Mediated Collusion*

Juan Ortner — Boston University
Takuo Sugaya — Stanford GSB
Alexander Wolitzky — MIT

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Abstract

Cartels and bidding rings are often facilitated by intermediaries, who recommend prices/bids to firms and can impose penalties (such as reverting to competitive behavior in future interactions) if these recommendations are not followed. Motivated by such cases, we study correlated equilibria in first-price auctions with complete information, where bidders who disobey their recommendations are penalized. Cartel-optimal profit is greater when more information about submitted bids is disclosed at auction and when the maximum penalty is larger. When only the winner's identity is disclosed (or the winner's identity and bid), cartels do not benefit from mediation. Our main result characterizes the cartel-optimal equilibrium with two symmetric bidders when both bids are disclosed. When the maximum penalty is not too small, the optimal bid distribution is atomless on a connected subset of $\mathbb{R}^2$, and is characterized by a double-continuum of binding downward incentive constraints. The optimal equilibrium displays tied bids and high winning bids with positive probability, even when the maximum penalty is very small. The stationary mediation schemes we consider are always more profitable for the cartel than bid rotation.

Keywords: collusion, bidding rings, mediation, correlated equilibrium, bid rotation

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1 Introduction

Cartels and bidding rings are often facilitated by intermediaries, which exert control over the information available to cartel participants. For example, Harrington (2006) and Marshall and Marx (2012) document that numerous industrial cartels uncovered by the European Commission were supported by industry groups, consultancies, or accounting firms that intentionally limited participants' information concerning each other's operations in various ways.[1] In an auction context, Asker (2010) studies a bidding ring of stamp dealers, who relied on an intermediary (a New York taxi driver) to privately collate bids in an internal, "knockout" auction, before bidding on behalf of the winning bidder in a target auction. A particularly striking case comes from Kawai, Nakabayashi, and Ortner (2022), who study a long-running bidding ring among construction firms in the town of Kumatori, Japan. Collusion in this ring was facilitated by a trade association—the Kumatori Contractors Cooperative—which privately recommended bids in procurement auctions to the various ring members. The ring members then submitted their own bids, which were later publicly disclosed following the auction.[2] In this setting, while the trade association could not directly control the firms' bids, the auction environment provided some scope for the association to punish firms who did not bid according to their recommendations, in addition to limiting the firms' information about each other's recommended bids. For instance, if a bidder placed a bid below the one the association recommended to her, the association could announce that a deviation occurred, and could recommend that the bidders revert to competitive bidding in future auctions.

In mediated auctions of this form, the intermediary (or *mediator*) is more powerful than a mere correlating device (Aumann, 1974; also called a *bid coordination mechanism* by Marshall and Marx, 2007), because she has some scope to punish bidders who disobey their recommendations. However, because these punishments are bounded—e.g., by the difference in a bidder's continuation value under collusive and competitive bidding—the mediator

[1] One well-known intermediary in this context was the Swiss consulting firm AC Treuhand (see, e.g., Marshall and Marx, 2012, pp. 138–140).

[2] First-price sealed-bid auctions where all bids are disclosed following the auction are standard in procurement settings (e.g., Marshall and Marx, 2012, pp. 200–202). A classic empirical study of bid-rigging in such auctions is Porter and Zona (1993).

cannot fully control the bidders' bids: she is weaker than a *bid submission mechanism* that can directly submit ring members' bids or prevent some members from bidding (Marshall and Marx, 2007). The object of this paper is to analyze optimal collusion with such a mediator.[3]

We study correlated equilibria in a symmetric first-price procurement auction with complete information, where a mediator privately recommends a bid to each bidder and can penalize bidders who deviate from their recommendations. (The complete information assumption is for tractability; as we will see, optimal mediated collusion can be quite complicated even with complete information.) A leading interpretation is that the bidders are playing a stationary collusive equilibrium in a repeated auction environment, where, following a deviation, the mediator can direct the bidders to revert to competitive bidding in future auctions. With this interpretation, this size of the penalty equals the difference between a bidder's continuation payoff under collusive and competitive bidding. Other interpretations are also possible: for example, Marshall and Marx (2012, p. 138) document several industrial cartels that required firms to post bonds to a common fund, where the bonds were forfeited if the firms deviated from the collusive agreement, while Clark and Houde (2013, p. 118) study a large gasoline cartel whose leaders relied on harassment, threats, and intimidation (in addition to price wars) to enforce compliance.

In this setting, we characterize the cartel-optimal equilibrium as a function of the size of the penalty and the type of bid information that is disclosed following the auction. We consider three different specifications of the bid disclosure policy: *winner's identity disclosed*, where only the identity of the winning bidder is disclosed; *winner's bid disclosed*, where the identity and the bid of the winning bidder are disclosed; and *all bids disclosed*, where the full vector of bids is disclosed.[4] In the first two cases, the cartel-optimal equilibrium is relatively straightforward to characterize: the optimal equilibrium involves a deterministic

[3] In practice, the line between bid coordination and submission can be blurry. For example, while the bidding ring studied by Asker (2010) superficially resembles a bid submission mechanism, the intermediary had no means of preventing dealers from directly bidding in the target auction as well as the knockout, so economically this setting is more like bid coordination.

[4] While disclosing all bids is the standard practice in public procurement auctions in the United States, auctions where only the winner's identity and/or bid are disclosed are also well-studied and often arise in practice (e.g., Skrzypacz and Hopenhayn, 2004; Bergemann and Hörner, 2018), and are also natural benchmarks for comparing the benefits and drawbacks of disclosing additional bid information. The three disclosure policies we consider are the same ones studied by Bergemann and Hörner (2018), who give examples of important auctions run under each policy.

winning bid, and in fact can be implemented by the cartel even without the mediator's assistance. In the third case—which is the core of our analysis—the optimal equilibrium is more complex. Here, firms mix over a continuum of bids in a correlated manner, and the winning bid is stochastic. Mediation is essential: the cartel obtains strictly higher profits when firms privately communicate with a mediator, as compared to the case where firms can only communicate in public, or cannot communicate at all. Our model is thus one where explicit (mediated) collusion is more profitable than tacit collusion. The intuition is that, when the optimal bidding strategy involves randomization, a mediator must be present to know what bids were recommended, in order to know whether the observed bids match the recommendations, and hence whether punishment is warranted. (This intuition also explains why mediation does not improve payoffs when losing bids are not disclosed. In that case, deviations can only be punished when they change the identity of the winner or the winning bid; but these events are already observable without a mediator.)

Our analysis of the disclosed-bids case (with mediation) proceeds as follows. We first derive simple bounds on optimal cartel profit, as a function of the penalty size and the number of bidders. These bounds are tight for a large cartel, i.e., in the limit where the number of bidders is large. We then turn to small cartels. Our main result (Proposition 6) fully characterizes the cartel-optimal equilibrium with two bidders, in the case where the constraint that a bidder is not tempted to deviate from her recommended bid to a *higher* bid is slack. Such *upward incentive constraints* are always slack when the penalty is not too small, as well as when the mediator has the ability to deter upward deviations by placing a shill bid just above the recommended winning bid.[5] When upward constraints are slack, we show that the optimal equilibrium is characterized by a double-continuum of binding downward incentive constraints. Denoting the penalty size by x, the available surplus (i.e., the reserve price or consumer valuation) by 1, and the optimal cartel profit level by π^*, we show that the optimal bid distribution is atomless and is supported on a connected subset of the set $[2x, 1]^2$, so that $\pi^* \in (2x, 1)$. Even though the bid distribution is atomless, the bidders place the same bid with positive probability: i.e., there is a positive probability of a *random*

[5] Similar upward deviation-deterring bids also arise in Marshall and Marx (2007) and Bernheim and Madsen (2017).

tied bid. When a bidder is recommended a bid p above π^*, she expects to win the auction with positive probability, and she is indifferent between following the recommendation and placing any bid in an interval $[\chi(p), p)$ (and facing the punishment, x), where $\chi(p) \in [2x, \pi^*]$ is the lowest bid that the other bidder is ever recommended when one bidder is recommended bid p. When a bidder is recommended a bid below π^*, she expects to win the auction for sure, and she strictly prefers to follow the recommendation. Thus, each firm sometimes places a bid that it is certain will win the auction, but—in contrast to a *bid rotation scheme*, where the cartel agrees on a winner in advance (McAfee and McMillan, 1992; Kawai et al., 2022)—a firm never places a bid that it is certain will lose. Methodologically, we cast the problem of finding the cartel-optimal equilibrium as an infinite-dimensional linear program, which we are able to solve analytically by solving a pair of ordinary differential equations that characterize the boundary of the support of the optimal bid distribution as well as the optimal multipliers on the downward incentive constraints.

When the penalty size x is small and the mediator cannot place shill bids, upward incentive constraints also bind, and the optimal equilibrium is more complicated. In this case, we mostly rely on numerical solutions. However, a notable analytic result that can be obtained in a discretized version of the model is that, no matter how small x is, the support of the optimal bid distribution contains the bid pair $(1, 1)$. Thus, the optimal equilibrium involves high prices with positive probability, even when the maximum penalty is very small. To see why this is a striking result, recall that the Nash equilibrium bid pair $(0, 0)$ is the unique correlated equilibrium in the first-price auction with complete information (Jann and Schottmuller 2015; Feldman, Lucier, and Nisan, 2016), and by upper hemi-continuity all equilibrium bid distributions in our model converge to this degenerate distribution as $x \to 0$. Thus, while the optimal bid distribution converges to a point mass on $(0, 0)$ as $x \to 0$, for any $x > 0$ there is a positive probability that the winning bid is arbitrarily close to 1. This result has the empirical implication that observing high prices in a single auction does not allow an observer to conclude that a cartel has substantial enforcement power, or that prices will remain high in future auctions.

The static auction game with mediator-imposed penalties that we focus on captures stationary, symmetric equilibria of a repeated auction game. Analyzing optimal, non-stationary, repeated-game equilibria is beyond the scope of this paper. However, we do show that our optimal stationary equilibrium always yield higher cartel profit than non-stationary bid-rotation equilibria (in our main setting where all bids are disclosed at auction). Thus, optimal stationary mediation outperforms this canonical class of non-stationary strategies.

The paper is organized as follows. Following a brief discussion of related literature, Section 2 describes our model of mediated collusion. Section 3 establishes the benchmark profit that can be obtained by a cartel without mediation. Section 4 analyzes optimal mediated collusion when losing bids are not disclosed (where it turns out that the cartel does not benefit from mediation, so the optimal profit level is the same as in Section 3). The heart of the paper is Section 5, which analyzes optimal mediated collusion when all bids are disclosed. Finally, in contrast to our static main analysis, Section 6 explicitly considers a repeated-game setting, and shows that optimal stationary equilibria outperform bid rotation. Section 7 concludes. Omitted proofs are deferred to Appendix A.

1.1 Related Literature

We contribute to the literature on mediation and correlated equilibrium in moral hazard problems and repeated games, as well as to the literature on correlated and communication equilibrium in auctions. In the former literature, Rahman and Obara (2010) and Rahman (2012) derive general results on incentive compatibility in mediated partnerships. Several papers study how cartels can benefit from creating uncertainty about their members' current-period prices. These include Sugaya and Wolitzky (2017, 2018a), who show how cartel members can benefit from less precise monitoring of their competitors' past prices, since past prices can be informative of current prices[6]; Bernheim and Madsen (2017), who show how mixed equilibria outperform pure ones in repeated auctions with asymmetric costs; and Kawai, Nakabayashi, and Ortner (2022), who characterize optimal mediated bid rotation equilibria, where the mediator randomizes the recommended winning bid.[7] However, none of these papers attempts to characterize optimal correlated equilibria. We also mention Awaya

[6] Sugaya and Wolitzky (2018b) derive conditions under which optimal correlated equilibria in repeated games coincide with optimal Nash equilibria (see also Neyman, 1997; Ui, 2008). These conditions are not satisfied by the first-price auction game; otherwise, the cartel could not benefit from mediation in our model.

[7] The authors document that the Kumatori Contractors Cooperative relied on bid rotation equilibria.

and Krishna (2016), who exhibit a class of repeated Bertrand games where explicit communication increases cartel-optimal profits. In their model, this occurs because communication facilitates improved monitoring. Instead, in our model, (mediated) communication increases profits by creating uncertainty about competitors' current-period bids, which reduces a firm's gain from deviating.

A few papers study communication equilibria in one-shot first-price auctions (among other auction formats). With complete information, Jann and Schottmuller (2015) and Feldman, Lucier, and Nisan (2016) show that the Nash equilibrium is the only correlated equilibrium in the first-price auction. With independent private values, Marshall and Marx (2007) show that a cartel cannot obtain the first-best surplus in any communication equilibrium; moreover, Lopomo, Marx, and Sun (2011) show that with two symmetric bidders, two types, and discrete bids, no communication equilibrium outperforms Nash. With general information structures, Bergemann, Brooks, and Morris (2017) characterize the best equilibrium for the bidders that can arise over all information structures; while Bergemann, Brooks, and Morris (2021) analyze a related model where firms are assumed to know their own values. The classic papers of Graham and Marshall (1987), Mailath and Zemsky (1991), and McAfee and McMillan (1992) are less related, as they assume that the ring controls its members' bids (i.e., they consider bid submission mechanisms). Such enforcement power arises endogenously in repeated auctions with patient players (e.g., Athey and Bagwell, 2001; Athey, Bagwell, and Sanchirico, 2004; Aoyagi 2003, 2007; Skrzypacz and Hopenhayn, 2004; Blume and Heidhues, 2006; Hörner and Jamison, 2007; Harrington and Skrzypacz, 2011). In contrast, we focus on the case where players are impatient, so the maximum penalty size imposes a binding constraint.

More speculatively, mediated collusion can be related to recent interest in algorithmic collusion (e.g., Harrington, 2018; Schwalbe, 2018; Calvano et al., 2020; Klein, 2021). Even if firms do not explicitly communicate through a mediator, in principle the same outcome could be obtained if firms use a common pricing algorithm or online price recommendation tool that mimics the mediator's stochastic recommendations. Such an algorithm could support firm profits that are even higher than those attainable under tacit collusion, or under explicit collusion with only public communication. While we suspect that current pricing algorithms do not randomize in this way, the relationship between algorithms and mediation seems like an interesting issue for future study.[8]

2 Preliminaries

2.1 Model

Consider $n \geq 2$ firms (bidders) competing in a first-price procurement auction, with a reserve price normalized to 1. We assume that the firms have the same, commonly known production cost, which we normalize to 0. Firms simultaneously place bids $\mathbf{p} = (p_i)_{i=1}^n$, where without loss $p_i \in [0, 1]$ for each firm i. If a unique firm i places the lowest bid p_i, this firm wins the auction and receives a payment of p_i, while the other firms receive payment 0. In case of a tie, the winner is chosen uniformly at random.[9]

The firms are assisted by a mediator, who has two roles. Before the auction, the mediator privately recommends a bid to each firm. After the auction, the mediator may impose a *penalty* of size $x \in (0, 1)$ on each firm.[10] A firm's payoff is its payment in the auction, less the penalty if it is applied. In sum, the mediator cannot directly control the firms' bids, but it can impose a (limited) penalty on firms who deviate from their recommended bids, for example by triggering reversion to competitive bidding in future auctions (in a dynamic elaboration of the model, which we spell out in Section 2.3).

The usefulness of penalties for supporting collusion depends on the bid information that is disclosed at auction. We will consider three disclosure regimes: *winner's identity disclosed*, where only the identity of the winning bidder is disclosed; *winner's bid disclosed*, where the identity and the bid of the winning bidder are disclosed; and *all bids disclosed*, where the full vector of bids is disclosed. In each case, a *strategy profile* consists of a joint distribution of recommended bids (a probability distribution on $[0, 1]^n$), a bidding strategy for each firm

[8] For an example of a price-recommendation website that is suspected of facilitating collusion, see "Company That Makes Rent-Setting Software for Apartments Accused of Collusion, Lawsuit Says" (ProPublica, 2022).

[9] With symmetric firms, uniform tie-breaking is a harmless simplifying assumption.

[10] Since the penalty will never be imposed on-path in an optimal equilibrium, our analysis is the same if the mediator has the option of penalizing only a subset of firms—e.g., by expelling them from the cartel—or is restricted to penalizing all firms simultaneously—e.g., by dissolving the cartel.

i (a mapping from recommended bids $p_i \in [0,1]$ to probability distributions over actual bids $p_i' \in [0,1]$), and a punishment strategy for the mediator (a mapping from vectors of recommended bids and the disclosed bid information to a probability of penalizing one or more bidders). The mediator is assumed to be able to commit to its strategy. Thus, a strategy profile is a (Bayes Nash) *equilibrium* if each firm's strategy maximizes its expected payoff, given the strategies of the other firms and the mediator.[11] We are interested in the *optimal equilibrium* from the firms' perspective: that is, the equilibrium that maximizes *cartel profit*, which is defined as the sum of the firms' expected payoffs, or equivalently the expected winning bid (taking for granted that the penalty is not imposed on path). As we will see, optimal cartel profit is increasing and concave in the penalty size x.

We briefly comment on two of our assumptions.

First, the mediator has commitment power. As we will see, the bid distribution that maximizes expected cartel profit can involve a stochastic winning bid. An intermediary who receives a share of cartel profit is thus tempted to recommend higher winning bids more frequently, upsetting the optimal equilibrium. We instead interpret the mediator as an outsider that is paid a flat fee, independent of the recommended bids or any deviations by the firms.[12] Alternatively, the mediator can represent an automated system, such as a website or software package that recommends prices.

Second, the firms compete in an auction, even though their costs are identical and are commonly known among the firms. Obviously, if a procurer who likewise knew the firms' costs designed a mechanism to minimize the procurement price, she would simply fix the (reserve) price at cost. There are several reasons why this observation does not render our analysis irrelevant. First, it is likely that in many procurement settings the bidders know considerably more about each others' costs than the auctioneer does. This seems especially likely when the bidders are colluding and can communicate through a trusted intermediary, as in our model. In these settings, fixing the price at cost is infeasible for the auctioneer.

[11] With *all bids disclosed*, this solution concept is the same as (interim) ε-*correlated equilibrium* in the game without the mediator, with $\varepsilon = x$. Thus, our main technical contribution can be described as characterizing the optimal ε-correlated equilibrium of the symmetric first-price auction with complete information (or, equivalently, of symmetric Bertrand competition).

[12] For example, the intermediary in the bidding ring studied by Asker (2010) was paid \$30 per hour, plus an additional \$50 per auction from each participant.

Second, even if the auctioneer does have a good sense of the firms' costs, she may have limited discretion over the reserve price or weak incentives to set it optimally, so that, as in our model, the reserve price ends up above cost.[13] Third, our main reason for focusing on the complete-information case is tractability: while mediated collusion with incomplete information is clearly an important topic, a thorough treatment of this problem will likely require building on aspects of our simpler complete-information analysis.

