Risk-taking over the Life Cycle: Aggregate and Distributive Implications of Entrepreneurial Risk

Dejanir H. Silva† Robert M. Townsend‡

June 2021

Abstract

We study the risk-taking behavior of entrepreneurs over the life cycle under limited idiosyncratic insurance. The model quantitatively accounts for the levels of aggregate and idiosyncratic risk premium, the patterns of inequality, and the life-cycle profiles of consumption and risk-taking observed in the data. A reform that relaxes the risk constraints leads to a reduction in the idiosyncratic risk premium and an investment boom. Consistent with a Kuznets curve, inequality increases in the short run and declines in the long run. The initial generation of entrepreneurs benefits from better insurance, but future generations of entrepreneurs are worse-off after the reform.

Keywords: Entrepreneurship, risk taking, risk premium, insurance, inequality

JEL Classification: G11, G51, E44.
1. Introduction

Entrepreneurship is inherently a risky activity. Entrepreneurial risk is especially important in the context of developing economies, where the bulk of production takes place in privately-owned businesses and risk-sharing opportunities are particularly limited. Given that entrepreneurs hold under-diversified portfolios, the presence of such risks has far reaching implications. At the individual level, entrepreneurial risk distorts investment and savings decisions. The importance of business income is highly heterogeneous and varies substantially over the life cycle, which shapes the patterns of wealth inequality. At the aggregate level, imperfect insurance leads to an idiosyncratic risk premium which affects the capital stock and the degree of economic development. The pervasive effects of limited risk sharing highlight the importance of policy interventions that may alleviate the consequences of entrepreneurs’ lack of diversification.

To assess the aggregate impact of such interventions, it is crucial to determine the quantitative importance of the limits to risk sharing. One challenge is the lack of rich data covering the risk and return of private businesses or entrepreneurs’ exposure to the business. This information is essential to determine the importance of these frictions. For instance, despite holding concentrated portfolios, entrepreneurs may engage in self-insurance or participate in informal insurance arrangements, which limits the equilibrium impact of under-diversification. The degree of partial insurance can potentially be inferred from the compensation for holding idiosyncratic risk implicit in business returns, but this requires information on entrepreneurs that is not typically available.

In this paper, we study the aggregate and distributive implications of entrepreneurial risk in the context of a quantitative life-cycle model with limited idiosyncratic insurance. Importantly, we discipline the model using panel data on small business owners in Thailand, including the returns of entrepreneurial activity as well as entrepreneurs’ risk-taking and consumption. The availability of detailed information on entrepreneurs in a developing country provides an ideal setting to assess the importance of limits to idiosyncratic risk sharing.

---

1 For evidence on the under-diversification of entrepreneurs see e.g. Moskowitz and Vissing-Jørgensen (2002) and Herranz et al. (2009) for a study focused on small businesses.

2 Informal insurance arrangements are widespread in developing economies, as documented, for example, by Kinnan and Townsend (2012).

A large literature studies the asset-pricing and macroeconomic implications of firm-level risk (see e.g. Christiano et al. 2014, Gârleanu et al. 2015, Dou 2016, Herskovic et al. 2016, Di Tella 2017, Herskovic et al. 2018), but these studies usually rely on data of public companies to discipline their quantitative exercises.
The model is able to capture main features of the data, including the life-cycle profile of entrepreneurs’ risk-taking and consumption, the patterns of inequality between- and within-age groups, as well as the level of aggregate and idiosyncratic risk premia. We study a counterfactual given by a reform that improves the extent of idiosyncratic risk sharing. We find that a reform that reduces the idiosyncratic risk premium by 100 basis points leads to an increase in the capital stock by 10% and a reduction of inequality in the long run. We also consider the transitional dynamics to the new steady state. In the short run, the reform leads to an increase in inequality, which slowly goes down to its new long run level. Therefore, the model generates a form of Kuznets dynamics, where inequality initially increases and then goes down as the economy grows. Taking into account the transitional dynamics is important when considering the impact of the reform on welfare. The initial cohort of entrepreneurs benefit the most from the intervention, and future generations of entrepreneurs are actually worse-off relative to an economy without the reform.

We start by providing motivating evidence on entrepreneurial activity in the context of a developing country. First, we extend the work of Samphantharak and Townsend (2018) to obtain a decomposition of entrepreneurial returns into an aggregate and idiosyncratic risk premia. We find that roughly half of expected returns is accounted for the compensation to hold idiosyncratic risk. Moreover, idiosyncratic volatility is three times larger than aggregate volatility, so the Sharpe ratio for the idiosyncratic component is smaller than the one for the aggregate component. The different compensations for holding aggregate and idiosyncratic risk is inconsistent with an autarky allocation, where only total volatility would matter, indicating that entrepreneurs may partially insure these risks. Second, we obtain a measure of entrepreneurs’ risk-taking, the fraction of financial wealth exposed to business, and show that there is substantial variation in risk-taking over the life cycle. In particular, young entrepreneurs are 50% more exposed to the business than old entrepreneurs. These differences in risk-taking cannot be explained by differences in expected returns, suggesting difference in risk tolerance across age groups.

To capture these motivating facts, we propose a general equilibrium model with two main ingredients: limited idiosyncratic insurance and finite lives with imperfect altruism. The economy is populated by entrepreneurs and wage earners. Entrepreneurs are the only ones with access to a

---

4This decomposition is computed by running Fama and MacBeth (1973) regressions, where average returns are regressed on a measure of aggregate risk (a “market” portfolio) and on idiosyncratic variance. See Section 2 for details.
production technology, which is exposed to aggregate and idiosyncratic shocks. Aggregate shocks are public information, so there are no frictions in sharing aggregate risk. We follow Di Tella (2017) and assume that idiosyncratic shocks are private information. Entrepreneurs face a moral hazard problem, as they can divert a fraction of capital every period. Under the optimal contract, entrepreneurs are subject to a skin-in-the-game constraint, so they must bear a fraction of the idiosyncratic risk in equilibrium. The moral hazard parameter controls the degree of partial insurance. We identify this moral hazard parameter by matching the decomposition of expected returns into aggregate and idiosyncratic components performed in the data. Entrepreneurs are able to diversify a significant fraction of the risk, roughly one quarter of the idiosyncratic volatility, but they still have to absorb the majority of the idiosyncratic shocks.

Importantly, we assume that entrepreneurial households also receive some labor income, presumably from members of the household working outside the business, consistent with what we observe in our data. An implication of this fact is that entrepreneurs’ effective risk aversion depends on the ratio of human wealth, the present discounted value of future labor income, to financial wealth, the value of the investment in the business and other safe assets minus any liabilities. We find that the human-financial wealth ratio declines over the life cycle in the data. This mechanism endogenously creates heterogeneity in risk aversion, which will be important to capture the observed life-cycle profiles.

The model matches quantitatively the decline in risk-taking over the life cycle by having young entrepreneurs being relatively risk tolerant, given a higher human-financial ratio early in life.\footnote{This mechanism is reminiscent of the work on portfolio choice with labor income (Bodie et al. 1992, Heaton and Lucas 1997, Koo 1998, Viceira 2001). See Huggett and Kaplan (2016) for a similar approach on valuing human wealth.} The savings behavior of entrepreneurs also has an important life-cycle component. We find that the consumption-wealth ratio initially declines with age and then increases by the end of the life cycle. The model captures this pattern through the interplay of two different effects. First, the decline in the human-financial wealth ratio reduces the consumption-wealth ratio. Second, the marginal propensity to consume (MPC) increases with age due to imperfect altruism. This effect is particularly strong when entrepreneurs get older, which explains the increase in the consumption-wealth ratio late in life.

Entrepreneurial risk has important implications for inequality. We find that, on average, wealth
increases initially with age and then declines later in life. We find a similar inverted-U shape for the standard deviation of wealth conditional on age, capturing the evolution of within-group inequality. The initial increase in inequality is explained by what we call a generalized \( r - g \) effect. A positive difference between the expected return on wealth, which incorporates the aggregate and idiosyncratic risk premium, and the growth rate of the economy lead entrepreneurs to accumulate more wealth on average as they age, which also amplifies the effect of the idiosyncratic shocks their wealth is exposed to. A countervailing force comes from having the MPC to increase with age, which tends to reduce wealth inequality for older entrepreneurs. Even though this was not targeted during the calibration, the model roughly captures the patterns of between- and within-group inequality observed in the data.

At the aggregate level, the idiosyncratic risk premium and the capital stock are simultaneously determined. In equilibrium, the marginal product of capital (MPK) is pinned down by the expected return on the business. An increase in the idiosyncratic risk premium leads to a higher MPK and a smaller capital stock. In contrast, the idiosyncratic risk premium depends on entrepreneurs’ risk aversion, the degree of partial insurance, and entrepreneurs exposure to risk, which is increasing in the capital stock. We find that an improvement in risk sharing leads to a decline in the idiosyncratic risk premium and an increase in the capital stock in the long run. The effect is quantitatively large, where a reform that reduces the risk premium by 100 basis points lead to an increase of the capital stock of 10%. This is consistent with the view that the level of financial development, captured here by the extent of idiosyncratic risk sharing, is an important determinant of the level of economic development.\(^6\)

We consider as well the dynamic implications of relaxing risk constraints. Solving for the transitional dynamics is challenging, as one must compute the evolution over time of the distribution of different outcomes by age. We propose a new method to compute the transitional dynamics, based on a combination of perturbation and finite-difference methods, which is potentially useful in other applications involving heterogeneous agents. We find that improving idiosyncratic insurance leads to an investment boom that lasts for a decade, which is accompanied by a sharp increase in the value of private businesses in the short run. Inequality falls in the long run, as entrepreneurs are less exposed to risk after the reform. However, inequality increases in the short

\(^6\text{See Levine (2005) for a review of literature on the connections between financial and economic development.\)}}
run, due to a revaluation effect. The increase in the value of the business benefits entrepreneurs with larger investments initially, which are the ones relatively richer before the reform. It takes a long time for inequality to converge to its new long-run level, as the effects of the initial increase in inequality persists due to intergenerational links. This pattern is consistent with Kuznets’s (1955) hypothesis over the relationship between inequality and economic development.\(^7\)

The transitional dynamics is also important to assess the welfare effects of improvements in risk sharing. We find that entrepreneurs are actually worse-off in the long run after the reform, despite of the benefits of better diversification. The reason is that the reform reduces the expected return on entrepreneurial activity, so it becomes harder for entrepreneurs to accumulate wealth, affecting the bequests and welfare of future generations. In contrast, the welfare of the initial generation of entrepreneurs improves with the reform. They received their bequest before the intervention, and the value of their businesses increases substantially in the short run. Therefore, most of the gains of the reform are reaped by the initial generation of entrepreneurs and by wage earners, who receive higher wages given the larger capital stock. This demonstrates how the effect of risk constraints interacts in rich ways with the demographic structure.

This paper is related to several strands of literature. First, the work in macro-finance studying how firm-level uncertainty affects asset prices and the real economy (Gârleanu et al. 2015, Di Tella 2017, Dou 2016, Herskovic et al. 2016, Iachan et al. 2021).\(^8\) While this literature focus mostly on business-cycle fluctuations, we study how firm-level uncertainty affects the economy in the long run. Second, the literature on how the lack of diversification of entrepreneurs’ portfolios affects several firm outcomes, including real investment (Panousi and Papanikolaou 2012), capital structure (Chen et al. 2010, Herranz et al. 2015), and risk-taking (Chen and Strebulaev 2019). This work is mostly in partial equilibrium and abstracts from the aggregate implications of entrepreneurial risk.

We also contribute to recent work on the importance of heterogeneous returns to understand the patterns of inequality. This literature documents substantial heterogeneity in portfolio returns (Fagereng et al. 2019, Fagereng et al. 2020, Bach et al. 2020), it finds that private business wealth

---

\(^7\)See Greenwood and Jovanovic (1990) for an early treatment of the Kuznets dynamics and its connection with financial development.

\(^8\)A related literature studies the asset-pricing implications of uninsurable labor income risk in infinite-horizon (Constantinides and Duffie 1996, Heaton and Lucas 1996, Constantinides and Ghosh 2017) and life-cycle models (Constantinides et al. (2002) and Storesletten et al. 2007) .
is one the main sources of wealth at the top (Smith et al. 2019, Smith et al. 2020), and that return heterogeneity is important to quantitatively match the observed levels of inequality (Gomez 2017, Hubmer et al. 2021). We study how the presence of an idiosyncratic risk premium in private business affects inequality both between- and within-age groups, and how changes in this premium affect the dynamics of inequality. Greenwald et al. (2021) show that financial wealth inequality increases after a decline in interest rates. We find a similar effect for a reduction in the idiosyncratic risk premium in the short run, but inequality actually falls in the long run, showing that discount-rate effects may have very different implications over long horizons.

Our result that young entrepreneurs invest a larger fraction of their wealth on the business is closely related to the findings in the literature on portfolio choice over the life cycle (Jagannathan et al. 1996, Viceira 2001, Cocco et al. 2005), who found that young households should invest more in the stock market than old households. In particular, we show that variations in the human-financial wealth ratio accounts quantitatively for the pattern of entrepreneurial risk-taking over the life cycle. Related work uses business income as a source of background risk to explain stock market investment (Heaton and Lucas 2000a, Heaton and Lucas 2000b). In contrast, we endogeneize the decision to invest in the business and show how non-business income plays an important role in accounting for cross-sectional differences in risk taking.

A large (micro) development literature studies risk sharing (Townsend 1994, Morduch 1995) and the risk and return of production activities in developing economies (Udry and Anagol 2006, De Mel et al. 2008). Karlan et al. (2014) conducted a randomized control trial extending credit and insurance to farmers and found that the lack of insurance is the binding constraint to investment. Their results are consistent with our findings that relaxing risk constraints have large impacts on investment. The (macro) development literature studies the aggregate implications of credit constraints (Buera and Shin 2013, Midrigan and Xu 2014, Moll 2014). Our approach is complementary to theirs, as we focus instead on the role of risk constraints. Our work is closer to the original model of uninsurable investment risk by Angeletos (2007), which we extend to allow for partial idiosyncratic insurance, a rich demographic structure, and aggregate risk. These extensions are crucial to capture the patterns of consumption and risk-taking observed in the micro data as well as to derive the dynamics of inequality in response to a relaxation of risk constraints.

See also De Nardi and Fella (2017) for a survey of the literature on quantitative Bewley models of wealth inequality.
2. Motivating evidence

In this section, we provide motivating evidence on entrepreneurial activity in the context of a developing country. First, we consider a decomposition of total entrepreneurial returns into an aggregate and idiosyncratic component. Second, we document large variations in risk taking over the life cycle by entrepreneurs, measured by the share of wealth invested in a private business.

2.1. Data

We use data from the Townsend Thai Monthly Survey, an intensive monthly survey initiated in 1998 in four provinces of Thailand. Two provinces, Chachoengsao and Lopburi, are semi-urban in a more developed central region near the capital, Bangkok. The other two provinces are rural, Buriram and Srisaket, and are located in the less developed northeastern region by the border of Cambodia. In each of the four provinces, the survey is conducted in four villages, chosen at random within a given township. A detailed discussion of the Townsend Thai Monthly Survey can be found in Samphantharak and Townsend (2010).

The utilized sample covers 716 households and 14 years of monthly data, starting in January 1999. During this time, these economies were subject to all sorts of aggregate and idiosyncratic shocks. Rice cultivation is affected by seasonal variation in rainfall and temperature. Restrictions on exports to the EU affected shrimp ponds. The productivity of milk cows varies substantially both over time for a given animal and over the herd. The exposure to this different set of shocks highlights the importance of entrepreneurial risk, and it also enable us to disentangle the role of aggregate and idiosyncratic shocks.

The data collected in the Townsend Thai Monthly Survey includes information on the net income generated by the business as well as total assets and liabilities of households. Importantly, the data are rich enough to allow us to construct a detailed balance sheet for these private businesses. We can then compute the return on assets (ROA), measured as business income over assets net of liabilities.\footnote{See Samphantharak and Townsend (2018) for a discussion of alternative return measures and the robustness of the return decomposition to such alternative measures.} We also measure the fraction of entrepreneurs’ wealth invested in the business and the fraction invested in safe (real or financial) assets, providing us with a measure of risk-taking for these entrepreneurs. Finally, the data also includes information on the household’s
labor income and consumption, which gives a characterization of the savings behavior and the
importance of non-business income for entrepreneurial households. See the appendix for a more
detailed description of the construction of these variables.\footnote{The household is the unit of measurement, even though households typically consist of multiple members doing separate or partially overlapping activities. We do treat the household as a whole as unitary (see e.g. Doepke and Tertilt (2016) for a discussion of unitary models of the household).}

2.2. \textit{Idiosyncratic risk and expected returns}

We consider next a decomposition of the expected return on a business into an aggregate and
an idiosyncratic component. This decomposition is based on the results of Samphantharak and
Townsend (2018). We provide here only a brief discussion of the estimation procedure and refer
to their work for the details.

The (total) expected return on the business is estimated by taking the time-series average of
realized returns for each entrepreneur. Similarly, we obtain a measure of the variance of returns
by computing the time-series variance of returns for each entrepreneur. Using the Townsend Thai
Monthly Survey, Samphantharak and Townsend (2018) decompose the total expected returns into
an aggregate component and an idiosyncratic component by running Fama and MacBeth (1973)
regressions. In the first stage, they run for each entrepreneur a time-series regression of returns
on a measure of "market" (or aggregate) return, given by the average return on all business in
the entrepreneur’s region.\footnote{The aggregate return on the region is computed using a leave-one-out mean, so the household’s own return is not included as a regressor.} In the second stage, they run a cross-section regression of the average
return on the exposure to aggregate risk and on the idiosyncratic variance. Given theses estimates,
one can decompose expected returns into a compensation for aggregate risk and a compensation
for idiosyncratic risk.

Table 1 reports the expected return, volatility, and Sharpe ratio for total returns, the aggregate
component, and the idiosyncratic component of expected returns.\footnote{One important distinction with Samphantharak and Townsend (2018) is that they focus on the return of the entrepreneurs’ entire portfolio, which depends on the household’s portfolio composition, instead of considering only the return on the business. We use our measure of entrepreneurs’ risk-taking to infer the decomposition for business returns from the one for portfolio returns.} We derive three main lessons
from this table. First, entrepreneurial risk is large, both in absolute terms and relative to the level of returns, given an annual volatility of 20\% and a Sharpe ratio of 0.20. Second, idiosyncratic shocks
account for most of the total risk, representing more than 90\% of the variance. Third, a large
Table 1: Aggregate and idiosyncratic components of risk and return

This table presents the decomposition of total returns into an aggregate component and an idiosyncratic component. The aggregate component is measured as the coefficient on the market return on the cross-sectional regression times the market beta averaged across entrepreneurs. The idiosyncratic component is measured as the coefficient on idiosyncratic risk on the cross-sectional regression times the idiosyncratic variance averaged across entrepreneurs.

<table>
<thead>
<tr>
<th></th>
<th>Risk premium</th>
<th>% of returns</th>
<th>Volatility</th>
<th>% of variance</th>
<th>Sharpe ratio</th>
</tr>
</thead>
<tbody>
<tr>
<td>Total returns</td>
<td>4.0%</td>
<td>100%</td>
<td>20.1%</td>
<td>100%</td>
<td>0.20</td>
</tr>
<tr>
<td>Aggregate component</td>
<td>1.9%</td>
<td>46.2%</td>
<td>5.2%</td>
<td>6.8%</td>
<td>0.35</td>
</tr>
<tr>
<td>Idiosyncratic component</td>
<td>2.2%</td>
<td>53.8%</td>
<td>19.4%</td>
<td>93.2%</td>
<td>0.11</td>
</tr>
</tbody>
</table>


A share of total expected returns is a compensation for bearing idiosyncratic risk, where roughly half of the total risk premium is explained by the idiosyncratic component. However, given that idiosyncratic shocks account for 90% of the risk and again only 50% of the average return, they have a substantially smaller Sharpe ratio than aggregate shocks.

These results are informative about the degree of risk sharing available to entrepreneurs. The positive compensation for bearing idiosyncratic risk is inconsistent with the benchmark case of complete markets, as entrepreneurs would perfectly diversify idiosyncratic risk and expected returns would reflect only compensation for bearing aggregate risk. The differences in Sharpe ratio indicate that entrepreneurs receive a smaller compensation for holding idiosyncratic risk, which is inconsistent with the polar case of autarky, as entrepreneurs would only care about total risk in that case. Therefore, the decomposition in Table 1 suggests that entrepreneurs have access to some form of partial insurance. This is consistent with the results in Samphantharak and Townsend (2018) who document an active role for gifts and loans, measured in the Thai data, as a way of sharing risk among entrepreneurs. Specifically, they find that entrepreneurs who suffer a negative shock are more likely to receive a transfer from a friend or relative, and the opposite pattern when they receive a positive shock, showing that entrepreneurs are able to at least partially insure against these shocks.
Table 2: Risk taking, consumption, and labor income over the life cycle

This table presents the life-cycle profiles for business exposure, consumption-wealth ratio, and labor income. Business exposure is measured as the average across all entrepreneurs of the value of the business relative to (net) financial wealth. The consumption-wealth ratio is the average across all entrepreneurs of consumption relative to financial wealth. Labor income corresponds to the average across all entrepreneurs of the household labor income for a given age group normalized by the average labor income for entrepreneurs across all ages.

<table>
<thead>
<tr>
<th></th>
<th>Age groups</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>25-35</td>
</tr>
<tr>
<td>Business exposure</td>
<td>29.6%</td>
</tr>
<tr>
<td>Consumption-wealth ratio</td>
<td>11.0%</td>
</tr>
<tr>
<td>Labor income</td>
<td>1.04</td>
</tr>
</tbody>
</table>

Source: Townsend Thai Monthly Survey.

2.3. Risk-taking and savings over the life cycle

We consider next how entrepreneurs’ risk-taking and savings behavior vary over the life cycle. Table 2 reports the life-cycle profiles for risk taking, measured by the share of entrepreneurs’ wealth invested in the business, the consumption-wealth ratio, and labor income. Entrepreneurial risk taking declines sharply with age. The business exposure of the oldest group is 36% smaller than the business exposure of the youngest group. The consumption-wealth ratio, which captures entrepreneurs’ saving behavior, is U-shaped over the life cycle: it initially declines with age, and then eventually it increases with age by the end of the life cycle. Table 2 also reports the evolution of labor (or non-business) income for entrepreneurial households, typically representing sources of income coming from household members other than the head. We observe the typical pattern of a hump-shaped labor income profile. In the analysis that follows, the presence of non-business income will be important to generate the life cycle patterns we observe in the data.

3. A life-cycle model of entrepreneurial risk taking

In this section, we consider a model of entrepreneurial activity with two main ingredients: i) imperfect idiosyncratic insurance and ii) finite lives with imperfect altruism. These two ingredients play a crucial role in capturing the evidence described in Section 2.

\textsuperscript{14}See, for instance, Lagakos et al. (2018) for a discussion of the life-cycle profile for labor income in both developed and developing countries.
3.1. Environment

Time is continuous and the economy is populated by two types of households: entrepreneurs and wage earners. Population grows at rate $g$, and the share of entrepreneurs in the population is constant and given by $\chi_e$. The set of entrepreneurs and wage earners alive at period $t$ are denoted by $E_t$ and $W_t$, respectively. Entrepreneurs live for $T$ periods and leave bequests to their offspring. For simplicity, we assume that wage earners have infinite horizon. While all households receive labor income, only entrepreneurs have access to an investment technology which is exposed to both aggregate and idiosyncratic shocks. Households can buy and sell aggregate insurance in a frictionless market, but entrepreneurs have access only to imperfect idiosyncratic insurance. Households can also borrow and lend at a risk-free interest rate. We now describe in detail the technology, preferences, and financial frictions faced by both types of households.

