the context of complete markets for contingent claims. It was found that with at least as many commodities as states, prestate forward markets with poststate spot markets may support Pareto optimal allocations. Thus the existence of forward markets in some commodities rather than markets for contingent claims should not be taken as prima facie evidence of some inefficiency.[9]

[9]The ultimate intent of a paper of this sort is to explain why futures contracts with subsequent spot markets is a prominent institutional configuration. If agents were indifferent between complete markets for contingent claims and futures contracts with subse-

APPENDIX A

PROOF of Proposition 1:

As P'' is of rank S, C-S columns may be deleted from P'' while leaving a square matrix P of rank S. Then without loss of generality commodities with prices in P are numbered one through S. Let $\hat{Y}_{is} = \sum_{c=1}^{C} P^*_{sc} Z_{isc}$ with associated S x 1 vector $\hat{Y}_i$. Then each consumer i is endowed implicitly with forward contracts $\{E_{ic};\ c = 1,2,\ldots,S\}$ with associated S x 1 vector E_i such that

$$\text{(A1)} \qquad \sum_{c=1}^{S} P^*_{sc} E_{ic} = \hat{Y}_{is} \qquad s = 1,2,\ldots,S$$

or in matrix notation $PE_i = \hat{Y}_i$. Let $Y^*_{is} = \sum_{c=1}^{C} P^*_{sc} X^*_{isc}$ with associated S x 1 vector Y^*_i. Then if the optimal allocation $\{X^*_{isc}\}i = 1,2,\ldots,I$ is to be achieved consumer i must enter spot markets holding forward contracts $\{F_{ic};\ c = 1,2,\ldots,S\}$ with associated S x 1 vector F_i such that

$$\text{(A2)} \qquad \sum_{c=1}^{S} P^*_{sc} F_{ic} = Y^*_{is} \qquad s = 1,2,\ldots,S$$

or in matrix notation $PF_i = Y^*_i$.

Choose spot prices $P_{sc} = P^*_{sc}$ and choose forward prices $f_c = f^*_c$, $c = 1, 2, \ldots S$, where $f^*_c = \sum_{s=1}^{S} r^*_{sc}$. Define a S x S diagonal matrix D with diagonal elements r^*_{sC} and zeros elsewhere.

First it is shown that individual budget constraints are satisfied. From (A1) and

quent spot markets, and if there were a cost associated with the former contracts which is not associated with the latter, then one structure would emerge endogenously. A cost which might be associated with contingent but not with futures contracts could be the cost of state verification. It is in this sense that the requirement that P'' be of rank S is somewhat disappointing. If P'' is of rank S, then no two rows of P'' can be identical. Agents will be fully informed by the spot market prices of which state has occurred. State verification is costless and is no obstacle to the making of contingent contracts. Futures contracts with subsequent spot markets may allow agents to do just as well, but there is nothing in the model to lead them to choose one structure over the other.

(A2) $DPQ^*_i = D(Y^*_i - \hat{Y}_i)$ where $Q^*_i = F_i - E_i$, with typical row s,

$$\text{(A3)} \qquad \sum_{c=1}^{S} r^*_{sc} Q^*_{ic} = \sum_{c=1}^{C} r^*_{sc}(X^*_{isc} - Z_{isc})$$

Summing over the rows (A3) yields

$$\text{(A4)} \qquad \sum_{c=1}^{S} f^*_c Q^*_{ic} = \sum_{s=1}^{S} \sum_{c=1}^{C} r^*_{sc}(X^*_{isc} - Z_{isc})$$

By hypothesis the right side of (A4) equals zero, and hence so does the left side.