2.2 Canonical Equilibria

Two preliminary observations will simplify the search for optimal equilibria. First, for any equilibrium, there exists an equilibrium with the same joint distribution of bids where, on path, the bidders always follow their recommendations and the penalty is never imposed. This follows by a revelation principle-like argument. Second, any level of cartel profit that is attainable in any equilibrium can be attained in a symmetric equilibrium, where the cdf of recommended bids $F(\mathbf{p})$ satisfies $F(p_1, \ldots, p_n) = F\left(p_{\phi(1)}, \ldots, p_{\phi(n)}\right)$ for every permutation ϕ on $\{1, \ldots, n\}$. This follows because, given any asymmetric equilibrium where bidders follow their recommendations, the strategy profile that results from randomly permuting the bidders' recommendations (and similarly permuting the bidders' identities in the mediator's punishment strategy) is symmetric (by construction), is an equilibrium (as each bidder's incentive constraint is an average of those in the original equilibrium), and has the same expected winning bid as in the original equilibrium. Given these observations, we henceforth restrict attention to symmetric equilibria where, on path, bidders follow their recommendations and the penalty is never imposed. We call such an equilibrium *canonical*.

[13] Casual evidence suggests that reserve prices in public procurement auctions are not always chosen optimally to minimize bidders' rents. For instance, in Japanese public procurement, reserve prices are typically determined based on the engineer's cost estimate (Hatsumi and Ishii, 2022), and, in some municipalities, are rarely binding (Chassang and Ortner, 2019). Similarly, procurement auctions run by Michigan's Department of Transportation (MDOT) do not have a formal reserve price, but MDOT has the right to reject all bids if the lowest one is higher than 110% of the engineer's cost estimate (Somaini, 2020).

2.3 Repeated Game Interpretation

A leading interpretation of the model is that the penalty x represents a firm's lost continuation payoff from switching from collusive to competitive play in a stationary, symmetric repeated-game equilibrium. To spell this out, consider any canonical equilibrium of our one-shot game with a penalty size of x. Let $\pi_i(x)$ denote a firm's expected payoff. (Recall that this is the same for each firm i.) If the firms repeatedly participate in identical auctions with a common discount factor δ, it is a (stationary, symmetric) equilibrium of the repeated game for the firms to play the symmetric equilibrium of the one-shot game in every period, where the penalty size x satisfies the fixed-point equation

$$x = \frac{\delta}{1-\delta}\pi_i(x), \tag{1}$$

and the prescribed equilibrium play is enforced by the threat of reversion to the static Nash equilibrium $\mathbf{p} = \mathbf{0}$ following any deviation. Conversely, in any stationary, symmetric repeated-game equilibrium, play in every period corresponds to a canonical equilibrium of our one shot game, with a value of x that satisfies equation (1).

In some versions of our model, it will turn out that optimal cartel profits are linear in the penalty size x, and hence $\pi_i(x)$ is linear in x. In this case, the highest value of $\pi_i(x)$ that satisfies equation (1) is given by a corner solution, $\pi_i(x) \in \{0, 1/n\}$ (depending on the values of δ and the coefficient on x in the formula for $\pi_i(x)$), so the model becomes trivial. However, enriching the interpretation of the penalty recovers an interior solution to (1). For example, if the penalty consists of some exogenous component y in addition to the lost continuation payoff (e.g., bonds posted by the firms, the threat of harassment or intimidation), then equation (1) becomes $x = y + (\delta/(1-\delta))\pi_i(x)$, which has an interior solution whenever δ is sufficiently small. Another possibility is that the reserve price or production cost may be stochastic. In this case, the variable $\pi_i(x)$ in equation (1) should be interpreted as the expected profit prior to the realization of the stochastic variable. With this interpretation, $\pi_i(x)$ can be concave in x even if optimal cartel profit is linear in x in our baseline model. This again yields an interior solution for (1).[14]

[14] Kawai, Nakabayashi, and Ortner (2022) work out a model along these lines.

While our main analysis is static and takes the penalty size x as a primitive, we explicitly consider the repeated game in Section 6, where we compare optimal stationary, symmetric equilibria (which correspond to the static equilibria in our main analysis) with non-stationary bid-rotation strategies, where each firm is supposed to win the auction every n periods.

3 Unmediated Collusion

We first establish the benchmark profit that the firms can obtain without the assistance of a mediator. We assume that the firms can agree in advance on a punishment scheme as a function of the information that is disclosed at auction (the winner's identity, the winner's identity and bid, or all bids), with a maximum penalty of x. This benchmark lets us isolate the value for the cartel of employing a mediator who makes private bid recommendations.

Proposition 1 *Without a mediator, the following hold:*

1. *With winner's identity disclosed, optimal cartel profit equals x. Moreover, the winning bid cannot exceed x with positive probability in any equilibrium.*
2. *With winner's bid disclosed, optimal cartel profit equals $\min\left\{\frac{n}{n-1}x, 1\right\}$. Moreover, the winning bid cannot exceed $\min\left\{\frac{n}{n-1}x, 1\right\}$ with positive probability in any equilibrium.*
3. *With all bids disclosed, optimal cartel profit is at least $\min\left\{\frac{n}{n-1}x, 1\right\}$ and at most $\min\{2x, 1\}$. The lower bound of $\min\left\{\frac{n}{n-1}x, 1\right\}$ is exact if $n = 2$ or if attention is restricted to equilibria where all firms make positive expected profits.*

The intuition for part 1 is that, since a deviation is detected only if it succeeds in switching the deviating firm from losing the auction to winning it, the maximum winning bid (which equals the gain from a successful deviation) cannot exceed the available punishment. Disclosing the winning bid lets the cartel support somewhat higher bids, because now a firm that deviates to a winning bid is punished even if it would also have won by following its recommendation. However, disclosing the losing bids does not further increase optimal cartel profit, at least for equilibria where all firms make positive expected profits.[15]

[15] See Appendix B for an example where $n = 3$ but only two firms make serious bids and cartel profit is above $\frac{3}{2}x$.

Proof. 1. An equilibrium yielding cartel profit x is as follows: Bidder 1 bids x, and every other bidder draws their bid from the uniform distribution on $[x, x+\varepsilon]$, where ε is any constant satisfying $\bar{\varepsilon} \in (0, \min\{(n-1)x, 1-x\})$. The bidders are all punished iff bidder 1 loses the auction.

To see that this is an equilibrium, note that if bidder 1 bids $p \in [x, x+\varepsilon]$, she wins with probability $\left(\frac{x+\varepsilon-p}{\varepsilon}\right)^{n-1}$. Since $x = \text{argmax}_{p\in[x,x+\varepsilon]} \left(\frac{x+\varepsilon-p}{\varepsilon}\right)^{n-1} p$ and bidding any $p \neq x$ only increases the probability of punishment, bidder 1's optimal bid is x. Meanwhile, any other bidder has an equilibrium payoff 0, against a payoff of at most $x - x = 0$ from deviating to a winning bid and getting punished.

That the winning bid cannot exceed x with positive probability (and hence optimal cartel profit cannot exceed x) is implied by Proposition 2.1.

2. An equilibrium yielding cartel profit $\frac{n}{n-1}x$ is as follows: Everyone bids $\frac{n}{n-1}x$. The bidders are all punished iff the winning bid differs from $\frac{n}{n-1}x$.

This is an equilibrium because a bidder's equilibrium expected payoff is $\frac{1}{n-1}x$ (by uniform tie-breaking), against a payoff of at most $\frac{n}{n-1}x - x = \frac{1}{n-1}x$ from deviating to a different bid and getting punished.

That the winning bid cannot exceed $\min\left\{\frac{n}{n-1}x, 1\right\}$ with positive probability (and hence optimal cartel profit cannot exceed $\min\left\{\frac{n}{n-1}x, 1\right\}$) is implied by Proposition 2.2.

3. The lower bound follows because the equilibrium in the winner's bid observed case remains an equilibrium when all bids are observed. The proof of the upper bound is deferred to the appendix. ■

It may be surprising that, without mediation, the cartel does not strictly benefit from the disclosure of the losing bids. Indeed, this finding contrasts with the classic intuition of Stigler, who argued that, "The system of sealed bids, publicly opened with full identification of each bidder's price and specifications, is the ideal instrument for the detection of price-cutting," (Stigler, 1964, p. 48). Proposition 1 qualifies this intuition by noting that, without mediation, disclosing all bids is no more favorable to collusion than is disclosing only the winner's bid and identity.[16] However, we will see that mediated cartels obtain strictly higher

[16] This result relies on the assumption that the bidders' costs are common knowledge, so that optimal unmediated collusion involves a fixed winning bid. However, we conjecture that the same result would also hold with incomplete information if the bidders held a knockout auction before the target auction, as in this

profits when all bids are disclosed, so Stigler's intuition is vindicated for mediated cartels.

4 Mediated Collusion with Unobserved Losing Bids

This section characterizes optimal mediated collusion when losing bids are not disclosed at auction. We will see that in this case optimal cartel profit is the same as in Proposition 1. Thus, the cartel does not benefit from mediation when losing bids are not disclosed.

To establish this result, it turns out to be sufficient to consider a limited class of deviations for the bidders: *uniform downward deviations*, where, for some cutoff bid $p^* \in [0, 1]$, a bidder follows her recommendation p_i whenever $p_i \leq p^*$, while she deviates by bidding p^* whenever $p_i > p^*$.[17] However, neither the ineffectiveness of mediation nor the sufficiency of uniform downward deviations will carry over when losing bids are disclosed.

Proposition 2 *With a mediator, the following hold:*

1. *With winner's identity disclosed, optimal cartel profit equals x. Moreover, the winning bid cannot exceed x with positive probability in any equilibrium.*

2. *With winner's bid disclosed, optimal cartel profit equals $\min\left\{\frac{n}{n-1}x, 1\right\}$. Moreover, the winning bid cannot exceed $\min\left\{\frac{n}{n-1}x, 1\right\}$ with positive probability in any equilibrium.*

Proof. 1. Since Proposition 1.1 shows that the cartel can obtain profit x even without mediation, it suffices to show that the winning bid cannot exceed x with mediation. To see this, fix any canonical equilibrium with bid profile cdf $F(\mathbf{p})$, and let $W(p)$ denote the corresponding cdf of the winning bid, $p = \min\{p_1, \ldots, p_n\}$. Since the equilibrium is symmetric, each bidder's equilibrium expected payoff equals

$$\int_0^1 \frac{p}{n} dW(p).$$

case the prescribed winning bid in the target auction would again be commonly known by the bidders.

[17] Jann and Schottmuller (2015) and Feldman, Lucier, and Nisan (2016) show that the Nash equilibrium of the symmetric, complete-information, first-price auction is also the unique correlated equilibrium. This result is similar to the $x = 0$ case of Proposition 2.1, and likewise is proved by considering uniform deviations. Uniform deviations also suffice to characterize optimal equilibria in some other auction games, such as the one considered by Bergemann, Brooks, and Morris (2017).

In contrast, for any $p^* \in [0,1]$ such that $W(p)$ is continuous at p^*, a uniform downward deviation with cutoff $p^* \in [0,1]$ gives an expected payoff of

$$\int_0^{p^*} \frac{p}{n} dW(p) + (1 - W(p^*)) \left(p^* - \frac{n-1}{n} x \right). \tag{2}$$

Indeed, if the lowest recommended bid is $p < p^*$, with probability $\frac{1}{n}$ the firm wins with bid p, and with probability $\frac{n-1}{n}$ the firm loses but is not punished (as even if it deviated, the deviation is not detected). If instead all recommended bids are strictly above p^*, then the firm wins with bid p^*, and the firm is punished with probability $\frac{n-1}{n}$ (since with probability $\frac{1}{n}$ the firm would have won even absent its deviation, so its deviation is not detected).[18] Since this deviation must be unprofitable in equilibrium, we have

$$\begin{aligned} \int_0^{p^*} \frac{p}{n} dW(p) + (1 - W(p^*)) \left(p^* - \frac{n-1}{n} x \right) &\leq \int_0^1 \frac{p}{n} dW(p) \qquad \iff \\ \int_{p^*}^1 \left(p^* - \frac{n-1}{n} x - \frac{p}{n} \right) dW(p) &\leq 0. \end{aligned}$$

Now, let $\bar{p} = \max \operatorname{supp}(W)$ denote the highest winning bid. Since the above inequality must hold for $p^* = \bar{p} - \varepsilon$ for any $\varepsilon > 0$ such that $W(p)$ is continuous at p^*, taking a sequence $\varepsilon \downarrow 0$ gives

$$\bar{p} - \frac{n-1}{n} x - \frac{\bar{p}}{n} \leq 0 \qquad \iff \qquad \bar{p} \leq x,$$

as desired.

2. Since Proposition 1.2 shows that the cartel can obtain profit $\min \left\{ \frac{n}{n-1} x, 1 \right\}$ even without mediation, it suffices to show that the winning bid cannot exceed $\min \left\{ \frac{n}{n-1} x, 1 \right\}$ with mediation. The proof of this fact follows the same line as in the winner's identity disclosed case. The difference is that a uniform downward deviation with cutoff $p^* \in [0,1]$ (where $W(p)$ is continuous at p^*) now gives an expected payoff of only

$$\int_0^{p^*} \frac{p}{n} dW(p) + (1 - W(p^*)) (p^* - x), \tag{3}$$

[18] The event that the lowest recommended price equals p^* occurs with probability 0 because $W(p)$ is continuous at p^*, and thus does not affect (2).

where the difference from (2) is that, now, if all recommended bids are strictly above p^*, the deviator is punished with probability 1 (rather than with probability $\frac{n-1}{n}$ as in the winner's identity observed case). We thus have

$$\int_0^{p^*} \frac{p}{n} dW(p) + (1 - W(p^*))(p^* - x) \leq \int_0^1 \frac{p}{n} dW(p) \quad \forall p^* \text{ s.t. } W(p) \text{ is continuous at } p^*.$$

Taking a sequence $p^* \uparrow \bar{p}$ gives

$$\bar{p} - x - \frac{\bar{p}}{n} \leq 0 \qquad \Longleftrightarrow \qquad \bar{p} \leq \frac{n}{n-1} x.$$

■

5 Mediated Collusion with All Bids Disclosed

Our main analysis concerns mediated collusion when all bids are disclosed at auction. We first derive simple upper and lower bounds on optimal cartel profit, where the upper bound comes from considering uniform downward deviations as in the previous section, and the lower bound comes from considering equilibria where at most one firm at a time bids below a pre-specified target (a class of strategies we call "almost-uniform bids"). Unlike in the previous section, the profit bound implied by uniform downward deviations is not generally tight; however, the bound is tight for a large cartel (where $n \to \infty$). We then turn to optimal equilibria for a small cartel, focusing on the case of two firms for tractability. Here we fully characterize the optimal equilibrium when upward incentive constraints are slack: this is our main result. As we show, upward incentive constraints are slack if $x \geq 1/3$. Alternatively, if the mediator herself (or a proxy) can enter the auction and place a shill bid just above the lowest recommended bid, then upward deviations are always unprofitable. Such shill bids always lose in equilibrium, so entering such a bid may be costless for the mediator in many auction settings. Finally, we consider the case where both downward and upward constraints bind (again with $n = 2$): i.e., the case where $x < 1/3$ and the mediator cannot place shill bids. Here we prove that, no matter how small x is, the support of the distribution of winning bids contains the reserve price of 1. We also further characterize the optimal equilibrium numerically, showing that the support of the bid distribution takes a relatively simple form. We end with a figure comparing cartel profit across the various settings and equilibria we consider, for two firms and any value for x.

5.1 Profit Bounds and Large Cartels

The upper bound on cartel profit that results from considering uniform downward deviations is as follows.

Proposition 3 *With all bids disclosed, cartel profit cannot exceed*

$$\frac{n}{n-1}\left(x+((n-1)x)^{\frac{n-1}{n}}(n-1-x)^{\frac{1}{n}}\right)-nx.$$

For example, when $n=2$, cartel profit cannot exceed $2\sqrt{x-x^2}$.

The proof shows that in the optimal bid distribution that deters all uniform downward deviations, with probability 1 at least $n-1$ firms bid 1: that is, at most one firm at a time bids below 1. The intuition is that increasing all losing bids to 1 increases the marginal distribution of each bidder's bid—and hence increases the probability that she is punished following any uniform downward deviation—without affecting the distribution of the winning bid (and hence without affecting a bidder's equilibrium payoff or her probability of winning the auction following a uniform downward deviation). However, this bid distribution does not deter non-uniform downward deviations, where a bidder deviates to a bid $p<1$ when she is recommended a bid of 1, but does not deviate when she is recommended a bid in between p and 1. For this reason, the resulting upper bound on cartel profit is slack.

We obtain a lower bound on optimal cartel profit by considering equilibria of a similar form, where there exists a bid $\bar{p}\in[0,1]$ such that, with probability 1, at least $n-1$ firms bid $\bar{p}$, and the remaining firm's bid does not exceed $\bar{p}$. We say that such an equilibrium has *almost-uniform bids*. This lower bound suffices to establish that the cartel strictly benefits from mediation when all bids are disclosed, whenever unmediated cartel profit is below the first-best level of 1.

Proposition 4 *With all bids disclosed, the optimal equilibrium among those with almost-uniform bids gives cartel profit equal to*

$$\min\left\{\frac{(n-1)\,x}{n-1-x}\left(1-\frac{n}{n-1}\log\left(\frac{nx}{n-1}\right)\right),\frac{(2n-1)\,x}{2n-2}\left(1-\frac{n}{n-1}\log\left(\frac{n}{2n-1}\right)\right),1\right\}.$$

In particular, this profit level strictly exceeds $\frac{n}{n-1}x$ *whenever* $\frac{n}{n-1}x<1$*, so the cartel strictly benefits from mediation whenever unmediated cartel profit is below* 1.

In addition, if the mediator can deter upward deviations by placing a shill bid, the optimal equilibrium with almost-uniform bids gives cartel profit equal to $\min\left\{\frac{(n-1)x}{n-1-x}\left(1-\frac{n}{n-1}\log\left(\frac{nx}{n-1}\right)\right),1\right\}$.

Intuitively, the best equilibrium with almost-uniform bids is characterized by the maximum bid $\bar{p}$, together with the condition that a firm that is recommended bid $\bar{p}$ is indifferent between bidding $\bar{p}$ and any bid p in between $\bar{p}$ and $\frac{n}{n-1}x$, which is the lowest bid in the support of the equilibrium bid distribution. If only downward incentive constraints bind, the optimal maximum bid $\bar{p}$ equals 1, and cartel profit equals $\min\left\{\frac{(n-1)x}{n-1-x}\left(1-\frac{n}{n-1}\log\left(\left(\frac{nx}{n-1}\right)\right)\right),1\right\}$.[19] This case arises when $x\geq\frac{n-1}{2n-1}$, as well as when the mediator can deter upward deviations by placing a shill bid. If instead $x<\frac{n-1}{2n-1}$ (and shill bids are infeasible), then the maximum bid cannot equal 1, because a bidder who is recommended the minimum bid of $\frac{n}{n-1}x$ would prefer to deviate upward to a bid just below 1. In this case, the maximum bid is set to satisfy this upward incentive constraint—so that $\frac{n}{n-1}x=\bar{p}-x$, or $\bar{p}=\frac{2n-1}{n-1}x$—and cartel profit equals $\frac{(2n-1)x}{2n-2}\left(1-\frac{n}{n-1}\log\left(\frac{n}{2n-1}\right)\right)$. Note that $\frac{2n-1}{2n-2}\left(1-\frac{n}{n-1}\log\left(\frac{n}{2n-1}\right)\right)>\frac{n}{n-1}$, so in either case profits are greater than $\frac{n}{n-1}x$, the optimal cartel profit without mediation.