3.1.1. Technology

Entrepreneur $i$ combines capital $k_{i,t}$ and hired labor $l_{i,t}$ to produce a final homogeneous good $\tilde{y}_{i,t}$, the numeraire in this economy, using the technology:

$$\tilde{y}_{i,t} = A_t k_{i,t}^\alpha l_{i,t}^{1-\alpha},$$

and we denote scaled output by $y_t = \tilde{y}_t / A_t$.\(^\text{15}\)

Productivity $A_t$ is subject to aggregate shocks and follows a geometric Brownian motion:

$$\frac{dA_t}{A_t} = \mu_A dt + \sigma_A dZ_t,$$

where $Z_t$ is a standard Brownian motion.

Entrepreneurs can adjust their capital stock by either investing in new capital or buying capital from another entrepreneur. The investment technology is risky and subject to adjustment costs.

\(^{15}\)We adopt this convention throughout the paper: variables that grow with aggregate productivity $A_t$ are denoted with a tilde and the corresponding scaled variable are denoted without a tilde.
In particular, given a total investment of $i_{i,t}A_t k_{i,t}$, capital evolves according to

$$\frac{d k_{i,t}}{k_{i,t}} = (\Phi(i_{i,t}) - \delta) \, dt + \sigma_d dZ_{i,t},$$

where $Z_{i,t}$ is an idiosyncratic Brownian motion for entrepreneur $i$, which is independent across entrepreneurs. The investment function $\Phi(\cdot)$ satisfies $\Phi(0) = 0$, $\Phi'(\cdot) > 0$, and $\Phi''(\cdot) < 0$.

The concavity of $\Phi(\cdot)$ captures the presence of adjustment costs, as it makes costly to sharply increase the expected growth rate of capital. Importantly, investment is risky and subject to idiosyncratic shocks. Entrepreneurs can also adjust their capital stock by buying capital from other entrepreneurs at the price $\bar{q}_t = q_t A_t$. The market value of the business is given by $\bar{q}_i k_{i,t}$. In equilibrium, $q_t$ will be non-stochastic, so the relative price of capital $\bar{q}_t$ moves proportionally with aggregate productivity shocks.

Note that the evolution of the aggregate capital stock, $k_t = \int_{E_t} k_{i,t} \, di$, is not affected by idiosyncratic shocks, as these shocks get diversified\footnote{This is an implication of the exact law of large numbers for a continuum of random variables, as in Sun (2006).} in the aggregate:

$$d k_t = \left( \int_{E_t} (\Phi(i_{i,t}) - \delta) \, k_{i,t} \, di \right) \, dt.$$

The return of investing in the project can be written as the sum of the dividend yield, the business net income after labor and investment expenses relative to its market value, and capital gains:

$$d R_{i,t} = \frac{\bar{y}_i t - \bar{w}_i i_{i,t} - i_{i,t} A_t k_{i,t}}{\bar{q}_i k_{i,t}} \, dt + \frac{d(\bar{q}_i k_{i,t})}{\bar{q}_i k_{i,t}}$$

$$= \mu_{i,t}^{R} dt + \sigma_A dZ_t + \sigma_d dZ_{i,t},$$

where $\bar{w}_t = w_t A_t$ denotes the wage rate.

Using Ito’s lemma to compute the expected capital gain, the expected return on the project is

$$\mu_{i,t}^{R} \equiv \frac{\bar{y}_i t - \bar{w}_i i_{i,t} - i_{i,t} k_{i,t}}{\bar{q}_i k_{i,t}} + \frac{\bar{q}_i}{\bar{q}_t} + \mu_A + \Phi(i_{i,t}) - \delta.$$

\footnote{This is an implication of the exact law of large numbers for a continuum of random variables, as in Sun (2006).}
3.1.2. Preferences and labor supply

Entrepreneurs live for $T$ periods. They have isoelastic preferences over consumption $c_{i,t}$ with curvature parameter $\gamma$ and derive utility of leaving bequests:

$$
E_i \left[ \int_{s_i}^{s_i+T} e^{-\rho (t-s_i)} \frac{c_{i,t}^{1-\gamma}}{1-\gamma} dt + e^{-\rho (T-s_i)} (1 - \psi) \gamma V^* \frac{\bar{n}_{i,s_i+T}^{1-\gamma}}{1-\gamma} \right],
$$

(6)

where $\bar{n}_{i,t}$ denotes financial wealth, and $s_i$ denotes the birth date of entrepreneur $i$.\(^{17}\)

The parameter $\psi$ measures the strength of the bequest motive. If $\psi = 1$, entrepreneurs give no weight to their offspring, and there are no intergenerational linkages. If $\psi = 0$, the behavior of entrepreneurs coincides with the one of an agent with infinite horizon.\(^{18}\) The case $0 < \psi < 1$ captures a form of imperfect altruism.

In addition to business income, entrepreneurial households are allowed to receive labor income. This is consistent with the observation that households have multiple sources of income in our data. Labor is supplied inelastically, it is denoted by $\tilde{l}_{i,t}$, and it can vary deterministically over the life cycle.\(^{19}\) As discussed in Section 2, there is significant variation in the importance of labor income over the life cycle.

3.1.3. Financial friction

Entrepreneurs face a moral hazard problem, similar to the one in He and Krishnamurthy (2012) and Di Tella (2017). In particular, we follow Di Tella (2017) and focus attention on short-term contracts. Though aggregate shocks are perfectly observable by all households, the idiosyncratic shock is only observed by the entrepreneur. An entrepreneur can divert capital, but a fraction $1 - \phi$ of the diverted capital is lost in the process. The parameter $\phi \in (0, 1)$ controls the severity of the moral hazard problem.

Following the literature on dynamic moral hazard, we show in Appendix B.1 that the solution to the contracting problem can be implemented by a market structure where entrepreneurs have

\(^{17}\)Financial wealth consists of the entrepreneur’s assets, including the amount invested in the business, net of any liabilities.

\(^{18}\)The constant $V^*$ equals the value function coefficient for an infinite horizon agent, and it is given in Appendix A.1.

\(^{19}\)Notice that $\tilde{l}_{i,t}$ denotes the amount of labor supplied by the household of an entrepreneur born at period $s_i$, while $l_{i,t}$ is the amount of labor demanded by the entrepreneur to run her project.
access to a riskless asset with return \( r_t \) and both aggregate and idiosyncratic insurance. There is no limit on aggregate insurance, as aggregate shocks are perfectly observable, but the quantity of idiosyncratic insurance is limited for incentive purposes. Formally, each entrepreneur \( i \) pays \( p_t^{ag} \hat{\theta}_{i,t}^{ag} \) to reduce aggregate volatility by the amount \( \hat{\theta}_{i,t}^{ag} \), where \( p_t^{ag} \) denotes the price of aggregate insurance. In contrast, each entrepreneur \( i \) can buy idiosyncratic insurance \( \hat{\theta}_{i,t}^{id} \) at zero cost in equilibrium, as providers of insurance can perfectly diversify across entrepreneurs. However, the amount of idiosyncratic insurance is limited by the skin-in-the-game constraint:

\[
\hat{\theta}_{i,t}^{id} \leq (1 - \phi) \bar{q}_i k_{i,t} \sigma_{id}.
\] (7)

This particular market structure represents one possible implementation of the allocation under the optimal contract. For instance, instead of formal insurance contracts, this implementation may capture the presence of informal insurance arrangements in developing countries. As documented by e.g. Kinnan and Townsend (2012), kinship networks play an important role in allowing households to partially share idiosyncratic risk. The moral hazard friction enable us to capture not only the polar extremes of autarky (\( \phi = 1 \)) and complete markets (\( \phi = 0 \)), but also the case of partial insurance which seems to better represent the level of risk sharing observed in the data.

The skin-in-the-game constraint will be binding in equilibrium, causing entrepreneurs to be insurance-constrained.\(^{20}\) To focus on the implications of entrepreneurial risk, we abstract from (ad-hoc) borrowing constraints.\(^{21}\) In Appendix D.1, we consider an extension with ex-ante heterogeneity among entrepreneurs and limited pledgeability and show that our main results hold in an environment with both risk and borrowing constraints. Entrepreneurs can borrow freely against the present discounted value of their future labor income, that is, their human wealth:

\[
\tilde{n}_{i,t} \geq -\tilde{h}_{i,t},
\] (8)

where \( \tilde{n}_{i,t} \) denotes the entrepreneur’s financial wealth, \( \tilde{h}_{i,t} \equiv \mathbb{E}_t \left[ \int_t^{t+T} \frac{\pi_z}{m} \delta z \tilde{I}_{i,z} dz \right] \) denotes human

---

\(^{20}\)Alternatively, this can be interpreted as an equity constraint, where entrepreneurs are unable to freely sell claims on their business to a diversified set of investors, as in e.g. Chen et al. (2010) and Panousi and Papanikolaou (2012).

\(^{21}\)This focus is also partly motivated by recent evidence that finds a modest impact of microcredit on entrepreneurship, suggesting that borrowing constraints may not be the key limiting factor for entrepreneurship in the aggregate. See e.g. the evidence from six different randomized control trials discussed in Banerjee et al. (2015).
wealth, and $\pi_t$ denotes a stochastic discount factor (SDF) for this economy, which evolves according to $d\pi_t = -r_t \pi_t dt - p_t^{\text{ag}} \pi_t dZ_t$. Note that we can write the entrepreneur’s financial wealth as $\tilde{n}_{i,t} = \tilde{q}_t k_{i,t} + \tilde{b}_{i,t}$, the sum of the value of the business and the amount invested in the riskless asset, denoted by $\tilde{b}_{i,t}$. Condition (8) then can be written as $-\tilde{b}_{i,t} \leq \tilde{q}_t k_{i,t} + \tilde{h}_{i,t}$, so the entrepreneur can borrow freely against the value of the business and the human wealth.

### 3.1.4. Entrepreneurs’ problem

The problem of entrepreneur $i$ with age $a_i$ is to choose a vector of stochastic processes $(\tilde{c}_{i,t}, \tilde{\varphi}_t, \tilde{\mu}_t, \tilde{q}_t, \tilde{h}_t, \tilde{l}_t, \tilde{a}_t)$, taking the processes for prices $(\tilde{q}_t, \tilde{w}_t, r, p_t^{\text{ag}})$ as given, to solve the following program:

$$V_t(\tilde{n}_i, a_i) = \max_{\tilde{c}_{i,t}, \tilde{\varphi}_t, \tilde{\mu}_t, \tilde{q}_t, \tilde{h}_t, \tilde{l}_t, \tilde{a}_t} \mathbb{E}_t \left[ \int_0^{T-a_i} e^{-\rho_t \tilde{c}_{i,t}^{1-\gamma}} dZ_t + e^{-\rho_t (T-a_i)} (1-\psi) \gamma V_{t+T-a_i} \tilde{h}_t^{1-\gamma} \right],$$

subject to (7), (8), non-negativity constraints $c_{i,t}, k_{i,t} \geq 0$, and the law of motion of $\tilde{n}_{i,t}$

$$dn_{i,t} = \left[ (\tilde{n}_{i,t} - \tilde{q}_t k_{i,t}) r_t + \tilde{q}_t k_{i,t} \mu_{i,t} - p_t^{\text{ag}} \tilde{\varphi}_t + \tilde{w}_t \tilde{l}_{i,t} - \tilde{c}_{i,t} \right] dt + \left( \tilde{q}_t k_{i,t} \sigma_A - \tilde{\varphi}_t \right) dZ_t + \left( \tilde{q}_t k_{i,t} \sigma_d - \tilde{\mu}_t \right) dZ_{i,t},$$

given initial financial wealth $\tilde{n}_{i,0} = \tilde{n}_i > \tilde{h}_{i,0}$. Note that the relevant state variables in this problem are the entrepreneur’s financial wealth and age.

The term in brackets in the expression above represents the expected growth rate of financial wealth. The entrepreneur invests $\tilde{n}_{i,t} - \tilde{q}_t k_{i,t}$ in the riskless asset, which gives a return $r_t$, and she invests the amount $\tilde{q}_t k_{i,t}$ in the risky business technology, which gives an expected return $\mu_{i,t}$. The cost of aggregate insurance is $p_t^{\text{ag}} \tilde{\varphi}_t$. The entrepreneur receives labor income $\tilde{w}_t \tilde{l}_{i,t}$ and consumes $\tilde{c}_{i,t}$. The last two terms represent the exposure to aggregate and idiosyncratic risk, which corresponds to the risk exposure through the business net of insurance.

### 3.1.5. Wage earners’ problem

In contrast to entrepreneurs, wage earners do not have access to a production technology. To focus on the behavior of entrepreneurs and simplify exposition, we assume that wage earners

---

22Even though markets are (endogenously) incomplete, households agree on the valuation of variables not exposed to idiosyncratic risk. Therefore, there is no ambiguity in defining the relevant SDF for aggregate payoffs.

23Note that insurance contracts have zero value on origination, as the present discounted value of the premium equals the present discounted value of the transfers received, when using the appropriate SDF to discount payoffs.
have infinite horizon, such that we can abstract from life-cycle considerations. Allowing for finite
lives and a bequest motive does not change our main results, as shown in Appendix D.2. Wage
earners and entrepreneurs share a common per-period utility function, that is, utility is isoelastic
with risk-aversion coefficient $\gamma$. As commonly assumed in models with heterogeneous rate of
return, as e.g. Kiyotaki and Moore (1997), we allow for different discount rates between wage
earners and entrepreneurs.

The problem of wage earner $j \in \mathcal{W}_t$ can then be written as

$$
\hat{V}_t^{\infty}(\hat{n}_j) = \max_{\hat{\epsilon}_j, \hat{\theta}_j^{\rho}} \mathbb{E}_t \left[ \int_t^{\infty} e^{-\rho(z-t)} \frac{\hat{c}_j^{1-\gamma}}{1-\gamma} \, dz \right],
$$

subject to $\hat{n}_{j,t} \geq \hat{h}_{j,t}$, where $\hat{h}_{j,t}$ denotes wage earner $j$’s human wealth, non-negativity constraint
$\hat{c}_{j,t} \geq 0$, and the law of motion of financial wealth $\hat{n}_{j,t}$

$$
d\hat{n}_{j,t} = \left[ \hat{n}_{j,t}r_t + p_t^{\rho_S} \hat{\theta}_{j,t}^{\rho_S} + \hat{w}_t \hat{h}_{j,t} - \hat{c}_{j,t} \right] dt - \hat{\theta}_{j,t}^{\rho_S} dZ_t,
$$
given initial financial wealth $\hat{n}_{j,0} = \hat{n}_j > -\hat{h}_{j,0}$. Note that $\hat{\theta}_t^{\rho_S}$ can take positive or negative
values, so wage earners can choose to either buy aggregate insurance from entrepreneurs or provide
aggregate insurance to entrepreneurs.\footnote{Wage earners can also provide idiosyncratic insurance. In our notation, we have already imposed that they can
diversify the exposure to idiosyncratic risk, so their financial wealth is only exposed to aggregate risk in equilibrium.}

3.1.6. Equilibrium

We provide below a definition of the competitive equilibrium.

**Definition 1.** A competitive equilibrium is a set of aggregate stochastic processes: the aggregate capital
stock $k$, the interest rate $r$, the wage rate $\hat{w}$, the relative price of capital $\hat{q}$, and the price of aggregate ins-
urance $p^{\rho_S}$; and a set of stochastic processes for each entrepreneur $i \in \mathcal{E}_t$ and each wage earner $j \in \mathcal{W}_t$:
consumption $\hat{c}_i$, financial wealth $\hat{n}_i$, capital $k_i$, labor $l_i$, aggregate insurance $\hat{\theta}_{i,t}^{\rho_S}$, and idiosyncratic insurance
$\hat{\theta}_{id,t}^{\rho_S}$ for entrepreneurs, and consumption $\hat{c}_j$ and aggregate insurance $\hat{\theta}_{j,t}^{\rho_S}$ for wage earners such that:

(a) Aggregate capital stock satisfies the law of motion (4), given the initial capital stock $k_0$.

(b) $(\hat{c}_i, \hat{\theta}_{id,t}^{\rho_S}, \hat{\theta}_{j,t}^{\rho_S}, k_i, l_i, t_i)$ solve entrepreneurs’ problem (9), given $(\hat{q}, \hat{w}, r, p^{\rho_S})$. 

16
(c) \((\tilde{c}_j, \tilde{\theta}_j)\) solve wage earners’ problem \((10)\), given \((\tilde{w}, r, p^\alpha)\).

(d) Markets clear for all \(t \geq 0\):

i. Market for consumption goods

\[
\int_{\mathcal{E}_t} \tilde{c}_{i,t}di + \int_{W_t} \tilde{c}_{J,t}d\tilde{j} + \int_{\mathcal{E}_t} t_{i,t}A_t k_{i,t}di = \int_{\mathcal{E}_t} \tilde{y}_{i,t}di.
\]

ii. Markets for capital and labor

\[
\int_{\mathcal{E}_t} k_{i,t}di = k_t, \quad \int_{\mathcal{E}_t} l_{i,t}di = \int_{\mathcal{E}_t} \tilde{t}_{i,t}di + \int_{W_t} \tilde{t}_{j,t}dj.
\]

iii. Markets for aggregate insurance and riskless bonds

\[
\int_{\mathcal{E}_t} \tilde{\zeta}_{t}di + \int_{W_t} \tilde{\theta}_{t}d\tilde{j} = 0, \quad \int_{\mathcal{E}_t} [\tilde{n}_{i,t} - \tilde{q}_{i,t}]di + \int_{W_t} \tilde{n}_{j,t}dj = 0.
\]

The first market clearing condition corresponds to the goods market, where the value of consumption by entrepreneurs and wage earners plus total investment equals total output produced in the economy. The second set of market clearing conditions corresponds to factor markets, where the total demand for capital and labor equal their corresponding supply. Finally, the third set of market clearing conditions corresponds to aggregate insurance and bonds, where the total demand for aggregate insurance and riskless assets must be equal to zero.

3.2. Solution to entrepreneurs’ problem

We consider next the solution to the entrepreneurs’ problem. We focus on a stationary equilibrium, where scaled aggregate variables are constant, that is, \(w_t = w\) and \(q_t = q\). In Section 6, we consider a non-stationary environment where aggregate variables are allowed to be time-varying.

3.2.1. Maximizing expected returns

Notice that \((l_{i,t}, l_{i,t})\) enters the maximization problem in Equation \((9)\) only through the expected return on the business given in Equation \((5)\). Hence, entrepreneurs choose these variables to max-
imize expected returns. Labor demand assumes the usual form:
\[ w = (1 - \alpha) \left( \frac{k_{i,t}}{l_{i,t}} \right)^{\alpha}. \] (11)

The capital-labor ratio is equalized across entrepreneurs and coincides with the aggregate capital-labor ratio \( K_t \equiv k_t/\bar{l}_t \), where \( k_t \) denotes the aggregate capital stock and \( \bar{l}_t \) denotes the aggregate labor supply. In a stationary equilibrium, capital grows at the same rate as labor supply, which we assume grows with population at rate \( g \).

The investment rate \( i_{i,t} \) is given by
\[ \Phi'(i_{i,t}) = \frac{1}{q} \Rightarrow i_{i,t} = (\Phi')^{-1} \left( \frac{1}{q} \right) \equiv i(q), \] (12)
where \( i(q) \) is increasing in \( q \) given the concavity of \( \Phi(\cdot) \).

Note that, from Equation (5), a small increase in the investment rate \( i_{i,t} \) raises the expected capital gain by \( \Phi'(i_{i,t}) \), but it reduces the current dividend-price ratio by \( 1/q \). Expected returns are then maximized when \( \Phi'(i_{i,t}) = 1/q \), that is, Equation (12) is satisfied.

Given that the capital-labor ratio and the investment rate are the same for all entrepreneurs, the expected return on the business is equalized across entrepreneurs. Realized returns are, of course, still heterogeneous due to imperfect idiosyncratic insurance. Substituting Equations (11) and (12) into Equation (5), we obtain
\[ \mu^R = \frac{\alpha K^{\alpha-1} - i(q)}{q} + \mu_A + \Phi(i(q)) - \delta, \] (13)
where \( \mu^R_{i,t} = \mu^R \) in a stationary equilibrium.

3.2.2. Human and total wealth

The following lemma shows that the relevant notion of wealth for entrepreneurs is total wealth, \( \omega_{i,t} = n_{i,t} + h_{i,t} \), the sum of financial wealth and human wealth. In particular, the entrepreneur’s value function depends on total wealth \( \omega_{i,t} \) and on age \( a_i = t - s_i \).

**Lemma 1.** Suppose the economy is in a stationary equilibrium.
(a) Human wealth evolves according to

\[
\frac{\partial h(a)}{\partial a} = (r + \sigma_A - \mu_A) h(a) - w \bar{A}(a),
\] (14)

given \( h(T) = 0 \). Human wealth is then given by

\[
h(a) = \int_0^{T-a} e^{-(r + \sigma_A - \mu_A) a} w \bar{A}(a + z) dz.
\] (15)

(b) The (scaled) value function is given by\(^{25}\)

\[
V(n, a) = \xi(a) \left( \frac{n + h(a)}{1 - \gamma} \right) ^ {1 - \gamma - \frac{1}{\gamma}},
\] (16)

where \( \xi(a) \) equals the ratio of consumption to total wealth.

The entrepreneur’s effective risk aversion is given by

\[
- \frac{V_{nn} h}{V_n} = \frac{\gamma}{1 + \frac{h(a)}{n}}.
\] (17)

(c) Demand for capital and the aggregate and idiosyncratic insurance solve the mean-variance problem:

\[
\max_{k_{i,t}, \sigma_{i,t}^2, g_{i,t}^d} \frac{q_{i,t}}{n_{i,t}} \left( \mu^R - r \right) + \frac{h_{i,t}}{n_{i,t}} \sigma_A - \frac{\theta_{i,t}^2}{n_{i,t}} \left( \frac{q_{i,t} + h_{i,t}}{n_{i,t}} \sigma_A - \frac{\theta_{i,t}^2}{n_{i,t}} \right)^2 + \left( \frac{q_{i,t}}{n_{i,t}} \sigma_{id} - \frac{\theta_{i,t}^2}{n_{i,t}} \right)^2,
\] (18)

subject to (7).

**Proof.** See Appendix A.1. \( \square \)

The first part of Lemma 1 provides the law of motion of (scaled) human wealth. Human wealth is the present discounted value of future labor income. As it depends only on the entrepreneur’s age in a stationary equilibrium, we drop the dependence on the household \( i \), \( h_{i,t} = h(a_t) \). Implicitly, the expression for the law of motion of human wealth gives the appropriate discount rate to compute human wealth. As wages move with aggregate productivity, labor income is risky,

\(^{25}\)The scaled value function is related to the original value function by the condition \( \tilde{V}_t(n, a) = \tilde{A}_t^{1-\gamma} V \left( \frac{a}{\tilde{A}_t}, a \right) \).
so the discount rate for human wealth incorporates the aggregate risk premium \( p^{\delta} \sigma_A \).\(^{26}\) Human wealth being a risky asset is consistent with the work of Benzoni et al. (2007), who show that human wealth becomes highly correlated with stocks when aggregate output and labor income are cointegrated, as in our model.\(^{27}\)

The second part of Lemma 1 gives the value function, an age-dependent CRRA function of total wealth. Importantly, the entrepreneur’s effective (value-function) risk aversion depends on \( h_{i,t}/n_{i,t} \), which we refer to as the human-financial wealth ratio. This ratio varies significantly over the life cycle in our data and will be important to explain the differences in risk-taking across age groups.

The final part of the lemma shows that the portfolio choice of an entrepreneur reduces to a simple mean-variance problem given the effective risk aversion \( \gamma/(1 + h_{i,t}/n_{i,t}) \). This is a consequence of the continuous-time formulation, as the mean-variance objective comes from a direct rearrangement of the Hamilton-Jacobi-Bellman (HJB) equation of the entrepreneurs’ problem. The first two terms in Equation (18) capture the expected excess return on total wealth, which comes from the return on the business and on human wealth. The third term represents the cost of aggregate insurance. The last term is the product of the effective risk aversion and the aggregate and idiosyncratic variance. Importantly, the maximization of the mean-variance objective is subject to the skin-in-the-game constraint (7). The Lagrange multiplier to this constraint, which we refer to as the shadow price of idiosyncratic insurance, plays an important role in the characterization of entrepreneurs’ risk-taking decision.