Now suppose that Q^*_i were not a maximizing forward position for some consumer i given prices $\{f^*_c\}$, $\{P^*_{sc}\}$. That is, suppose there existed some choice Q^{**}_i of forward contracts and associated consumption $\{X^{**}_{isc}\}$ in spot markets such that

$$\text{(A5)} \qquad \sum_{s=1}^{S} \pi_s U^i(X^{**}_{is}) > \sum_{s=1}^{S} \pi_s U^i(X^*_{is})$$

Since these choices are feasible, the budget constraint in forward markets is satisfied, i.e., $\sum_{c=1}^{S} f^*_c Q^{**}_{ic} = 0$, and there is sufficient income to purchase $\{X^{**}_{isc}\}$ in spot markets, i.e., $PQ^{**}_i = Y^{**}_i - \hat{Y}_i$ where Y^{**}_i is the S x 1 vector associated with $Y^{**}_{is} = \sum_{c=1}^{C} P_{sc} X^{**}_{isc}$. With virtually the same manipulations that yielded (A4), one obtains

$$\text{(A6)} \qquad \sum_{c=1}^{S} f^*_c Q^{**}_{ic} = \sum_{s=1}^{S} \sum_{c=1}^{C} r^*_{sc}(X^{**}_{isc} - Z_{isc})$$

But

$$\text{(A7)} \qquad \sum_{c=1}^{S} f^*_c Q^{**}_{ic} = 0$$

and therefore

$$\text{(A8)} \qquad \sum_{s=1}^{S} \sum_{c=1}^{C} r^*_{sc}(X^{**}_{isc} - Z_{isc}) = 0$$

so that $\{X^{**}_{isc}\}$ was feasible under the budget constraint of the first regime. This is the desired contradiction.

It remains to show that forward markets clear. From (A1) and (A2)

$$\text{(A9)} \quad \sum_{i=1}^{I} Q_i^* = P^{-1} \sum_{i=1}^{I} (Y_i^* - \hat{Y}_i)$$

But

(A10)

$$\sum_{i=1}^{I} (Y_{is}^* - \hat{Y}_{is}) = \sum_{c=1}^{C} P_{sc}^* \sum_{i=1}^{I} (X_{isc}^* - Z_{isc})$$

From (3) the right side equals zero. Substitution into (A9) yields the desired result.

Finally, each spot market s is in equilibrium at the prices $\{P_{sc}^*\}$. For at these prices consumers achieve the same distribution of incomes across states as in the equilibrium of the first regime. That is, each consumer is on the same budget hyperplane in each state. As spot markets were implicitly in equilibrium at these same prices (see Section I) they will continue to be so (see (A15) below).

PROOF of Proposition 2:

The idea underlying the proof is that the consumption allocation of the second regime can be obtained as an equilibrium allocation of the first regime. Without loss of generality assume the first S commodities are traded forward in the second regime. Let $\{f_c^*\}$ and $\{Q_{ic}^*\}$ $c = 1,2,\ldots,S$; $i = 1,2,\ldots,I$ denote the equilibrium forward prices and forward positions, respectively, of the second regime. Then let the claim prices $\{r_{sc}^*\}$ be chosen such that

$$\text{(A11)} \quad f_c^* = \sum_{s=1}^{S} r_{sc}^* \qquad c = 1,2,\ldots,S$$

$$\text{(A12)} \quad r_{sc}^* = P_{sc}^* r_{sC}^* \qquad s = 1,2,\ldots,S; \quad c = 1,2,\ldots,C$$

(Note that by substituting (A12) into (A11) one obtains the system

$$\text{(A13)} \quad f_c^* = \sum_{s=1}^{S} P_{sc}^* r_{sC}^* \qquad c = 1,2,\ldots,S$$

With P of full rank, there exist a unique solution for $\{r_{sC}^* s = 1,2,\ldots,S\}$ in (A13) so (A11) and (A12) are well defined.)

As $\{X_{isc}^*\}$ is the final allocation, it must be, as in the proof of Proposition 1, that (A4) holds. But by hypothesis the left side of (A4) equals zero. Hence $\{X_{isc}^*\}$ satisfies the budget constraint (1) under $\{r_{sc}^*\}$.

Now suppose $\{X_{isc}^*\}$ were not maximizing under the first regime. Suppose there exist some $\{X_{isc}^{**}\}$ such that (A8) and (A5) hold. But then define $\{Q_{ic}^{**}\}$ such that there is sufficient income to purchase $\{X_{isc}^{**}\}$. That is, (A6) applies. From (A8), the right side of (A6) equals zero. Hence $\sum_{c=1}^{S} f_c^* Q_{ic}^{**} = 0$. This contradicts $\{Q_{ic}^*\}$ as maximizing.