Combining Propositions 3 and 4 lets us characterize optimal profit for a large cartel, where $n\to\infty$. This exercise requires taking a stance on how x varies with n. We assume that x is non-increasing in n, so that either x approaches a positive lower bound $\underline{x}>0$ as $n\to\infty$ (e.g., the penalty is harassment by other cartel members, which is similarly unpleasant in small and large cartels), or $x\to 0$ as $n\to\infty$ (e.g., the penalty is reversion to competitive play, hence each firm loses their $1/n$ share of cartel profits).

[19] Note that this is lower than the bound from Proposition 3, because here the corresponding bid distribution deters *all* downward deviations, not just uniform ones.

Proposition 5 *Suppose that the penalty size with n firms is given by a non-increasing function $x(n)$ satisfying $\lim_{n\to\infty} x(n) = \underline{x} \geq 0$. Let $\pi^*(n)$ denote optimal cartel profit with n firms and penalty size $x(n)$. Then $\lim_{n\to\infty} \pi^*(n) = \underline{x}(1-\log \underline{x})$, with convention $0\log 0 = 0$.*

Proof. Note that $\lim_{n\to\infty} \frac{n}{n-1}\left(\underline{x} + ((n-1)\underline{x})^{\frac{n-1}{n}}(n-1-\underline{x})^{\frac{1}{n}}\right) - n\underline{x} = \underline{x}(1-\log \underline{x})$. Hence, by Proposition 3, $\limsup_{n\to\infty} \pi^*(n) \leq \underline{x}(1-\log \underline{x})$. Next, note that $\lim_{n\to\infty} \frac{(n-1)\underline{x}}{n-1-\underline{x}}\left(1-\frac{n}{n-1}\log\left(\frac{n\underline{x}}{n-1}\right)\right) = \underline{x}(1-\log \underline{x})$. Hence, by Proposition 4, if $\underline{x} \geq \frac{1}{2}$ (so the first term in the bound in Proposition 4 is the minimizer when n is large), then $\liminf_{n\to\infty} \pi^*(n) \geq \underline{x}(1-\log \underline{x})$. In the appendix, we show that if $\underline{x} < \frac{1}{2}$ then again $\liminf_{n\to\infty} \pi^*(n) \geq \underline{x}(1-\log \underline{x})$. Thus, the lim inf and lim sup must both equal $\underline{x}(1-\log \underline{x})$. ■

There is a simple intuition for the limit profit of $\underline{x}(1-\log \underline{x})$. This is the profit that results when one firm is pre-selected to win the auction and is recommended to bid 1 with probability $\underline{x}$ and otherwise to bid between $\underline{x}$ and 1 according to the unit-elastic cdf $F(p) = 1-\underline{x}/p$, while the other firms place losing bids. Under this scheme, losing bidders cannot profitably deviate, because bidding just below 1 wins with probability $\underline{x}$ but results in punishment, and (by construction) bidding anywhere between $\underline{x}$ and 1 gives the same expected payoff. So this scheme is an equilibrium whenever upward incentive constraints hold. Moreover, the corresponding winning bid distribution is the limit as $n \to \infty$ of both the distribution in the proof of Proposition 3 and that in the proof of Proposition 4. For fixed n, both of these distributions put more weight on higher bids, because a bidder that is recommended to bid 1 wins with positive probability, and so is less tempted to deviate as compared to the case where she always lose. However, as $n \to \infty$, a bidder that is recommended to bid 1 wins with probability approaching 0, so both distributions converge to the one just described.[20]

An interesting implication of Proposition 5 is that if the reserve price r is taken as a free parameter (rather than being normalized to 1), cartel profit in the $n \to \infty$ limit equals $\underline{x}(1-\log(\underline{x}/r))$, which diverges as $r \to \infty$ whenever $\underline{x} > 0$. The intuition is that when the winning bid is randomized over a wide range, a small penalty is enough to deter deviations by losing bidders.

[20] The distribution in the proof of Proposition 3 is not an equilibrium for any finite n, because a bidder that is recommended to bid 1 has a profitable non-uniform downward deviation. However, as the probability that each bidder is recommended to bid 1 converges to 1, the expected payoff difference between uniform and non-uniform deviations vanishes, so this distribution is an ε-equilibrium, for ε converging to 0 as $n \to \infty$.

While the bounds in Propositions 3 and 4 coincide only in the limit, they are already quite close together when $n = 2$ and $x \geq 1/3$ (so upward incentive constraints are slack). In this case, the upper bound exceeds the lower bound by less than 5%.

5.2 Optimal Equilibrium with Downward Incentive Constraints

We now assume that $n = 2$ and (to rule out the case where first-best profit is attainable) $x \leq 1/2$. We characterize the optimal equilibrium, when only downward incentive constraints are considered. We will see that the resulting strategy profile is a genuine equilibrium—and hence the optimal one—if $x \geq 1/3$, or if shill bids are feasible.

5.2.1 The Optimal Equilibrium

To understand the structure of the optimal equilibrium, first recall the optimal equilibrium with almost-uniform bids, ignoring upward incentive constraints. In this equilibrium, the higher of the two bids is always equal to 1, and the lower bid is distributed on the interval $[2x, 1]$, so that a firm that is recommended a bid of 1 is indifferent among all bids in this interval. Observe that one way to improve cartel profit relative to this equilibrium is to recommend the bid profile $p_1 = p_2 = p$ with small probability, for any price $p < 1$ that is greater than the cartel profit in the original equilibrium. This follows because the resulting bid distribution yields higher expected profit by construction, and it remains an equilibrium because a firm that is recommended a bid of p expects to win the auction with high probability. The same logic implies that to obtain the optimal cartel profit π^*, the bid profile $p_1 = p_2 = p$ should be recommended with positive weight for every price p in the interval $[\pi^*, 1]$: i.e., the optimal equilibrium features a *random tied bid*. Moreover, to increase the weight that can be placed on tied bids, for each bid $p \geq \pi^*$ there should exist a bid $\chi(p) \in [2x, \pi^*]$ such that a firm that is recommended a bid of p is indifferent among all bids in the interval $[\chi(p), p]$. To support this indifference condition, the bid $\chi(p)$ is also the lowest bid in the support of the conditional distribution $F_j(p_j | p_i = p)$; that is, for $p \geq \pi^*$, the function $\chi(p)$ describes the lower boundary of the support of the optimal joint bid distribution.[21] For $p \geq \pi^*$, the

[21] The function $\chi : [\pi^*, 1] \to [2x, \pi^*]$ is the inverse of the function ω referenced in Proposition 6.

conditional distribution $F_j(p_j|p_i = p)$ is thus supported on the interval $[\chi(p), 1]$; it turns out to have a positive density for all $p_j \neq p_i$, with an atom at $p_j = p_i$. Finally, given the function $\chi(p)$, the double-continuum of binding downward incentive constraints (from each $p \geq \pi^*$ to each $p' \in [\chi(p), p)$), together with symmetry, determines the optimal distribution with this support.[22]

Specifically, let $\hat{\pi}$ denote the unique solution for π in the interval $[2x, 1]$ to the equation

$$2x \log 2x + (\pi - x) \log \frac{\pi(\pi - x)}{x} = 0. \tag{4}$$

We show that $\hat{\pi}$ is the optimal cartel profit when only downward incentive constraints are considered, and we characterize the corresponding joint distribution of bids $F(p_1, p_2)$. Formally, we establish the following result.

Proposition 6 *With all bids disclosed, $n = 2$, and $x \leq 1/2$, the optimal equilibrium when only downward incentive constraints are considered gives cartel profit $\hat{\pi}$. The optimal joint distribution of bids $F(p_1, p_2)$ (which is characterized in the appendix) is atomless and is supported on a connected subset of $[2x, 1]^2$. The distribution of the winning bid $p^L = \min\{p_1, p_2\}$ and the losing bid $p^H = \max\{p_1, p_2\}$ have the following properties:*

1. *The winning bid distribution has support $[2x, 1]$ and is atomless with a divergent density at 1.*

2. *When the winning bid p^L is below $\hat{\pi}$, the losing bid p^H is supported on an interval $\left[\omega\left(p^L\right), 1\right]$, where $\omega(p)$ is a decreasing function satisfying $\omega(2x) = 1$ and $\omega(\hat{\pi}) = \hat{\pi}$. In particular, the bid gap $p^H - p^L$ is always positive.*

3. *When the winning bid p^L is above $\hat{\pi}$, the losing bid p^H is supported on the interval $\left[p^L, 1\right]$. Moreover, the conditional distribution of p^H has an atom at $p^H = p^L$, so the bid gap $p^H - p^L$ equals zero (i.e., the bids are tied) with positive probability.*

[22] As this discussion indicates, while the optimal distribution maximizes the expected winning bid (by definition), it does not maximize the winning bid distribution in terms of first-order stochastic dominance. For example, the distribution given by $\Pr(\max\{p_1, p_2\} = 1) = 1$ and the same conditional distribution $F_j(p_j|p_i = 1)$ as in the optimal distribution (which is also the same conditional distribution as in the proof of Proposition 4) gives a higher probability that $\min\{p_1, p_2\} = 1$.

In addition, $F(p_1, p_2)$ also satisfies all upward incentive constraints—and hence is the optimal equilibrium—if either $x \geq 1/3$ or the mediator can place a shill bid.

Figure 1 is a heat map for the optimal joint bid distribution $F(p_1, p_2)$ when $x = .35$ and recommended bids are restricted to multiples of .01.[23] The qualitative features described above are readily apparent. The marginal $F_i(p_i)$ is supported on the interval $[2x, 1]$; the conditional $F_j(p_j|p_i)$ is supported on an interval that includes 1 and that is wider when p_i is higher; and more probability mass is assigned to bid pairs (p_1, p_2) where either $p_1 = p_2$ or $\max\{p_1, p_2\} = 1$ than to other pairs. (When recommended bids are continuous, this corresponds to the conditional $F_j(p_j|p_i)$ having an atom at $p_j = p_i$, and to the marginal density $f_i(p_i)$ diverging to infinity as $p_i \to 1$.) Optimal cartel profit $\hat{\pi}$ is equal to the smallest price p such that $(p, p) \in \operatorname{supp} F(p_1, p_2)$, which is approximately .93. (This can be seen by counting down seven grid points along the diagonal, starting from $(1, 1)$.) Winning bids below $\hat{\pi}$ are *isolated* (i.e., the bid gap is bounded away from zero), while winning bids above $\hat{\pi}$ are tied with positive probability.

5.2.2 Empirical Implications of the Optimal Equilibrium

We now highlight some features of the optimal bid distribution and explain how they relate to screens for collusive bidding that have been proposed in the literature.

First, winning bids below $\hat{\pi}$ are isolated. The optimal equilibrium thus displays the "missing bids" pattern documented by Chassang et al. (2022). As Chassang et al. (2022) show, this bid pattern is inconsistent with competitive bidding and thus indicates collusion.

Second, winning bids above $\hat{\pi}$ are tied with positive probability. The optimal equilibrium may thus fail collusive screens that flag auctions with clustered bids.[24] Moreover, the feature that the tied bid is *random* is a marker of *mediated* collusion, as this feature is inconsistent with independent randomization by the bidders. Looking across auctions, the exactly parallel bid movements that result from random tied bids are a familiar collusive marker: e.g.,

[23] The distribution is a true equilibrium: i.e., deviations to all bids in $[0, 1]$ are unprofitable. For heat maps for some other values for x, see Appendix C.

[24] One such a screen is the *coefficient of variation* proposed by Imhof et al. (2018), which flags auctions where the variance of bids is low. Imhof et al. (2018) document a successful application of this screen by the Swiss Competition Authority.

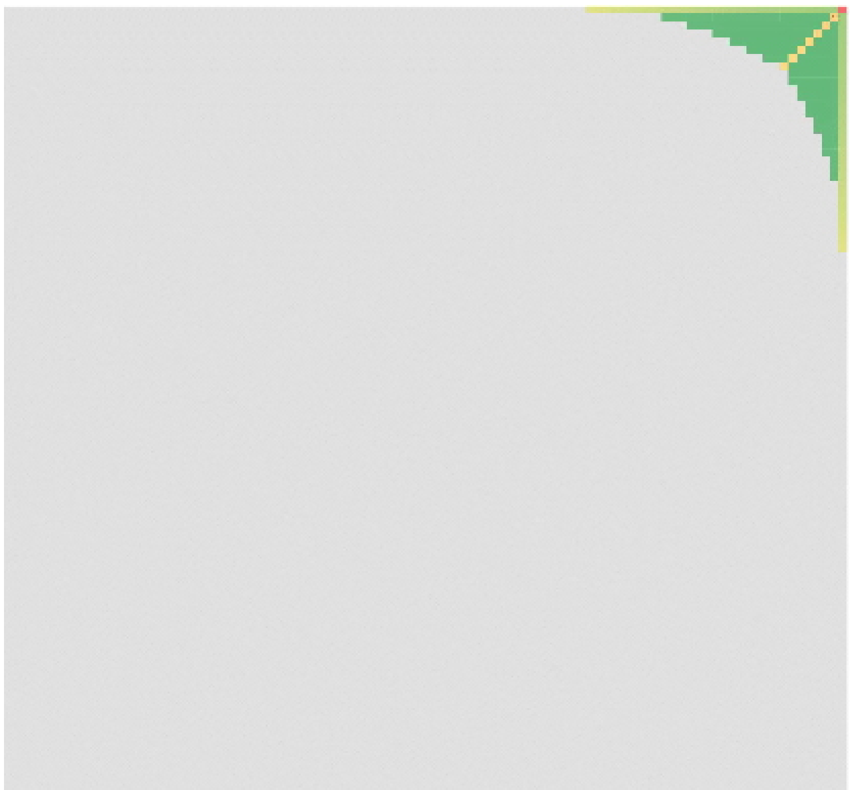

Figure 1: The optimal bid distribution when $x = .35$. The figure is generated by restricting recommended bids to multiples of .01. The color scheme is as follows: gray cells have 0 mass; green cells have small positive mass; yellow cells have mass .01; red cells have mass at least .1; and colors in between green and yellow, (resp., yellow and red), interpolate between between mass 0 and .01 (resp., .01 and .1). The only red cell in the figure is at $p_1 = p_2 = 1$; this single bid pair is recomended with probability approximately .42. The second-heaviest cell is $p_1 = p_2 = .99$, which is recommended with probability approximately .03.

Harrington (2008) writes that, "there is a common wisdom that parallel price movements are a collusive marker." However, Harrington continues, "Though there is a fair amount of documentation of identical bids at auction... in very few cases has collusion been found. More broadly, evidence that parallel pricing is a feature of collusion is ambiguous." Given our theory, it would be interesting to study if this ambiguity is linked to the presence or absence of mediation, i.e., if mediated cartels exhibit more parallel price movements.

Putting these features together, the prediction that low winning bids are isolated while high winning bids are clustered and move in parallel seems quite distinctive to our theory.

Third, while random tied bids are a marker of mediated collusion, the overall correlation between the two bidders' bids in the optimal equilibrium is close to zero: we have calculated that for every value of $x \in (0, 1)$, the absolute value of the correlation between the bids is less than .01. This lack of correlation contrasts some other collusive models, such as bid rotation, where this correlation is strongly negative, or bid clustering to deter upward deviations as in Marshall and Marx (2007), where it is strongly positive.

Fourth, whether the optimal equilibrium passes or fails price-variance screens for collusion, which flag price sequences that are unusually stable over time (Abrantes-Metz et al., 2006), depends on the penalty size. For auctions with large penalties (or, equivalently, low reserve prices), the winning bid is always close to 1, so the time-series variance of the winning bid is close to zero. In contrast, for auctions with small penalties (i.e., large reserve prices), the winning bid displays considerable variability as a result of randomization by the mediator, so the time-series variance of the winning bid is large. For example, it follows easily from Proposition 6 that the support of the optimal bid distribution expands when the penalty x shrinks, so that "less enforcement power" for the cartel is associated with a wider range of possible bids. This implies that price-variance screens for collusion may fail to detect mediated cartels with weak enforcement power.[25]

[25] A related effect arises in Bernheim and Madsen (2017), where efficient unmediated collusion involves mixed strategies, which reduce a competitor's gain from deviating. Unmediated cartels employing such strategies may also evade price-variance screens.

5.2.3 Outline of the Derivation of Proposition 6

To prove Proposition 6, we set up the problem of finding the optimal equilibrium with only downward incentive constraints as an infinite-dimensional linear program, and solve it using duality. We first construct the function $\chi(p)$ as the solution to an ordinary differential equation (equation (13)). Next, we define the marginal distribution of a firm's bid, $F_i(p_i)$, as the solution to an integral equation involving the function $\chi(p)$ (equation (14)). The characterization of the optimal joint distribution $F(p_1, p_2)$ is then completed by specifying the conditional distributions $F_j(p_j|p_i = p)$ so that the downward incentive constraint from any $p \geq \hat{\pi}$ to any $p' \in [\chi(p), p)$ binds (equation (15)), and using symmetry. Finally, we prove that this joint distribution is indeed optimal by constructing multipliers $\lambda(p'|p)$ on the downward incentive constraints, which are feasible for the dual linear program and yield the same value in the dual program as $F(p_1, p_2)$ does in the primal program. (Here, $\lambda(p'|p)$ denotes the multiplier on the constraint that it is unprofitable for a firm that is recommended a bid of p to deviate to a bid of p'.) By weak duality, the existence of such multipliers implies that $F(p_1, p_2)$ is optimal in the primal program, which completes the proof.