3.2.3. Risk-taking and savings over the life cycle

The next proposition characterizes the risk-taking and savings decisions of entrepreneurs.

**Proposition 1.** Suppose the economy is in a stationary equilibrium.

(a) The shadow price of idiosyncratic insurance is given by

\[
p^{id} = \frac{\mu^R - r - p^{\delta} \sigma_A}{\phi \sigma_{id}}.
\]  

\(^{26}\)We can write Equation (14) alternatively as \( w(\alpha) + \mu + \frac{1}{h(\alpha)} \frac{\partial h(\alpha)}{\partial x} - r = p^{\delta} \sigma_A \), so the expected excess return on human wealth equals the aggregate risk premium \( p^{\delta} \sigma_A \).

\(^{27}\)Notice that we abstract from idiosyncratic risk on labor income, which allows us to focus on the implications of uninsurable entrepreneurial risk.
(b) Demand for capital is given by
\[ q_{k,t} = \frac{1}{\gamma} \frac{1 + h_{i,t}}{n_{i,t}} p^{id} \phi \sigma_{id}. \]  

(20)

(c) The demand for aggregate insurance is given by
\[ \theta_{i,t}^{ig} = \left( \frac{q_{k,t}}{n_{i,t}} + \frac{h_{i,t}}{n_{i,t}} \right) \sigma_A - \frac{1}{\gamma} \frac{b_{i,t}}{n_{i,t}} p^{ig}. \]  

(21)

(d) The consumption-wealth ratio is given by
\[ \frac{c_{i,t}}{n_{i,t}} = \frac{\bar{r}}{1 - \psi e^{-\bar{r}(T - a_i)}} \left( 1 + \frac{h_{i,t}}{n_{i,t}} \right), \]

where \( \bar{r} \equiv \frac{1}{\gamma} \rho + \left( 1 - \frac{1}{\gamma} \right) r^{MV} \) and \( r^{MV} \equiv r + \frac{(p^{ig})^2 + (p^{id})^2}{2\gamma}. \)

Proof. See Appendix A.1. \hfill \Box

The first part of Proposition 1 shows that the shadow price of idiosyncratic insurance, the Lagrange multiplier on the skin-in-the-game constraint, is equalized across entrepreneurs. Moreover, it equals the return per unit of risk (the Sharpe ratio) of an investor who fully insures the project against aggregate risk, so it is exposed only to idiosyncratic risk. In equilibrium, this Sharpe ratio is positive, so the skin-in-the-game constraint is always binding, that is, \( \theta_{i,t}^{id} = (1 - \phi) q_{k,t} \sigma_{id}. \)

Intuitively, entrepreneurs purchase as much idiosyncratic insurance as possible given its zero cost.

The demand for capital depends on the effective risk aversion as well as the price and quantity of idiosyncratic risk. The cross-sectional dispersion in risk-taking is captured by differences in effective risk aversion, which are driven by the human-to-financial wealth ratio. As this ratio has an important life-cycle component, risk-taking varies substantially over the life cycle. The average scale of the business depends on the ratio of the shadow price of idiosyncratic risk and the fraction of idiosyncratic volatility entrepreneurs cannot insure. An important implication of this result is that the decision about the scale of the business does not depend directly on the level of aggregate risk. The possibility of sharing risk through the insurance contract leads to a separation between the choice of the scale of the business and how much aggregate risk the entrepreneur is willing to hold. Therefore, the decision of how much to invest in the business is essentially a decision of
how much idiosyncratic risk to bear.

Equation (21) gives the demand for aggregate insurance. The demand is linear and decreasing in the price of aggregate insurance with a slope given by the inverse of the effective risk aversion and intercept given by the total exposure to aggregate risk, which comes both from the business and the human wealth. Using the fact that $p^{RS} = \gamma \sigma_A$ in a stationary equilibrium, the demand for aggregate insurance simplifies to $\theta_{i,t}^{RS} = (qk_{i,t} - n_{i,t})\sigma_A$. Poor entrepreneurs end up buying aggregate insurance ($\theta_{i,t}^{RS} > 0$), while rich entrepreneurs provide insurance ($\theta_{i,t}^{RS} < 0$). This arrangement can be implemented by having richer entrepreneurs sending transfers to poor entrepreneurs as an indemnity after a negative aggregate shock, with transfers in the opposite direction after the economy receives a positive aggregate shock. Note that if entrepreneurs on average have positive savings on the risk-free asset, such that $\int \xi_i (qk_{i,t} - n_{i,t}) di < 0$, then they are net sellers of insurance. In this case, wage earners are borrowers in equilibrium. As they effectively hold a leveraged position on their risky human wealth in this case, wage earners are disproportionately exposed to risk and demand aggregate insurance in equilibrium.

Finally, the expression for the consumption-wealth ratio is given in Equation (22). The first term $\bar{\tau}/(1 - \psi e^{-\tau(T-t)})$ represents the marginal propensity to consume (MPC). It is increasing in age, as it is typical in finite-horizon problems, and the bequest motive parameter $\psi$ controls the strength of this effect. If $\psi = 0$, the MPC is constant, recovering a standard result in infinite-horizon problems. If $\psi = 1$, then the MPC gets arbitrarily large as the entrepreneur approaches the end of life, so the stock of wealth is fully consumed at the final age $T$, as in Merton (1969). While $\psi$ is important to determine how the MPC varies over the life cycle, $\bar{\tau}$ is important to determine the average MPC. If $\gamma = 1$, then $\bar{\tau} = \rho$, the entrepreneur’s discount rate. In general, $\bar{\tau}$ is a linear combination of $\rho$ and $r^{MV}$, the risk-adjusted expected return on total wealth $\omega_{i,t}$. The term $r^{MV}$ can be written as a mean-variance objective $r^{MV} = \frac{1}{dt} \left[ E[\frac{d\omega_{i,t}}{\omega_{i,t}}] - \frac{1}{2} \theta V[\frac{d\omega_{i,t}}{\omega_{i,t}}] \right]$. Hence, if $\gamma > 1$, an increase in risk-adjusted returns raises the average MPC. Finally, the consumption-wealth

---

28The result $p^{RS} = \gamma \sigma_A$ is obtained by combining Equation (21) with the market clearing condition for aggregate insurance, as shown in Appendix B.3.

29The MPC is defined as the change in consumption in response to an increase in financial wealth, that is, the MPC is given by $\frac{\Delta c_i}{\Delta m_i} = \frac{\bar{\tau}}{1 - \phi e^{-\tau(T-t)}}$.

30This is an extension of the usual result that interest rates have income and substitution effect on savings decisions, where the income effect dominates for $\gamma > 1$. In an environment with risky returns, the relevant notion is the risk-adjusted return $r^{MV}$ instead of the riskless interest rate.
ratio depends on the human-financial wealth ratio, which potentially varies over the life cycle. Entrepreneurs with more human wealth consume more out of their current assets.

3.3. *Quantitative implications*

We consider next the quantitative implications of the model for the life-cycle behavior of entrepreneurs. First, we describe the calibration of the model. Then, we compare the model-implied evolution of consumption and risk-taking over the life cycle with their empirical counterparts.

3.3.1. *Technology, preferences, and demographics*

We adopt the following calibration, which is summarized in Table 3. The capital share is set to $\alpha = 0.33$. The average growth rate of productivity is set to $\mu = 0.003$, following the evidence provided by *Jeong and Townsend* (2007) for Thailand. The investment function assumes the functional form

$$\Phi(i) = \frac{\sqrt{\Phi_0^2 + 2\Phi_0 i - \Phi_0^2}}{\Phi_1}. $$

This corresponds to the case of quadratic adjustment costs, as the required investment rate so capital grows at the population growth rate $g$ is $\iota = \Phi_0(g + \delta) + 0.5\Phi_1(g + \delta)^2$. The coefficients of the investment function are chosen to match an investment rate of roughly 20% and a long-run relative price of capital of one. The depreciation rate is set to $\delta = 0.10$. The discount rate of wage earners is chosen to match a risk-free rate of $r = 3.77\%$, consistent with the average real rate for Thailand over the last two decades. The discount rate of entrepreneurs and the bequest motive parameters are chosen to match the consumption-wealth ratio at the beginning and end of life. The life horizon is set to $T = 55$, so it covers the life span from 25 to 80 years old, and the population growth is set to $g = 0.3\%$, the most recent value for population growth in Thailand. The parameter $\chi_c$ is chosen to match the average ratio of the value of the business relative to the entrepreneurs’ financial wealth.

3.3.2. *Risk, return, and moral hazard parameter*

We chose the risk aversion coefficient, the aggregate and idiosyncratic volatility, and the moral hazard parameter to match the decomposition of risk and return provided in Table 1. The volatility parameters, $\sigma_A$ and $\sigma_{id}$, are chosen to match the aggregate and idiosyncratic components of total volatility. From Equation (19), we can decompose the expected return on the business into an
Table 3. Calibrated parameters

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Choice</th>
</tr>
</thead>
<tbody>
<tr>
<td>Preferences</td>
<td></td>
</tr>
<tr>
<td>$\rho$</td>
<td>Entrepreneur’s rate of time preference</td>
</tr>
<tr>
<td>$\rho_w$</td>
<td>Wage earner’s rate of time preference</td>
</tr>
<tr>
<td>$\psi$</td>
<td>Bequest motive</td>
</tr>
<tr>
<td>$\gamma$</td>
<td>Risk aversion</td>
</tr>
<tr>
<td>Technology &amp; financial friction</td>
<td></td>
</tr>
<tr>
<td>$\mu_A$</td>
<td>Average productivity growth rate</td>
</tr>
<tr>
<td>$\sigma_A$</td>
<td>Aggregate volatility</td>
</tr>
<tr>
<td>$\sigma_{id}$</td>
<td>Idiosyncratic volatility</td>
</tr>
<tr>
<td>$\phi$</td>
<td>Moral hazard parameter</td>
</tr>
<tr>
<td>$\Phi_0$</td>
<td>Adjustment cost parameter (I)</td>
</tr>
<tr>
<td>$\Phi_1$</td>
<td>Adjustment cost parameter (II)</td>
</tr>
<tr>
<td>$\alpha$</td>
<td>Capital share in production function</td>
</tr>
<tr>
<td>$\delta$</td>
<td>Depreciation rate</td>
</tr>
<tr>
<td>Demographics</td>
<td></td>
</tr>
<tr>
<td>$\delta$</td>
<td>Population growth</td>
</tr>
<tr>
<td>$\chi_e$</td>
<td>Share of entrepreneurs in the population</td>
</tr>
<tr>
<td>$T$</td>
<td>Life span (adult life)</td>
</tr>
</tbody>
</table>

The aggregate and idiosyncratic component,

$$
\mu^R - r = \underbrace{p^{\alpha g} \sigma_A}_{\text{agg. risk premium}} + \underbrace{p^{\alpha d} \phi \sigma_{id}}_{\text{id. risk premium}}.
$$

The aggregate risk premium is given by $p^{\alpha g} \sigma_A = \gamma \sigma_A^2$, so we choose $\gamma$ to match the aggregate risk premium. The idiosyncratic risk premium is informative about the parameter $\phi$. If $\phi = 0$, then we recover the complete markets solution and there is no idiosyncratic risk premium. As we raise $\phi$, the importance of the idiosyncratic risk premium in total returns increases. Therefore, we calibrate this moral hazard parameter to match the idiosyncratic risk premium reported in Table 1.
3.3.3. Measuring human wealth

It remains to specify the labor supply parameters of the age profile. We assume that $\bar{I}_{i,t}$ is given by the following function of the entrepreneur’s age:

$$\bar{I}_{i,t} = \sum_{l=1}^{L} \Gamma_l e^{\phi_l a_i}$$

where we normalize the average labor supply of entrepreneurs to one, that is, $\int_{0}^{T} \bar{I}_{i,t} \frac{dI}{\int_{0}^{T} dI} = 1$. This particular functional form was chosen because it is flexible enough to capture the empirical labor income dynamics while being analytically tractable. We estimate the parameters $(\Gamma_l, \phi_l)_{l=1}^{L}$ by non-linear least squares such that the distribution of labor income across age groups matches the one observed in the data. We set the number of exponential terms to $L = 3$. For simplicity, we assume that wage earner’s labor supply is constant and given by $\bar{I}_{i,t} = 1$. Figure 1 shows that the functional form does a good job of approximating the empirical labor income profile.

From Equation (15), human wealth can be computed as follows:

$$h_{i,t} = \int_{0}^{T-a_i} e^{-\left(r+p_g\sigma_A-\mu_A\right)z} w \sum_{l=1}^{L} \Gamma_l e^{\phi_l (z+a_l)} dz.$$
Figure 2: Human-financial wealth ratio - Life Cycle Profile

![Graph showing human-financial wealth ratio over age]

Notice that if $T \to \infty$ and $\bar{I}_{i,z}$ was constant, then the expression above would boil down to the Gordon growth formula applied to human wealth: $h_{i,t} = \frac{w}{(r + p^{\sigma_A}) - \mu_A}$.

Given the discount rate and the labor income profile, we can compute the human-financial wealth $h_{i,t}/n_{i,t}$, both in the data and in the model. The evolution of the human-financial wealth ratio over the life cycle is determined by the discount rate, the labor income profile estimated above, and the entrepreneurs’ savings behavior. Figure 2 shows that the human-financial wealth ratio tends to decline over the life cycle. This is the result of labor income being relatively high at the beginning of the life cycle combined with the mechanical fact that households have less years of future income as time goes by. Human wealth is quantitatively as important as financial wealth at the beginning of the cycle. By the age of 60, human wealth is only half of the financial wealth. Even though this was not a direct target of the calibration, the model captures reasonable well the evolution of the human-financial wealth over the life cycle.

3.3.4. Implications for risk-taking and savings

We consider next the evolution of risk-taking and the consumption-wealth ratio over the life cycle. The left panel of Figure 3 shows that the share of wealth invested in the business declines with age, consistent with the evidence in Table 2. In a stationary environment, this pattern cannot be
Figure 3: Risk-taking and Savings - Life Cycle Profiles

(a) Share of wealth invested in the business

(b) Consumption-wealth ratio

explained by differences in returns, which are assumed to be constant. The model generates this pattern by having the effective risk aversion to be decreasing in the human-financial wealth ratio. Given the presence of human wealth, young entrepreneurs are relatively less risk averse than older entrepreneurs, so they tend to invest a larger fraction of their wealth on the business. Notice that the ratio between the share invested in the business at the beginning and at the end of life are entirely determined by the human-financial wealth ratio, which was calibrated independently of any information on the cross-section of entrepreneurs’ risk-taking.

The consumption-wealth ratio, which determines households savings behavior, is roughly U-shaped as a function of age, as this ratio is decreasing at the beginning of the life cycle and it is increasing by the end of the life cycle. This non-monotonic behavior is the result of two forces. First, the human-financial wealth ratio declines with age, which induces households to reduce consumption. Second, the MPC increases with age, which induces households to consume more. The first effect dominates at the beginning of the life cycle, and the second effect dominates as the entrepreneur gets older.

4. Distributive implications of entrepreneurial risk

In this section, we consider the implications of entrepreneurial risk for wealth inequality both between and within age groups. The presence of uninsurable idiosyncratic risk distorts the distribution of wealth in the economy in important ways. With perfect insurance, there is no wealth
inequality within age groups, and wealth across age groups varies only for standard life-cycle considerations. Imperfect insurance creates dispersion in wealth within age groups, and wealth across age groups depends on the magnitudes of risk and risk premia.

The main object of interest in this section is the joint distribution of (scaled) financial wealth and age, which we denote by \( f_t(n, a) \). Let \( f(a) \) be the (marginal) age distribution in the population. Given that population grows at rate \( g \) and that entrepreneurs live for \( T \) periods, \( f(a) \) follows an exponential distribution truncated at age \( T \). Given this joint distribution, one can compute the average wealth conditional on age \( a \), \( n_t(a) \), and the average wealth of all entrepreneurs, \( n_{e,t} \):

\[
    n_t(a) = \int_{-\infty}^\infty n f_t(n|a) dn, \quad n_{e,t} = \int_0^T n_t(a) f(a) da,
\]

where \( f_t(n|a) = f_t(n, a) / f(a) \).

We focus on the stationary distribution of financial wealth, such that \( f_t(n, a) = f(n, a) \) for all \( t \), which allow us to drop time subscripts, \( n_t(a) = n(a) \) and \( n_{e,t} = n_e \). Note that while the stationary distribution is independent of the realization of aggregate shocks, the presence of aggregate risk is still important, as it affects the expected return on the portfolio and the savings behavior of entrepreneurs.

We start by characterizing how wealth is distributed across age groups, that is, how \( n(a) \) varies with age and the role of risk and demographics in shaping this distribution. We then characterize the distribution of wealth conditional on age and how inequality varies with age. In both cases, the risk-taking and savings decisions discussed in Section 3 play an important role in the analysis.

4.1. Between-group inequality

The next proposition provides a characterization of between-group inequality and entrepreneurs’ average financial wealth.

**Proposition 2.** Suppose the economy is in a stationary equilibrium.
i. **Between-group inequality:** The share of wealth held by entrepreneurs of age $a$, $\frac{f(a)n(a)}{n_e}$, satisfies

$$\log \frac{f(a)n(a)}{n_e} = \log \frac{f(0)n(0)}{n_e} + \log \left( \frac{1 + \frac{h(0)}{n(0)}}{1 + \frac{h(a)}{n(a)}} \right) + \left( \frac{r + (p^g)^2}{\gamma} + \frac{(p^{id})^2}{\gamma} - (g + \mu_A) \right) a - \int_0^a \frac{\psi}{1 - \psi e^{-\gamma(T-a')}} da'. $$

where

$$n(0) = \frac{e^{\left( r + \frac{p^g}{\gamma} + \frac{\mu_A}{\gamma} \right) T}}{1 - e^{\left( r + \frac{p^g}{\gamma} + \frac{\mu_A}{\gamma} \right) T}} h(0),$$

and $mpc_e = \frac{1}{T} \int_0^T \frac{\bar{f} - e^{-\gamma(T-a)}}{1 - \psi e^{-\gamma(T-a)}} da$ is the average MPC across the life cycle.

ii. **Average financial wealth:** The average financial wealth of entrepreneurs is given by

$$n_e = f(0)(n(0) + h(0)) \int_0^T e^{\left( r + \frac{p^g}{\gamma} + \frac{\mu_A}{\gamma} \right) a} \frac{e^{-ra} - \psi e^{-rT}}{1 - \psi e^{-rT}} da - h_e,$$

where $h_e = \int_0^T f(a)h(a)da$.

**Proof.** See Appendix A.2.

The first part of Proposition 2 decomposes the distribution of wealth across age groups into three effects. First, a human-to-financial wealth effect. For older entrepreneurs, the human wealth has been mostly converted into financial wealth, that is, labor income accelerates the accumulation of financial wealth. The second term is a generalized “$r - g$” effect.\(^{31}\) The first component is the return on total wealth: $r + \frac{(p^g)^2}{\gamma} + \frac{(p^{id})^2}{\gamma}$. In the absence of risk, this term simplifies to $r$. Hence, the correct notion of return in this context is the return on total wealth taking into account the aggregate and idiosyncratic risk premium. The second component is the growth rate of the economy, $g + \mu_A$, the sum of population and productivity growth. The generalized “$r - g$” effect implies that the wealth share tends to increase with age if the return on total wealth exceeds the growth rate of the economy. The third term is the average MPC effect. It captures the fact that wealth accumulated at age $a$ depends on the entrepreneur’s past consumption decisions. In particular, this term is small at the beginning of life and increases as the entrepreneur gets older.

---

\(^{31}\)The importance of $r - g$ in determining wealth inequality has been emphasized by Piketty (2014). See Benhabib et al. (2011) for the implications to the tail of the wealth distribution and Jones (2015) for a review of the literature.
Therefore, entrepreneurs tend to accumulate wealth when young, but eventually consumption increases until wealth achieves a desired bequest level.

The wealth of newborn agents, that is the bequest they receive, depends on expected returns, consumption behavior, and the amount of human wealth. Notice that expression (24) can be written as

\[ n(0) = e^{\left(r + \frac{(\mu_{a}^2)}{\sigma} + \frac{(\mu_{d}^2)}{\sigma} - (g + \mu_{d} - \text{mpc})\right)} (n(0) + h(0)). \]

The exponential term captures the net accumulation rate of entrepreneurs’ wealth over their lifetime. In equilibrium, this number is less than one, as part of the total wealth is consumed and only the remaining part is transferred to the next generation. The amount of bequests is increasing in returns and decreasing in the growth rate of the economy and the average MPC.

Even though idiosyncratic risk is diversified by computing group averages, the presence of the idiosyncratic risk premium affects the wealth distribution through these three channels. The idiosyncratic risk premium increases the generalized \("r - g"\) term and it affects the level and dispersion of MPCs across age groups as \(\tilde{r}\) is a function of \(p^{id}\). Moreover, because the idiosyncratic risk premium affects financial wealth accumulation, it affects the evolution of the human-financial wealth ratio.

The second part of Proposition 2 characterizes the average financial wealth of entrepreneurs. For a given level of the capital stock, this captures the wealth distribution between the two types of households, as \(\frac{X_{fs}}{qK}\) equals the share of financial wealth held by entrepreneurs and \(1 - \frac{X_{fs}}{qK}\) the share of financial wealth held by wage earners. As \(n_e\) is an average of the financial wealth conditional on age \(n(a)\), the same effects that shape the distribution of wealth across age groups pin down the overall level of wealth held by entrepreneurs.

Figure 4 shows how financial wealth varies across age groups. Even though this was not part of the calibration targets, the model is able to capture the inverted-U pattern of financial wealth. At the beginning of the life cycle, both the human-financial wealth effect and the \("r - g"\) effect dominate the average MPC effect, so the wealth share initially increases with age. The average MPC effect dominates later in the life cycle, bringing down the wealth share.
4.2. **Within-group inequality**

We have considered so far the behavior of the aggregate wealth of entrepreneurs by age. However, even after conditioning on a given age group, wealth may vary substantially because of the presence of idiosyncratic risk. We turn next to the characterization of the wealth distribution conditional on age.

Let \( \mu_{n,t}(n,a) \) denote the expected change and \( \sigma_{n,t}(n,a) \) the instantaneous volatility of financial wealth for an entrepreneur with financial wealth \( n \) and age \( a \). The evolution of the distribution of wealth conditional on age can be characterized by the Kolmogorov Forward Equation, as shown in Lemma 2.

**Lemma 2** (Kolmogorov Forward Equation). The conditional distribution of financial wealth \( f_t(n|a) \) satisfies the partial differential equation

\[
\frac{\partial f_t(n|a)}{\partial t} + \frac{\partial f_t(n|a)}{\partial a} = -\frac{\partial [f_t(n|a)\mu_{n,t}(n,a)]}{\partial n} + \frac{1}{2} \frac{\partial^2 [f_t(n|a)\sigma^2_{n,t}(n,a)]}{\partial n^2},
\]

and the boundary condition \( f_t(e^{-\gamma T}n|0) = f_t(n|T) \), given an initial condition \( f_0(n|a) \).

**Proof.** See Appendix A.3.
Despite the complexity created by the dependency on age of the expected change and volatility of wealth, we are able to solve for the conditional distribution of financial wealth in closed-form for the special case where entrepreneurs leave no bequests, that is, $\psi = 1$. This allows us to characterize analytically the evolution of inequality over the life cycle.