Finally note that the markets for claims are in equilibrium. For let $\{\hat{X}_{isc}\}$ denote the forward position of consumer i in the second regime after trading in forward markets but before trading in spot markets. Then

$$\text{(A14)} \quad X_{isc}^* = (Z_{isc} + Q_{ic}^*) + (X_{isc}^* - \hat{X}_{isc}) \qquad s = 1,2,\ldots,S; \quad c = 1,2,\ldots,C$$

Summing (A14) over i yields

$$\text{(A15)} \quad \sum_{i=1}^{I} (X_{isc}^* - Z_{isc}) = \sum_{i=1}^{I} Q_{ic}^* + \sum_{i=1}^{I} (X_{isc}^* - \hat{X}_{isc}) \qquad s = 1,2,\ldots,S; \quad c = 1,2,\ldots,C$$

As forward and spot markets clear in the second regime, the right side of (A15) equals zero.

B

In what follows a security is defined to be a linear combination of unit claims on the SC contingent commodities. That is, a security of type τ entitles the holder to β_{sc}^{τ} units of commodity c in state s, $c = 1,2,\ldots,C$, $s = 1,2,\ldots,S$. Let $R_{s\tau}$ denote the return (in terms of commodity C) of security τ in state s so that given spot prices $\{P_{sc}\}$, $R_{s\tau} = \sum_{c=1}^{C} P_{sc}\beta_{sc}^{\tau}$.

PROPOSITION B-1: *Suppose that a Pareto optimal allocation $\{X_{isc}^*\}$ $i = 1,2,\ldots,I$ can be supported as a competitive equilibrium with complete markets for contingent claims and with no trade in spot markets with en-*

downments $\{Z_{isc}\}$ $i = 1, 2, \ldots, I$, and claim prices $\{r^*_{sc}\}$. Suppose also that there exist S securities where security of type τ has return $R^*_{s\tau}$ in state s as determined by the spot prices $P^*_{sc} = r^*_{sc}/r^*_{sC}$ such that the S x S matrix of security returns $R = [R^*_{s\tau}]$ is of rank S. Then $\{X^*_{isc}\}$ $i = 1, 2, \ldots, I$ can be supported with the same endowments as a competitive equilibrium with prestate markets for the S securities and poststate spot markets.

PROOF:

Let $P_{sc} = P^*_{sc}$. Let $\hat{Y}_{is} = \sum_{c=1}^{C} P^*_{sc} Z_{isc}$ with associated S x 1 vector $\hat{Y}_i$. Then each consumer i is endowed implicitly with $\tilde{E}_{i\tau}$ units of security of type τ with associated S x 1 vector $\tilde{E}_i$ defined by

$$R\tilde{E}_i = \hat{Y}_i \tag{A16}$$

Let $Y^*_{is} = \sum_{c=1}^{C} P^*_{sc} X^*_{isc}$ with associated S x 1 vector Y^*_i. Then if the allocation $\{X^*_{isc}\}$ is to be attained, consumer i must enter spot markets with securities $\tilde{F}_{i\tau}$ with associated S x 1 vector $\tilde{F}_i$ defined by

$$R\tilde{F}_i = Y^*_i \tag{A17}$$

Subtracting (A17) from (A16), premultiplying by the S x S diagonal matrix D with elements r^*_{sC} and summing over rows yields

(A18)
$$\sum_{\tau=1}^{S} \tilde{f}^*_\tau (\tilde{F}_{i\tau} - \tilde{E}_{i\tau}) = \sum_{s=1}^{S} \sum_{c=1}^{C} r^*_{sc}(X^*_{isc} - Z_{isc})$$

where $\tilde{f}^*_\tau = \sum_{s=1}^{S} r^*_{sC} R^*_{s\tau}$ is taken as the price of security τ. By hypothesis the right side of (A18) equals zero so the $\tilde{Q}^*_i = \tilde{F}_i - \tilde{E}_i$ security trades are consistent with the budget constraint of consumer i in the prestate security markets.