In sum, the proof proceeds by first guessing equations that the value $\hat{\pi}$, the function $\chi(p)$, and the joint distribution $F(p_1, p_2)$ should satisfy; then constructing $\hat{\pi}$, $\chi(p)$, and $F(p_1, p_2)$ that satisfy these equations; and then constructing multipliers $\lambda(p'|p)$ that certify the optimality of $F(p_1, p_2)$. We should also explain the origin of the "guessed" equations for $\hat{\pi}$, $\chi(p)$, $F(p_1, p_2)$, and $\lambda(p'|p)$. We start with the properties that $\chi(\hat{\pi}) = \hat{\pi}$, $\chi(1) = 2x$, downward incentives constraints from any $p \geq \hat{\pi}$ to any $p' \in [\chi(p), p)$ bind, and the multipliers $\lambda(p'|p)$ take the form

$$\lambda(p'|p) = \begin{cases} \kappa(p') & \text{if } p \geq \hat{\pi} \text{ and } \chi(p) \leq p' < p, \\ 0 & \text{otherwise,} \end{cases}$$

for some function $\kappa(p')$. (In particular, $\lambda(p'|p) = \lambda(p'|\hat{p})$ for all $p, \hat{p} \geq \hat{\pi}$ such that $p' \in [\chi(p), p) \cap [\chi(\hat{p}), \hat{p})$. This property must hold in order for (p, p'), $(p, p' + \varepsilon)$, $(\hat{p}, p')$, and $(\hat{p}, p' + \varepsilon)$ to all satisfy the dual constraint with equality, which in turn is necessary for these points to all lie in the support of the optimal bid distribution.) For any candidate optimal profit level π, the equation $\chi(\pi) = \pi$, together with the above properties for $\chi(p)$ and $\lambda(p|\hat{p})$, "inductively" determines differential equations in p for $\chi(p)$ and $\lambda(p|\hat{p})$ (where $\hat{p} > p$ is an arbitrary price satisfying $p \in [\chi(\hat{p}), \hat{p})$), starting from $p = \pi$ and ending at $p = 1$. The optimal profit level $\hat{\pi}$ is then determined as the value for π such that initializing these equations at $\chi(\pi) = \pi$ yields the required terminal condition $\chi(1) = 2x$, and the optimal functions $\chi(p)$ and $\lambda(p|\hat{p})$ are given by the corresponding solutions. Next, given the optimal function $\chi(p)$, the joint distribution $F(p_1, p_2)$ is obtained in three steps. First, the binding downward incentive constraint from $p \geq \hat{\pi}$ to $p' \in [\chi(p), p)$ implies that, for all $p \geq \hat{\pi}$, $F_j(p'|p) = 1 - \chi(p)/p'$ for all $p' \in [\chi(p), p)$, and $F_j(p|p) = 1 - (\chi(p) - 2x)/p$. This pins down the conditional distributions $F_j(p'|p)$ for $p \geq \hat{\pi}$ and $p' \leq p$. Second, by symmetry, we have $f_j(p'|p)f_i(p) = f_j(p|p')f_i(p')$ for all p and $p' \leq p$, where $f_i(p)$ is the marginal density and $f_j(p'|p)$ is the conditional density. This symmetry condition pins down $F_j(p'|p)$ for $p' > p$. Third, the marginal distribution $F_i(p)$ is determined by the equation

$$(1 - F_j(p|p))f_i(p) = \int_p^1 f_j(p|p')f_i(p')dp' \qquad \forall p \in \operatorname{supp} F_i,$$

which again holds by symmetry. Together, the marginal distribution $F_i(p_i)$ and the conditional distributions $F_j(p_j|p_i)$ determine the joint distribution $F(p_1, p_2)$.

Extending Proposition 6 to $n \geq 3$ firms seems challenging. For any n, downward incentive constraints depend only on the joint distribution of a bidder's own recommended bid and the minimum (winning) recommended bid. When $n = 2$, the optimal *(own bid, winning bid)* distribution that satisfies downward incentive constraints is always implementable by a symmetric bid distribution. In contrast, when $n \geq 3$, we have verified numerically that the optimal such *(own bid, winning bid)* distribution is not always implementable by a symmetric distribution. Thus, when $n \geq 3$ global constraints on the set of implementable *(own bid, winning bid)* distributions bind. These constraints seem difficult to handle analytically.

5.3 Upward Incentive Constraints and Profit Comparisons

We finally consider optimal equilibria when $n = 2$ and both upward and downward incentive constraints bind. As we have seen, this case arises when $x < 1/3$ and shill bids are infeasible.

For example, this case may apply to auctions with a fixed set of officially registered bidders.

We first show that, for any penalty size $x > 0$, it is optimal to occasionally recommend a winning bid of 1. Thus, an arbitrarily small amount of cartel "enforcement power" implies a positive probability that the winning bid is as high as the reserve price. To avoid continuity issues, we establish this result in a discretized version of the model, where recommended bids are restricted to lie on the grid $\mathbf{N} = \left\{0, \frac{1}{N}, \frac{2}{N}, \ldots, 1\right\}^2$. We continue to let firms contemplate deviations to arbitrary (continuous) prices, so the cartel's problem in the discretized model is strictly more constrained than that in the continuous model.

Proposition 7 *In the discretized model with any grid size $1/N < x$, every optimal bid distribution puts positive probability on the bid profile $p_1 = p_2 = 1$.*

The intuition for Proposition 7 is as follows. First, an optimal equilibrium exists in the discretized model, by a standard compactness argument. Second, in every optimal equilibrium the firms are simultaneously recommended the maximum price in the support, $\bar{p}$, with positive probability. This follows because it is suboptimal to recommend a bid that is surely losing. Third, if $\bar{p} < 1$ then the bid distribution can be modified so that the probability mass on the bid recommendation pair $(\bar{p}, \bar{p})$ is split among the pairs $(\bar{p}, \bar{p})$, $\left(\bar{p}, \bar{p} + \frac{1}{N}\right)$, $\left(\bar{p} + \frac{1}{N}, \bar{p}\right)$, and $\left(\bar{p} + \frac{1}{N}, \bar{p} + \frac{1}{N}\right)$, in a way that increases profit and preserves incentive compatibility.

We next illustrate numerically how the optimal bid distribution depends on x.[26] Figures 2–5 displays heat maps for the optimal bid distribution for four values of x: .1, .2, .3, and .32.[27] When $x = .32$, the shape of the optimal distribution is similar to that when $x > 1/3$ (cf. Figure 1), except that the support of the conditional $F_j(p_j|p_i)$ is now larger for the smallest values of p_i in the support of the marginal $F_i(p_i)$ than it is for slightly larger values of p_i. Intuitively, upward incentive constraints now bind for the smallest recommended bids, and recommending a wider range of opposing bids relaxes these constraints.

[26] An earlier version of this paper outlined how the optimal bid distribution with binding upward incentive constraints can be characterized as the solution to a system of differential equations. Since the characterization in this case is quite complicated, we omit it here.

[27] These figures are all given for the case where shill bids are infeasible, so upward incentive constraints bind. Analogous figures with only downward incentive constraints—which illustrate the bid distribution derived in Proposition 6—are given in Appendix C.

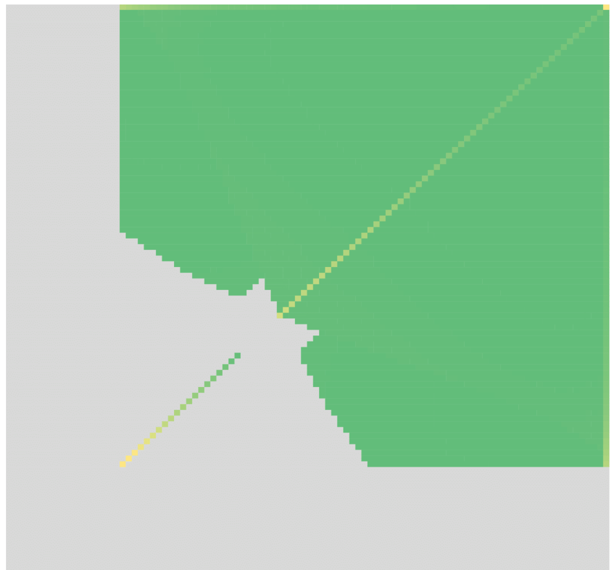

Figure 2: The optimal bid distribution when $x = .1$. The color scheme in Figures 2–5 is the same as in Figure 1. When $x = .1$, the bid pair $p_1 = p_2 = 1$ is recommended only with probability approximately .02.

Figure 3: The optimal bid distribution when $x = .2$. Bid pair $p_1 = p_2 = 1$ is recommended with probability .11.

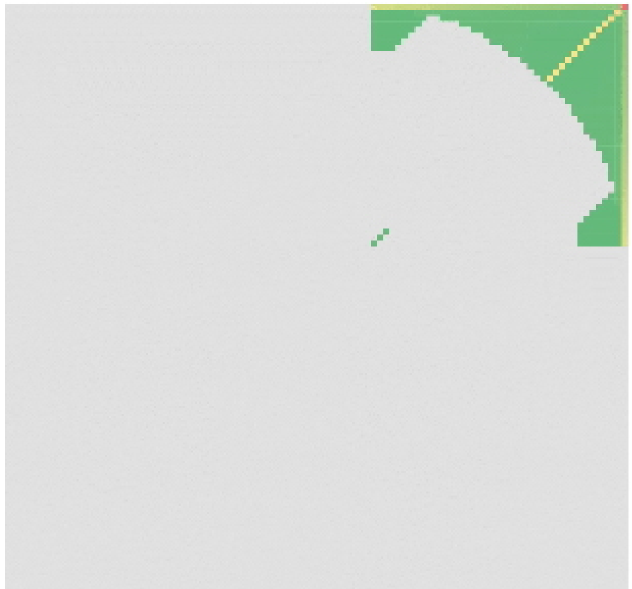

Figure 4: The optimal bid distribution when $x = .3$. Bid pair $p_1 = p_2 = 1$ is recommended with probability .29.

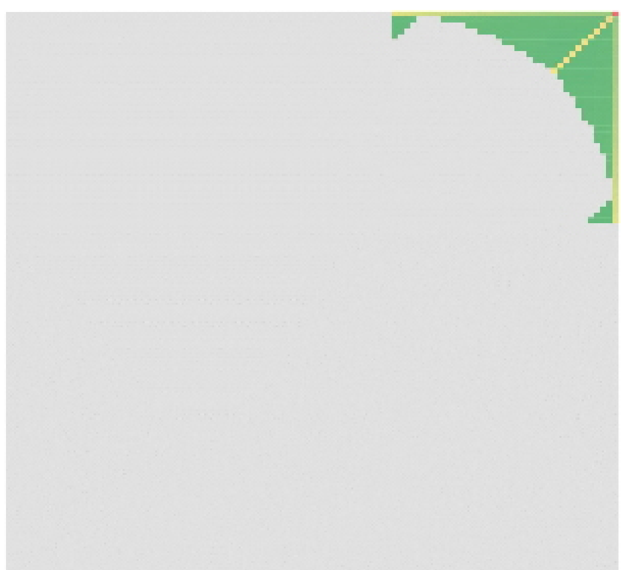

Figure 5: The optimal bid distribution when $x = .32$. Bid pair $p_1 = p_2 = 1$ is recommended with probability .34. Note that the support of the bid distribution is now connected.

When $x \in \{.1, .2, .3\}$, the optimal bid distribution again has a similar shape, except that now $(p, p) \in \operatorname{supp} F(p_1, p_2)$ for an interval of bids p including $2x$. Intuitively, occasionally recommending a tied bid at p is another way to deter upward deviations from p, and when x is small this can be the most efficient way to deter such deviations. As the figures show, if x is above a threshold (approximately equal to .31), $F(p_1, p_2)$ is supported on a single connected subset of $\mathbb{R}^2$, which includes the point $(1, 1)$; while if x is below this threshold, $F(p_1, p_2)$ is supported on the union of two connected subsets of $\mathbb{R}^2$, of which one includes $(1, 1)$ and the other consists exclusively of tied bid pairs. Note also that as x shrinks, the distribution $F(p_1, p_2)$ converges (in distribution) toward a point mass on $p_1 = p_2 = 0$, but the set $\operatorname{supp} F(p_1, p_2) \cup \{(p_1, p_2) : p_1 = p_2\}$ only expands, consistent with Proposition 7.

Figure 6 compares cartel profit across the different settings we have considered, for two firms and any value for x. The top (orange) curve is the upper bound $2\sqrt{x - x^2}$ derived in Proposition 3. The next (green) curve is the solution to equation (4), which by Proposition 6 is optimal cartel profit with only downward incentive constraints. The next two curves are optimal profit with all incentives constraints (the dark blue curve, which coincides with the green curve for $x \geq 1/3$ and is computed numerically for $x < 1/3$) and the optimal equilibrium with almost-uniform bids with only downward incentive constraints (the yellow curve, which is below the dark blue curve for x above approximately .2 and is always below the green curve). The next (grey) curve is the optimal equilibrium with almost-uniform bids with all incentive constraints, derived in Proposition 4 (which coincides with the yellow curve for $x \geq 1/3$). The last (light blue) curve is the line $2x$, which equals optimal cartel profit without mediation by Proposition 1. The value of mediation for the cartel is thus the gap between the light blue curve and the dark blue curve (if shill bids are infeasible) or the green curve (if shill bids are feasible).

6 Non-Stationary Strategies: Comparison to Bid-Rotation

So far, we have analyzed optimal mediated collusion in a one-shot auction game augmented with a penalty of exogenous size x. As explained in Section 2.3, this analysis captures collusion in a stationary, symmetric equilibrium of a repeated auction game. However,

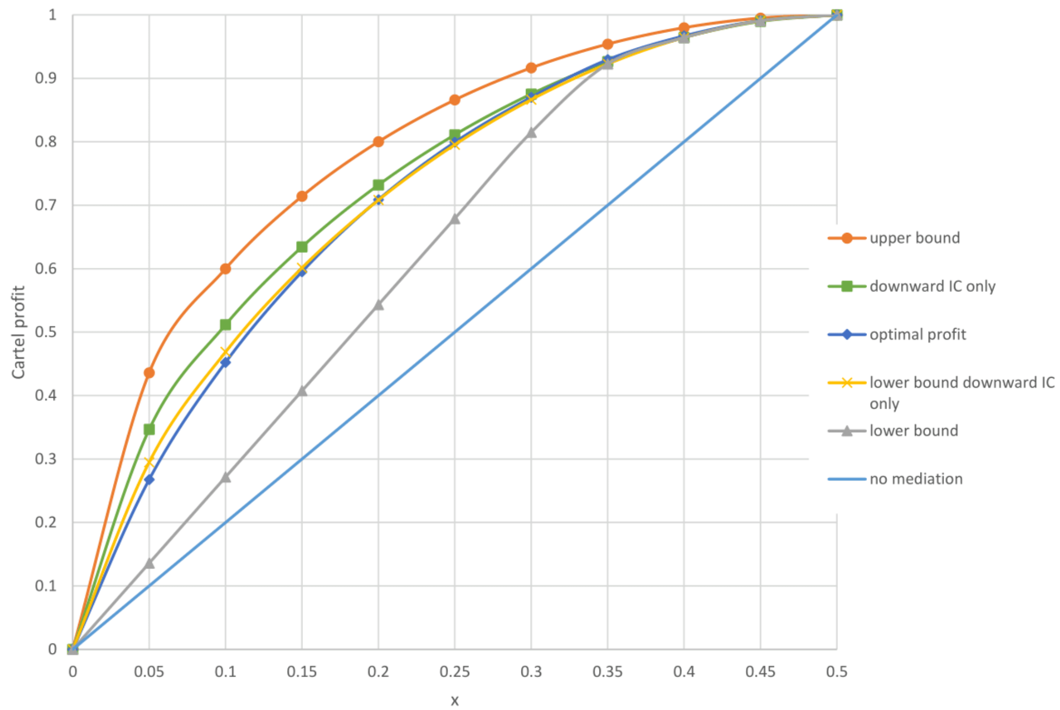

Figure 6: Cartel profit with two bidders.

optimal repeated-game equilibria are typically non-stationary. Characterizing optimal, non-stationary, mediated, repeated-game equilibria is a daunting problem that is beyond the scope of this paper. However, it is interesting to compare stationary, symmetric equilibria with a particularly simple and realistic class of non-stationary strategies, namely *bid rotation* equilibria, where each firm $i \in \{1, \ldots, n\}$ is supposed to win the auction with probability 1 in every period $t = i \bmod n$, but the mediator can randomize the recommended winning bid to deter deviations by the designated losing bidders. Bid rotation equilibria are studied by Kawai, Nakabayashi, and Ortner (2022), who show that the Kumatori Contractors Cooperative relied on equilibria of this form. As compared to stationary equilibria, bid rotation equilibria have the advantage of promising an extra future reward for firms that are supposed to lose in the current period; however, they have the disadvantage that each firm knows whether it is supposed to win or lose in any period. Somewhat surprisingly, it turns out that when all bids are disclosed the latter effect always dominates: optimal stationary equilibria give higher cartel profit than optimal bid rotation equilibria.[28] Moreover, to establish this result, it suffices to consider the stationary, almost-uniform bids equilibria introduced in Section 5.1, rather than (more complex) optimal stationary equilibria. For simplicity, we state this result for the case with only downward incentive constraints.

Proposition 8 *In the repeated game model of Section 2.3 with all bids disclosed, for any number of bidders n and any discount factor δ, optimal cartel profit in a stationary, symmetric equilibrium with almost-uniform bids is greater than cartel profit in any bid rotation equilibrium.*

Proof. By equation (1) and Proposition 4, optimal cartel profit in a stationary, symmetric equilibrium with almost-uniform bids equals 1 if $\delta \geq \frac{1}{2}$, and otherwise is given by the greatest solution (π, x) to the system of equations

$$\begin{aligned} \pi &= \frac{(n-1)\,x}{n-1-x}\left(1-\frac{n}{n-1}\log\left(\frac{nx}{n-1}\right)\right), \\ x &= \frac{\delta}{1-\delta}\frac{\pi}{n}. \end{aligned}$$

Meanwhile, optimal cartel profit in a bid rotation equilibrium is given by the greatest solution (π, x) to the system of equations

$$\begin{aligned} \pi &= \min\left\{x\,(1-\log x)\,, 1\right\}, \\ x &= \frac{\delta^{n-1}}{1-\delta^{n}}\pi. \end{aligned}$$

Here, the first equation holds because, for a given penalty $x \leq 1$ for a losing bidder, the highest winning-bid distribution that deters a deviation by this bidder is

$$F(p) = \begin{cases} 1-\frac{x}{p} & \text{if } p \in [x, 1)\,, \\ 1 & \text{if } p = 1. \end{cases}$$

[28] When losing bids are not disclosed, $\pi_i(x)$ is linear in x in both the optimal stationary, symmetric equilibrium and the optimal bid rotation equilibrium. Hence, for each equilibrium there is a cutoff discount factor δ^* such that first-best profit is attainable if $\delta \geq \delta^*$ and only zero profit is attainable if $\delta < \delta^*$. It can be shown that the cutoff discount factor for the optimal stationary equilibrium is lower than that for the optimal bid rotation equilibrium under *winner's bid disclosed* with any n, as well as under *winner's identity disclosed* with $n \geq 3$.

The second equation holds because the losing bidder whose turn to win the auction is the farthest away (i.e., in $n-1$ periods) has an equilibrium continuation payoff of $\frac{\delta^{n-1}}{1-\delta^n}\pi$.

We show in the appendix that, for any $\delta < \frac{1}{2}$, the solution π to the first system is greater than the solution to the second. When $n \geq 3$, this follows because $\frac{\delta}{(1-\delta)n} \geq \frac{\delta^{n-1}}{1-\delta^n}$ (for any $\delta < \frac{1}{2}$), so the optimal stationary equilibrium with almost-uniform bids gives both a higher profit π for any given penalty size x, and a higher penalty size x for any given profit π. When $n = 2$, $\frac{\delta}{(1-\delta)n} < \frac{\delta^{n-1}}{1-\delta^n}$, but a simple calculation shows that the stationary equilibrium still gives higher profit. ■

We also note that, when $n = 2$, optimal stationary equilibrium profit under mediation is always weakly greater—and sometimes strictly greater—than equilibrium profit in any, potentially non-stationary, unmediated equilibrium. This follows because optimal stationary equilibrium profit equals 1 for all $\delta \geq 1/2$ and is strictly positive for some $\delta < 1/2$, while unmediated cartel profit equals 0 for all $\delta < 1/2$.[29] When $n > 2$, the same comparison holds if we restrict attention to pure equilibria in the unmediated game. This follows because optimal stationary equilibrium profit equals 1 for all $\delta \geq (n-1)/n$ and is strictly positive for some $\delta < (n-1)/n$, while unmediated cartel profit in any pure equilibrium equals 0 for all $\delta < (n-1)/n$.[30] Thus, if $n = 2$ or if firms play pure strategies in the absence of mediation, our conclusion that the cartel benefits from mediation when all bids are disclosed does not rely on restricting attention to stationary equilibria.