**Proposition 3** (Within-group inequality: no bequests). Suppose $\psi = 1$ and $r + \frac{(\mu^d)^2}{\gamma} + \frac{(\mu^d)^2}{\gamma} > \mu_A$.

i. **Shifted log-normal distribution**: The distribution of financial wealth conditional on age is given by a shifted log-normal distribution with support $(-h(a), \infty)$, that is, the distribution of total wealth $n + h(a)$ is log-normal.

ii. **Mean and variance by age**: The expected value and variance of $n$ conditional on age are given by

\[
\mathbb{E}[n|a] = h(0) e^{(r + \frac{(\mu^d)^2}{\gamma} + \frac{(\mu^d)^2}{\gamma} - \mu_A) a} e^{-ra} - e^{-rT} \left[ 1 - e^{-rT} \right] - h(a) \tag{27}
\]

\[
\mathbb{V}[n|a] = e^{\left( \frac{(\mu^d)^2}{\gamma} a \right)^2} \left[ h(0) e^{(r + \frac{(\mu^d)^2}{\gamma} + \frac{(\mu^d)^2}{\gamma} - \mu_A) a} e^{-ra} - e^{-rT} \left[ 1 - e^{-rT} \right] \right]^2. \tag{28}
\]

iii. **Inverted-U shape of inequality over the life cycle**: there exists $0 < \hat{a} < T$ such that $\mathbb{V}[n|a]$ is increasing in $a$ for $a < \hat{a}$ and decreasing for $a > \hat{a}$.

**Proof.** See Appendix A.4. \hfill \square

Proposition 3 gives a complete characterization of the distribution of wealth conditional on age. Wealth has a shifted log-normal distribution, with an age-dependent shifter $-h(a)$. Since entrepreneurs are allowed to borrow, financial wealth clearly cannot be log-normally distributed, as $n$ can take on negative values. However, financial wealth cannot go below the natural borrowing limit $-h(a)$, so total wealth assumes only positive values. Total wealth follows a log-normal distribution with mean and variance dependent on age.

The expression for the conditional mean $\mathbb{E}[n|a] = n(a)$ is essentially the same as the one in Equation (23), after rearranging and specializing to the case $\psi = 1$. As we have seen, average wealth tends to increase with age at the beginning of life, and it goes down by the end of the life cycle. This is the result of the increase in MPC balancing out the effect of wealth being accumulated over time.

32
Expression (28) shows how the variance of wealth evolves over the life cycle. In the case with no bequests, the variance is zero at ages $a = 0$ and $a = T$. Since entrepreneurs leave no bequests, everyone starts with zero financial wealth. The MPC becomes arbitrarily large by the end of life, so the flow of consumption exhausts the whole stock of wealth. The dispersion of wealth increases at the beginning of the life cycle, as some entrepreneurs receive a series of positive shocks, while others suffer a sequence of negative shocks. This force is magnified with the exposure to idiosyncratic risk $p_{id} / \gamma$, but also with the magnitude of portfolio returns net of the growth rate, a version of the "$r - g$" effect. The increase in MPC provides a countervailing force, as the impact of the proportional shocks is reduced as the level of wealth is brought down at the end of the life cycle.

The results in Proposition 3 require the assumption of no bequests. In the general case where $\psi > 0$, we must resort to numerical methods. In Appendix B.2, we show how to adapt the methods in Achdou et al. (2017) to deal with the case of an economy with bequests, where the distribution at age 0 depends on the distribution at age $T$. Figure 5 shows the stationary distribution of financial wealth for selected ages. The mean and the dispersion of the distribution initially increases with age, then eventually both start to decline as entrepreneurs get older.

The same inverted-U pattern on wealth inequality can be found in the data. Figure 6 shows the evolution of within-group inequality over the life cycle in our data. To make the units easier to interpret, we divide financial wealth by the average wealth for all entrepreneurs. The figure shows

\footnote{Note that $\sigma(n, a) = \frac{p_{id}}{\gamma}(n + h(a))$, so the term $p_{id} / \gamma$ measures the exposure of entrepreneur’s wealth to idiosyncratic risk.}

33
that the standard deviation of $n/n_e$ increases sharply until roughly age 60 and then declines until the end of the life cycle. Quantitatively, the model generates a substantial increase in inequality over the life cycle, over 50% from early in life until the peak, but still below the amount of inequality observed in the data. Taking into account additional sources of heterogeneity, as differences in expected returns or volatilities, may help to bring the quantitative predictions closer to the one in the data.

5. Aggregate implications of entrepreneurial risk

In this section, we study the impact of entrepreneurial risk on the long-run level of the aggregate capital stock. We find that the degree of financial development, captured by the magnitude of the idiosyncratic risk premium, is tightly linked to the level of economic development of the economy.

5.1. Equilibrium characterization

We present next the determination of aggregate variables in a stationary equilibrium. We start by solving for the price of aggregate insurance, interest rate, and the relative price of capital, which are determined by standard conditions that do not depend on the degree of idiosyncratic insurance. We then consider the determination of the aggregate capital stock and the idiosyncratic
risk premium. The detailed derivations are provided in Appendix B.3.

5.1.1. Aggregate risk premium, interest rate, and the relative price of capital

Equating supply and demand for aggregate insurance, we obtain the price of aggregate insurance,

\[ p^\rho_s = \gamma \sigma_A, \]  

(29)

where the expression above holds even outside of a stationary equilibrium.

The interest rate is given by the condition

\[ r = \rho_w + \gamma \mu_A - \frac{\gamma (\gamma + 1)}{2} \sigma_A^2. \]  

(30)

The three terms in Equation (30) capture, respectively, the impact of impatience, intertemporal substitution, and precautionary savings. Note that the relevant impatience parameter is the one for the infinite-horizon agent.

In a stationary equilibrium, the capital-labor ratio is constant, so capital grows at the constant rate \( g \). Given the investment function \( \Phi(\iota) = \sqrt{\Phi_0^2 + 2 \Phi_1 (\iota - \Phi_0)} \), the required investment rate to achieve a growth rate of capital \( g \) is given by \( \iota = \Phi_0 (g + \delta) + 0.5 \Phi_1 (g + \delta)^2 \). Using the expression for optimal investment condition, Equation (12), we obtain the relative price of capital:

\[ q = \Phi_0 + \Phi_1 (g + \delta). \]  

(31)

Note that \( p^\rho_s, r, \) and \( q \) are independent of the moral hazard parameter \( \phi \), so they coincide with the corresponding value at the complete markets economy.\(^{33}\)

---

\(^{33}\)The result that the price of aggregate risk does not depend on the degree of market incompleteness is reminiscent of the findings of Krueger and Lustig (2010), who identified conditions under which uninsurable labor income risk does not affect the premium for aggregate risk.
5.1.2. Aggregate capital stock and idiosyncratic risk premium

The aggregate capital stock and the idiosyncratic risk premium are jointly determined. From Equation (19), we obtain an expression for the expected return on the business

\[ r + p^a \sigma_A + p^{id} \Phi \sigma_{id} = \frac{\alpha K^{a-1} - \ell(q)}{q} + \delta + \mu_A. \]

(32)

The left-hand side captures the required rate of return of investing in the business, which includes a premium for holding aggregate and idiosyncratic risk. The right-hand side gives the actual expected return of investing in the business, a function of the marginal product of capital (MPK) net of adjustment costs. Notice that Equation (32) generalizes the standard textbook relation between MPK and interest rates to an environment with growth, risk, and adjustment costs. In the absence of these three elements, the expression above boils down to \( r = \alpha K^{a-1} - \delta \).

Expression (32) gives an inverse relation between the idiosyncratic risk premium \( p^{id} \Phi \sigma_{id} \) and the capital-labor ratio \( K \). This downward-sloping relationship is represented by the solid blue line in Figure 7, which we refer to as the MPK schedule.

We need another condition relating \( K \) and \( p^{id} \). Aggregating the demand for capital (20) across all entrepreneurs, we obtain

\[ p^{id} = \gamma \Phi \sigma_{id} \frac{qK}{\chi_c(n_c + h_c)}, \]

(33)

where \( n_c \) and \( h_c \) denote the average financial and human wealth of entrepreneurs.

In the same way the price of aggregate risk depends on the product of risk aversion and risk, \( \gamma \sigma_A \), the price of idiosyncratic risk also depends on the product of the risk aversion \( \gamma \) and the idiosyncratic risk (net of insurance) \( \Phi \sigma_{id} \). However, the price of idiosyncratic risk depends on an additional term: the idiosyncratic risk exposure, that is, the ratio of physical assets to total wealth of entrepreneurs. This term captures the fact that entrepreneurs require a larger idiosyncratic risk premium when they have a larger fraction of their wealth invested in the business. From Equation (25), we obtain \( n_c + h_c \) as a function of the capital stock and the price of idiosyncratic risk. After substituting \( n_c + h_c \) into Equation (33), we obtain an implicit relationship between \( p^{id} \)
Figure 7: Idiosyncratic risk premium and capital stock

Note: the upward-sloping solid curve shows the pricing schedule in the stationary equilibrium for $\phi_0 = 0.73$ (initial steady state) and the dashed curve shows the pricing schedule in the stationary equilibrium for $\phi_0 = 0.365$ (new steady state). The idiosyncratic risk premium is expressed in basis points.

and $K$. Figure 7 plots this relationship as the solid upward-sloping curve, which we refer to as the pricing schedule.

The idiosyncratic risk premium and the capital stock in the economy are determined by the intersection of the MPK and pricing schedules. Hence, the degree of financial development in the economy and the amount of idiosyncratic risk are important determinants of the level of economic development.

5.2. The price of aggregate and idiosyncratic risk in the data

The model replicates the level of risk premium and volatility on aggregate and idiosyncratic risk observed in the data, as given in Table 1. A striking fact is that, despite idiosyncratic volatility being three times larger than aggregate volatility, the idiosyncratic risk premium is only slightly larger than the aggregate risk premium. This leads to a Sharpe ratio three times larger for aggregate risk. Equations (29) and (33) help to shed light on this pattern.

The Sharpe ratio of aggregate risk corresponds to $p^{ag}$ and it is given by $\gamma \sigma_A$. The Sharpe ratio
of idiosyncratic risk is given by $p^{id}\phi$, as the volatility reported here does not take into account any insurance taken by the entrepreneur. If we were to naively price the idiosyncratic risk by analogy with the aggregate risk, the Sharpe ratio would be $\gamma\sigma_{id}$, that is, more than three times larger than the one for aggregate risk. Two factors explain why the price of idiosyncratic risk is actually three times smaller than the one for aggregate risk: idiosyncratic insurance and the risk exposure. The pricing equation for idiosyncratic risk shows the role of these two components:

$$p^{id}\phi = \gamma\sigma_{id}\phi^2 \frac{qK/\chi e}{n_e(1 + h_e/n_e)}.$$

The parameter of the skin-in-the-game constraint is given by $\phi = 0.73$. This term by itself reduces the price of idiosyncratic risk by 45%. However, the bulk of the adjustment comes from the risk exposure factor, as $qK/\chi e(n_e + h_e) \approx 0.15$. This is the result of a share invested in the business of around 27% and human-financial wealth ratio of around 0.7 on average. Intuitively, the reason for a much smaller price of idiosyncratic risk is that the entrepreneurs are proportionally less exposed to idiosyncratic risk, either because of insurance mechanisms, or because only a fraction of total wealth is exposed to this risk. Notice the importance of explicitly introducing human wealth and heterogeneous agents into the model for the pricing of idiosyncratic risk. In an environment without human wealth and without wage earners, the risk exposure factor would be necessarily equal to one, thus ignoring an important determinant of the idiosyncratic risk premium.

5.3. Long-run effects of relaxing risk constraints

So far we have shown that the idiosyncratic risk premium is an important determinant of entrepreneurial returns with consequences for risk-taking, savings behavior, and wealth accumulation. We now turn to the aggregate implications of relaxing the risk constraints that give rise to this risk premium.

The parameter $\phi$ measures the strength of contractual frictions in the economy. High values of $\phi$ are meant to capture situations where the access to insurance arrangements, formal or informal, is rather limited. In this case, entrepreneurs are forced to hold most of the idiosyncratic risk of their businesses, potentially limiting their choice of scale. As the institutional arrangements improve, in particular mechanisms to monitor entrepreneurs’ activities, such frictions are expected to be
Figure 8: Financial development and inequality in the long-run

(a) Between-group inequality  
(b) Within-group inequality

Note: Inequality in a stationary equilibrium for \( \phi_0 = 0.73 \) (initial steady state) and \( \phi_1 = 0.365 \) (new steady state). Panel (a) shows average financial wealth by age. Panel (b) shows the standard deviation of financial wealth by age. All variables are normalized by entrepreneurs’ average wealth in the initial steady state.

reduced and entrepreneurs would hold a smaller fraction of the risk.\(^{34}\) Hence, we see the process of financial development as leading to a reduction in the skin-in-the-game parameter \( \phi \).

Figure 7 shows the impact of an intervention or reform that reduces the parameter \( \phi \). The MPK schedule is unchanged with a reduction in \( \phi \), but the pricing scheduled is shifted down. The reason is that entrepreneurs are now able to share a larger share of the risk of the project, so they demand a smaller compensation to hold this risk. In the long-run, the price of capital does not change, so this reduction in the required return to invest in the business is matched by a higher capital stock and smaller marginal product of capital. Hence, financial development and economic development go together in this economy. Reducing the skin-in-the-game parameter in half, from \( \phi_0 = 0.73 \) to \( \phi = 0.365 \), the idiosyncratic risk premium falls by about 150 basis point, leading to an increase of roughly 15% in the capital stock. Therefore, relaxing risk constraints have a quantitatively large impact on the aggregate capital stock.

Financial development also has important implications for wealth accumulation and inequality. The left panel on Figure 8 shows the average financial wealth for each age group relative to the average across all entrepreneurs on the initial stationary equilibrium. Financial wealth falls for

\(^{34}\)To focus on entrepreneurs, we assume that insurance providers can perfectly diversify idiosyncratic risk. If insurance providers also hold under-diversified portfolios, as in Gärkeanu et al. (2015), a reduction in the risk premium can also be caused by an improvement in their ability to diversify, see e.g. Khorrami (2019).
all age groups in the new stationary equilibrium. From Equation (23), one can see that a reduction 
in the price of idiosyncratic insurance tends to reduce bequests and wealth accumulation over the 
life cycle. In the long-run, inequality falls in response to the reduction in \( \phi \), as indicated by the 
right panel on Figure 8. Equation (28) provides some intuition on the effects of \( p^{id} \) on wealth in-
equality. Both the reduction in the amount of risk entrepreneurs are exposed as well as a reduction 
on the level of financial wealth contributes to the reduction in the standard deviation of wealth.

6. Dynamic effects of relaxing risk constraints

The results above are based on a comparison of two stationary equilibria (or steady states). Therefore, they ignore what happens during the transitional dynamics. To compute the welfare implications of relaxing risk constraints, it is important to explicitly take into account what happens 
during the transition. Moreover, some of the effects may take a long time to materialize. For this 
reason, we turn next to the analysis of the response of the economy over time to changes in \( \phi \).

6.1. Computing the transitional dynamics

We consider a small open economy version of the model, where the interest rate is kept at the 
level of the original stationary equilibrium.\(^{35}\) This allow us to focus on the dynamic implications 
of fluctuations in the idiosyncratic risk premium on the capital stock. We still have to solve for 
the path of the capital stock and its relative price, \( K \) and \( q \), the consumption-wealth ratio of ent-
trepreneurs by age, \( \xi_t(a) \), and the level of total and human wealth, \( \omega_t(a) \) and \( h_t(a) \). The next 
proposition provides the conditions characterizing the evolution of these variables.

**Proposition 4.** The evolution of \((q_t, K_t, \{\xi_t(a), h_t(a), \omega_t(a)\})\) is characterized by a pair of ordinary dif-
ferential equations (ODEs)

\[
\begin{align*}
\dot{K}_t &= [\Phi(i(q_t)) - \delta - g]K_t \\
\dot{q}_t &= \left[ r + \gamma \sigma^2_A + \gamma \Phi' \sigma^2_A \frac{q_t K_t}{\chi \omega_{c,t}} + \delta - \mu_A - \Phi(i(q_t)) - \frac{\alpha K_t^{\alpha-1}}{q_t} \right] q_t
\end{align*}
\]

\(^{35}\)Alternatively, one could assume that wage earners have Epstein-Zin utility with linear intertemporal preferences.
and three partial differential equations (PDEs)
\[
\frac{\partial \zeta_t(a)}{\partial t} = -\frac{\partial \zeta_t(a)}{\partial a} + \zeta_t^2(a) - \bar{r}_t \zeta_t(a)
\]
\[
\frac{\partial h_t(a)}{\partial t} = -\frac{\partial h_t(a)}{\partial a} + (r + \gamma \sigma_A^2 - \mu_A) h_t(a) - (1 - \alpha) K_t \tilde{h}_t(a)
\]
\[
\frac{\partial \omega_t(a)}{\partial t} = -\frac{\partial \omega_t(a)}{\partial a} + \left[ r + \gamma \sigma_A^2 - \mu_A + \gamma \Phi^2 \sigma_{id}^2 \left( \frac{q_t K_t}{\chi e \omega_{c,t}} \right)^2 - \zeta_t(a) \right] \omega_t(a),
\]
subject to the boundary conditions described in the appendix.

Proof. See Appendix A.5.

In the expressions above, we have eliminated the price of aggregate insurance using \( p_t^{\text{ag}} = \gamma \sigma_A \) and the price of idiosyncratic insurance using \( p_t^{id} = \gamma \phi \sigma_{id} \frac{q_t K_t}{\chi e \omega_{c,t}} \), where \( \omega_{c,t} = n_{c,t} + h_{c,t} \). We obtain the first ODE by aggregating (3) and the second one by rearranging the time-dependent version of (32). The first PDE comes from the HJB equation for entrepreneurs, the second one corresponds to the time-dependent version of (14), and the last PDE can be obtained by averaging the budget constraint of entrepreneurs.

The transitional dynamics for heterogeneous agents models are often computed using a shooting algorithm, as in e.g. Guerrieri and Lorenzoni (2017) or Achdou et al. (2017). Given the large number of forward-looking variables, such algorithm is impractical in our setting. We propose an alternative form of solving for the transitional dynamics, by combining perturbation and finite-difference methods. First, we use finite differences to discretize the system of ODE/PDEs. We end up with a (large) non-linear boundary value problem, since \((K_t, \omega_t(a))\) have initial conditions, while \((q_t, \zeta_t(a), h_t(a))\) have terminal conditions. We then linearize the system around the new stationary equilibrium. Notice that since we do not assume that aggregate or idiosyncratic shocks are small, in contrast to the approach in e.g. Ahn et al. (2018) that linearizes around the economy without aggregate shocks, we are able to capture time-varying risk premia and precautionary savings effects using this method.\(^{36}\) The final step consists of solving the resulting linear rational expectation model, which can be done by standard techniques, as the one proposed by Blanchard and Kahn (1980). The method is able to solve the model efficiently with total computation time in

\(^{36}\) For the use of perturbation methods to solve heterogeneous agents models with aggregate risk, see also Reiter (2009) and Winberry (2018).
Figure 9: Transitional dynamics: aggregate variables

(a) Capital Stock  
(b) Price of Capital  
(c) Entrepreneurs’ Financial Wealth  
(d) Idiosyncratic Risk Premium

Note: Transitional dynamics from a stationary equilibrium with \( \phi_0 = 0.73 \) to a stationary equilibrium with \( \phi_1 = 0.365 \). Capital stock, relative price of capital, and entrepreneurs’ financial wealth are expressed as percentage deviations from the initial steady state. The idiosyncratic risk premium is expressed as absolute deviation from the initial steady state in basis points.

the order of seconds, given our calibration and grid size. We discuss the method in more detail in Appendix C.

6.2. Short-run dynamics: the overshooting effect

Figure 9 shows the evolution of the capital stock, the relative price of capital, entrepreneurs’ financial wealth, and the idiosyncratic risk premium, all as deviations from the initial steady state. We consider again a reform that reduces the moral hazard parameter by half, from \( \phi_0 = 0.73 \) under our calibration to \( \phi_1 = 0.365 \). In response to a relaxation of risk constraints, we observe an investment boom and a sharp increase in the value of businesses that last for roughly a decade, as shown in Panels (a) and (b) of Figure 9. Importantly, the short-run response of the price of
capital exceeds by a large margin its long-run level, that is, there is an *overshooting* effect. The intuition for this result is the following. As entrepreneurs must hold less of the risk of the project, they require a smaller premium to invest in the business. In the long-run, this reduction in returns is obtained by a larger capital stock and smaller marginal product of capital. However, in the short-run the capital stock is fixed, so the only way returns can go down is through expected capital losses. Since the price of capital must be expected to go down and its long-run level is unaffected by the financial friction, then it must jump up on impact. This increase in marginal $q$ induces entrepreneurs to invest more, generating the investment boom. This logic is reminiscent of Dornbusch’s (1976) overshooting model, where exchange rates react more strongly to shocks in the short run to create expected capital losses to domestic investors. This is analogous to how the price of capital overshoots its long-run level in our model to create expected capital losses to entrepreneurs.\(^{37}\) The overshooting effect has important implications for wealth accumulation and inequality.

As discussed in Subsection 5.3, the financial wealth of entrepreneurs goes down in the long run. In contrast, their financial wealth actually increases in the short run. The reason for the contrast between the short-run and long-run responses is a *revaluation* effect. Since entrepreneurs own the capital stock in this economy, as the relative price of capital goes up in the short-run, their wealth also jumps on impact. However, after this initial increase in wealth, they start to accumulate wealth at a slower pace, since expected returns on the business go down with the reduction in the idiosyncratic risk premium. The reduction in the speed at which they accumulate wealth translate in a smaller level of wealth in the long run. This long-run effect takes a long time to materialize though, as shown in Panel (c) of Figure 9, given it is in part transmitted to the future generations through lower bequests. Even thirty years after the shock, the reduction in entrepreneurs’ financial wealth is only 30% of the long-run effect.

The slow dynamics of wealth affects the behavior of the idiosyncratic risk premium. The short-run response of the idiosyncratic risk premium exceeds its long-run level by more than ten basis points, as shown in Panel (d) of Figure 9. After a decade, the difference of the risk premium to the new steady state level is cut by more than half, and then the risk premium increases slowly as

\(^{37}\) Note that the presence of adjustment costs is crucial for the overshooting result. In the absence of adjustment costs, capital jumps to the new steady-state level and there are no movement in prices. Similarly, there is no overshooting in Dornbusch’s model when the price level can immediately jump to its steady-state level.
entrepreneurs’ wealth declines, in line with the risk exposure effect discussed in Subsection 5.2.

6.3. Kuznets dynamics

Relaxing the risk constraints has important implications for the dynamics of inequality. Figure 10 shows the evolution of between- and within-group wealth inequality. The left panel shows the evolution of average financial wealth for each age group in different points in time, normalized by average wealth across all entrepreneurs in the initial equilibrium. Because of the revaluation effect, financial wealth goes up on impact for all age groups, but the effect is stronger for younger entrepreneurs, as they are proportionally more exposed to the business. Over time the financial wealth goes down due to the reduction in expected returns.

The right panel on Figure 10 shows the standard deviation of financial wealth for different age groups. Again the short-run response is distinct than the long-run response. While wealth inequality goes down in the long run, wealth inequality goes up in the short run. The reason for this initial increase in inequality is that the revaluation effect does not affect all entrepreneurs in the same way. The ones with initially more wealth also hold initially more capital. As the value of capital increases, wealthier entrepreneurs benefit the most from the reform, increasing the dispersion in financial wealth in the economy.
The evolution of inequality interacts in interesting ways with the demographic structure. After the initial increase in inequality, wealth dispersion goes down for all age groups over time, but the cohort starting their professional life just after the reform are the ones mostly affected. For instance, ten years after the reform, the drop in inequality is three times larger for 35-year old entrepreneurs, who lived their entire professional life under the new regime, relative to 80-year old entrepreneurs, who lived most of their life under the old regime. Similarly, the drop in inequality is more pronounced to 35-year old entrepreneurs than to 25-year old entrepreneurs, who just inherited their wealth from old entrepreneurs. A similar argument explains why, twenty years after the reform, 45-year old entrepreneurs are the ones with the largest drop in wealth dispersion.