Moreover $\{\tilde{Q}^*_i\}$ is maximizing for consumer i. The argument is virtually identical to the one given in Proposition 1 with E_i, F_i, Q^*_i, f^*_c, and P replaced by $\tilde{E}_i$, $\tilde{F}_i$, $\tilde{Q}^*_i$, $\tilde{f}^*_\tau$, and R, respectively.

Also, security markets clear. The excess demand for security τ is

$$\sum_{i=1}^{I} (\tilde{F}_{i\tau} - \tilde{E}_{i\tau}) \tag{A19}$$

From (A16), $\tilde{E}_i = R^{-1}\hat{Y}_i$, using the fact that R is of full rank. That is,

$$\tilde{E}_{i\tau} = \sum_{s=1}^{S} \alpha_{\tau s} \hat{Y}_{is} \tag{A20}$$

where the $\{\alpha_{\tau s}\}$ are expressions involving the terms of R. These may be regarded as constants. Similarly one obtains

$$\tilde{F}_{i\tau} = \sum_{s=1}^{S} \alpha_{\tau s} Y^*_{is} \tag{A21}$$

Then substituting (A20) and (A21) into (A19) and recalling the definitions of Y^*_{is} and $\hat{Y}_{is}$ one obtains

$$\sum_{s=1}^{S} \alpha_{\tau s} \sum_{c=1}^{C} P^*_{sc} \sum_{i=1}^{I} (X^*_{isc} - Z_{isc}) \tag{A22}$$

which equals zero by the market-clearing conditions of the first regime.

Finally it may be argued as in Proposition 1 that spot markets clear.

PROPOSITION B-2: *Suppose that there exists a competitive equilibrium with prestate markets in S securities and with poststate spot markets with spot prices $\{P^*_{sc}\}$ and a consumption allocation $\{X^*_{isc}\}$ $i = 1, 2, \ldots, I$ such that the matrix of security returns $R = [R^*_{s\tau}]$ is of rank S. Then the consumption allocation $\{X^*_{isc}\}$ $i = 1, 2, \ldots, I$ is Pareto optimal.*

PROOF:

Let $\{\tilde{f}^*_\tau\}$ and $\{\tilde{Q}^*_{i\tau}\}$ $i = 1, 2, \ldots, I$ denote the equilibrium security prices and security trades, respectively. Then choose claim prices $\{r^*_{sc}\}$ to satisfy

$$\tilde{f}^*_\tau = \sum_{s=1}^{S} r^*_{sC} R^*_{s\tau} \qquad \tau = 1, 2, \ldots, S \tag{A23}$$

$$r^*_{sc} = P^*_{sc} r^*_{sC} \tag{A24}$$

(As R is of full rank, these equations are well defined.) Then as in Proposition 2, it can be shown that $\{X^*_{isc}\}$ is a maximizing choice of each consumer i in claims markets of the first regime. Finally let $\{\hat{X}_{isc}\}$ denote the implicit forward position of consumer i after trading in security markets. Then

(A25)

$$X^*_{isc} = Z_{isc} + \sum_{\tau=1}^{S} \beta^{\tau}_{sc} \tilde{Q}^*_{i\tau} + (X^*_{isc} - \hat{X}_{isc})$$

$$s = 1,2,\ldots,S; \qquad c = 1,2,\ldots,C$$

Summing over i in (A25) yields

$$\text{(A26)} \quad \sum_{i=1}^{I} (X^*_{isc} - Z_{isc}) = \sum_{\tau=1}^{S} \beta^{\tau}_{sc} \sum_{i=1}^{I} \tilde{Q}^*_{i\tau} + \sum_{i=1}^{I} (X^*_{isc} - \hat{X}_{isc})$$

$$s = 1,2,\ldots,S; \qquad c = 1,2,\ldots,C$$

As security markets and spot markets clear, the right side of (A26) equals zero.

Two special cases of the propositions should be noted. First if there exist S commodities such that the corresponding S x S matrix P of spot prices is of full rank, setting $R = P$, Propositions 1 and 2 follow. Also with S elementary Arrow-Debreu securities (where a security of type s yields one unit of commodity C in state s and zero otherwise) $R = I$, the identity matrix, and the propositions apply.