7 Conclusion

This paper has introduced and analyzed the problem of how colluding firms maximize profit when they are assisted by an intermediary that privately recommend bids and can punish firms that disobey their recommendations. We focus on a static design problem, which captures the optimal stationary, symmetric equilibrium in a repeated auction game. We find that mediation and the disclosure of losing bids at auction are complements for the cartel:

[29] We omit the proof that unmediated cartel profit equals 0 when $n = 2$ and $\delta < 1/2$, which is fairly straightforward.

[30] The proof that unmediated cartel profit in any pure equilibrium equals 0 whenever $\delta < (n-1)/n$ is straightforward. Restricting to pure strategies in the repeated game plays a similar role as restricting to equilibria where all firms make positive expected profits in Proposition 1.3.

the cartel benefits from the disclosure of losing bids only when it is assisted by a mediator, and the cartel benefits from employing a mediator only when losing bids are disclosed. Among other results, we are able to fully characterize the cartel-optimal bid distribution with two symmetric firms, when bids are disclosed and upward incentive constraints are slack. When upward incentive constraints bind—which occurs when the maximum penalty is small and the auction environment precludes shill bidding—the optimal bid distribution can be found numerically. No matter how small the maximum penalty, the winning bid is close to the reserve price with positive probability. Moreover, the stationary mediation schemes we consider are always more profitable for the cartel than bid rotation.

Our characterization of the cartel-optimal equilibrium with disclosed bids relies on the strong assumption that there are two firms with identical and commonly-known production costs. It may be possible to relax these assumptions in future work, although the analytic characterization of the optimal equilibrium is likely quite complicated. Another direction for future theoretical work is introducing additional frictions in the auction environment, which could rationalize why governments often choose auction formats that disclose so much bid information, despite the risk of facilitating collusion. For example, if the government is concerned about corruption on the part of the auctioneer (e.g., Compte, Lambert-Mogiliansky, and Verdier, 2005), disclosing bids may be optimal, both for the usual reason that this can allow the government to monitor the auctioneer for corruption, and also because it allows the cartel mediator to monitor cartel members for collusion with the auctioneer. While the latter effect can increase cartel profit by facilitating collusion among bidders, this could still be cheaper for society than allowing collusion between bidders and a corrupt auctioneer, which cedes rents to the auctioneer in addition to the bidders.

Our results also suggest several directions for empirical work. First, since we find that mediation and the disclosure of losing bids at auction are complements for the cartel, it would be interesting to see if cartels are more likely to employ mediators in settings where losing bids are disclosed.[31] Second, as discussed in Section 5.2, it would be interesting to investigate

[31] The empirical variation required to test this hypothesis may exist, as some firms participate in many different cartels, of which some are mediated and some are unmediated. For examples, see, e.g., Kovacic, Marshall, and Meurer (2018). In practice, some cartel mediators also facilitate collusion by publicizing bids, for example by auditing cartel members' books. A mediator that can publicize bids is valuable for the cartel under all of the bid disclosure regimes we consider.

the implications of our results for screens for collusive bidding. For example, do mediated cartels exhibit more unexplained parallel price movements, as the theory predicts? Finally, it would also be interesting to analyze whether collusion that is coordinated through pricing algorithms or other automated systems resembles the bidding patterns we have characterized.

A Omitted Proofs

A.1 Proof of Proposition 1.3

For the upper bound, fix an equilibrium bid profile cdf F, and let $S \subseteq \{1, \ldots, n\}$ denote the set of firms that make positive expected profits. We consider separately the cases where $|S| = 1$ and where $|S| \geq 2$, and show that in each case cartel profit is at most $\min\{2x, 1\}$.

If $|S| = 1$, note that the winning firm $i \in S$ must bid at or below x with positive probability, as otherwise another firm could obtain a positive profit by bidding just above x and facing punishment. Since the winning firm must be indifferent among all bids in the support of its equilibrium strategy, its profit—and hence cartel profit—is at most x, which is less than $\min\{2x, 1\}$.

Now suppose that $|S| \geq 2$. For each firm $i \in S$, let $\pi_i > 0$ be the firm's expected payoff, and let $\bar{p}_i = \max \operatorname{supp}(F_i)$ be the firm's highest equilibrium bid. (Here F_i denotes the marginal of F on p_i.) Since $\pi_i > 0$ for each $i \in S$, there exists $\bar{p}$ such that $\bar{p}_i = \bar{p}$ for all $i \in S$, as otherwise there exists a firm $i \in S$ for which bidding $p_i = \bar{p}_i$ is optimal and yet this bid never wins. Let $\alpha_i = \Pr(p_j = \bar{p}\ \forall j \in S : j \neq i)$. We have $\frac{\alpha_i}{|S|}\bar{p} \geq \alpha_i \bar{p} - x$ (as firm i weakly prefers bidding $\bar{p}$ to bidding just below $\bar{p}$) and $\frac{\alpha_i}{|S|}\bar{p} \geq \pi_i$ (as bidding $\bar{p}$ gives firm i its equilibrium payoff of π_i, possibly following some punishment). By the first inequality, $\alpha_i \bar{p} \leq \frac{|S|}{|S|-1}x$. Hence, by the second inequality, $\pi_i \leq \frac{1}{|S|-1}x$. Therefore, cartel profit π satisfies $\pi = \sum_{i \in S} \pi_i \leq \min\left\{\frac{|S|}{|S|-1}x, 1\right\}$. Finally, since $\min\left\{\frac{|S|}{|S|-1}x, 1\right\}$ is decreasing in $|S|$, and $|S| \geq 2$ by hypothesis, we have $\pi \leq \min\{2x, 1\}$.

If $n = 2$, then the lower and upper bounds coincide. Moreover, if all firms make positive profits, then $|S| = n$, and hence $\pi \leq \min\left\{\frac{n}{n-1}x, 1\right\}$, so again the lower and upper bounds coincide.

A.2 Proof of Proposition 3

Fix a canonical equilibrium with bid profile cdf $F(\mathbf{p})$ and winning bid cdf $W(p)$. A uniform downward deviation with cutoff $p^* \in [0, 1]$ (where $W(p)$ is continuous at p^*) now gives an expected payoff of

$$\int_0^{p^*} \frac{p}{n} dW(p) + (1 - W(p^*))p^* - (1 - F_i(p^*))x, \tag{5}$$

where the difference from (2) and (3) is that now the firm is punished with probability $1 - F_i(p^*)$. Since this deviation must be unprofitable in equilibrium, we have

$$\begin{aligned} \int_0^{p^*} \frac{p}{n} dW(p) + (1 - W(p^*))p^* - (1 - F_i(p^*))x &\leq \int_0^1 \frac{p}{n} dW(p) \qquad \Longleftrightarrow \\ 1 - \frac{1}{p^*}\left(\int_{p^*}^1 \frac{p}{n} dW(p) + (1 - F_i(p^*))x\right) &\leq W(p^*). \end{aligned} \tag{6}$$

Note next that for all p, $W(p) = \Pr(\exists i : p_i \leq p) \leq nF_i(p)$, by union bound and symmetry, so $F_i(p) \geq \frac{1}{n}W(p)$. Using this in (6), we get that for all p^* where $W(\cdot)$ is continuous,

$$\begin{aligned} 1 - \frac{1}{p^*}\left(\int_{p^*}^1 \frac{p}{n} dW(p) + (1 - \frac{1}{n}W(p^*))x\right) &\leq W(p^*) \qquad \Longleftrightarrow \\ np^* - 1 - nx + \int_{p^*}^1 W(p)dp &\leq W(p^*)(np^* - p^* - x), \end{aligned} \tag{7}$$

where the second line follows since, by integration by parts, $\int_{p^*}^1 p dW(p) = 1 - W(p^*)p^* - \int_{p^*}^1 W(p)dp$. Moreover, note that since (7) holds for all p^* where W is continuous, it must also hold for all $p^* \in (x/(n-1), 1]$ where W is discontinuous.[32]

Now define an operator Φ, mapping cdf's on $[0,1]$ to cdf's on $[0,1]$, as

$$\Phi(W)(p) = \begin{cases} 0 & \text{if } p \leq \frac{x}{n-1}, \\ \max\left\{0, \min\left\{\frac{1}{np-p-x}\left(np - 1 - nx + \int_p^1 W(\tilde{p})d\tilde{p}\right), 1\right\}\right\} & \text{if } p \in (\frac{x}{n-1}, 1), \\ 1 & \text{if } p = 1. \end{cases}$$

By (7), for any winning bid distribution W,

$$W(p) \geq \Phi(W)(p) \qquad \forall p \in [0,1]. \tag{8}$$

[32]Suppose W is discontinuous at $\hat{p} \in (x/(n-1), 1]$, so that $W(\hat{p}^-) < W(\hat{p})$. Since (7) holds for all continuity points $p^* < \hat{p}$, taking the limit $p^* \uparrow \hat{p}$ on both sides of (7), we get $n\hat{p} - 1 - nx + \int_{\hat{p}}^1 W(p)dp \leq W(\hat{p}^-)(\hat{p}(n-1) - x) < W(\hat{p})(\hat{p}(n-1) - x)$, where the last inequality uses $\hat{p} > x/(n-1)$.

In particular, profits under $\Phi(W)$ are weakly larger than under W. Note also that the operator Φ is monotone: i.e., $W \geq_{FOSD} \hat{W} \implies \Phi(W) \geq_{FOSD} \Phi(\hat{W})$.

Consider the problem of finding the winning bid distribution W that maximizes cartel profit, subject to (8). Since any equilibrium winning bid distribution satisfies (8), the solution to this problem gives an upper bound for optimal cartel profit, π^*. We now show that the solution to this problem is a cdf W^* that satisfies $W^* = \Phi(W^*)$, and that cartel profit under W^* equals $\frac{n}{n-1}\left(x + ((n-1)x)^{\frac{n-1}{n}}(n-1-x)^{\frac{1}{n}}\right) - nx$.

Fix any winning bid distribution W satisfying $W \geq_{FOSD} \Phi(W)$, and consider the sequence of cdfs (W^k) with $W^0 = W$ and $W^{k+1} = \Phi(W^k)$ for all $k \geq 0$. Note that, since Φ is monotone, $W^{k+1} \geq_{FOSD} W^k$ for all k. We now show that sequence (W^k) converges in distribution to a cdf W^*, independent of the initial W. Hence, W^* solves our relaxed problem.

Since (W^k) is a decreasing sequence, we have that for all $p \in (x/(n-1), 1)$ and all $k \geq 1$,

$$\begin{aligned} W^{k+2}(p) &= \max\left\{0, \min\left\{\frac{1}{np-p-x}\left(np-1-nx+A^{k+1}(p)\right), 1\right\}\right\} \\ \geq W^{k+1}(p) &= \max\left\{0, \min\left\{\frac{1}{np-p-x}\left(np-1-nx+A^{k}(p)\right), 1\right\}\right\} \end{aligned}$$

where, for each k, $A^k(p) = \int_p^1 W^k(\tilde{p})d\tilde{p}$. Since $A^k(p)$ is decreasing in k and bounded (because $\left(W^k\right)$ is decreasing), it converges to some A^*. Hence, for all $p \in (x/(n-1), 1)$,

$$\lim_{k\to\infty} W^k(p) = W^*(p) = \max\left\{0, \min\left\{\frac{1}{np-p-x}\left(np-1-nx+A^*(p)\right), 1\right\}\right\}.$$

Moreover, it is clear that $\lim_{k\to\infty} W^k(p) = W^*(p) = 0$ for $p \leq x/(n-1)$, and that $\lim_{k\to\infty} W^k(1) = W^*(1) = 1$. Note next that, by dominated convergence, and since $\left(W^k\right)$ convergences pointwise to W^*,

$$A^* = \lim_{k\to\infty} A^k(p) = \lim_{k\to\infty} \int_p^1 W^k(\tilde{p})d\tilde{p} = \int_p^1 W^*(\tilde{p})d\tilde{p}.$$

Hence, $W^* = \Phi(W^*)$.

The unique solution to the equation $W^* = \Phi(W^*)$ is

$$W^*(p) = \begin{cases} 0 & \text{if } p < \underline{p}, \\ 1 - \frac{(n-1)x(n-1-x)^{\frac{1}{n-1}}}{(np-p-x)^{\frac{n}{n-1}}} & \text{if } p \in [\underline{p}, 1), \\ 1 & \text{if } p = 1, \end{cases}$$

where $\underline{p} = \frac{1}{n-1}\left(x + ((n-1)x)^{\frac{n-1}{n}}(n-1-x)^{\frac{1}{n}}\right)$ is the lowest point in the support of W^*. This follows because, for any p such that $W(p) \in (0,1)$, W is differentiable with derivative satisfying $W'(p) = n(1 - W(p))/(np - p - x)$, and $\lim_{p \to 1} W(p) = (n - 1 - nx)/(n - 1 - x)$, and solving this differential equation yields the desired equation. We thus have

$$W^*(\underline{p}) = 0 = \Phi(W^*)(\underline{p}) = \frac{1}{\underline{p}(n-1) - x}\left(n\underline{p} - 1 - nx + \int_{\underline{p}}^1 W^*(p)dp\right) \qquad \Longleftrightarrow$$
$$\int_{\underline{p}}^1 pdW^*(p) = n\left(\underline{p} - x\right) = \frac{n}{n-1}\left(x + ((n-1)x)^{\frac{n-1}{n}}(n-1-x)^{\frac{1}{n}}\right) - nx,$$

where the second line uses $\int_{\underline{p}}^1 W^*(p)dp = 1 - \int_{\underline{p}}^1 pdW^*(p)$. Thus, $\frac{n}{n-1}\left(x + ((n-1)x)^{\frac{n-1}{n}}(n-1-x)^{\frac{1}{n}}\right) - nx$ is the solution to the relaxed problem, and hence is an upper bound for π^*.

A.3 Proof of Proposition 4

Fix an equilibrium with almost-uniform bids. Let $\alpha = \Pr(p_j = \bar{p}\ \forall j \neq i | p_i = \bar{p})$, the probability that a bidder who is recommended bid $\bar{p}$ has the lowest bid, and let $\beta = \Pr(p_i = \bar{p}\ \forall i)$. Note that Bayes' rule gives $\beta = \frac{(n-1)\alpha}{n-\alpha}$, because

$$\alpha = \Pr(p_j = \bar{p}\ \forall j \neq i | p_i = \bar{p}) = \frac{\Pr(p_j = \bar{p}\ \forall j)}{\Pr(p_i = \bar{p})} = \frac{\beta}{\beta + \frac{n-1}{n}(1-\beta)},$$

where $\Pr(p_i = \bar{p}) = \beta + \frac{n-1}{n}(1-\beta)$ by symmetry and the assumption that at most one bidder bids below $\bar{p}$.

Let $\bar{F}$ denote the cdf of the random variable $\min_{j \neq i} p_j$ conditional on the event $p_i = \bar{p}$. Since a firm that is recommended bid $\bar{p}$ wins the auction with equilibrium probability α/n, the incentive constraint for this firm is

$$\frac{\alpha\bar{p}}{n} \geq \left(1-\bar{F}(p)\right)p - x \quad \text{for all } p < \bar{p}. \tag{9}$$

For fixed values of x and $\bar{p}$, the greatest distribution $\bar{F}$ (in the FOSD sense) that satisfies this constraint is given by $\alpha = \frac{n}{n-1}\frac{x}{\bar{p}}$ and

$$\bar{F}(p) = \begin{cases} 0 & \text{if } p < \frac{n}{n-1}x, \\ 1 - \frac{n}{n-1}\frac{x}{p} & \text{if } p \in \left[\frac{n}{n-1}x, \bar{p}\right), \\ 1 & \text{if } p \geq \bar{p}. \end{cases}$$

Therefore, in an optimal equilibrium with an almost-uniform bid of $\bar{p}$, the conditional distribution of $\min_{j\neq i} p_j$ is given by this distribution $\bar{F}$.

Now note that

$$\mathbb{E}\left[\min_{j\neq i} p_j | \min_{j\neq i} p_j < \bar{p}\right] = \frac{1}{1-\alpha}\int_0^{\bar{p}} p d\bar{F}(p) = \frac{1}{1-\alpha}\int_{\frac{n}{n-1}x}^{\bar{p}} \frac{n}{n-1}\frac{x}{p}dp = \frac{n}{n-1}\frac{x}{1-\alpha}\left(-\log\left(\frac{n}{n-1}\frac{x}{\bar{p}}\right)\right).$$

Therefore, cartel profit in an optimal equilibrium with an almost-uniform bid of $\bar{p}$ equals

$$\pi = \beta\bar{p} + (1-\beta)\frac{n}{n-1}\frac{x}{1-\alpha}\left(-\log\left(\frac{n}{n-1}\frac{x}{\bar{p}}\right)\right) = \frac{(n-1)x\bar{p}}{(n-1)\bar{p}-x}\left(1 - \frac{n}{n-1}\log\left(\frac{n}{n-1}\frac{x}{\bar{p}}\right)\right). \tag{10}$$

Note that (10) is increasing in $\bar{p}$ whenever $\bar{p} \geq \frac{n}{n-1}x$. Therefore, the optimal equilibrium with almost-uniform bids is given by maximizing $\bar{p}$, subject to the remaining incentive constraints. The remaining constraints are those for a firm that is recommended $p < \bar{p}$. In an equilibrium with with almost-uniform bids, such a firm knows that its opponent bids $\bar{p}$, so the most tempting deviation is to bid just below $\bar{p}$. In turn, this deviation is most tempting for a firm with the lowest possible recommendation, $\frac{n}{n-1}x$. Hence, the remaining binding incentive constraint is $\frac{n}{n-1}x \geq \bar{p} - x$, or equivalently $\bar{p} \leq \frac{2n-1}{n-1}x$. Therefore, if $x \geq \frac{n-1}{2n-1}$, the optimal equilibrium is given by $\bar{p} = 1$, and cartel profit is given by $\min\left\{\frac{(n-1)x}{n-1-x}\left(1-\frac{n}{n-1}\log\left(\frac{nx}{n-1}\right)\right), 1\right\}$. If instead $x < \frac{n-1}{2n-1}$, then the optimal equilibrium is given by $\bar{p} = \frac{2n-1}{n-1}x$, and cartel profit is given by $\frac{2n-1}{2n-2}x\left(1 - \frac{n}{n-1}\log\frac{n}{2n-1}\right)$.

A.4 Proof of Proposition 5

It remains to show that $\liminf_{n\to\infty} \pi^*(n) \geq \underline{x}(1-\log \underline{x})$, with convention $0 \log 0 = 0$. This is obvious if $\underline{x} = 0$, so suppose that $\underline{x} > 0$. We show that for any $\varepsilon > 0$, if n is sufficiently large then optimal cartel profit with n firms and penalty size $\underline{x}$ is at least $(\underline{x}-\varepsilon)(1-\log(\underline{x}-\varepsilon))$. Since optimal cartel profit is non-decreasing in x and $x(n)$ is non-increasing in n, this completes the proof.