Hence, a reduction in the amount of risk that entrepreneurs must hold and the consequent compression of the idiosyncratic risk premium leads to an investment boom and an increase in wealth inequality. As the economy approaches its new level of output, inequality starts to recede and it reaches a lower level in the long-run. The initial increase in inequality as the economy enters in a high growth phase and eventual reduction in inequality as the economy reaches a higher level of development is consistent with the idea of a Kuznets’s (1955) curve.38

6.4. Welfare implications

We turn next to the welfare implications of risk constraints. Remember that the value function of an entrepreneur of age \( a \) at time \( t \) is given by

\[
V_t(n, a) = \zeta_t^{-\frac{1}{\gamma}}(a) \frac{(n + h_t(a))^{1-\gamma}}{1 - \gamma}.
\]

Hence, the welfare of an entrepreneur depends on financial wealth \( n \), human wealth \( h_t(a) \), and the consumption-wealth ratio \( \zeta_t(a) \), which captures the path of expected future returns. We evaluate financial wealth at the average level of the age group, \( n = n_t(a) \), but one can use the inequality results previously discussed to infer the dispersion in welfare within age groups. Finally, we take a monotonic transformation of the value function to measure welfare in units of consumption.

---

38Moll (2012) found a related Kuznets curve by showing that the steady-state level of the top wealth share is a hump-shaped function of financial development. In contrast, we show that inequality initially increases in response to a shock that reduces inequality in the long run, so we focus on the transitional dynamics instead of long-run comparisons.
Hence, our measure of welfare will be

$$W_t(a) = \log \left[ u^{-1}(V_t(n_t(a), a)) \right] - \log \left[ u^{-1}(V^*(n^*(a), a)) \right]$$

$$= \frac{1}{\gamma(\gamma - 1)} \hat{\xi}_t(a) + \hat{\omega}_t(a),$$

where $u(c) = \frac{\gamma^{1-\gamma}}{1-\gamma}$, and a hat denotes log deviations from the initial steady state.

Figure 11 shows the welfare gains for each age groups in different points in time. The generation that is alive at the moment of the reform benefits the most from the reform, with gains concentrated on younger entrepreneurs. This is the result of the revaluation effect, which raises not only financial wealth but also human wealth. However, the negative impact on wealth accumulation affects future generations of entrepreneurs. As they receive smaller bequests, and it becomes harder to accumulate wealth, their welfare is adversely impacted. Figure 11 shows how demographics affect the welfare gains. For instance, ten years after the intervention, the welfare gains for the entrepreneurs who started their professional life after the reform, the ones with age 25 to 35 years old, have welfare gains that are smaller than the entrepreneurs of the same age at the time of the intervention. Thirty years after the reform, entrepreneurs with ages between 35 and 55...
are worse off in comparison to an equilibrium without the reform, with larger welfare losses for
the older entrepreneurs. Therefore, the initial generation of entrepreneurs reap most of the ben-
efits of the reform, illustrating the rich interactions of the financial friction and the demographic
structure.

7. Conclusion

In this paper, we provide a framework to analyze the aggregate and distributive implications
of entrepreneurial risk. We propose a life-cycle model of entrepreneurship with aggregate and
idiosyncratic risk under limited insurance. We show that entrepreneurial returns command a
positive idiosyncratic risk premium, which accounts for a large fraction of total returns. The model
captures quantitatively the empirical patterns of risk-taking and savings over the life cycle, the
inverted-U shape of wealth inequality, and the level of aggregate and idiosyncratic risk premia.

We also study the impact of relaxing risk constraints. We show that an improvement in id-
osyncratic insurance increases output and reduces inequality in the long-run and generates rich
transitional dynamics. The price of capital overshoots in the short run, generating a large in-
vestment boom, and an increase in the value of the business. This overshooting leads to an initial
increase in inequality. As the reduction in risk and expected returns have time to play out, inequal-
ity goes down in the long run with important intergenerational effects. Finally, we find that most
of the welfare gains are concentrated in the generations that are alive at the time of the change and
that future generations of entrepreneurs are actually worse-off.

An important direction of future research is the impact of risk constraints on misallocation
and aggregate productivity. Given imperfect insurance leads to different incentives to self-insure
than borrowing constraints, the aggregate consequences of financial frictions are potentially very
different than the more conventional borrowing constraints explored in the literature.
References


“Micro uncertainty and asset prices,” Available at SSRN 3220825, 2018.


Appendix A.  Proofs

A.1. Proofs of Lemma 1 and Proposition 1

Proof. We start by showing part (b) of Lemma 1, that is, we solve for \( h_t(a) \) and its dynamics. Then, we proceed to solve for the entrepreneurs’ value function and policy functions, deriving items (a) and (c) of Lemma 1 as well as the results in Proposition 1.

**Pricing human wealth.** Define the stochastic discount factor (SDF) for this economy as the process \( \pi_t \) satisfying the law of motion

\[
\frac{d\pi_t}{\pi_t} = -r_t dt - p_t^{\sigma} dZ_t.
\]

(A.1)

Without loss of generality, we assumed that the SDF is not exposed to idiosyncratic risk, as we only use the SDF to price human wealth which is not exposed to idiosyncratic risk. Integrating the process above, we obtain

\[
\frac{\pi_z}{\pi_t} = \exp \left( - \int_t^z \left( r_u + \frac{(p_u^{\sigma})^2}{2} \right) du - \int_t^z p_u^{\sigma} dZ_u \right).
\]

(A.2)

Similarly, integrating the process for \( A_t \)

\[
\frac{A_z}{A_t} = \exp \left( \int_t^z \left( \mu_A - \frac{\sigma_A^2}{2} \right) du + \int_t^z \sigma_A dZ_u \right).
\]

(A.3)

Hence, we can explicitly compute the following expectation

\[
\mathbb{E}_t \left[ \frac{\pi_z A_z}{\pi_t A_t} \right] = \mathbb{E}_t \left[ \exp \left( - \int_t^z \left( r_u - \mu_A + \frac{(p_u^{\sigma})^2 + \sigma_A^2}{2} \right) du - \int_t^z (p_u^{\sigma} - \sigma_A) dZ_u \right) \right]
\]

\[
= \exp \left( - \int_t^z \left( r_u - \mu_A + \frac{(p_u^{\sigma})^2 + \sigma_A^2}{2} \right) du + \frac{1}{2} \int_t^z (p_u^{\sigma} - \sigma_A)^2 du \right)
\]

\[
= \exp \left( - \int_t^z \left( r_u + p_u^{\sigma} \sigma_A - \mu_A \right) du \right),
\]

(A.4)

where we used Ito’s isometry in the second equality and the fact that \( p_t^{\sigma} \) is deterministic.
Human wealth is given by

$$h_t(a) = \mathbb{E}_t \left[ \int_t^{t+T-a} \pi_z \tilde{A}_t w_z \tilde{I}(a+z-t) dz \right]$$

$$= \int_t^{t+T-a} e^{-\int_t^z (r_u+p_u^{\nu} \sigma_A - \mu_A) du} w_z \tilde{I}(a+z-t) dz. \quad (A.5)$$

Consider the human wealth for someone born at date $s$, so $a = t-s$:

$$h_t(t-s) = \int_t^{s+T} e^{-\int_t^z (r_u+p_u^{\nu} \sigma_A - \mu_A) du} w_z \tilde{I}(z-s) dz. \quad (A.6)$$

Differentiating the expression above with respect to time yields

$$\frac{\partial h_t(a)}{\partial t} + \frac{\partial h_t(a)}{\partial a} = (r_t + p_t^{\nu} \sigma_A - \mu_A) h_t(a) - w_t \tilde{I}(a), \quad (A.7)$$

which gives (14) in a stationary equilibrium.

**The HJB equation.** The HJB equation for problem (9) is given by

$$\rho \tilde{V}_t(\tilde{n}, t-s; A_t) = \max_{\check{c}, \tilde{c_t}, \tilde{b_t}, \tilde{b_t}} \left\{ \frac{\check{c}^{\nu-\gamma}}{1-\gamma} + \frac{\mathbb{E}_t \left[ \tilde{d} \tilde{V}_s \right] }{\partial t} \right\}, \quad (A.8)$$

subject to (7) as well as the terminal and boundary conditions

$$\tilde{V}_t(\tilde{n}, T) = (1-\psi)^{\gamma} V^+ \tilde{n}^{1-\gamma}; \quad \lim_{\tilde{n} \to -h_t(a)} \tilde{V}_t(\tilde{n}, a) = \begin{cases} 0, & \text{if } \gamma < 1 \\ -\infty, & \text{if } \gamma \geq 1 \end{cases}, \quad (A.9)$$

where the terminal condition captures the effect of bequests and the boundary condition captures the fact that consumption is zero if the entrepreneur hits the natural borrowing limit.

Using Ito’s lemma, the HJB reduces to a partial differential equation for $\tilde{V}_t(\tilde{n}, a; A_t)$:

$$\rho \tilde{V}_t = \max_{\check{c}, \tilde{c_t}, \tilde{b_t}} \left\{ \frac{\check{c}^{\nu-\gamma}}{1-\gamma} + \frac{\partial \tilde{V}_t}{\partial t} + \frac{\partial \tilde{V}_t}{\partial \tilde{n}} \mu_{\check{n}, A} + \frac{\partial \tilde{V}_t}{\partial A_t} + \frac{\partial^2 \tilde{V}_t}{\partial \tilde{n}^2} (\tilde{c}_{\tilde{n}, \tilde{t}} + \tilde{c}_{\tilde{n}, \tilde{t}}^2) + \frac{\partial^2 \tilde{V}_t}{\partial \tilde{n} \partial A_t} \tilde{c}_{\tilde{n}, \tilde{t}} \sigma_A A_t + \frac{1}{2} \frac{\partial^2 \tilde{V}_t}{\partial A_t^2} \sigma_A^2 A_t^2 \right\}, \quad (A.10)$$

where $(\mu_{\check{n}, t}, \sigma_{\tilde{c}, t}, \sigma_{\tilde{d}, t})$ are the drift and diffusion terms for $\tilde{n}_t$, and the maximization is subject to (7).
First, we verify that the following guess for the value function solves the PDE

\[ \tilde{V}_t(\tilde{n}, a; A_t) = \frac{\tilde{\zeta}_t(a)^{-\gamma} (\tilde{n} + A_t h_t(a))^{1-\gamma}}{1-\gamma}. \]  

(A.11)

Plugging the derivatives of the equation above into the HJB equation, we obtain

\[
\frac{\rho}{1-\gamma} = \max_{c_{i,t}, k_{i,t} \in \mathbb{R}, \pi_{i,t}^{ag}, \pi_{i,t}^{id}} \left\{ \frac{\zeta_t(a)}{1-\gamma} \left( \frac{c_{i,t}}{\omega_{i,t}} \right)^{1-\gamma} - \frac{\gamma}{1-\gamma} \frac{1}{\tilde{\zeta}_t(a)} \left( \frac{\partial \tilde{\zeta}_t(a)}{\partial t} + \frac{\partial \tilde{\zeta}_t(a)}{\partial a} \right) + t + \frac{q_i k_{i,t} + h_{i,t}}{\omega_{i,t}} (\mu_t^R - r_t) - p_t^{ag} \frac{p_t^{ag}}{\omega_{i,t}} + h_{i,t} \sigma_A p_t^{ag} - c_{i,t} \frac{\sigma_{id}}{\omega_{i,t}} - \frac{\gamma}{2} \left( \frac{q_i k_{i,t} + h_{i,t}}{\omega_{i,t}} - \frac{\sigma_{id}}{\omega_{i,t}} \right)^2 \right\},
\]

(A.12)

where \( \omega_{i,t} = n_{i,t} + h_{i}(a) \) and \( p_{i,t}^{id} \) denotes the Lagrange multiplier on the skin-in-the-game constraint.

From the expression above, it is immediate that the optimal value of \((l_{i,t}, t_{i,t})\) maximizes the expected return on the business. The first-order conditions for \((l_{i,t}, t_{i,t})\) are given in (11) and (12), respectively. The expected return on the business will be equalized, allowing us to write \( \mu_{i,t}^R = \mu_t^R \).

The optimal quantity of capital, aggregate insurance, and idiosyncratic insurance solve the problem

\[
\max_{k_{i,t}, \pi_{i,t}^{ag}, \pi_{i,t}^{id}} \left\{ q_i k_{i,t} (\mu_t^R - r) - p_t^{ag} \frac{p_t^{ag}}{\omega_{i,t}} + h_{i,t} \sigma_A p_t^{ag} - \frac{\gamma}{2} \left( \frac{q_i k_{i,t} + h_{i,t}}{\omega_{i,t}} - \frac{\sigma_{id}}{\omega_{i,t}} \right)^2 + \frac{q_i k_{i,t} + h_{i,t}}{\omega_{i,t}} (\mu_t^R - r_t) - \frac{q_i k_{i,t} + h_{i,t}}{\omega_{i,t}} (\mu_t^R - r_t) \right\},
\]

subject to (7).

Multiplying the expression above by \( \omega_{i,t} / n_{i,t} \) gives (18).

**Policy functions.** The first-order condition for \( \theta_{i,t}^{id} \) is given by

\[
\gamma \left[ \frac{q_i k_{i,t}}{\omega_{i,t}} - \frac{\theta_{i,t}^{id}}{\omega_{i,t}} \right] = p_{i,t}^{id}. \tag{A.14}
\]

The equation above implies that the skin-in-the-game constraint is always binding, so \( p_{i,t}^{id} > 0 \).
and \( \theta_{id}^t = (1 - \phi)q_{ikid} \sigma_{id} \). If this was not the case, i.e. \( p_{id}^t \neq 0 \), then we would have \( \theta_{id}^t = q_{ikid} \sigma_{id} \), which violates the skin-in-the-game constraint.

The first-order conditions for capital and aggregate insurance are given by

\[
\mu_i^R - r + p_{id}^t (1 - \phi) \sigma_{id} = \gamma \left[ \left( \frac{q_{ikid} + h_{itd} \sigma_A}{\omega_{itd}} - \frac{\theta_{id}^g}{\omega_{itd}} \right) \sigma_A + \left( \frac{q_{ikid} \sigma_{id} - \theta_{id}^g}{\omega_{itd}} \right) \sigma_{id} \right]
\]

\[
p_{id}^g = \gamma \left( \frac{q_{ikid} + h_{itd} \sigma_A}{\omega_{itd}} - \frac{\theta_{id}^g}{\omega_{itd}} \right).
\]  \( \text{(A.15)} \)

Combining the expressions above, we obtain

\[
p_{id}^t = \frac{H_i^R - r_t - p_{id}^g \sigma_A}{\phi \sigma_{id}},
\]  \( \text{(A.16)} \)

which coincides with expression (19) after we write \( p_{id}^t = p_{id}^t \).

The demand for capital can be written as

\[
\frac{q_{ikid} + h_{itd} \sigma_A}{\omega_{itd}} = \frac{p_{id}^t}{\gamma \phi \sigma_{id}}.
\]  \( \text{(A.17)} \)

Multiplying by \( \omega_{itd}/n_{itd} \), we obtain expression (20). Solving for \( \theta_{id}^g \), in the optimality condition for aggregate insurance we obtain (21).

The first-order condition for consumption gives

\[
\frac{c_{it}}{\omega_{itd}} = \zeta_t(a).
\]  \( \text{(A.18)} \)

Plugging the expressions above back into the HJB, we obtain a PDE for \( \zeta_t(a) \)

\[
\frac{\partial \zeta_t(a)}{\partial t} + \frac{\partial \zeta_t(a)}{\partial a} = \zeta_t^2(a) - \bar{r}_t \zeta_t(a),
\]  \( \text{(A.19)} \)

where

\[
\bar{r}_t \equiv \frac{1}{\gamma} \rho + \left( 1 - \frac{1}{\gamma} \right) \left[ r + \left( \frac{p_{id}^t}{\gamma} \right)^2 + \left( \frac{p_{id}^g}{\gamma} \right)^2 \right].
\]  \( \text{(A.20)} \)

Define \( z_{s,t} \equiv \zeta_t^{-1}(t - s) \) as the wealth-consumption ratio for an entrepreneur born at date \( s \).
Differentiating with respect to $t$, we obtain
\[
\dot{z}_{s,t} = -\frac{1}{\zeta_t^2(a)} \left[ \frac{\partial \zeta_t}{\partial t} + \frac{\partial \zeta_t}{\partial a} \right] = \bar{r}_t z_{s,t} - 1. \tag{A.21}
\]

Solving the above differential equation, we get
\[
z_{s,t} = \int_t^{s+T} e^{-\int_t^u \bar{r}_z dz} du + e^{-\int_t^{s+T} \bar{r}_t dz} z_{s,s+T}, \tag{A.22}
\]

or in terms of $\zeta_t(a)$, we have
\[
\zeta_t(a) = \frac{1}{\int_t^{t+T-a} e^{-\int_t^u \bar{r}_z dz} du + e^{-\int_t^{t+T-a} \bar{r}_z dz} (1 - \psi)(V^*)^{\frac{1}{\gamma}}}, \tag{A.23}
\]

where we used the boundary condition $\zeta_t^{-1}(T) = (1 - \psi)(V^*)^{\frac{1}{\gamma}}$.

Assuming $(V^*)^{\frac{1}{\gamma}} = \frac{1}{\bar{r}}$ and a stationary equilibrium, where $\bar{r}_t = \bar{r}$, we obtain
\[
\zeta_t(a) = \frac{\psi}{1 - \psi e^{-\bar{r}(T-a)}}, \tag{A.24}
\]

which coincides with (22).

Notice that the assumption $(V^*)^{\frac{1}{\gamma}} = \frac{1}{\bar{r}}$ guarantees that the consumption-wealth ratio for $\psi = 0$ is the same as in the infinite horizon economy.

\[\square\]

A.2. Proof of Proposition 2

Proof. We start by deriving the law of motion of financial wealth for an entrepreneur of a given age. Using the value of capital, $k_{i,t}$, aggregate and idiosyncratic insurance, $(\theta_{i,t}^{ag}, \theta_{i,t}^{id})$, and the definition of the price of idiosyncratic risk, $p_{i,t}^{id}$, given in Proposition 1, we can write the law of motion of financial wealth as follows
\[
dh_{i,t} = \left[ r_t \hat{w}_{i,t} + \frac{1}{\gamma} \hat{w}_{i,t} \right] + \left[ \frac{1}{\gamma} \hat{w}_{i,t} \right] + \left[ \frac{1}{\gamma} \hat{w}_{i,t} \right] - \hat{h}_{i,t} + \hat{h}_{i,t} - \hat{c}_{i,t} \right] dt + \left( \hat{w}_{i,t} \frac{p_{i,t}^{ag}}{\gamma} - \hat{h}_{i,t} \sigma^A \right) dZ_t + \frac{p_{i,t}^{id}}{\gamma} \hat{w}_{i,t} dZ_{i,t}, \tag{A.25}
\]

where $\hat{w}_{i,t} = \hat{w}_{i,t} + \hat{h}_{i,t}$.

Using the fact that $p_{i,t}^{ag} = \gamma \sigma^A$ in equilibrium, we find that the aggregate risk exposure of
entrepreneurs is given \( \bar{n}_{i,t} \sigma_A \). Hence, scaled financial wealth, \( n_{i,t} = \bar{n}_{i,t} / A_t \), does not respond to aggregate shocks. The evolution of \( n_{i,t} \) can then be written as

\[
dn_{i,t} = \mu_{n,t}(n_{i,t}, a)dt + \sigma_{n,t}(n_{i,t}, a)dZ_{i,t},
\]

where

\[
\mu_{n,t}(n, a) = \left[ r_t + \left( \frac{p_t^{id}}{\gamma} + \frac{(p_t^{gs})^2}{\gamma} - \mu_A - \zeta_t(a) \right) (n + h_t(a)) - \mu_{h,t}(a) \right]
\]

\[
\sigma_{n,t}(n, a) = \frac{p_t^{id}}{\gamma} (n + h_t(a)),
\]

and \( \mu_{h,t}(a) \) is the drift of \( h_t(a) \).

**Derivation of Equation (23).** Notice that total wealth evolves according to

\[
\frac{d\omega_{i,t}}{\omega_{i,t}} = \left[ r_t + \left( \frac{p_t^{gs}}{\gamma} + \frac{(p_t^{id})^2}{\gamma} - \mu_A - \zeta_t(t - s_i) \right) dt + \frac{p_t^{id}}{\gamma} dZ_{i,t},
\]

where \( s_i \) is the birthdate of entrepreneur \( i \).

Let \( \bar{\omega}_{s,t} \equiv \frac{\int_{A_{i,s}} \omega_{i,t} dA_i}{\int_{A_{i,s}} dA_i} \) denote the average total wealth of entrepreneurs born at date \( s \). The law of motion of \( \bar{\omega}_{s,t} \) is given by

\[
\frac{d\bar{\omega}_{s,t}}{\bar{\omega}_{s,t}} = \left[ r_t + \left( \frac{p_t^{gs}}{\gamma} + \frac{(p_t^{id})^2}{\gamma} - \mu_A - \zeta_t(t - s) \right) \right] d\bar{\omega}_{s,t}
\]

where the idiosyncratic risk is diversified by averaging out across entrepreneurs of a given cohort.

It is convenient to express total wealth as a function of age instead of the entrepreneurs’ birthdate. Let \( \omega_t(a) \) denote the average total wealth of investors with age \( a \) at period \( t \). Using the fact that \( \bar{\omega}_{s,t} = \omega_t(t - s) \), we obtain the following PDE for \( \omega_t(a) \):

\[
\frac{\partial \omega_t(a)}{\partial t} + \frac{\partial \omega_t(a)}{\partial a} = \left[ r_t + \left( \frac{p_t^{gs}}{\gamma} + \frac{(p_t^{id})^2}{\gamma} - \mu_A - \zeta_t(a) \right) \right] \omega_t(a).
\]
In a stationary equilibrium, $\omega_t(a)$ does not depend on calendar time $t$, which allow us to write
\[
\frac{d \log \omega(a)}{da} = r + \frac{(p^{g})^2}{\gamma} + \frac{\gamma}{\gamma} - \mu_A - \zeta(a).
\] (A.32)

Integrating the expression above, we obtain
\[
\log \omega(a) = \log \omega(0) + \left[ r + \frac{(p^{g})^2}{\gamma} + \frac{\gamma}{\gamma} - \mu_A \right] a - \int_0^a \zeta(u) du.
\] (A.33)

Using the fact $\log \frac{\omega(a)}{\omega(0)} = \log \frac{f(a)\omega(a)}{f(0)\omega(0)} + ga$ and the identity $\omega(a) = n(a) \left( 1 + \frac{h(a)}{n(a)} \right)$, we obtain expression (23) after some rearrangement.

**Derivation of Equation (24).** The expression for $\omega(a)$ in levels can be written as
\[
\omega(a) = \omega(0)e^{\left[ r + \frac{(p^{g})^2}{\gamma} + \frac{\gamma}{\gamma} - \mu_A \right] a}e^{-\tau_a} - \phi e^{-\tau T} \over 1 - \phi e^{-\tau T}.\] (A.34)

Evaluating at $a = T$ gives
\[
\omega(T) = \omega(0)e^{\left[ r + \frac{(p^{g})^2}{\gamma} + \frac{\gamma}{\gamma} - \mu_A - mpc \right] T},\] (A.35)

where $mpc = \frac{1}{T} \int_0^T \zeta(a) da$.

The boundary condition at age $T$ implies $\omega(0) = e^{-\delta T} \omega(T) + h(0)$, then
\[
\omega(0) = \frac{h(0)}{1 - e^{\left[ r + \frac{(p^{g})^2}{\gamma} + \frac{\gamma}{\gamma} - \mu_A - \delta - mpc \right] T}}.
\] (A.36)

Using $\omega(0) = n(0) + h(0)$ and rearranging the resulting expression, we obtain (24).