C

PROOF of Proposition 3:

Under the assumptions of the example the indirect utility function is of one of the following two forms:

$$\text{(A27)} \quad V(Y_{is}, P_s) = \varphi(P_s)^{\mu}(Y_{is})^{\mu}$$

$$\text{(A28)} \quad V(Y_{is}, P_s) = \ln Y_{is} + \ln \varphi(P_s)$$

where $\varphi(P_s)$ is an expression in terms of α, σ, and P_s. Define

$$\text{(A29)} \quad G^i(Q_{i1}, f_1) = \sum_{s=1}^{S} \pi_s(P_{s1} - f_1) V_1\left(\sum_{c=1}^{2} P_{sc}Z_{isc} + Q_{i1}(P_{s1} - f_1), P_s\right)$$

Let f'' be defined by the equation $G^1(0, f'') = 0$ so that

(A30)

$$\sum_{s=1}^{S} \pi_s(P_{s1} - f'')\mu(P_{s1}Z_{1s1})^{\mu-1}\varphi(P_s)^{\mu} = 0$$

$$\text{(A31)} \quad \sum_{s=1}^{S} [\pi_s(P_{s1} - f'')]/[P_{s1}Z_{1s1}] = 0$$

for forms (A27) and (A28) of $V(\cdot,\cdot)$, respectively. Let f' be defined by the equation $G^2(0, f') = 0$ so that

$$\text{(A32)} \quad \sum_{s=1}^{S} \pi_s(P_{s1} - f')\mu(Z_2)^{\mu-1}\varphi(P_s)^{\mu} = 0$$

$$\text{(A33)} \quad \sum_{s=1}^{S} [\pi_s(P_{s1} - f')]/[Z_2] = 0$$

for forms (A27) and (A28) of $V(\cdot,\cdot)$, respectively. Now consider the following cases:

CASE (i): $\sigma > 1$

With $\sigma > 1$ it follows from (20) that $P_{s1}Z_{1s1}$ is strictly increasing in s. Therefore, both forms (A27) and (A28), $f'' > f'$. Let $\psi(f_1) = \sum_{i=1}^{2}\psi^i(f_1)$. Then $\psi(f_1)$ is continuous. As $d\psi^i(f_1)/df_1 < 0$, $i = 1, 2$, $\psi(f'') < 0$ and $\psi(f') > 0$; see Figure 3. Therefore, there exists some f^*, $f' < f^* < f''$, with $\psi(f^*) = 0$. Hence f^* is the unique equilibrium forward price with $\psi^1(f^*) > 0$.

CASE (ii): $0 < \sigma < 1$

With $0 < \sigma < 1$, $P_{s1}Z_{1s1}$ is strictly decreasing in s. Consequently $f'' < f'$ and there exist some f^*, $f'' < f^* < f'$, with $\psi^1(f^*) < 0$.

CASE (iii): $\sigma = 1$

With $\sigma = 1$, $P_{s1}Z_{1s1}$ is constant in s so that $f^* = f' = f''$ and $\psi^i(f^*) = 0$.

REFERENCES

K. J. Arrow, "The Role of Securities in the Optimal Allocation of Risk Bearing," *Rev. Econ. Stud.*, Apr. 1964, *31*, 91–96.

J. Drèze, "Econometrics and Decision Theory," *Econometrica*, Jan. 1972, *40*, 1–17.

S. Ekern and R. Wilson, "On the Theory of the

Firm in an Economy with Incomplete Markets," *Bell J. Econ.*, Spring 1974, *5*, 171–80.

John R. Hicks, *Value and Capital*, Oxford 1946, 135–39.

John M. Keynes, *A Treatise on Money*, London 1950, 142–47.

R. Radner, "A Note on Unanimity of Stockholders' Preferences among Alternative Production Plans: A Reformulation of the Ekern-Wilson Model," *Bell J. Econ.*, Spring 1974, *5*, 181–84.

S. A. Ross, "Options and Efficiency," *Quart. J. Econ.*, Feb. 1976, *90*, 75–89.