Fix $\varepsilon > 0$, and take any n such that $(1-\varepsilon+\varepsilon(1-\underline{x}))^{n-1} - \underline{x} < 0$. We construct an equilibrium where bidder 1 always wins as follows:

Bidder 1's recommendation p_1 is drawn with cdf $F(p_1) = 1 - \frac{\underline{x}-\varepsilon}{p_1}$.

For each bidder $i \neq 1$, bidder i's recommendation is drawn as follows, independently across bidders $i \neq 1$: With probability $1-\varepsilon$, p_i is drawn uniformly from $[1,2]$. With probability ε, a uniform $[0,1]$ random variable q_i is drawn, and then p_i is determined as follows: If $q_i > p_1$, then $p_i = q_i$; otherwise, p_i is drawn uniformly from $[1,2]$.

This distribution gives cartel profit $(\underline{x}-\varepsilon)(1-\log(\underline{x}-\varepsilon))$. It thus remains to check that it is an equilibrium.

Player 1 does not have incentive to deviate downward, since she always wins. To see that she also does not have incentive to deviate upward, note that if she deviates to $q > p_1$, her net payoff gain is $\Pr(\min_{i\neq 1} p_i \geq q)\, q - \underline{x} - p_1$. This gain is non-positive if $q < p_1 + \underline{x}$. If $q \geq p_1 + \underline{x}$, it is no more than

$$\Pr\left(\min_{i\neq 1} p_i \geq p_1 + \underline{x}\right) - \underline{x} - p_1 \leq (1-\varepsilon+\varepsilon(1-\underline{x}))^{n-1} - \underline{x} - p_1 \leq (1-\varepsilon+\varepsilon(1-\underline{x}))^{n-1} - \underline{x} < 0.$$

(The first inequality holds since, for each $i \neq 1$, $p_i \geq p_1 + \underline{x}$ only if (i) with probability $1-\varepsilon$, p_i is drawn uniformly from $[1,2]$, or (ii) with probability ε, q_i is drawn uniformly from $[0,1]$ and q_i is not included in $[p_1, p_1+\underline{x}]$.)

Player $i \neq 1$ does not have incentive to deviate upward since she always loses. To see that she also does not have incentive to deviate downward, suppose she is recommended $p_i \geq 1$. Then, for each $q \leq 1$,

$$\Pr(p_1 \geq q | p_i) = \frac{(1-\varepsilon)\Pr(p_1 \geq q) + \varepsilon \Pr(p_1 \geq q)\Pr(q_i \leq p_1 | p_1 \geq q)}{1-\varepsilon+\varepsilon\Pr(q_i \leq p_1)} \leq \frac{\Pr(p_1 \geq q)}{1-\varepsilon},$$

and hence the net payoff gain from bidding q is no more than

$$\frac{\Pr\left(p_1 \geq q\right)}{1-\varepsilon} q - \underline{x} = \frac{\underline{x}-\varepsilon}{(1-\varepsilon)\, q} q - \underline{x} = -\frac{\varepsilon\left(1-\underline{x}\right)}{1-\varepsilon} < 0,$$

where the first equality uses $\Pr\left(p_1 \geq q\right) = 1 - F(q) = \frac{\underline{x}-\varepsilon}{q}$.

Next, suppose she is recommended $p_i < 1$. Then, since $q_i = p_i$ for sure in this case, for each $q \leq 1$, (i) if $q \leq p_i$ then $\Pr\left(p_1 \geq q | p_i\right) = 0$, and (ii) otherwise,

$$\Pr\left(p_1 \geq q | p_i\right) = \frac{\Pr\left(p_i \geq p_1 \geq q\right)}{\Pr\left(p_i \geq p_1\right)} = \frac{\frac{\underline{x}-\varepsilon}{q} - \frac{\underline{x}-\varepsilon}{p_i}}{1 - \frac{\underline{x}-\varepsilon}{p_i}}.$$

Hence, the net payoff gain from bidding q is no more than

$$\frac{\frac{\underline{x}-\varepsilon}{q} - \frac{\underline{x}-\varepsilon}{p_i}}{1 - \frac{\underline{x}-\varepsilon}{p_i}} q - \underline{x} = \frac{1 - \frac{q}{p_i}}{1 - \frac{\underline{x}-\varepsilon}{p_i}} \left(\underline{x} - \varepsilon\right) - \underline{x}.$$

Since $p_i \geq \underline{x} - \varepsilon$ (the lowest possible value of p_1), this in turn is no more than $\frac{1-\frac{\underline{x}-\varepsilon}{p_i}}{1-\frac{\underline{x}-\varepsilon}{p_i}} \left(\underline{x} - \varepsilon\right) - \underline{x} \leq -\varepsilon$, as desired.

A.5 Proof of Proposition 6

Primal Linear Program Let $\mathcal{F}$ denote the set of all symmetric cdfs on $[0,1]^2$. The primal program $\mathbf{P}$ that characterizes the optimal equilibrium (considering only downward incentive constraints) is

$$\begin{aligned} &\sup_{F \in \mathcal{F}} \Pi(F) \\ \text{s.t.} \quad & IC(p, p'; F) \geq 0 \qquad \forall p \in \operatorname{supp}\left(F_i\right), p' < p, \end{aligned} \tag{11}$$

where $\Pi(F) := \int_{p,q \leq p} \left(2q - p\mathbf{1}_{p=q}\right) dF\left(p, q\right)$ is cartel profit when firms follow their recommendations, F_i is the marginal of F over p_i and, for all $p > p'$ and $F \in \mathcal{F}$,

$$IC(p, p'; F) := p\left(1 - \frac{1}{2}\left(F_j^-\left(p | p_i = p\right) + F_j\left(p | p_i = p\right)\right)\right) - p'\left(1 - F_j\left(p' | p_i = p\right)\right) + x,$$

where $F_j(p|p_i)$ denotes conditional probability under F, and $F_j^-(p|p_i) := \lim_{p_j \uparrow p} F_j(p_j|p_i)$.[33] To understand the constraint, note that a firm that follows a recommendation to bid p wins the auction with probability $1 - \frac{1}{2}\left(F_j^-(p|p_i = p) + F_j(p|p_i = p)\right)$ (by uniform tie-breaking), while if the firm bids $p' < p$ it wins with probability $1 - \frac{1}{2}\left(F_j^-(p'|p_i = p) + F_j(p'|p_i = p)\right)$ and is penalized.[34] However, since it is better to deviate to a bid just below any mass point of the conditional distribution $F_j(p_j|p_i)$, it is equivalent to impose the apparently tighter constraint where the latter probability is replaced by $1 - F_j(p'|p_i = p)$.

The program characterizing the optimal equilibrium with both downward and upward constraints differs from $\mathbf{P}$ only in that $IC(p, p'; F) \geq 0$ is also imposed for all $p' > p$. We will see that when $x \geq 1/3$ the solution to $\mathbf{P}$ also satisfies these extra constraints.

Optimal Bid Distribution We define a joint distribution of bids $F(p_1, p_2)$, which we will later verify is optimal. Recall that $\hat{\pi}$ is defined as the unique solution to equation (4) in the interval $[2x, 1]$. (This is well-defined because the LHS of (4) is strictly increasing in π over this range, and takes a negative value at $\pi = 2x$ and a positive value at $\pi = 1$.)

Given the definition of $\hat{\pi}$, the function $\chi(p)$ is defined in the following lemma.

Lemma 1 *There exists a unique function $\chi : [\hat{\pi}, 1] \to [0, 1]$ such that, for each $p \in [\hat{\pi}, 1]$,*

$$\frac{2x \log(\chi(p)) + (\hat{\pi} - x)\log(x + \hat{\pi} - \chi(p))}{x + \hat{\pi}} = \log p + \frac{2x \log(\hat{\pi}) + (\hat{\pi} - x)\log(x)}{x + \hat{\pi}} - \log(\hat{\pi}). \tag{12}$$

The function $\chi(p)$ is strictly decreasing and satisfies $\chi(\hat{\pi}) = \hat{\pi}$ and $\chi(1) = 2x$. Moreover, at each $p \in [\hat{\pi}, 1)$, $\chi(p)$ is differentiable, and its derivative $\chi'(p)$ satisfies

$$\frac{p\chi'(p)}{\chi(p)} = \frac{\chi(p) - (x + \hat{\pi})}{\chi(p) - 2x}. \tag{13}$$

Proof. The proofs of all lemmas used in the proof of Proposition 6 are deferred to the end of the proof. ■

[33] Formally, we require that there is a version of the conditional probability $F_j(p'|p)$ that satisfies the constraint for all p, or equivalently that the constraint holds for almost all p for every version of the conditional probability.

[34] Penalizing any firm that deviates from its recommendation is without loss, as this only relaxes incentive constraints.

Given the definitions of $\hat{\pi}$ and $\chi(p)$, the marginal distribution $F_i(p)$ is defined in the following lemma.

Lemma 2 *There exists a unique function $F_i : [0,1] \to [0,1]$ such that $F_i(2x) = 0$, $F_i(1) = 1$, and $F_i(p)$ is differentiable at each $p \in [0,1)$, with derivative $f_i(p)$ satisfying*

$$f_i(p) = \begin{cases} 0 & \text{if } p \in [0, 2x), \\ \frac{1}{p^2} \int_{\chi^{-1}(p)}^{1} \chi(p') f_i(p')\, dp' & \text{if } p \in [2x, \hat{\pi}), \\ \frac{1}{(\chi(p)-2x)p} \int_{p'>p}^{1} \chi(p') f_i(p')\, dp' & \text{if } p \in [\hat{\pi}, 1], \end{cases} \qquad \forall p \in [0,1). \tag{14}$$

Given the definitions of $\hat{\pi}$, $\chi(p)$, and $F_i(p)$, the conditional distribution $F_j(p'|p)$ is defined as follows: for all $p, p' \in [2x, 1]$,

$$F_j(p'|p) = \begin{cases} 0 & \text{if } p \geq \hat{\pi}, p' < \chi(p) \text{ or } p < \hat{\pi}, p' < \chi^{-1}(p), \\ 1 - \frac{\chi(p)}{p'} & \text{if } p \geq \hat{\pi}, p' \in [\chi(p), p), \\ 1 - \frac{\chi(p)-2x}{p} & \text{if } p \geq \hat{\pi}, p' = p, \\ 1 - \frac{\chi(p)-2x}{p} + \frac{1}{f_i(p)} \int_p^{p'} \frac{\chi(\hat{p})}{\hat{p}^2} f_i(\hat{p}) d\hat{p} & \text{if } p \geq \hat{\pi}, p' > p, \\ \frac{1}{f_i(p)} \int_{\chi^{-1}(p)}^{p'} \frac{\chi(\hat{p})}{\hat{p}^2} f_i(\hat{p}) d\hat{p} & \text{if } p < \hat{\pi}, p' > \chi^{-1}(p). \end{cases} \tag{15}$$

Note that, for all $p \in [2x, 1]$, $F_j(p'|p)$ is indeed a cdf. To see this, consider first $p \in [\hat{\pi}, 1]$. By (14) and (15), $F_j(p'|p)$ is increasing and right-continuous in p' for $p' \in [\chi(p), 1]$, with $F_j(\chi(p)|p) = 0$ and $F_j(1|p) = 1$. Similarly, for $p \in [2x, \hat{\pi})$, by (14) and (15) $F_j(p'|p)$ is increasing and continuous in p' for $p' \in [\chi^{-1}(p), 1]$, with $F_j(\chi^{-1}(p)|p) = 0$ and $F_j(1|p) = 1$.

Finally, let $F(p_1, p_2) \in \mathcal{F}$ be the (unique) joint cdf with marginal distribution $F_1(p_1)$ given by (14), and conditional distributions $F_2(p_2|p_1)$ given by (15). Note that

$$\text{supp}(F) = \left\{ (p_i, p_j) \in [0,1]^2 : p_i \in [2x, \hat{\pi}), p_j \in \left[\chi^{-1}(p), 1\right] \text{ or } p_i \in [\hat{\pi}, 1], p_j \in [\chi(p), 1] \right\}.$$

We verify that F is feasible for **P**. We first show that F is symmetric.

Lemma 3 $F(p, p') = F(p', p)$ *for all* $p, p' \in [0,1]$.

We now verify that $IC(p,p';F) \geq 0$ for all $p \geq 2x, p' < p$. This shows that F is feasible for **P**: i.e., F satisfies all downward incentive constraints. Moreover, if $x \geq 1/3$ then in addition $IC(p,p';F) \geq 0$ for all $p \geq 2x, p' > p$, so F is an equilibrium bid distribution.

Lemma 4 *$IC(p,p';F) \geq 0$ for all $p \geq 2x, p' < p$, with equality for all $p \geq \hat{\pi}, p' \in [\chi(p), p)$. Moreover, if $x \geq 1/3$ then in addition $IC(p,p';F) \geq 0$ for all $p \geq 2x, p' > p$.*

Dual Linear Program Let $\mathcal{B}[0,1]$ denote the set of Borel subsets of $[0,1]$, and let $\mathcal{M}$ denote the set of all bounded, measurable functions $\Lambda : [0,1] \times \mathcal{B}[0,1] \to \mathbb{R}_+$ such that, for every $p \in [0,1]$, the induced mapping $P \mapsto \Lambda(p,P)$ (henceforth written as $\Lambda(P|p)$) is a (finite) measure on $\mathcal{B}[0,1]$. Consider the dual linear program **D**, given by

$$\begin{aligned} &\inf_{\Lambda \in \mathcal{M},\, \mu \in \mathbb{R}_+} \mu \\ \text{s.t.} \quad & G(p,q;\Lambda,\mu) \geq 0 \quad \forall p, q \leq p, \end{aligned} \tag{16}$$

where, for all $p > q$, Λ, and $\mu \geq 0$,

$$\begin{aligned} G(p,p;\Lambda,\mu) &:= \int_{p' \leq p} \left(-\frac{p}{2} - x + p'\right) d\Lambda(p'|p) + \mu - p \quad \text{and} \\ G(p,q;\Lambda,\mu) &:= \int_{p' \leq p} \left(-x + p' \mathbf{1}_{p' \leq q}\right) d\Lambda(p'|p) + \int_{p' \leq q} \left(-q - x + p'\right) d\Lambda(p'|q) + 2(\mu - q). \end{aligned}$$

The interpretation of the dual program is that μ is cartel profit (which also equals the shadow value of probability mass on an optimal bid pair), $d\Lambda(p'|p)$ is the value of relaxing the incentive constraint that it is unprofitable for a bidder who is recommended a bid of p to deviate to a bid of p', and $G(p,q;\Lambda,\mu)$ is the cost of increasing the probability of recommending the bid pair (p,q), with $q < p$, where this cost is given by the effect of such an increase on the downward incentive constraints for each $p' < p$ (the first integral in the formula for $G(p,q;\Lambda,\mu)$), plus the effect on the downward incentive constraints for each $p' < q$ (the second integral in the formula for $G(p,q;\Lambda,\mu)$), plus twice the difference between equilibrium cartel profit (equal to μ) and profit at the bid pair (p,q) (equal to q).

Lemma 5 *Let F be feasible for **P**, and let (Λ,μ) be feasible for **D**. We have:*

(i) ***Weak duality:*** $\Pi(F) \leq \mu$.

(ii) ***Complementary slackness:*** *If F and Λ satisfy (1) $IC(p,p';F) = 0$ for all $p, p' < p$ with $p \in \operatorname{supp}(F_i)$ and $p' \in \operatorname{supp}(\Lambda(\cdot|p))$, and (2) $G(p,q;\Lambda,\mu) = 0$ for all $p, q \in \operatorname{supp}(F)$ with $q \leq p$, then $\Pi(F) = \mu$, F is optimal for* $\mathbf{P}$*, and (Λ, μ) is optimal for* $\mathbf{D}$.

Optimal Dual Variables Given Lemma 5, we prove Proposition 6 by finding a function Λ such that $(\Lambda, \hat{\pi})$ is feasible for $\mathbf{D}$, and F and $(\Lambda, \hat{\pi})$ satisfy the complementary slackness conditions in Lemma 5(ii).

For each $p \in [0,1]$, let $\Lambda(\cdot|p)$ be the unique Borel measure on $\mathcal{B}([0,1])$ such that

$$\Lambda([p', p''] \,| p) = \int_{p'}^{p''} \lambda(q|p) dq \quad \forall p', p'' \in [0,1], p' < p'',$$

where for each $p, p' \in [0,1]$,

$$\lambda(p'|p) = \begin{cases} \frac{2(\chi(p')-x)(-\chi'(p'))}{x\chi(p')} & \text{if } p \geq \hat{\pi}, p' \in [\hat{\pi}, p)), \\ \frac{2}{p'} & \text{if } p \geq \hat{\pi}, p' \in [\chi(p), \hat{\pi}), \\ 0 & \text{otherwise.} \end{cases} \tag{17}$$

We verify that $(\Lambda, \hat{\pi})$ is feasible for $\mathbf{D}$.

Lemma 6 *$\Lambda \in \mathcal{M}$, and $G(p,q;\Lambda,\hat{\pi}) \geq 0$ for all $p \geq q$, with equality if $p \geq \hat{\pi}$ and $q \in [\chi(p), p]$.*

Complementary Slackness Finally, we verify that F and $(\Lambda, \hat{\pi})$ satisfy the conditions in Lemma 5(ii). For each $p \in \operatorname{supp}(F_i) = [2x, 1]$, we have $\operatorname{supp}(\Lambda(\cdot|p)) = \emptyset$ if $p < \hat{\pi}$, and $\operatorname{supp}(\Lambda(\cdot|p)) = [\chi(p), p]$ if $p \geq \hat{\pi}$. By Lemma 4, we have $IC(p,p';F) = 0$ for all $p \in \operatorname{supp}(F_i)$ and $p' \in \operatorname{supp}(\Lambda(\cdot|p))$. By Lemma 6, $G(p,q;\Lambda,\hat{\pi}) = 0$ for all $(p,q) \in \operatorname{supp}(F)$ with $p \geq q$. Hence, by Lemma 5(ii), $\Pi(F) = \mu$, F is optimal for $\mathbf{P}$, and (Λ, μ) is optimal for $\mathbf{D}$.

A.5.1 Proof of Lemma 1

To see that $\chi(p)$ is well defined, note that the LHS of (12) is strictly concave in χ over the range $(0, \hat{\pi} + x)$, and attains its maximum at $\chi = 2x$. Moreover, for $\chi = \hat{\pi}$, the LHS of (12) is no more than the RHS for all $p \in [\hat{\pi}, 1]$. Hence, as long as

$$\frac{2x \log(2x) + (\hat{\pi} - x)\log(\hat{\pi} - x)}{x + \hat{\pi}} \geq \log p + \frac{2x \log(\hat{\pi}) + (\hat{\pi} - x)\log(x)}{x + \hat{\pi}} - \log(\hat{\pi}) \quad \forall p \in [\hat{\pi}, 1], \tag{18}$$

equation (12) admits a unique solution $\chi \in [2x, \hat{\pi} + x]$. Since the LHS of (18) is independent of p and the RHS is increasing in p, it suffices that (18) holds for $p = 1$. In turn, this holds by the definition of $\hat{\pi}$. Finally, that $\chi(p)$ is decreasing and differentiable, with derivative satisfying (13) on $[\hat{\pi}, 1)$, follows from the implicit function theorem.