**Derivation of Equation (25).** Multiplying Equation (A.34) by $f(a)$, integrating over age, and using the fact that $f(a) = e^{-ga} f(0)$, we obtain
\[
n_e + h_e = f(0)\omega(0) \int_0^T e^{\left[ r + \frac{(p^{g})^2}{\gamma} + \frac{\gamma}{\gamma} - (g + \mu_A) \right] a} e^{-\tau_a} - \phi e^{-\tau T} \over 1 - \phi e^{-\tau T} da,
\] (A.37)

which gives Equation (25) after some rearrangement.
A.3. Proof of Lemma 2

Proof. We derive the Kolmogorov Forward Equation as the limit of a discrete-time economy. The discrete-time approximation goes as follows. Time takes values on the discrete set \( \{t^1, \ldots , t^L\} \), where \( \Delta t = t^{l+1} - t^l \) is the constant time step. Scaled financial wealth \( n_{ij} \) takes values on a discrete grid, \( n_{ij} \in \{n^1, n^2, \ldots , n^K\} \) with a constant step size \( \Delta n = n^{j+1} - n^j \). Age is also assumed to take values in a discrete grid \( \{a^1, \ldots , a^K\} \), where \( \Delta a = a^{k+1} - a^k \), \( a^1 = 0 \), and \( a^K = T \). For simplicity, assume \( \Delta a = \Delta t \). The probability of moving up, down, or staying at the same point of the grid are chosen to approximate \((A.26)\) and are given, respectively, by

\[
\begin{align*}
p_u(n^j, a^k) &= \frac{1}{2} \left[ \frac{\sigma(n^j, a^k)^2}{\sigma^2} + \frac{\mu(n^j, a^k)}{\sigma^2} \Delta n \right] \quad \text{(A.38)} \\
p_d(n^j, a^k) &= \frac{1}{2} \left[ -\frac{\sigma(n^j, a^k)^2}{\sigma^2} - \frac{\mu(n^j, a^k)}{\sigma^2} \Delta n \right] \quad \text{(A.39)} \\
p_s(n^j, a^k) &= 1 - \frac{\sigma(n^j, a^k)^2}{\sigma^2} \quad \text{(A.40)}
\end{align*}
\]

where \( \sigma = \max_{1 \leq j \leq K} \sigma(n^j, a^k) \), \( \Delta n = \sigma \sqrt{\Delta t} \), and \( \Delta a = \Delta t \).

Notice that the expected change in \( n_{ij} \), where \( n_{ij} = n^j \) and \( a_i = a^k \), is given by

\[
\mathbb{E} [n_{i,t+1} - n_{i,t}] = p_u(n^j, a^k) \Delta n + p_d(n^j, a^k)(-\Delta n) = \mu(n^j, a^k)\Delta t,
\]

and

\[
\mathbb{E} [(n_{i,t+1} - n_{i,t})^2] = p_u(n^j, a^k)\Delta n^2 + p_d(n^j, a^k)(-\Delta n)^2 = \sigma(n^j, a^k)^2\Delta t.
\]

Let \( m(n^j, a^k, t^l) \) denote the mass of agents with financial wealth \( n^j \), age \( a^k \), at period \( t^l \). Summing over \( n^j \), we obtain the mass of agents with age \( a^k \), \( M_{k,l} = \sum_{j=1}^{J} m(n^j, a^k, t^l) = e^{\sigma(t^l-(k-1)\Delta t)} \). Summing over \( (n^j, a^k) \), we obtain the total population \( M_t = \sum_{k=1}^{K} M_{k,t} \), so \( M_{l+1} = e^{\sigma\Delta t} M_t \). The law of motion of \( m \), for \( k > 1 \) and \( 1 < j < J \), is given by

\[
m(n^j, a^k, t^l + \Delta t) = p_u(n^j - \Delta n, a^k - \Delta a)m(n^j - \Delta n, a^k - \Delta a, t^l) + p_d(n^j, a^k - \Delta a)m(n^j, a^k - \Delta a, t^l) \\
+ p_s(n^j + \Delta n, a^k - \Delta a)m(n^j + \Delta n, a^k - \Delta a, t^l).
\]

(A.43)
The boundary conditions are defined as follows. For \( j = 1 \) and \( j = J \), we will assume a reflecting boundary, that is, if \( n \) moves up from \( n_l \) or down from \( n_1 \), it is immediately reflected back to its initial position

\[
m(n_l^l, a^k, t^l + \Delta t) = p_u(n_l^l - \Delta n, a^k - \Delta a)m(n_l^l - \Delta n, a^k - \Delta a, t^l) + p_s(n_l^l, a^k - \Delta a)m(n_l^l, a^k - \Delta a, t^l) + p_u(n_l^l, a^k - \Delta a)m(n_l^l, a^k - \Delta a, t^l),
\]

and analogously for \( j = 1 \).

Finally, for \( k = 1 \), we have

\[
m(e^{-\Delta T} n_l^l, a^k, t^l + \Delta t) = e^{\Delta T} \left[ p_u \left( n_l^l - \Delta n, a^K \right) m \left( n_l^l - \Delta n, a^K, t^l \right) + p_s \left( n_l^l, a^K \right) m \left( n_l^l, a^K, t^l \right) + p_d \left( n_l^l + \Delta n, a^K \right) m \left( n_l^l + \Delta n, a^K, t^l \right) \right].
\]

since each one of the \( e^{\Delta T} \) heirs inherit \( e^{-\Delta T} n_l^l \), where we assumed \( e^{-\Delta T} n_l^l \) belongs to the grid.

Let \( f(n_l^l, a^k, t^l) \equiv \frac{m(n_l^l, a^k, t^l)}{M_l} \) denote the share of agents in state \( (n_l^l, a^k) \) in period \( t^l \). Dividing both sides of (A.43) by \( M_l \) and taking a Taylor expansion, we obtain

\[
(1 + g \Delta t)(f + f_t \Delta t) = \frac{1}{2} \left( \sigma_a^2 - (\sigma_a^2)_{u} n \Delta n + 0.5(\sigma_a^2)_{uu} n^2 - (\sigma_a^2)_{u} \Delta t + \mu_n - (\mu_n)_{u} n \Delta n \right) \left( f - f_u \Delta t - f_u n \Delta n + 0.5 f_{uu} n^2 \right) + \frac{1}{2} \left( \sigma_n^2 + (\sigma_n^2)_{u} n \Delta n + 0.5(\sigma_n^2)_{uu} n^2 - (\sigma_n^2)_{u} \Delta t - \mu_n + (\mu_n)_{u} n \Delta n \right) \left( f - f_u \Delta t + f_u n \Delta n + 0.5 f_{uu} n^2 \right) + \left( 1 - \frac{\sigma_n^2 - (\sigma_n^2)_{u} \Delta t}{\sigma^2} \right) \left( f - f_u \Delta t \right) + o(\Delta t).
\]

Simplifying the expression above and taking the limit \( \Delta t \to 0 \), we obtain

\[
f_t + f_u + g f = \frac{1}{2} \left( \sigma_n^2 \right)_{nn} f - (\mu_n)_{n} f + (\sigma_n^2)_{n} f_n - \mu_n f + \frac{1}{2} \sigma_n^2 f_{nn},
\]

or, more explicitly, we can write the expression as follows

\[
\frac{\partial f(n, a, t)}{\partial t} + \frac{\partial f(n, a, t)}{\partial a} + g f = -\frac{\partial [f(n, a, t) \mu_n(n, a)]}{\partial n} + \frac{1}{2} \frac{\partial [f(n, a, t) \sigma_n^2(n, a)]}{\partial n^2}.
\]

Let \( f_t(n|a) \) denote the conditional density at date \( t \), so \( f_t(n, a) = f_t(n|a)f(a) \). We can write the Kolmogorov Forward Equation in terms of the conditional density:

\[
f(a) \frac{\partial f_t(n|a)}{\partial t} + f(a) \frac{\partial f_t(n|a)}{\partial a} + f_t(n|a)f'(a) = -f(a) \frac{\partial [f_t(n|a) \mu_n(n, a)]}{\partial n} + f(a) \frac{1}{2} \frac{\partial [f_t(n|a) \sigma_n^2(n, a)]}{\partial n^2} - g f_t(n, a).
\]
Dividing by \( f(a) \) and using the fact that \( f'(a) = -g f(a) \), we obtain

\[
\frac{\partial f_t(n|a)}{\partial t} + \frac{\partial f_t(n|a)}{\partial a} = -\frac{\partial [f_t(n|a)\mu_n(n,a)]}{\partial n} + \frac{1}{2} \frac{\partial [f_t(n|a)\sigma^2_n(n,a)]}{\partial n^2}.
\]  

(A.50)

In a stationary equilibrium, we can ignore the dependence on calendar time to obtain

\[
\frac{\partial f(n|a)}{\partial a} = -\frac{\partial [f(n|a)\mu_n(n,a)]}{\partial n} + \frac{1}{2} \frac{\partial [f(n|a)\sigma^2_n(n,a)]}{\partial n^2}.
\]  

(A.51)

A.4. Proof of Proposition 3

Proof. The law of motion of (log) total wealth is

\[
d \log \omega_{i,t} = \left[ r + \frac{(p^g)^2}{\gamma} + \frac{(p^{id})^2}{\gamma} - \frac{\bar{r}}{1 - e^{-r(T - (t - s))}} - \mu_A - \frac{1}{2} \left( \frac{p^{id}}{\gamma} \right)^2 \right] dt + \frac{p^{id}}{\gamma} dZ_{i,t},
\]  

(A.52)

where \( s \) denotes the birth date of entrepreneur \( i \).

Integrating the expression above, we obtain

\[
\log \omega_{i,t} = \log \omega_{i,s} + \int_s^t \left[ r + \frac{(p^g)^2}{\gamma} + \frac{(p^{id})^2}{\gamma} - \frac{\bar{r}}{1 - e^{-r(T - (t - s))}} - \mu_A - \frac{1}{2} \left( \frac{p^{id}}{\gamma} \right)^2 \right] dt' + \frac{p^{id}}{\gamma} (Z_{i,t} - Z_{i,s}),
\]

(A.53)

where \( Z_{i,t} - Z_{i,s} \sim \mathcal{N}(0, a) \) and \( a = t - s \).

Hence, \( \log \omega_{i,t} \sim \mathcal{N}(m(a), v(a)) \), where the mean and variance are given by

\[
m(a) = \log h(0) + \left[ r + \frac{(p^g)^2}{\gamma} + \frac{(p^{id})^2}{\gamma} - \mu_A - \frac{1}{2} \left( \frac{p^{id}}{\gamma} \right)^2 \right] a + \log \frac{1 - e^{-r(T - a)}}{1 - e^{-RT}}
\]  

(A.54)

\[
v(a) = \left( \frac{p^{id}}{\gamma} \right)^2 a,
\]  

(A.55)

using the fact that \( \omega_{i,s_i} = h(0) \) when \( \psi = 1 \).

Normalized financial wealth \( n_{i,t} = \omega_{i,t} - h_{i,t} \) has a shifted log-normal distribution conditional
on $s_i = s$, with support $(-h(a), \infty)$. The expected value and variance of $n_{i,t}$ is given by

$$
\mathbb{E}[n|a] = h(0) e^{\left( r + \frac{(p^a)^2}{\gamma} + \frac{(p^{id})^2}{\gamma} - \mu_A \right) a \frac{1 - e^{-r(T-a)}}{1 - e^{-rT}} - h(a)}
$$

(A.56)

$$
\mathbb{V}[n|a] = \left[ e^{\left( \frac{\mu_A}{\gamma} \right)^2 a} + 1 \right] \left[ h(0) e^{\left( r + \frac{(p^a)^2}{\gamma} + \frac{(p^{id})^2}{\gamma} - \mu_A \right) a \frac{e^{-ra} - e^{-rT}}{1 - e^{-rT}}} \right]^2.
$$

(A.57)

We show next that $\mathbb{V}[n|a]$ has an inverted U shape. Define the following functions:

$$
v_1(a) = \left[ e^{\left( \frac{\mu_A}{\gamma} \right)^2 a} + 1 \right]^{\frac{1}{2}} e^{\left( r + \frac{(p^a)^2}{\gamma} + \frac{(p^{id})^2}{\gamma} - \mu_A \right) a}; \quad v_2(a) = \frac{e^{-ra} - e^{-rT}}{1 - e^{-rT}}.
$$

(A.58)

The derivative of the product of $v_1(a)$ and $v_2(a)$ will be positive if

$$
v'_1(a)v_2(a) + v_1(a)v'_2(a) > 0 \iff \frac{v'_1(a)}{v_1(a)} > -\frac{v'_2(a)}{v_2(a)},
$$

(A.59)

for $a \neq 0$ and $a \neq T$.

Notice that $-\frac{v'_2(a)}{v_2(a)}$ is positive, monotonically increasing, and approaches $\infty$ as $a$ approaches $T$:

$$
-\frac{v'_2(a)}{v_2(a)} = \frac{1}{1 - e^{-r(T-a)}}.
$$

(A.60)

The term $v'_1(a)/v_1(a)$ is positive, monotonically decreasing, and approaches $+\infty$ as $a \to 0$:

$$
\frac{v'_1(a)}{v_1(a)} = \frac{1}{2} \left( \frac{p^{id}}{\gamma} \right)^2 \frac{e^{\left( \frac{\mu_A}{\gamma} \right)^2 a}}{e^{\left( \frac{\mu_A}{\gamma} \right)^2 a} - 1} + r + \frac{(p^a)^2}{\gamma} + \frac{(p^{id})^2}{\gamma} - \mu_A.
$$

(A.61)

Hence, there exists a unique $0 < \hat{a} < T$ such that $v'_1(a)v_2(a) + v_1(a)v'_2(a) > 0$ for all $a < \hat{a}$ and $v'_1(a)v_2(a) + v_1(a)v'_2(a) < 0$ for all $a > \hat{a}$. Hence, $\mathbb{V}[n|a]$ is equal to zero at $a = 0$, it increases monotonically for $a < \hat{a}$, where it achieves the maximum, and it decreases towards zero for $\hat{a} < a \leq T$. 

$\square$
A.5. Proof of Proposition 4

Proof. Aggregating Equation (3) and using the fact that labor supply grows at rate $g$, we obtain

$$
\dot{K}_t = [\Phi(l(q_t)) - \delta - g] K_t,
$$

(A.62)
given the initial condition $K_0 = K^*.$

From (A.16) and (5), we obtain the expression

$$
r + p_t^{aq} \sigma_A + p_t^{id} \phi \sigma_{id} = \frac{\alpha K_t^{a-1} - t(q_t)}{q_t} + \frac{\dot{q}_t}{q_t} + \Phi(l(q_t)) - \delta + \mu_A.
$$

(A.63)

Using $p_t^{aq} = \gamma \sigma_A$ and $p_t^{id} = \gamma \phi \sigma_{id} \frac{q_t K_t}{\chi \omega_{c,t}}$ and solving for $\dot{q}_t$, we obtain

$$
\dot{q}_t = \left[ r + \gamma \sigma_A^2 + \gamma \phi^2 \sigma_{id}^2 \frac{q_t K_t}{\chi \omega_{c,t}} + \delta - \mu_A - \Phi(l(q_t)) - \frac{\alpha K_t^{a-1} - t(q_t)}{q_t} \right] q_t,
$$

(A.64)

where $\omega_{c,t} = n_{c,t} + h_{c,t}.$

The ODE above is subject to the terminal condition

$$
\lim_{t \to \infty} q_t = q,
$$

(A.65)

where $q$ is the value of $q_t$ in the new stationary equilibrium.

The first PDE was derived in the proof of Lemma 1 and it was given in (A.19). The boundary conditions are $\zeta_t(T) = (1 - \psi)^{-1} (V^*)^{-\frac{1}{\gamma}}$ and

$$
\lim_{t \to \infty} \zeta_t(a) = \zeta(a),
$$

(A.66)

where $\zeta(a)$ is the value in the new stationary equilibrium.

The PDE for the human wealth was given in (A.7). The boundary conditions are $h_t(T) = 0$ and

$$
\lim_{t \to \infty} h_t(a) = h(a),
$$

(A.67)

where $h(a)$ is the value in the new stationary equilibrium.
The PDE for total wealth was derived in the proof of Proposition 2 and it was given in (A.31). The first boundary condition is \( \omega_t(0) = e^{-gT} \omega_t(T) + h_t(0) \). The initial condition for \( \omega_t(a) \) is given by

\[
\omega_0(a) = n^*(a) + (q_0 - q^*)k^*(a) + h_0(a),
\]

where variables with an asterisk denote values before the change in \( \phi \).

The initial condition captures two types of revaluation effects. First, the value of the capital stock changes, since the price of capital jumps on impact. Second, the value of human wealth also changes since expected future wages respond on impact.

\[ \square \]

### Appendix B. Derivations

#### B.1. The Optimal Contract

In this appendix, we consider the contracting problem in more detail. In particular, we show that the market structure assumed in Section 3, where entrepreneurs have access to a riskless bond and both aggregate and idiosyncratic insurance, corresponds to a specific implementation of the optimal contract allocation. The derivation follows closely the work of Di Tella (2017) and it is provided for completeness.

##### B.1.1. Moral hazard

We assume that the aggregate productivity shock \( Z_t \) and the individual cumulative return \( R_{i,t} \) are publicly observable, but the idiosyncratic investment shock \( Z_{i,t} \) is privately observed by entrepreneur \( i \). Moreover, the entrepreneur may secretly divert capital at rate \( \xi_{i,t} \). The return on the business is then given by

\[
dR_{i,t} = \left[ \frac{y_{i,t} - w_{i,t} - i_{i,t}k_{i,t}}{q_{i,t}k_{i,t}} + \frac{\delta_{i,t}}{q_{i,t}} + \mu_A + \Phi(i_{i,t}) - \delta - \xi_{i,t} \right] dt + \sigma_A dZ_t + \sigma_{id} dZ_{i,t}.
\]

Because \( Z_{i,t} \) and \( \xi_{i,t} \) are not publicly observable, a principal contracting with the entrepreneur cannot determine whether a low return is the result of a negative investment shock or positive
stealing. The optimal contract will ensure that it is incentive-compatible for the entrepreneur to choose \( \zeta_{i,t} = 0 \) at all times.\(^{39}\) Note that the expected return coincides with the one in condition (5) in the case of no stealing.

Diverted capital can be sold in the market, but a fraction \( 1 - \phi \) is lost in the process. The proceeds of the sale is invested in a hidden account, which is remunerated at the risk-free rate \( r_i \). The entrepreneur’s hidden savings \( S_{i,t} \) evolve as follows:

\[
dS_{i,t} = r_i S_{i,t} dt + \phi q_{l,i} S_{i,t} \zeta dt. \tag{B.2}
\]

**B.1.2. The optimal contract problem**

Consumption, investment, and factor demands are contractible. A contract between a principal and an entrepreneur is then given by \((\tilde{c}_{i,t}, \tilde{r}_i, k_i, \tilde{l}_i, \tilde{F}_i)\), where all variables are adapted to the filtration generated by \((Z, R_i)\), and \( \tilde{F}_{i,t} \) denotes the transfer to the principal. Entrepreneurs cannot commit to long term contracts. At any point in time, an entrepreneur can settle her promises with the principal, transfer the funds from the hidden account to her bank account, and offer a new contract to the principal. Therefore, contracts are effectively short-term and the terms of the contract are redefined at every period.

The continuation value to the principal is given by

\[
\tilde{J}_{i,t} = \mathbb{E}_t \left[ \int_t^{s_i + T} \frac{\pi_{z,t}}{\pi_t} \tilde{F}_{i,z} dz \right], \tag{B.3}
\]

where the expectation is taken under no stealing, \( \zeta = 0 \), and \( \pi_t \) corresponds to the principal’s SDF, which evolves according to \( d\pi_t = -r_t \pi_t dt - p_t^{\rho} \pi_t dZ_t \), given the processes for \( r_t \) and \( p_t^{\rho} \).\(^{40}\)

To compute the law of motion of \( \tilde{J}_{i,t} \), let \( G_{i,t} \) denote a martingale defined as follows

\[
G_{i,t} = \int_{s_i}^t \pi_{z,t} d\tilde{F}_{i,t} + \mathbb{E}_t \left[ \int_{t}^{s_i + T} \pi_{z,t} d\tilde{F}_{i,z} \right], \tag{B.4}
\]

\(^{39}\)This result is typical of cash flow diversion models, see e.g. DeMarzo and Sannikov (2006) and DeMarzo and Fishman (2007).

\(^{40}\)When contracting with a wage earner, the relevant SDF is \( \pi_t = \exp^{\gamma t} \zeta_{i,t}^{-\gamma} \). In a stationary equilibrium, consumption follows the process \( d\tilde{c}_{i,t} = \mu A \tilde{c}_{i,t} dt + \sigma A \tilde{c}_{i,t} dZ_t \), then \( d\pi_t = -\left[ \rho_{W} + \gamma \mu A - \gamma (\gamma + 1) \frac{\sigma^2}{2} \right] dt - \gamma \sigma_A dZ_t \), where the drift corresponds to the interest rate \( r_t \) and the diffusion term corresponds to the the price of aggregate risk \( p_t^{\rho} \).
where $G_{i,t} = E_{t}[G_{i,t}]$. By the martingale representation theorem, there exists $\sigma_{G_{i,t}}^Z$ and $\sigma_{G_{i,t}}^R$ such that

$$
\pi_t \tilde{F}_{i,t} dt + d(\pi_t \tilde{F}_{i,t}) = \pi_t \sigma_{G_{i,t}}^Z dZ_t + \pi_t \sigma_{G_{i,t}}^R (dR_{i,t} - E_t[dR_{i,t}]).
$$

(B.5)

Applying Ito’s lemma on $\pi_t \tilde{F}_{i,t}$ and combining with the expression above, we obtain

$$
d\tilde{I}_{i,t} = \left[ r_t \tilde{I}_{i,t} + p_t^{ag} (\sigma_{F_{i,t}}^Z + \sigma_{F_{i,t}}^A) - \tilde{F}_{i,t} \right] dt + \sigma_{F_{i,t}}^Z dZ_t + \sigma_{F_{i,t}}^R (dR_{i,t} - E_t[dR_{i,t}]),
$$

(B.6)

where $\sigma_{F_{i,t}}^Z = \sigma_{G_{i,t}}^Z + p_t^{ag} \tilde{I}_{i,t}$ and $\sigma_{F_{i,t}}^R = \sigma_{G_{i,t}}^R$.

The financial wealth of entrepreneur $i$ is defined as $\tilde{n}_{i,t} = \tilde{b}_{i,t} + \tilde{q}_t k_{i,t} - \tilde{I}_{i,t}$, which corresponds to the sum of holdings of the risk-free asset $\tilde{b}_{i,t}$ and the value of the business $\tilde{q}_t k_{i,t}$, net of the promised payments to the principal $\tilde{I}_{i,t}$. The law of motion of financial wealth is given by

$$
d\tilde{n}_{i,t} = \left[ r_t \tilde{b}_{i,t} + \tilde{w}_t \tilde{I}_{i,t} - \tilde{F}_{i,t} - \tilde{c}_{i,t} \right] dt + \tilde{q}_t k_{i,t} dR_{i,t} - d\tilde{I}_{i,t}.
$$

(B.7)

Combining (B.6) and (B.7), and assuming $\zeta_{i,t} = 0$, we obtain

$$
d\tilde{n}_{i,t} = \left[ r_t (\tilde{n}_{i,t} - \tilde{q}_t k_{i,t}) + \tilde{q}_t k_{i,t} \mu_{1,t} - p_t^{ag} (\sigma_{F_{i,t}}^Z + \sigma_{F_{i,t}}^A) + \tilde{w}_t \tilde{I}_{i,t} - \tilde{c}_{i,t} + \left( -\tilde{q}_t k_{i,t} + \phi \tilde{q}_t k_{i,t} + \frac{\tilde{q}_t d}{\sigma_{id}} \right) \zeta_{i,t} \right] dt
$$

$$
+ \left( \tilde{q}_t k_{i,t} \sigma_A - \left( \sigma_{F_{i,t}}^Z + \sigma_{F_{i,t}}^A \right) \right) dZ_t + \left( \tilde{q}_t k_{i,t} \sigma_{id} - \sigma_{F_{i,t}}^R \right) dZ_{i,t}.
$$

(B.8)

By imposing $\zeta_{i,t} = 0$ and defining $\tilde{b}_{1,t} \equiv \sigma_{F_{i,t}}^Z + \sigma_{F_{i,t}}^R \sigma_A$ and $\tilde{b}_{id} \equiv \sigma_{F_{i,t}}^R \sigma_{id}$, we obtain the law of motion of financial wealth presented in Section 3. Note that the transfers to the principal $\tilde{F}_{i,t}$ only affect the law of motion of $\tilde{n}_{i,t}$ through the diffusion terms of the principal’s continuation value $\tilde{J}_{i}$. Therefore, we can write the contract in terms of $(\tilde{b}_{1,t}, \tilde{b}_{id})$ instead of $\tilde{F}_{i,t}$.\footnote{This reformulation also avoids the issue that the path of transfers $\tilde{F}_{i,t}$ is not uniquely determined, as an entrepreneur can, for instance, borrow from the principal and invest in the risk-free asset without affecting her utility.}

The entrepreneur’s problem can then be written as

$$
\rho \tilde{V}_{i} = \max_{\tilde{c}_{i,t}, \tilde{b}_{1,t}, \tilde{b}_{id}, \tilde{q}_t} \frac{c_{1,t}^{1-\gamma}}{1-\gamma} + E_t [dV_t],
$$

subject to the law of motion of financial wealth, $\tilde{n}_{i,t} \geq -\tilde{n}_{i,t}$, and the incentive-compatibility (IC)
constraint
\[ 0 \in \arg \max_{s_{i,t} \geq 0} \left\{ \frac{c_{i,t}^{1-\gamma}}{1-\gamma} + \mathbb{E}_t [dV_t] \right\}. \]  

Applying Ito’s lemma to the value function, we can write the IC constraint as follows
\[ -\tilde{q}_t k_{i,t} + \phi \tilde{q}_t k_{i,t} + \frac{\tilde{d}_t \sigma_{id}}{\sigma_{id}} \leq 0 \Rightarrow \tilde{d}_t \sigma_{id} \leq (1-\phi) \tilde{q}_t k_{i,t} \sigma_{id}, \]  

where we used the fact that \( V_{n,t} \) is positive.

Therefore, the optimal contract problem, where entrepreneurs choose transfers to a principal, is equivalent to problem (9), where entrepreneurs have access to aggregate and idiosyncratic insurance subject to the skin-in-the-game constraint (7).

B.2. Numerical solution of the KFE

We compute the solution of the KFE using a finite-difference scheme. We consider first the case of a stationary equilibrium and then discuss the solution in the case of a time-dependent KFE.

B.2.1. Stationary KFE.

Consider the case of a stationary solution to the KFE. We solve the PDE (26) using a finite-differences method. As in the proof of Lemma 2, we assume that \( n \) takes values in the grid \( \{ n_1, n_2, \ldots, n_I \} \) and age \( a \) takes values in the grid \( \{ a_1, a_2, \ldots, a_K \} \), with \( a_1 = 0 \) and \( a_K = T \). We adopt the following (upwind) difference scheme for \( 1 < i < I \) and \( k < K \)
\[ \frac{f^{i,k+1} - f^{i,k}}{\Delta n} = -\frac{(\mu_{i}^{i,k})^+ f^{i,k} - (\mu_{i-1,k})^+ f^{i-1,k} - (\mu_{i+1,k})^- f^{i+1,k} - (\mu_{i,k})^- f^{i,k}}{\Delta n} + \frac{(\sigma_{i}^{i,k})^2 f^{i+1,k} - 2(\sigma_{i-1,k} f^{i,k} + (\sigma_{i-1,k})^2 f^{i-1,k})}{2 \Delta n^2}, \]  

where \( f^{i,k} \equiv f(n_i, a_k), \mu_{i}^{i,k} \equiv \mu_n(n^i, a^k), \sigma_{i}^{i,k} \equiv \sigma_n(n^i), (x)^+ = \max\{x, 0\}, \) and \( (x)^- = \min\{x, 0\}. \)

Rearranging the expression above and collecting terms, we obtain
\[ f^{i,k+1} = \pi_{i-1,k} f^{i-1,k} + \pi_{i,k} f^{i,k} + \pi_{i+1,k} f^{i+1,k}, \]  

(B.13)
where

\[ \pi^{i,k}_u = (\mu^{i,k}_n) + \frac{\Delta a}{\Delta n} + \frac{(\sigma^{i,k}_n)^2}{2} \frac{\Delta a}{\Delta n^2}, \quad \pi^{i,k}_d = -(\mu^{i,k}_n) - \frac{\Delta a}{\Delta n} + \frac{(\sigma^{i,k}_n)^2}{2} \frac{\Delta a}{\Delta n^2}, \quad \pi^{i,k}_s = 1 - \left( |\mu^{i,k}_n| \frac{\Delta a}{\Delta n} + \frac{\sigma^{i,k}_n}{2} \frac{\Delta a}{\Delta n^2} \right). \]

The above scheme converges if, for all \((i, k)\), the following variant of the Courant-Friedrichs-Lewy (CFL) condition holds

\[ |\mu^{i,k}_n| \frac{\Delta a}{\Delta n} + \frac{(\sigma^{i,k}_n)^2}{2} \frac{\Delta a}{\Delta n^2} \leq 1. \]  

(B.14)

We adopt a reflecting boundary at \(n_1\) and \(n_I\)

\[ f^{1,k+1} = (\pi^{1,k}_d + \pi^{1,k}_s)f^{1,k} + \pi^{2,k}_d f^{2,k} \]  

(B.15)

\[ f^{1,k+1} = \pi^{l,k}_u f^{l-1,k} + (\pi^{l,k}_s + \pi^{l,k}_u) f^{l,k}. \]  

(B.16)

In matrix form, we can write

\[ \frac{f^{k+1}}{\text{explicit method}} = \Pi^k f^k, \quad \frac{f^{k+1}}{\text{implicit method}} = [2I - \Pi^k]^{-1} f^k, \]  

(B.17)

where

\[ \Pi^k = \begin{bmatrix}
\pi^{1,k}_d + \pi^{1,k}_s & \pi^{2,k}_d & 0 & 0 & \cdots & 0 & 0 & 0 \\
\pi^{1,k}_u & \pi^{2,k}_s & \pi^{3,k}_d & 0 & \cdots & 0 & 0 & 0 \\
0 & \pi^{2,k}_u & \pi^{2,k}_s & \pi^{3,k}_d & \cdots & 0 & 0 & 0 \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\
0 & 0 & 0 & \cdots & \pi^{l-2,k}_s & \pi^{l-1,k}_d & 0 \\
0 & 0 & 0 & \cdots & \pi^{l-2,k}_u & \pi^{l-1,k}_s & \pi^{l,k}_d \\
0 & 0 & 0 & \cdots & \pi^{l-2,k}_u & \pi^{l-1,k}_s & \pi^{l,k}_d + \pi^{l,k}_u
\end{bmatrix}. \]  

(B.18)

Notice that \((\Pi^k)^t\) is a stochastic matrix, so we can interpret the coefficients as probabilities. It remains to specify the boundary condition in the age dimension. An entrepreneur with age \(T\) and financial wealth \(n\) leaves as bequest \(e^{-\delta T} n\) for each children. The quantity \(e^{-\delta T} n\) may not be in the grid, so we linearly interpolate between the points in the grid. For any point \(n_i\) in the grid, there
exists coefficients \((i', b_i)\) such that

\[
e^{-gT}n_i = n_i' + b_i \Delta n = (1 - b_i)n_i + b_in_{i+1},
\]

(B.19)

where \(0 \leq b_i < 1\).

We can interpret \(b_i\) as the probability of receiving a bequest of size \(n_{i+1}\) and \(1 - b_i\) as the probability of receiving a bequest of size \(n_i\). The boundary condition can be written as follows:

\[
f(n_{i'}, 0) = (1 - b_i)f(n_i, T), \quad f(n_{i'+1}, 0) = b_if(n_i, T).
\]

(B.20)

Collecting the coefficients in a matrix \(B\), we obtain the following condition in matrix form

\[
f^1 = Bf^k.
\]

(B.21)

Combining the expressions above with the difference scheme for the \(f\), we obtain

\[
f^k = \Pi Bf^k \iff [I - \Pi B]f^k = 0,
\]

(B.22)

where

\[
\Pi = \Pi^1 \times \Pi^2 \times \ldots \Pi^{K-1}.
\]

(B.23)

Since \(B'\) and \((\Pi^k)'\) are stochastic matrices, we have that \(B'(\Pi)'\) is also a stochastic matrix. Hence, the matrix has a unit eigenvalue, and a solution to the system of equations above exists.

**B.2.2. Time-dependent KFE.**

The discretized time-dependent KFE can be written as

\[
f^{i,k}_{t+1} = \frac{\Delta t}{\Delta a} f^{i,k-1}_{t} + \pi_{u,i}^{i,k} f^{i-1,k}_{t} + \pi_{s,i}^{i,k} f^{i,k}_{t} + \pi_{d,i}^{i,k} f^{i+1,k}_{t},
\]

(B.24)
for $1 < i \leq I$ and $1 < k \leq K$, where

$$\pi_{u,t}^{i,k} = (\mu_{n,t}^{i,k}) + \frac{\Delta t}{\Delta n} + \frac{(\sigma_{n,t}^{i})^2}{2 \Delta n^2} \frac{\Delta t}{\Delta n^2} \ (B.25)$$

$$\pi_{d,t}^{i,k} = -(\mu_{n,t}^{i,k}) - \frac{\Delta t}{\Delta n} + \frac{(\sigma_{n,t}^{i})^2}{2 \Delta n^2} \ (B.26)$$

$$\pi_{s,t}^{i,k} = 1 - \frac{\Delta t}{\Delta a} - \left( |\mu_{n,t}^{i,k}| \frac{\Delta t}{\Delta n} + (\sigma_{n,t}^{i})^2 \frac{\Delta t}{\Delta n^2} \right). \ (B.27)$$

Note that the difference equation above corresponds to the implicit scheme for the stationary KFE if $f_{t+1}^{i,k} = f_{t}^{i,k}$. The boundary conditions are given by

$$f_{t+1}^{i,1} = (1 - b_t) f_{t+1}^{i,K} + b_t f_{t+1}^{i+1,K} \ (B.28)$$

$$f_{t+1}^{i,k} = \frac{\Delta t}{\Delta a} f_{t}^{i,k-1} + (\pi_{d,t}^{i,k} + \pi_{s,t}^{i,k}) f_{t}^{i,k} + \pi_{d,t}^{i,k} f_{t}^{i,k} \ (B.29)$$

$$f_{t+1}^{i,k} = \frac{\Delta t}{\Delta a} f_{t}^{i,k-1} + \pi_{u,t}^{i,k} f_{t}^{i,k-1} + (\pi_{s,t}^{i,k} + \pi_{u,t}^{i,k}) f_{t}^{i,k}. \ (B.30)$$

Let $f_t^k \equiv [f_t^{1,k}, f_t^{2,k}, \ldots, f_t^{l,K}]'$ and $f_t \equiv [f_t^{1}, f_t^{2}, \ldots, f_t^{l}]'$. The recursion for $f_t$ can be written as

$$\begin{bmatrix}
  f_t^{2,1} \\
  f_t^{3,1} \\
  \vdots \\
  f_t^{K,1} \\
\end{bmatrix} = \begin{bmatrix}
  \Pi_t^2 & 0 & \cdots & 0 & \frac{\Delta t}{\Delta a} B \\
  \frac{\Delta t}{\Delta a} I_l & \Pi_t^3 & \cdots & 0 & 0 \\
  \vdots & \vdots & \ddots & \vdots & \vdots \\
  0 & 0 & \cdots & \Pi_t^{K-1} & 0 \\
\end{bmatrix} \begin{bmatrix}
  f_t^{2} \\
  f_t^{3} \\
  \vdots \\
  f_t^{K} \\
\end{bmatrix}. \quad (B.31)$$

In matrix form, we can write the recursion for both the explicit scheme above and an implicit scheme:

$$f_{t+1} = A_t f_t, \quad f_{t+1} = [2I - A_t]^{-1} f_t. \quad (B.32)$$

### B.3. Equilibrium prices and capital stock in a stationary equilibrium

We derive next the equilibrium prices and the capital stock in a stationary equilibrium.
B.3.1. Price of aggregate insurance, interest rate, and the relative price of capital

Price of aggregate insurance. Combining the demand for aggregate insurance (21) and the corresponding market clearing condition, we obtain

\[ \int_{E_t} \left[ (qk_{i,t} + h_{i,t}) \sigma_A - (n_{i,t} + h_{i,t}) \rho^{\sigma} \right] \gamma \, di + \int_{W_t} \left[ h_{j,t} \sigma_A - (n_{j,t} + h_{j,t}) \rho^{\sigma} \right] \gamma \, dj = 0. \tag{B.33} \]

Rearranging the expression above, we can solve for the price of aggregate insurance \( p^{\sigma} \)

\[ p^{\sigma} = \frac{\int_{E_t} (qk_{i,t} + h_{i,t}) di + \int_{W_t} h_{j,t} dj}{\int_{E_t} (n_{i,t} + h_{i,t}) di + \int_{W_t} (n_{j,t} + h_{j,t}) dj} \gamma \sigma_A = \gamma \sigma_A, \tag{B.34} \]

using the fact that \( \int_{E_t} n_{i,t} di + \int_{W_t} n_{j,t} dj = \int_{E_t} qk_{i,t} di \). Notice that this result does not rely on the assumption of a stationary equilibrium.

Interest rate. The financial wealth of wage earners evolve according to

\[ d\bar{n}_{j,t} = \left( r + \gamma \sigma_A^2 + \bar{n}_{j,t} - \bar{c}_{j,t} \right) dt + \sigma_A dZ_t. \tag{B.35} \]

using the fact that the demand for aggregate insurance is given by \( \theta_{j,t} = -n_{j,t} \sigma_A \) in equilibrium.

Combining the expression above with the law of motion for human wealth, we obtain the law of motion of total wealth:

\[ \frac{d\bar{\omega}_{i,t}}{\bar{\omega}_{i,t}} = \left[ r + \gamma \sigma_A^2 - \frac{c_{j,t}}{n_{j,t} + h_{j,t}} \right] dt + \sigma_A dZ_t. \tag{B.36} \]

In a stationary equilibrium, wage earners’ scaled total wealth, \( \omega_{j,t} = \bar{\omega}_{j,t} / A_t \), is constant. Therefore, the drift of \( \omega_{j,t} \) must be zero. We can compute the drift of \( \omega_{j,t} \) using Ito’s lemma:

\[ \frac{d\omega_{i,t}}{\omega_{i,t}} = \frac{d\bar{\omega}_{i,t}}{\bar{\omega}_{i,t}} - \frac{dA_t}{A_t} + \left( \frac{dA_t}{A_t} \right)^2 - \frac{dA_t}{A_t} \frac{d\bar{\omega}_{i,t}}{\bar{\omega}_{i,t}} \tag{B.37} \]

\[ = \left[ r + \gamma \sigma_A^2 - \frac{c_{j,t}}{n_{j,t} + h_{j,t}} - \mu_A \right] dt. \tag{B.38} \]

The interest rate must then satisfy the condition

\[ r + \gamma \sigma_A^2 - \left[ \frac{1}{\gamma} \rho_w + \left( 1 - \frac{1}{\gamma} \right) \left( r + \frac{\gamma \sigma_A^2}{2} \right) \right] - \mu_A = 0, \tag{B.39} \]
using the fact that the consumption-wealth is given by

\[
\frac{c_{j,t}}{n_{j,t} + h_{j,t}} = \frac{1}{\gamma} \rho_w + \left(1 - \frac{1}{\gamma}\right) \left(r + \left(p^a s^2 \right)^2 \right),
\]  

(B.40)

which is a special case of (22), as we set \( p^i t = 0 \) and \( T \to \infty \).

Rearranging expression (B.39), we obtain

\[
r = \rho_w + \gamma \mu_A - \gamma (\gamma + 1) \frac{\sigma_A^2}{2}.
\]  

(B.41)

Relative price of capital. Plugging the expression for \( \Phi'(i) \) into the first-order condition for \( i \) in equation (12), we obtain

\[
\frac{1}{\sqrt{\Phi_0^2 + 2\Phi_1 i}} = \frac{1}{q} \Rightarrow i = \frac{q^2 - \Phi_0^2}{2\Phi_1}.
\]  

(B.42)

In a stationary equilibrium, the capital-labor ratio is constant. Thus, capital grows at the population rate \( g \), which gives us the condition

\[
\Phi(i) - \delta = g \Rightarrow q = \Phi_0 + \Phi_1 (g + \delta),
\]  

(B.43)

where we used the fact that \( \Phi(i) = \frac{q - \Phi_0}{\Phi_1} \), obtained by plugging the expression for \( i \) into the functional form for \( \Phi(i) \).

B.3.2. Entrepreneurs’ human and total wealth

The value of human wealth for an entrepreneur with age \( a \) is given by

\[
h(a) = \int_0^{T-a} e^{-(r+p^a s_A - \mu_A)z} \sum_{l=1}^{L} \Gamma_l e^{\phi_l(z+a)} dz \bar{w}_e
\]  

\[
= (1 - \alpha) K^0 \bar{t}_e \sum_{l=1}^{L} \Gamma_l e^{\phi_l a} \frac{1 - e^{-(r+p^a s_A - \mu_A - \phi_l)(T-a)}}{r + p^a s_A - \mu_A - \phi_l}.
\]  

(B.44)

Average human wealth for entrepreneurs is given by

\[
h_e = \int_0^T \frac{g e^{-g a}}{1 - e^{-g T}} h(a) da = (1 - \alpha) K^0 \bar{t}_e \sum_{l=1}^{L} \Gamma_l \frac{g e^{(\phi_l - g) a} 1 - e^{-(r+p^a s_A - \mu_A - \phi_l)(T-a)}}{1 - e^{-g T}} \frac{1 - e^{-(r+p^a s_A - \mu_A - \phi_l)(T-a)}}{r + p^a s_A - \mu_A - \phi_l}.
\]  

(B.45)
Expressing condition (23) in levels, and after some rearrangement, we obtain entrepreneur’s
total wealth by age

$$\omega(a) = \omega(0)e^{\left(r + \frac{\langle \sigma_a^2 \rangle}{\gamma} + \frac{\langle \mu_i^2 \rangle}{\gamma} - \mu_A\right) a e^{-rT} - \psi e^{-rT} - \frac{1}{1 - \psi}}$$

(B.46)

where \( \omega(0) \) is given by

$$\omega(0) = \frac{h(0)}{1 - e^{\left(r + \frac{\langle \sigma_a^2 \rangle}{\gamma} + \frac{\langle \mu_i^2 \rangle}{\gamma} - g - \mu_A - r\right) a}}.$$  

(B.47)

Average total wealth for entrepreneurs is given by

$$\omega_e = \int_0^T \frac{g e^{-ga}}{1 - e^{-gT}} \omega(a) da = \omega(0)f(0) \int_0^T e^{\hat{r}_1 a} - \psi e^{-\hat{r}_1} e^{\hat{r}_2 a} \frac{1}{1 - \psi} da,$$

(B.48)

where \( \hat{r}_1 = r + \gamma \sigma_A^2 + \frac{\langle \mu_i^2 \rangle}{\gamma} - \mu_A - g - \hat{r} \) and \( \hat{r}_2 = \hat{r}_1 + \hat{r} \).

### B.3.3. Capital stock and idiosyncratic risk premium

Rearranging the expression for the shadow price of idiosyncratic risk (19), and using the definition
of expected returns (13), we obtain the MPK schedule, as discussed in Section 5:

$$r + p^{\sigma_A} + p^{\mu_i} \Phi(\mu_i) = \frac{\alpha K^{\alpha - 1} - \lambda(q)}{q} + \mu_A + \Phi(\mu(q)) - \delta.$$  

(B.49)

where \( r, p^{\sigma_A}, \) and \( q \) are functions of parameters, as derived above.

Integrating condition (20) across all entrepreneurs, we obtain

$$p^{id} = \gamma \phi \sigma_{id} \frac{qK}{\chi e \omega_e},$$

(B.50)

where \( \chi e \) is the fraction of entrepreneurs in the population.