A.5.2 Proof of Lemma 2

For all $p \in [\hat{\pi}, 1]$, define $\alpha(p) = (\chi(p) - 2x)p$. Consider the differential equation

$$h'(p)\alpha(p) + \alpha'(p)h(p) = -\chi(p)h(p), \quad \text{with } h(\hat{\pi}) = 1. \tag{19}$$

The solution to (19) is $h(p) = \exp\left(-\int_{\hat{\pi}}^{p} \gamma(p')dp'\right)$, with $\gamma(p) = (\chi(p) + \alpha'(p))/\alpha(p)$.

We now construct density function $f_i(p)$. For $p \in [\hat{\pi}, 1)$, $f_i(p) = K \times h(p)$ for some constant $K > 0$ to be determined shortly. For $p \in [2x, \pi)$,

$$f_i(p) = \frac{1}{p^2}\int_{\chi^{-1}(p)}^{1} \chi(p')Kh(p')dp' = \frac{1}{p^2}\int_{\chi^{-1}(p)}^{1} \chi(p')f_i(p')dp'.$$

(Note that $h(p)$ is defined over $[\chi^{-1}(p), 1)$ for all $p \in [2x, \pi)$, since $\chi^{-1}(p) \in [\hat{\pi}, 1)$ for all p in this range). Finally, for $p < 2x$, $f_i(p) = 0$.

We now show that $f_i(p)$ satisfies (14). By construction, $f_i(p)$ satisfies (14) for all $p < \hat{\pi}$.

Next, note that $f_i(p)$ solves (19) for $p \in [\hat{\pi}, 1)$, with $f_i(\hat{\pi}) = K$. Hence, for all $p \in [\hat{\pi}, 1)$,

$$\int_p^1 (f_i'(p')\alpha(p') + \alpha'(p) f_i(p'))dp' = -\alpha(p) f_i(p) = -\int_p^1 \chi(p') f_i(p') dp'$$
$$\iff f_i(p) = \frac{1}{(\chi(p) - 2x)p} \int_p^1 \chi(p') f_i(p') dp',$$

where the first line uses $\alpha(1) = 0$. We now pin down constant $K > 0$. Since $F_i(\cdot)$ should satisfy $\int_{2x}^1 f_i(p)dp = F_i(1) - F_i(2x) = 1$, we have that K is the solution to:

$$1 = K \left[\int_{2x}^{\hat{\pi}} \frac{1}{p^2} \int_{\chi^{-1}(p)}^1 \chi(p') h(p') dp' dp + \int_{\hat{\pi}}^1 \frac{1}{(\chi(p) - 2x)p} \int_p^1 \chi(p') h(p') dp' dp \right].$$

A.5.3 Proof of Lemma 3

Without loss, let $p < p'$. We consider two cases: $p \in [2x, \hat{\pi})$ and $p \in [\hat{\pi}, 1]$.

Suppose that $p \in [2x, \hat{\pi})$. We have

$$\begin{aligned}
F(p, p') &= \int_{\hat{p} \le p} F_j(p'|\hat{p}) f_i(\hat{p}) d\hat{p} = \int_{\chi(p')}^{p} \int_{\chi^{-1}(\hat{p})}^{p'} \frac{\chi(\tilde{p})}{\hat{p}^2} f_i(\tilde{p}) d\tilde{p} d\hat{p} \\
&= \int_{\chi^{-1}(p)}^{p'} \int_{\chi(\tilde{p})}^{p} \frac{\chi(\tilde{p})}{\hat{p}^2} d\hat{p} f_i(\tilde{p}) d\tilde{p} = \int_{\chi^{-1}(p)}^{p'} \left(1 - \frac{\chi(\tilde{p})}{p}\right) f_i(\tilde{p}) d\tilde{p} \\
&= \int_{\tilde{p} \le p'} F_j(p|\tilde{p}) f_i(\tilde{p}) d\tilde{p} = F(p', p),
\end{aligned}$$

where the second equality follows from (15) and the fact that $F(p'|\hat{p}) = 0$ for all $\hat{p} < \chi(p')$; the third equality follows from reversing the order of integration; and the fifth equality again follows from (15).

Now suppose that $p \in [\hat{\pi}, 1]$. We have

$$\begin{aligned}
F(p, p') &= \int_{\hat{p} \le p} F_j(p'|\hat{p}) f_i(\hat{p}) d\hat{p} \\
&= \int_{\hat{p} \le p} F_j(p|\hat{p}) f_i(\hat{p}) d\hat{p} + \int_{\hat{p} \le p} (F_j(p'|\hat{p}) - F_j(p|\hat{p})) f_i(\hat{p}) d\hat{p} \\
&= F(p, p) + \int_{\hat{p} \le p} (F_j(p'|\hat{p}) - F_j(p|\hat{p})) f_i(\hat{p}) d\hat{p}. \qquad (20)
\end{aligned}$$

Similarly, we have

$$\begin{aligned}
F\left(p', p\right) &= \int_{\hat{p} \leq p'} F_j\left(p|\hat{p}\right) f_i\left(\hat{p}\right) d\hat{p} \\
&= \int_{\hat{p} \leq p} F_j\left(p|\hat{p}\right) f_i\left(\hat{p}\right) d\hat{p} + \int_p^{p'} F_j\left(p|\hat{p}\right) f_i\left(\hat{p}\right) d\hat{p} \\
&= F\left(p, p\right) + \int_p^{p'} F_j\left(p|\hat{p}\right) f_i\left(\hat{p}\right) d\hat{p} \\
&= F\left(p, p\right) + \int_p^{p'} \left(1 - \frac{\chi\left(\hat{p}\right)}{p}\right) f_i\left(\hat{p}\right) d\hat{p}, \qquad (21)
\end{aligned}$$

where the last equality uses (15). Since $F_j\left(p'|\hat{p}\right) = 0$ for all $\hat{p} < \chi\left(p'\right)$ and $F_j\left(p|\hat{p}\right) = 0$ for all $\hat{p} < \chi\left(p\right)$, and again using (15), we have

$$\begin{aligned}
\int_{\hat{p} \leq p} \left(F_j\left(p'|\hat{p}\right) - F_j\left(p|\hat{p}\right)\right) f_i\left(\hat{p}\right) d\hat{p} &= \int_{\chi(p')}^{\chi(p)} F_j\left(p'|\hat{p}\right) f_i\left(\hat{p}\right) d\hat{p} + \int_{\chi(p)}^{p} \left(F_j\left(p'|\hat{p}\right) - F_j\left(p|\hat{p}\right)\right) f_i\left(\hat{p}\right) d\hat{p} \\
&= \int_{\chi(p')}^{\chi(p)} \int_{\chi^{-1}(\hat{p})}^{p'} \frac{\chi\left(\tilde{p}\right)}{\hat{p}^2} f_i\left(\tilde{p}\right) d\tilde{p} d\hat{p} + \int_{\chi(p)}^{p} \int_p^{p'} \frac{\chi\left(\tilde{p}\right)}{\hat{p}^2} f_i\left(\tilde{p}\right) d\tilde{p} d\hat{p} \\
&= \int_p^{p'} \int_{\chi(\tilde{p})}^{\chi(p)} \frac{\chi\left(\tilde{p}\right)}{\hat{p}^2} d\hat{p} f_i\left(\tilde{p}\right) d\tilde{p} + \int_p^{p'} \int_{\chi(p)}^{p} \frac{\chi\left(\tilde{p}\right)}{\hat{p}^2} d\hat{p} f_i\left(\tilde{p}\right) d\tilde{p} \\
&= \int_p^{p'} \left(1 - \frac{\chi\left(\tilde{p}\right)}{p}\right) f_i\left(\tilde{p}\right) d\tilde{p}.
\end{aligned}$$

Together with (20) and (21), we have $F\left(p, p'\right) = F\left(p', p\right)$.

A.5.4 Proof of Lemma 4

The lemma follows from checking the various cases in (15). In particular, if $p \geq \hat{\pi}$ and $p' \in [\chi(p), p)$, then $F_j\left(p|p\right) = 1 - \frac{\chi(p) - 2x}{p}$ and $F_j\left(p'|p\right) = 1 - \frac{\chi(p)}{p'}$, and hence

$$\begin{aligned}
IC(p, p'; F) &= p\left(1 - \frac{1}{2}\left(F_j^-\left(p|p_i = p\right) + F_j\left(p|p_i = p\right)\right)\right) - p'\left(1 - F_j\left(p'|p_i = p\right)\right) + x \\
&= p\left(\frac{\chi\left(p\right) - x}{p}\right) - p'\left(\frac{\chi\left(p\right)}{p'}\right) + x = 0.
\end{aligned}$$

If $p \geq \hat{\pi}$ and $p' < \chi(p)$, then $F_j(p'|p) = 0$, and hence

$$IC(p, p'; F) = p\left(\frac{\chi(p) - x}{p}\right) - p' + x = \chi(p) - p' > 0.$$

Finally, if $p < \hat{\pi}$ and $p' < p$, then $F_j(p|p) = F_j(p'|p) = 0$, and hence $IC(p, p'; F) = x > 0$.

Now suppose that $x \geq 1/3$. If $p \in [2x, \hat{\pi})$ and $p' > p$, then

$$IC(p, p'; F) = p - p'(1 - F_j(p'|p)) + x \geq 3x - 1 \geq 0,$$

where the first inequality uses $p \geq 2x$ and $p' \leq 1$, and the second uses $x \geq 1/3$. If $p \geq \hat{\pi}$ and $p' > p$, then

$$\begin{aligned} IC(p, p'; F) &= p\frac{\chi(p) - x}{p} - p'\left(\frac{\chi(p) - 2x}{p} - \frac{1}{f_i(p)}\int_p^{p'} \frac{\chi(\hat{p})}{p^2} f_i(\hat{p}) d\hat{p}\right) + x \\ &= \chi(p)\frac{p - p'}{p} + 2x\frac{p'}{p} + \frac{p'}{f_i(p)}\int_p^{p'} \frac{\chi(\hat{p})}{p^2} f_i(\hat{p}) d\hat{p} \\ &\geq \hat{\pi}\frac{\hat{\pi} - 1}{\hat{\pi}} + 2x + \frac{p'}{f_i(p)}\int_p^{p'} \frac{\chi(\hat{p})}{p^2} f_i(\hat{p}) d\hat{p} > 0, \end{aligned}$$

where the weak inequality uses $p \geq \hat{\pi}$, $p' \in (p, 1]$, and $\chi(p) \leq \hat{\pi}$, and the strict inequality follows as $\hat{\pi} > 2x$ and hence $\hat{\pi} - 1 + 2x > 4x - 1 > 0$ (since $x \geq 1/3$).

A.5.5 Proof of Lemma 5

For each $p, q \in [0, 1]^2$ and $p' < p$, define

$$\phi(p, q, p') = p - p' + x - \frac{p}{2}\left(\mathbf{1}_{q \leq p} + \mathbf{1}_{q < p}\right) + p'\mathbf{1}_{q < p'}.$$

Intuitively, $\phi(p,q,p')$ is the payoff loss incurred by a bidder who deviates from p to p' when the opponent bids q. Note that

$$\begin{aligned}
IC(p,p';F) &= \int_q \phi(p,q,p')dF_j(q|p) \qquad \forall p > p', \\
G(p,p;\Lambda,\mu) &= -\int_{p'\le p} \phi(p,p,p')d\Lambda(p'|p) + \mu - p \qquad \forall p, \qquad \text{and} \\
G(p,q;\Lambda,\mu) &= -\int_{p'\le p} \phi(p,q,p')d\Lambda(p'|p) - \int_{p'\le q} \phi(q,p,p')d\Lambda(p'|q) + 2(\mu - q) \qquad \forall p > q.
\end{aligned}$$

Now let F be feasible for $\mathbf{P}$, and let (Λ,μ) be feasible for $\mathbf{D}$. Note that

$$\begin{aligned}
0 &\le \int_{p,q\le p} G(p,q;\Lambda,\mu)\, dF(p,q) \\
&= -\int_{p,q\le p}\left(\int_{p'\le p}\phi(p,q,p')d\Lambda(p'|p) + \mathbf{1}_{q<p}\int_{p'\le q}\phi(q,p,p')d\Lambda(p'|q)\right)dF(p,q) \\
&\quad + \int_{p,q\le p}\left(2(\mu-q) - \mathbf{1}_{q=p}(\mu-p)\right)dF(p,q) \\
&= -\int_{p,q\le p}\left(\int_{p'\le p}\phi(p,q,p')d\Lambda(p'|p) + \mathbf{1}_{q<p}\int_{p'\le q}\phi(q,p,p')d\Lambda(p'|q)\right)dF(p,q) + \mu - \Pi(F),
\end{aligned} \tag{22}$$

where the last equality uses $\int_{p,q\le p}(2-\mathbf{1}_{q=p})dF(p,q) = 1$ (by symmetry of F) and $\int_{p,q\le p}(2q - \mathbf{1}_{q=p}p)dF(p,q) = \Pi(F)$.

Since F is symmetric,

$$\begin{aligned}
\int_{p,q\le p}\mathbf{1}_{q<p}\int_{p'\le q}\phi(q,p,p')d\Lambda(p'|q)\,dF(p,q) &= \int_{p,q<p}\int_{p'\le q}\phi(q,p,p')d\Lambda(p'|q)\,dF(p,q) \\
&= \int_{p,q>p}\int_{p'\le p}\phi(p,q,p')d\Lambda(p'|p)\,dF(p,q).
\end{aligned}$$

Using this in (22), we get

$$
\begin{aligned}
0 &\le -\int_{p,q}\int_{p'\le p}\phi(p,q,p')d\Lambda\left(p'|p\right)dF(p,q)+\mu-\Pi(F)\\
&= -\int_{p}\int_{q}\int_{p'\le p}\phi(p,q,p')d\Lambda\left(p'|p\right)dF(q|p)dF_i(p)+\mu-\Pi(F)\\
&= -\int_{p}\int_{p'\le p}\int_{q}\phi(p,q,p')dF(q|p)d\Lambda\left(p'|p\right)dF_i(p)+\mu-\Pi(F)\\
&= -\int_{p}\int_{p'\le p}IC(p,p';F)d\Lambda\left(p'|p\right)dF_i(p)+\mu-\Pi(F)\\
&\le \mu-\Pi(F), \qquad (23)
\end{aligned}
$$

where the first two equalities follow from Fubini's theorem, which applies as ϕ, Λ, and F are bounded and measurable. This establishes part (i).

Suppose next that F and (Λ,μ) satisfy the conditions in part (ii). Then, the inequalities in (22) and (23) hold with equality, and so $\Pi(F)=\mu$. By part (i), it follows that F is optimal for **P**, and (Λ,μ) is optimal for **D**.

A.5.6 Proof of Lemma 6

To show that $\Lambda\in\mathcal{M}$, it suffices to show that Λ is bounded and measurable: given this, $\Lambda\left(\cdot|p\right)$ is a Borel measure on $\mathcal{B}\left([0,1]\right)$ by construction. Boundedness follows because, for each $p\in[0,1]$, we have

$$
\begin{aligned}
\Lambda([0,1]\,|p) &\le \int_{\chi(p)}^{\hat{\pi}}\frac{2}{q}dq+\int_{\hat{\pi}}^{p}\frac{2\left(\chi\left(q\right)-x\right)\left(-\chi'\left(q\right)\right)}{x\chi(q)}dq\\
&\le \int_{\chi(p)}^{\hat{\pi}}\frac{2}{q}dq+\int_{\hat{\pi}}^{p}\frac{2(-\chi'(q))}{x}dq\\
&\le 2(\log(\hat{\pi})-\log(2x))+\frac{2}{x}(\hat{\pi}-2x),
\end{aligned}
$$

where the first inequality is by definition of λ, the second follows since $\chi\left(q\right)\ge x$, and the third follows since $\chi\left(p\right)\ge 2x$ and $\chi\left(\hat{\pi}\right)-\chi\left(p\right)\le\hat{\pi}-2x$. Measurability follows because $\Lambda([p',p'']\,|p)$ is continuous on $[0,1]^3$ in the weak topology, which suffices for continuity on $[0,1]\times\mathcal{B}\left([0,1]\right)$.

For the proof that G satisfies the desired properties, we ease notation by writing G as a function of p, q only, suppressing the arguments $\Lambda, \hat{\pi}$. We use the following equations, which follow immediately from the definitions of G and Λ.

1. For $p \geq \hat{\pi}$,
$$G(p,p) = \int_{\chi(p)}^{p} \left(-\frac{p}{2} - x + p'\right) \lambda(p'|p)\, dp' + \hat{\pi} - p. \tag{24}$$

2. For $p \geq q \geq \hat{\pi}$,
$$G(p,q) = \int_{\chi(p)}^{p} \left(-x + p' \mathbf{1}_{\{p' \leq q\}}\right) \lambda(p'|p)\, dp' + \int_{\chi(q)}^{q} \left(-q - x + q'\right) \lambda(q'|q)\, dq' + 2(\hat{\pi} - q). \tag{25}$$

3. For $p \geq \hat{\pi} \geq q \geq \chi(p)$,
$$G(p,q) = \int_{\chi(p)}^{p} \left(-x + p' \mathbf{1}_{\{p' \leq q\}}\right) \lambda(p'|p)\, dp' + 2(\hat{\pi} - q). \tag{26}$$

4. For $p \geq \hat{\pi} \geq \chi(p) \geq q$,
$$G(p,q) = 2(\hat{\pi} - q). \tag{27}$$

5. For $p \geq \hat{\pi}$ and $q \in [\hat{\pi}, p)$,
$$\lambda(q|p) = \frac{(\chi(q) - x)(-\chi'(q))\, \lambda(\chi(q)|p)}{x}. \tag{28}$$

We prove the lemma through a series of claims.

Claim 1. For any sequence $(p_n, q_n) \to (\hat{\pi}, \hat{\pi})$ such that $p_n \geq \hat{\pi} \geq q_n$ for all n, we have $\lim_{n\to\infty} G(p_n, q_n) = 0$.

Proof. This follows from two observations. First, $G(\hat{\pi}, \hat{\pi}) = 0$, by (24) and $\chi(\hat{\pi}) = \hat{\pi}$. Second, for any $\pi \in [0,1]$ and any sequence $(p_n, q_n) \to (\pi, \pi)$ such that $p_n \geq \pi \geq q_n$ and $p_n \neq q_n$ for all n, we have $\lim_{n\to\infty} G(p_n, q_n) = 2G(\pi, \pi)$.

Claim 2. $G(p,q) = 0$ for all p, q such that $1 > p \geq \hat{\pi} \geq q \geq \chi(p)$.

Proof. By Claim 1, it suffices to show that $\partial G(p,q)/\partial p = \partial G(p,q)/\partial q = 0$ whenever $p > q$ and $1 > p \geq \hat{\pi} \geq q \geq \chi(p)$. By (26),

$$\begin{aligned}
\frac{\partial G(p,q)}{\partial p} &= -x\lambda(p|p) + (\chi(p) - x)(-\chi'(p))\lambda(\chi(p)|p), \qquad \text{and} \\
\frac{\partial G(p,q)}{\partial q} &= q\lambda(q|p) - 2.
\end{aligned}$$

By (17) and (28), both of these derivatives equal 0.