Note that \( \omega_e \) is a function of \( p^{id} \) and \( K \). Plugging the expression for (B.48) into the equation
above, we obtain \( p^{id} \) implicitly as a function of \( K \). This relationship between \( p^{id} \) and \( K \) corresponds
to the pricing schedule discussed in Section 5. Therefore, we obtain the equilibrium capital stock
and equilibrium price of idiosyncratic risk by finding the pair \( (K, p^{id}) \) that simultaneously satisfies
the MPK schedule and the pricing schedule.
Appendix C. Transitional dynamics

Let $\hat{x}_t \equiv \log \frac{x_t}{x}$ for any variable $x_t$, where variables without a time subscript indicate the value in the new stationary equilibrium. The system of differential equations can then be written as

$$\dot{\hat{x}}_t = \frac{q}{\Phi_1} (e^\hat{x}_t - 1)$$  

(C.1)

$$\dot{\hat{t}}_t = \gamma \Phi^2 \sigma^2 \frac{q K}{\chi e \omega e} (e^\hat{t}_t + \hat{K}_t - \hat{\omega}_{c,t}) - \frac{q}{\Phi_1} (e^\hat{t}_t - 1) - \frac{a K^{a-1} q (e^\hat{t}_t - t(q e^\hat{t}_t) - a K^{a-1} t(q))}{q}$$  

(C.2)

$$\frac{\partial \hat{h}_t(a)}{\partial t} = -\frac{\partial \hat{h}_t(a)}{\partial a} - (1 - \alpha) \frac{K^a (a)}{h(a)} (e^\hat{K}_t - \hat{h}_t(a)) - 1$$  

(C.3)

$$\frac{\partial \hat{\omega}_t(a)}{\partial t} = -\frac{\partial \hat{\omega}_t(a)}{\partial a} + \gamma (\Phi \sigma^2) \frac{q K}{\chi e \omega e} (e^{2(\hat{t}_t + \hat{K}_t - \hat{\omega}_{c,t}) - 1}) - \zeta(a) (e^{\hat{t}_t(a) - 1})$$  

(C.4)

$$\frac{\partial \hat{\xi}_t(a)}{\partial t} = -\frac{\partial \hat{\xi}_t(a)}{\partial a} + \zeta(a) (e^{\hat{t}_t(a) - 1}) - (\gamma - 1) \frac{(\Phi \sigma^2)^2}{2} \frac{q K}{\chi e \omega e} (e^{2(\hat{t}_t + \hat{K}_t - \hat{\omega}_{c,t}) - 1}).$$  

(C.5)

Linearizing the system above, we obtain

$$\dot{\hat{x}}_t = \frac{q}{\Phi_1} \dot{\hat{t}}_t$$  

(C.6)

$$\dot{\hat{t}}_t = \gamma \Phi^2 \sigma^2 \frac{q K}{\chi e \omega e} (\hat{t}_t + \hat{K}_t - \hat{\omega}_{c,t}) + \frac{a K^{a-1} - t(q)}{q} \dot{\hat{t}}_t + (1 - \alpha) \frac{a K^{a-1}}{q} \dot{\hat{K}}_t$$  

(C.7)

$$\frac{\partial \hat{h}_t(a)}{\partial t} = -\frac{\partial \hat{h}_t(a)}{\partial a} - (1 - \alpha) \frac{K^a (a)}{h(a)} (\hat{\xi}_t - \hat{h}_t(a))$$  

(C.8)

$$\frac{\partial \hat{\omega}_t(a)}{\partial t} = -\frac{\partial \hat{\omega}_t(a)}{\partial a} + \gamma (\Phi \sigma^2) \frac{q K}{\chi e \omega e} (\hat{t}_t + \hat{K}_t - \hat{\omega}_{c,t}) - \zeta(a) \hat{\xi}_t(a)$$  

(C.9)

$$\frac{\partial \hat{\xi}_t(a)}{\partial t} = -\frac{\partial \hat{\xi}_t(a)}{\partial a} + \zeta(a) \hat{\xi}_t(a) - (\gamma - 1) (\Phi \sigma^2) \frac{q K}{\chi e \omega e} (\hat{t}_t + \hat{K}_t - \hat{\omega}_{c,t}),$$  

(C.10)

where

$$\hat{\omega}_{c,t} = \int_0^T \frac{\omega(a)}{\omega e} \hat{\omega}_t(a) da.$$  

(C.11)

We now discretize the system using a finite-differences method. The time variable $t$ and age $a$ will take values in the equally spaced grid $\{t_1, t_2, \ldots, t_N\}$ and $\{a_1, a_2, \ldots, a_K\}$, respectively. We adopt the following notation: $\hat{z}_n^a = \hat{z}_n(a_k)$ denotes the consumption-wealth ratio at time $t_n$ and age $a_k$, and an analogous notation holds for the remaining variables. The time and age steps are
denoted by $\Delta t = t_{n+1} - t_n$ and $\Delta a = a_{k+1} - a_k$. The discretized version of the ODEs are given by

\[
\begin{align*}
\frac{\hat{K}_{n+1} - \hat{K}_n}{\Delta t} &= \frac{q}{\Phi_1} \hat{q}_n \\
\frac{\hat{q}_{n+1} - \hat{q}_n}{\Delta t} &= \gamma \Phi^2 \sigma^2_{id} \frac{qK}{\omega_e} (\hat{q}_n + \hat{K}_n - \hat{\omega}_{e,n}) + \frac{\alpha K^{n-1} - \xi(q)}{q} \hat{q}_n + \alpha (1 - \alpha) \frac{K^{n-1}}{q} \hat{K}_n.
\end{align*}
\]

Discretizing the PDEs, we obtain at the interior points

\[
\begin{align*}
\frac{\hat{h}^{k+1}_n - \hat{h}^k_n}{\Delta t} &= -\frac{\hat{h}^{k+1}_n - \hat{h}^k_n}{\Delta a} - (1 - \alpha) \frac{K^{k+1}}{h^k} (a \hat{K}_n - \hat{h}^k_n) \\
\frac{\hat{\omega}_{n+1} - \hat{\omega}_n}{\Delta t} &= -\frac{\hat{\omega}_n - \hat{\omega}_{n-1}}{\Delta a} + 2 \gamma (\phi \sigma_{id})^2 \left( \frac{qK}{\omega_e} \right)^2 (\hat{q}_n + \hat{K}_n - \hat{\omega}_{e,n}) - \zeta^k \xi^k \\
\frac{\hat{\xi}_{n+1}^k - \hat{\xi}_n^k}{\Delta t} &= -\frac{\hat{\xi}_{n+1}^k - \hat{\xi}_n^k}{\Delta a} + \zeta^k \xi^k - (\gamma - 1) (\phi \sigma_{id})^2 \left( \frac{qK}{\omega_e} \right)^2 (\hat{q}_n + \hat{K}_n - \hat{\omega}_{e,n}).
\end{align*}
\]

where, using the Trapezoidal rule, $\hat{\omega}_{e,n}$ is given by

\[
\hat{\omega}_{e,n} = \left[ \sum_{k=2}^{K-1} \hat{\omega}_n^k \frac{f(a_k)\omega(a_k)}{\omega_e} + \hat{\omega}_n^k \frac{f(a_1)\omega(a_1)}{2\omega_e} + \alpha \hat{K} \frac{f(a_k)\omega(a_k)}{2\omega_e} \right] \Delta a.
\]

It remains to specify the boundary conditions. We have initial conditions for state variables and terminal conditions for jump variables. For the first two equations, we have that capital starts at the old steady state and the relative price of capital converges to the new one.

\[
\hat{k}_1 = \hat{k}^*; \quad \lim_{n \to \infty} \hat{q}_n = 0, \quad (C.18)
\]

where $\hat{k}^*$ is the log-deviation of the capital stock at the old steady state relative to the new one.

The boundary conditions associated with $\hat{\xi}_t(a)$ and $\hat{h}_t(a)$ are the following

\[
\begin{align*}
\hat{\xi}_n^k &= 0; \quad \lim_{n \to \infty} \hat{\xi}_n^k = 0, \forall k \\
\hat{h}_n^k &= 0; \quad \lim_{n \to \infty} \hat{h}_n^k = 0, \forall k.
\end{align*}
\]

The consumption-wealth ratio and human wealth are forward-looking variables, so they have terminal conditions instead of initial conditions. Notice that $h_t(T) = 0$, while $\hat{\xi}_t(T)$ is determined by the bequest motive, so deviations from the new steady state are equal to zero. We then only
need to solve for the vectors \( \hat{\zeta}_n = \left[ \hat{\zeta}_1^n, \hat{\zeta}_2^n, \ldots, \hat{\zeta}_{K-1}^n \right] \) and \( \hat{h}_n = \left[ \hat{h}_1^n, \hat{h}_2^n, \ldots, \hat{h}_{K-1}^n \right] \), since the value at the final age is pinned down by the boundary condition.

The boundary condition for total wealth is given by

\[
\hat{\omega}_1^k = \hat{\omega}_1^{k*}; \quad \hat{\omega}_n^1 = \frac{f^K \omega^K}{f_1 \omega_1} \hat{\omega}_n^K + \frac{h^1}{\omega_1} \hat{h}_n^1.
\] (C.21)

Note that the value of \( \hat{\omega}_n^1 \) is pinned down by the boundary condition, given \( \hat{\omega}_n^K \) and \( \hat{h}_n^1 \). In this case, we have to solve for \( \hat{\omega}_n = [\hat{\omega}_n^2, \ldots, \hat{\omega}_n^K]' \), a \( K - 1 \)-dimensional vector. The determination of \( \hat{\omega}_1^{k*} \) will be discussed below.

Given the boundary conditions we can assemble the system in matrix form. First, we can write the difference equations for the state variables in matrix form

\[
\begin{bmatrix}
\hat{K}_{n+1} \\
\hat{\omega}_{n+1}
\end{bmatrix} =
\begin{bmatrix}
1 & 0_{K-1}' & \frac{q}{\phi_1} \Delta t & 0_{K-1} & 0_{K-1}' \\
1 & A_\omega_1 & 1 & A_\omega_{K-1} & A_\omega h & A_\omega \zeta
\end{bmatrix}
\begin{bmatrix}
\hat{K}_n \\
\hat{\omega}_n \\
\hat{q}_n \\
\hat{h}_n \\
\hat{\zeta}_n
\end{bmatrix}.
\] (C.22)

where \( a_\omega = 2\gamma (\phi \sigma_d)^2 \left( \frac{q^K}{\chi \omega_e} \right)^2 \Delta t \), and \( A_\omega_1, A_\omega h \), and \( A_\omega \zeta \) are \( K - 1 \times K - 1 \) matrices given by

\[
A_\omega_1 =
\begin{bmatrix}
1 - a_\omega \frac{f^2 \omega_1^2}{\omega_1} \Delta a - \frac{\Delta a}{\Delta a} & -a_\omega \frac{f^2 \omega_1^3}{\omega_1} \Delta a & \cdots & -a_\omega \frac{f^{K-1} \omega^{K-1} \omega_1}{\omega_1} \Delta a & -a_\omega \frac{f^K \omega^K \omega_1}{\omega_1} \Delta a + \frac{f^{K} \omega^K \Delta a}{f_1 \omega_1} \\
-a_\omega \frac{f^2 \omega_1^2}{\omega_1} \Delta a + \frac{\Delta a}{\Delta a} & 1 - a_\omega \frac{f^2 \omega_1^3}{\omega_1} \Delta a & \cdots & -a_\omega \frac{f^{K-1} \omega^{K-1} \omega_1}{\omega_1} \Delta a & -a_\omega \frac{f^K \omega^K \omega_1}{\omega_1} \Delta a \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
-a_\omega \frac{f^2 \omega_1^2}{\omega_1} \Delta a & -a_\omega \frac{f^2 \omega_1^3}{\omega_1} \Delta a & \cdots & 1 - a_\omega \frac{f^{K-1} \omega^{K-1} \omega_1}{\omega_1} \Delta a - \frac{\Delta a}{\Delta a} & -a_\omega \frac{f^K \omega^K \omega_1}{\omega_1} \Delta a \\
-a_\omega \frac{f^2 \omega_1^2}{\omega_1} \Delta a & -a_\omega \frac{f^2 \omega_1^3}{\omega_1} \Delta a & \cdots & -a_\omega \frac{f^{K-1} \omega^{K-1} \omega_1}{\omega_1} \Delta a & 1 - a_\omega \frac{f^K \omega^K \omega_1}{\omega_1} \Delta a - \frac{\Delta a}{\Delta a}
\end{bmatrix}
\] (C.23)

\[
A_\omega h =
\begin{bmatrix}
-a_\omega \frac{f^2 h_1^1}{\omega_1} \Delta a + \frac{h_1^1 \Delta a}{\Delta a} & 0 & \cdots & 0 \\
-a_\omega \frac{f^2 h_1^1}{\omega_1} \Delta a & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
-a_\omega \frac{f^2 h_1^1}{\omega_1} \Delta a & 0 & \cdots & 0 \\
-a_\omega \frac{f^2 h_1^1}{\omega_1} \Delta a & 0 & \cdots & 0 \\
\end{bmatrix}, \quad A_\omega \zeta =
\begin{bmatrix}
0 & -\zeta^2 \Delta t & 0 & \cdots & 0 & 0 \\
0 & 0 & -\zeta^2 \Delta t & \cdots & 0 & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & 0 & -\zeta^{K-1} \Delta t \\
0 & 0 & 0 & \cdots & 0 & 0
\end{bmatrix}.
\] (C.24)
The difference equations for the jump variables can be written as

\[
\begin{bmatrix}
\hat{q}_{n+1} \\
\hat{r}_{n+1} \\
\hat{\xi}_{n+1}
\end{bmatrix} =
\begin{bmatrix}
A_{qK} & A_{q\omega} \tilde{\omega}' & A_{q\phi} & A_{qK-1} \\
A_{hK} & 0_{K-1,K-1} & 0_{K-1} & A_{hh} & 0_{K-1,K-1} \\
0_{\xi 1,K-1} & -a_\xi 1_{K-1,\tilde{\omega}'} & 0_{\xi 1,K-1} & A_{\zeta h} & A_{\zeta \xi} \\
\end{bmatrix}
\begin{bmatrix}
\hat{K}_n \\
\hat{\omega}_n \\
\hat{\phi}_n \\
\hat{\xi}_n
\end{bmatrix},
\quad (C.25)
\]

where

\[
A_{qK} = \left[ \gamma \phi \sigma_d^2 \frac{qK}{\lambda e} + \alpha(1-\alpha) \frac{K^{a-1}}{\bar{q}} \right] \Delta t, \quad A_{q\omega} = -\gamma \phi \sigma_d^2 \frac{qK}{\lambda e} \Delta t, \quad \tilde{\omega} = \left[ \frac{f_2^2}{\alpha e}, \frac{f_3^2}{\alpha e}, \ldots, \frac{f^K}{\alpha e} \right]^T \Delta t,
\]

\[
A_{q\phi} = 1 + \left[ \gamma \phi \sigma_d^2 \frac{qK}{\lambda e} + \alpha \frac{K^{a-1}}{q} \right] \Delta t, \quad A_{qh} = -\gamma \phi \sigma_d^2 \frac{qK}{\lambda e} \Delta t \frac{f^1}{2\alpha e} \Delta t \Delta h',
\]

\[
A_{hK} = -(1-\alpha) aK \left[ \frac{l(a^1)}{h(a^1)} \right]' \Delta t, \quad \tilde{\omega} = \left[ \frac{f_2^2}{\alpha e}, \frac{f_3^2}{\alpha e}, \ldots, \frac{f^K}{\alpha e} \right]^T \Delta t,
\]

\[
A_{\zeta h} = -a_\xi \frac{f^1 h^1}{2\alpha e} \Delta a 1_{K-1} \epsilon_{i,K-1},
\]

\[
A_{\zeta \xi} = \begin{bmatrix}
1 + w \frac{l(a^1)}{h(a^1)} \Delta t + \frac{\Delta t}{\lambda a} & -\frac{\Delta t}{\lambda a} & 0 & \cdots & 0 & 0 \\
0 & 1 + w \frac{l(a^2)}{h(a^2)} \Delta t + \frac{\Delta t}{\lambda a} & -\frac{\Delta t}{\lambda a} & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & 1 + w \frac{l(a^{K-2})}{h(a^{K-2})} \Delta t + \frac{\Delta t}{\lambda a} & -\frac{\Delta t}{\lambda a} \\
0 & 0 & 0 & \cdots & 0 & 1 + w \frac{l(a^{K-1})}{h(a^{K-1})} \Delta t + \frac{\Delta t}{\lambda a}
\end{bmatrix}
\]

\[
A_{\zeta \xi} = \begin{bmatrix}
1 + \xi^1 \Delta t + \frac{\Delta t}{\lambda a} & -\frac{\Delta t}{\lambda a} & 0 & \cdots & 0 & 0 \\
0 & 1 + \xi^2 \Delta t + \frac{\Delta t}{\lambda a} & -\frac{\Delta t}{\lambda a} & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & 1 + \xi^{K-2} \Delta t + \frac{\Delta t}{\lambda a} & -\frac{\Delta t}{\lambda a} \\
0 & 0 & 0 & \cdots & 0 & 1 + \xi^{K-1} \Delta t + \frac{\Delta t}{\lambda a}
\end{bmatrix}
\]

Let \( x_n = [\hat{K}_n, \hat{\omega}_n, \hat{\phi}_n]' \) denote the vector of predetermined variables, \( y_n = [\hat{q}_{n}, \hat{r}_{n}, \hat{\xi}_{n}]' \) the vector of jump variables, and \( z_n = [x_n', y_n']' \) a vector containing the state and jump variables. We can then write the system in matrix form

\[
z_{n+1} = Az_n,
\]

for a matrix of coefficients \( A \).
Provided the Blanchard and Kahn (1980) conditions are satisfied, there exists a unique pair of matrices \((P, H)\) such that
\[
x_{n+1} = Px_n; \quad y_n = Hx_n. \tag{C.33}
\]

The initial value of entrepreneurs’ total wealth by age is given by
\[
\omega_0(a) = \omega^*(a) + (q_0 - q^*)k^*(a) + h_0(a) - h^*(a), \tag{C.34}
\]
where \(\omega^*(a)\) and \(h^*(a)\) denote total wealth and human wealth at the initial steady state, respectively, and \(k^*(a)\) denotes the amount of capital held by entrepreneurs of age \(a\) in the old steady state, which is given by
\[
k^*(a) = \frac{\omega^*(a)}{\omega_T} K^*.
\]

Log-linearizing the expression above around the new steady state, we obtain
\[
\dot{\omega}_0(a) = \dot{\omega}^*(a) + \frac{\dot{q}K}{\omega_T} (\dot{q}_0 - \dot{q}^*) + \frac{\dot{h}(a)}{\omega(a)} (\dot{h}_0(a) - \dot{h}^*(a)). \tag{C.35}
\]

We can write the initial condition for \(x_1\) as follows
\[
x_1 = x^* + G(y_1 - y^*) \Rightarrow x_1 = [I - GH]^{-1}(x^* - Gy^*), \tag{C.36}
\]
where \(x^* = [\bar{k}^*, (\bar{\omega}^*)']', y^* = [\bar{q}^*, (\bar{h}^*)', (\bar{\xi}^*)']',\) and \(G\) is a \(K \times 2K - 1\) matrix given by
\[
G = \begin{bmatrix}
0 & 0_{K-1}' & 0_{K-1}' \\
\frac{\dot{q}K}{\omega_T} 1_{K-1} & G_{w_h} & 0_{K-1,k-1}
\end{bmatrix}, \tag{C.37}
\]
where \(G_{w_h}\) has entries \((k, k + 1)\) equal to \(\frac{h(a_{k+1})}{\omega(a_{k+1})}\) for \(k = 1, \ldots, K - 1\), and zero otherwise.

**Appendix D. Extensions**

**D.1. Heterogeneous productivity and limited pledgeability**

In this subsection, we extend the basic environment in three dimensions: i) heterogeneous productivity, ii) heterogeneous idiosyncratic volatility, and iii) limited pledgeability of physical assets.
We focus on the case of a stationary equilibrium.

D.1.1. Heterogeneous productivity

Entrepreneurs draw at birth an idiosyncratic productivity $B_i$ which is fixed for the entrepreneur’s lifetime, it has mean one in the cross-section $\mathbb{E}_i[B_i] = 1$, and the draw is independent of the previous generation productivity. The production technology for entrepreneur $i$ is given by

$$\tilde{y}_{i,t} = A_i B_i k_i^\alpha t_i^{1-\alpha}. \tag{D.1}$$

As before, labor will be chosen to maximize returns, so the labor demand is given by

$$w = (1 - \alpha) B_i \left( \frac{k_i}{l_i} \right)^\alpha. \tag{D.2}$$

Expected return for entrepreneur $i$ is given by

$$\mu_i = \frac{\alpha B_i^2 \left( \frac{1 - \alpha}{w} \right)^{\alpha} - \lambda(q)}{q} + \mu_A + \Phi(\lambda(q)) - \delta. \tag{D.3}$$

D.1.2. Limited pledgeability and business exposure

Let $b_{i,t} = n_{i,t} - q_i k_i$ denote the amount of safe assets held by entrepreneur $i$. The natural borrowing limit can be written as

$$b_{i,t} \geq -h_{i,t} - q k_{i,t}. \tag{D.4}$$

This allows the entrepreneur to borrow freely against physical assets or human wealth. Let’s now assume that there is limited pledgeability of physical assets, that is, entrepreneurs can only borrow a fraction of $1 - \lambda^{-1}$ of the value of the assets:

$$b_{i,t} \geq -h_{i,t} - (1 - \lambda^{-1})q k_{i,t} \Rightarrow q k_{i,t} \leq \lambda \omega_{i,t}. \tag{D.5}$$

Hence, the entrepreneur faces a portfolio problem subject to leverage constraints. Given that the leverage constraint is linear in total wealth, we are able to obtain a closed-form solution for
the portfolio problem with leverage constraints. The HJB for an entrepreneur can be written as

$$\frac{\rho}{1 - \gamma} = \max_{c_{i,t}, k_{i,t}, l_{i,t}, \lambda, \sigma_{A,t}} \left\{ \frac{\xi^T_t(a)}{1 - \gamma} \left( \frac{c_{i,t}}{\omega_{i,t}} \right)^{1 - \gamma} - \frac{\gamma}{1 - \gamma} \frac{1}{\xi^T_t(a)} \left( \frac{\partial \xi^T_t(a)}{\partial l_t} + \frac{\partial \xi^T_t(a)}{\partial a} \right) + r_t + \frac{q_i k_{i,t}}{\omega_{i,t}} \left( \mu_{i,t} - r_t \right) \right\}.$$ 

subject to $q_i k_{i,t} \leq \lambda \omega_{i,t}$ and $\theta_{i,t}^{id} \leq (1 - \phi) q_i k_{i,t} \sigma_{i,t}$. The optimal capital demand is given by

$$\frac{q_i k_{i,t}}{n_{i,t}} = \frac{p_{i,t}^{id}}{\gamma \phi \sigma_{i,t}} \left( 1 + \frac{h_{i,t}}{n_{i,t}} \right),$$ \hspace{1cm} (D.6)

where the price of idiosyncratic risk $p_{i,t}^{id}$ is given by

$$p_{i,t}^{id} = \min \left\{ \frac{h_{i,t}^R - r - p_{i,t}^{gs} \sigma_A}{\phi \sigma_{i,t} \gamma}, \gamma \phi \sigma_{i,t} \right\}.$$

To aggregate equation (D.6) across types, it is convenient to first take logs, average for a given age group, and then convert the expression back to levels. Define $k(a) = \exp \left( \mathbb{E}_i [\log k_{i,t} | a_i = a] \right)$ and $\omega(a) = \exp \left( \mathbb{E}_i [\log (n_{i,t} + h_{i,t}) | a_i = a] \right)$ as the relevant cross-sectional average of capital and total wealth conditional on age. The aggregate exposure to the business for entrepreneurs of age $a$ is then given by

$$\frac{q_i k(a)}{n(a)} = \frac{1 + \frac{h(a)}{n(a)}}{\gamma} \frac{p_{i,t}^{id}}{\phi \sigma_{i,t} \gamma}.$$

(D.7)

where $n(a) = \omega(a) - h(a)$ and

$$\sigma_{i,t} = \exp \left( \mathbb{E}_i [\log \sigma_{i,t}] \right), \quad p_{i,t}^{id} = \exp \left( \mathbb{E}_i [\log p_{i,t}^{id}] \right).$$ \hspace{1cm} (D.8)

Therefore, we obtain the same expression for the business exposure after aggregation as in the baseline model, showing that our results extend to the case with limited pledgeability and heterogeneity in productivity and risk. An analogous derivation shows that our results for the consumption-wealth ratio extend to this case as well.

---

42 A similar result on the optimal portfolio share with leverage constraints can be found for investors without labor income in, for instance, Grossman and Vila (1992) and Detemple and Murthy (1997).
D.2. Finite-horizon wage earners

In this subsection, we consider the case where wage earners have finite horizon and imperfect altruism in the same way as entrepreneurs. For simplicity, we abstract from limited pledgeability and ex-ante heterogeneity on entrepreneurs, and once again focus on a stationary equilibrium. The optimal consumption and demand for insurance for wage earners are now given by

$$c_{j,t} = \frac{\bar{r}_w}{1 - \psi e^{-\bar{r}_w(T-a)}}(\omega)_{j,t}, \quad \theta^g_{j,t} = h_{j,t} \sigma_A - \frac{p^g}{\gamma} \omega_{j,t},$$

where

$$\bar{r}_w = \frac{1}{\gamma} \rho_w + \left(1 - \frac{1}{\gamma} \right) \left( r + \frac{(p^g)^2}{2\gamma} \right).$$

The price of aggregate insurance, wages, and the relative price of capital are the same as in the baseline model:

$$p^g = \gamma \sigma_A, \quad w = (1 - \alpha)K^a, \quad q = \Phi_0 + \Phi_1(g + \delta).$$

Finite lives for wage earners changes the determination of the interest rate. The interest rate is now jointly determined with the capital-labor ratio and the price of idiosyncratic risk by conditions (32), (33), and the market clearing condition for consumption

$$\int_0^T \frac{\bar{r}_w \omega(a)}{1 - \psi e^{-\bar{r}_w(T-a)}} f(a) da + \int_0^T \frac{\bar{r}_w \omega_w(a)}{1 - \psi e^{-\bar{r}_w(T-a)}} f(a) da = \alpha K^a - t K,$$

where

$$\omega(a) = \omega(0)e^{\left(\gamma \sigma_A^2 + \frac{\mu_A^2}{\gamma} - \bar{r}_w - r\right) a 1 - \psi e^{-\bar{r}_w(T-a)}} , \quad \omega_w(a) = \omega_w(0)e^{\left(\gamma \sigma_A^2 - \mu_A - \bar{r}_w\right) a 1 - \psi e^{-\bar{r}_w(T-a)}} .$$

Assuming finite lives for wage earners would change the calibration of $\rho_w$ but otherwise would not affect our main results.