Claim 3. $G(p,q) = 0$ for all p, q such that $1 > p \geq q \geq \hat{\pi}$.

Proof. Suppose first that $p > q$. By Claim 1, it suffices to show that $\partial G(p,q)/\partial p = \partial G(p,q)/\partial q = 0$ whenever $1 > p > q \geq \hat{\pi}$. By (25),

$$\begin{aligned}
\frac{\partial G(p,q)}{\partial p} &= -x\lambda(p|p) + (\chi(p) - x)(-\chi'(p))\lambda(\chi(p)|p), \qquad \text{and} \\
\frac{\partial G(p,q)}{\partial q} &= q\lambda(q|p) - \int_{\chi(q)}^{q} \lambda(q'|q)\,dq' - x\lambda(q|q) + (\chi(q) - x + q)(-\chi'(q))\lambda(\chi(q)|q) - 2.
\end{aligned}$$

By (17) and (28), $\partial G(p,q)/\partial p = 0$. To show that $\partial G(p,q)/\partial q = 0$, note that $G(q,\chi(q)) = 0$ by Claim 2, so (26) gives

$$\begin{aligned}
G(q,\chi(q)) &= \int_{\chi(q)}^{q} \left(-x + q'\mathbf{1}_{\{q' \leq \chi(q)\}}\right)\lambda(q'|q)\,dq' + 2(\hat{\pi} - q) = 0 \qquad \Longleftrightarrow \\
\int_{\chi(q)}^{q} \lambda(q'|q)\,dq' &= 2\frac{\hat{\pi} - \chi(q)}{x}. \qquad (29)
\end{aligned}$$

Note also that

$$-x\lambda(q|q) + (\chi(q) - x)(-\chi'(q))\lambda(\chi(q)|q) = 0, \qquad (30)$$

since $\partial G(q,q')/\partial q = 0$ for all $q > q' \geq \chi(q)$ by Claim 1. In total, we have

$$
\begin{aligned}
\frac{\partial G(p,q)}{\partial q} &= q\lambda(q|p) - \int_{\chi(q)}^{q} \lambda(q'|q)\, dq' - x\lambda(q|q) + (\chi(q) - q - x)(-\chi'(q))\lambda(\chi(q)|q) - 2 \\
&= q\lambda(q|p) - 2\frac{\hat{\pi} - \chi(q)}{x} - x\lambda(q|q) + (\chi(q) - q - x)(-\chi'(q))\lambda(\chi(q)|q) - 2 \\
&= -2\frac{\hat{\pi} - \chi(q)}{x} + q(\lambda(q|p) + \chi'(q)\lambda(\chi(q)|q)) - 2 \\
&= -2\frac{\hat{\pi} - \chi(q)}{x} + q\left(\frac{2(\chi(q) - x)(-\chi'(q))}{x\chi(q)} + \chi'(q)\frac{2}{\chi(q)}\right) - 2 \\
&= 2\left(-\frac{\hat{\pi} - \chi(q)}{x} - \frac{\chi(q) - 2x}{x} q \frac{\chi'(q)}{\chi(q)} - 1\right) \\
&= 2\left(-\frac{\hat{\pi} - \chi(q)}{x} - \frac{\chi(q) - 2x}{x}\frac{\chi(q) - (\hat{\pi} + x)}{\chi(q) - 2x} - 1\right) = 0,
\end{aligned}
$$

where the second line is by (29), the third line is by (30), the fourth line uses $\lambda(q|p) = \frac{2(\chi(q)-x)(-\chi'(q))}{x\chi(q)}$ and $\lambda(\chi(q)|q) = \frac{2}{\chi(q)}$ (by (17)), the fifth line collects terms, and the sixth line is by (13). Finally, the claim also holds when $p = q$, as $\lim_{q\uparrow p} G(p,q) = 2G(p,p)$ for all p.

Claim 4. $G(p,q) > 0$ if either $p \in (\hat{\pi}, 1), q < \chi(p)$ or $p < \hat{\pi}, q \leq p$.

Proof. If $p \in (\hat{\pi}, 1)$ and $q < \chi(p)$, then $\partial G(p,q)/\partial q = -2 < 0$ by (27). Since $G(p, \chi(p)) = 0$ by Claim 2, this implies that $G(p,q) > 0$.

If $p < \hat{\pi}$ and $q \leq p$, then $d\Lambda(p'|p) = d\Lambda(q'|q) = 0$ for all $p' < p, q' < q$. Hence, $G(p,q) = 2(\hat{\pi} - q) > 0$.

Claims 1–4 establish the desired conclusion when $p < 1$. The $p = 1$ case follows by continuity, since $\chi(p)$ and $G(p,q)$ are continuous in p for all $q < 1$.

A.6 Proof of Proposition 7

For any positive integer $N > 1/x$, let $\mathcal{F}^N$ denote the set of all symmetric cdfs on $\mathbf{N} = \left\{0, \frac{1}{N}, \frac{2}{N}, \ldots, 1\right\}^2$. Let $\mathbf{P}^N$ denote the primal program with $\mathcal{F}^N$ in place of $\mathcal{F}$. Note that in $\mathbf{P}^N$ firms can still contemplate deviations to arbitrary prices, so $\mathbf{P}^N$ is a strictly more constrained program than $\mathbf{P}$. Let π^N and π denote the values of $\mathbf{P}^N$ and $\mathbf{P}$, respectively. Let $\mathcal{F}_{IC} \subset \mathcal{F}$ and $\mathcal{F}_{IC}^N \subset \mathcal{F}^N$ denote the set of symmetric cdfs that satisfy incentive compatibility in $\mathbf{P}$ and $\mathbf{P}^N$. Note that $\mathcal{F}_{IC}^N = \mathcal{F}^N \cap \mathcal{F}_{IC}$. We show that there exists a solution to $\mathbf{P}^N$, and that in every such solution the highest winning bid equals 1 with positive probability.

Let f be an optimal pmf in $\mathbf{P}^N$. Without loss, assume that f is symmetric. Such an f exists as $\mathcal{F}_{IC}^N$ is compact and profits are continuous in f. Let $\bar{p}$ denote the highest bid in the support of the marginal of f. Suppose toward a contradiction that $\bar{p} < 1$. Let $\beta = f(\bar{p}, \bar{p})$ and $\gamma_p = f(\bar{p}, p)$ for $p \in [0, \bar{p})$. Let $\gamma = \sum_{p<\bar{p}} \gamma_p$ and $\tilde{\gamma}_p = \sum_{p \leq p' < \bar{p}} \gamma_{p'}$ (and let $\tilde{\gamma}_{\bar{p}} = 0$).

Lemma 7 *$\beta > 0$.*

Proof. Suppose that $\beta = 0$. Then $\bar{p}$ is never a winning bid. Let $p^* < \bar{p}$ denote the highest bid that ever wins. Consider the bid distribution that everywhere replaces all bids above p^* with bids of p^*. Clearly, this variation leaves profit unchanged. We argue that it remains an equilibrium.

First, conditional on each recommended bid, the distribution of the opponent's bid has shifted down. This makes any upward deviation weakly less profitable. So upward IC still holds.

Next, if one's recommended bid is below p^*, the payoff from following the recommendation as well as from any downward deviation are unchanged. So downward IC holds for recommendations below p^*.

It remains to establish downward IC for a recommended bid of p^*. Let Pr denote probability under the original distribution, and $\widehat{\Pr}$ denote probability under the new distribution. We must show that, for all $p < p^*$,

$$\frac{1}{2}\widehat{\Pr}(p_i = p_j = p^*) p^* \geq \widehat{\Pr}(p_i = p^* \wedge p_j \geq p) p - \widehat{\Pr}(p_i = p^*) x. \tag{31}$$

First, note that

$$\begin{aligned}
& \frac{1}{2}\widehat{\Pr}(p_i = p_j = p^*) \\
= \; & \frac{1}{2}\left(\Pr(p_i = p_j = p^*) + \Pr(p_i = p^* \wedge p_j > p^*) + \Pr(p_i > p^* \wedge p_j = p^*)\right) \\
= \; & \frac{1}{2}\left(\Pr(p_i = p_j = p^*) + 2\Pr(p_i = p^* \wedge p_j > p^*)\right) \\
= \; & \frac{1}{2}\Pr(p_i = p_j = p^*) + \Pr(p_i = p^* \wedge p_j > p^*), \qquad (32)
\end{aligned}$$

where the second equality uses symmetry. Second, note that

$$\widehat{\Pr}(p_i = p^* \wedge p_j \geq p) = \Pr(p_i = p^* \wedge p_j \geq p) + \Pr(p_i > p^* \wedge p_j \geq p) \quad \text{and} \tag{33}$$

$$\widehat{\Pr}(p_i = p^*) = \Pr(p_i = p^*) + \Pr(p_i > p^*). \tag{34}$$

Third, by $IC_{p^*,p}$, we have

$$\left(\frac{1}{2}\Pr(p_i = p_j = p^*) + \Pr(p_i = p^* \wedge p_j > p^*)\right) p^* \geq \Pr(p_i = p^* \wedge p_j \geq p)\, p - \Pr(p_i = p^*)\, x,$$

and by $IC_{p',p}$ for all $p' > p^*$, we have

$$0 \geq \Pr(p_i > p^* \wedge p_j \geq p)\, p - \Pr(p_i > p^*)\, x.$$

Adding these inequalities, we have

$$\begin{aligned} & \left(\frac{1}{2}\Pr(p_i = p_j = p^*) + \Pr(p_i = p^* \wedge p_j > p^*)\right) p^* \\ \geq\ & \left(\Pr(p_i = p^* \wedge p_j \geq p) + \Pr(p_i > p^* \wedge p_j \geq p)\right) p - \left(\Pr(p_i = p^*) + \Pr(p_i > p^*)\right) x. \end{aligned} \tag{35}$$

Finally, (32), (33), (34), and (35) imply (31). ■

We now construct another distribution $\hat{f}$, which we will show is an equilibrium bid distribution that yields strictly higher profit than f. This will establish the desired contradiction, completing the proof. First, note that $2(\beta + \gamma)x \geq \beta\bar{p}$, by $IC_{\bar{p},\bar{p}}$. Let $\eta = 1/N < x$, and fix $\rho \in \left(0, \min\left\{\frac{\bar{p}}{\bar{p}+\eta}, \frac{\bar{p}+\eta}{\bar{p}+2\eta}\right\}\right)$. Define

$$\hat{f}(p,q) = \begin{cases} f(p,q) & \text{if } \max\{p,q\} < \bar{p} \text{ or } \max\{p,q\} > \bar{p}+\eta \\ (1-\rho) f(p,q) & \text{if } \min\{p,q\} < \bar{p} = \max\{p,q\} \\ \rho f(\min\{p,q\}, \bar{p}) & \text{if } \min\{p,q\} < \bar{p}, \max\{p,q\} = \bar{p}+\eta \\ \left(1 - \frac{\bar{p}+2\eta}{\bar{p}+\eta}\rho\right)\beta & \text{if } p = q = \bar{p} \\ \frac{\eta\rho}{\bar{p}+\eta}\beta & \text{if } \min\{p,q\} = \bar{p}, \max\{p,q\} = \bar{p}+\eta \\ \frac{\bar{p}\rho}{\bar{p}+\eta}\beta & \text{if } p = q = \bar{p}+\eta \end{cases}.$$

Since $\beta > 0$, industry profit under $\hat{f}$ exceeds that under f. Thus, it remains to verify incentive compatibility. Define

$$\begin{aligned}
\widehat{IC}_{p,p'} &= \left(\sum_{\tilde{p}>p} \hat{f}(\tilde{p},p) + \frac{1}{2}\hat{f}(p,p)\right) p - \left(\sum_{\tilde{p}\geq p'} \hat{f}(\tilde{p},p)\right) p' + \sum_{\tilde{p}} \hat{f}(\tilde{p},p)\, x \quad \text{and} \\
IC_{p,p'} &= \left(\sum_{\tilde{p}>p} f(\tilde{p},p) + \frac{1}{2}f(p,p)\right) p - \left(\sum_{\tilde{p}\geq p'} f(\tilde{p},p)\right) p' + \sum_{\tilde{p}} f(\tilde{p},p)\, x \geq 0.
\end{aligned}$$

Lemma 8 *Downward IC from $\bar{p}+\eta$ is satisfied: $\widehat{IC}_{\bar{p}+\eta,p} \geq 0$ for all $p \leq \bar{p}+\eta$.*

Proof. First,

$$\begin{aligned}
\widehat{IC}_{\bar{p}+\eta,\bar{p}+\eta} &= \frac{1}{2}\frac{\bar{p}\rho}{\bar{p}+\eta}\beta(\bar{p}+\eta) - \frac{\bar{p}\rho}{\bar{p}+\eta}\beta(\bar{p}+\eta) + \left(\frac{\bar{p}\rho}{\bar{p}+\eta}\beta + \frac{\eta\rho}{\bar{p}+\eta}\beta + \rho\gamma\right) x \\
&= \rho\left(-\frac{1}{2}\beta\bar{p} + (\beta+\gamma)\, x\right) \geq 0,
\end{aligned}$$

where the inequality holds because $2(\beta+\gamma)\, x \geq \beta\bar{p}$.

Second, for $p \in (\bar{p}, \bar{p}+\eta)$, it is clear that $\widehat{IC}_{\bar{p}+\eta,p} \geq \widehat{IC}_{\bar{p}+\eta,\bar{p}+\eta}$, and hence $\widehat{IC}_{\bar{p}+\eta,p} \geq 0$.

Third, for $p \leq \bar{p}$, we have

$$\begin{aligned}
\widehat{IC}_{\bar{p}+\eta,p} &= \frac{1}{2}\frac{\bar{p}\rho}{\bar{p}+\eta}\beta(\bar{p}+\eta) - \left(\frac{\bar{p}\rho}{\bar{p}+\eta}\beta + \frac{\eta\rho}{\bar{p}+\eta}\beta + \rho\tilde{\gamma}_p\right) p \\
&\quad + \left(\frac{\bar{p}\rho}{\bar{p}+\eta}\beta + \frac{\eta\rho}{\bar{p}+\eta}\beta + \rho\gamma\right) x \\
&= \rho\left(\frac{1}{2}\beta\bar{p} - (\beta+\tilde{\gamma}_p)\, p + (\gamma+\beta)\, x\right) \geq 0,
\end{aligned}$$

where the inequality holds by $IC_{\bar{p},p}$. ■

Lemma 9 *Downward IC from $\bar{p}$ is satisfied: $\widehat{IC}_{\bar{p},p} \geq 0$ for $p \leq \bar{p}$.*

Proof. By definition,

$$\begin{aligned}
\widehat{IC}_{\bar{p},p} &= \left(\frac{1}{2}\beta - \frac{1}{2}\frac{\rho\bar{p}\beta}{\bar{p}+\eta} - \frac{\rho}{\bar{p}+\eta}\beta\eta + \frac{\rho}{\bar{p}+\eta}\beta\eta\right)\bar{p} \\
&\quad - \left(\beta - \frac{\rho\bar{p}\beta}{\bar{p}+\eta} - 2\frac{\rho}{\bar{p}+\eta}\beta\eta + \frac{\rho}{\bar{p}+\eta}\beta\eta + (1-\rho)\tilde{\gamma}_p\right)p \\
&\quad + \left((1-\rho)\gamma + \beta - \frac{\rho\bar{p}\beta}{\bar{p}+\eta} - 2\frac{\rho}{\bar{p}+\eta}\beta\eta + \frac{\rho}{\bar{p}+\eta}\beta\eta\right)x \\
&= (1-\rho)\left(\frac{1}{2}\beta\bar{p} - \left(\beta+\tilde{\gamma}_p\right)p + (\gamma+\beta)x\right) \\
&\quad + \rho\left(\left(\frac{1}{2}\beta - \frac{1}{2}\frac{1}{\bar{p}+\eta}\beta\bar{p}\right)\bar{p} - \left(\beta - \frac{\beta\bar{p}}{\bar{p}+\eta} - \frac{1}{\bar{p}+\eta}\beta\eta\right)p + \left(\beta - \frac{\beta\bar{p}}{\bar{p}+\eta} - \frac{1}{\bar{p}+\eta}\beta\eta\right)x\right).
\end{aligned}$$

Since the first line is non-negative by $IC_{\bar{p},p}$, the proof is completed by noting that

$$\begin{aligned}
&\left(\frac{1}{2}\beta - \frac{1}{2}\frac{1}{\bar{p}+\eta}\beta\bar{p}\right)\bar{p} - \left(\beta - \frac{\beta\bar{p}}{\bar{p}+\eta} - \frac{1}{\bar{p}+\eta}\beta\eta\right)p + \left(\beta - \frac{\beta\bar{p}}{\bar{p}+\eta} - \frac{1}{\bar{p}+\eta}\beta\eta\right)x \\
\geq\ & \frac{1}{2}\beta\frac{\bar{p}\eta}{\bar{p}+\eta} > 0.
\end{aligned}$$

■

Lemma 10 *Upward IC from $p \leq \bar{p}$ to $p' \geq \bar{p}$ is satisfied: $\widehat{IC}_{p,p'} \geq 0$ for $p < \bar{p}$ and $p' \geq \bar{p}$.*

Proof. For $p = \bar{p}$, this holds because $\eta \leq x$. So suppose that $p < \bar{p}$. Since $(1-\rho)\gamma_p + \rho\gamma_p = \gamma_p$, $\hat{f}(p) = f(p)$ for all $p < \bar{p}$. Moreover, given recommendation p, the conditional probability that the opponent's recommendation is no less than $\bar{p}$ is

$$\frac{(1-\rho)\gamma_p + \rho\gamma_p}{\hat{f}(p)} = \frac{\gamma_p}{f(p)},$$

and hence $\widehat{IC}_{p,\bar{p}}$ follows from $IC_{p,\bar{p}}$.

For $p' \in (\bar{p}, \bar{p}+\eta)$, it is clear that $\widehat{IC}_{p,p'} \geq \widehat{IC}_{p,\bar{p}+\eta}$. Thus, it remains to show that $\widehat{IC}_{p,\bar{p}+\eta} \geq 0$. Given recommendation p, the conditional probability that the opponent's recommendation is no less than $\bar{p}+\eta$ is

$$\frac{\rho\gamma_p}{\hat{f}(p)} = \rho\frac{\gamma_p}{f(p)}.$$

Thus, $\widehat{IC}_{p,\bar{p}+\eta} \geq 0$ is implied by $IC_{p,\bar{p}} \geq 0$ if

$$(\bar{p}+\eta)\,\rho\frac{\gamma_p}{f(p)} \leq \bar{p}\frac{\gamma_p}{f(p)},$$

which follows from $\rho \leq \frac{\bar{p}}{\bar{p}+\eta}$. ■

Lemma 11 *IC from $p < \bar{p}$ to $p' < \bar{p}$ is satisfied: $\widehat{IC}_{p,p'} \geq 0$ for all $p < \bar{p}, p' < \bar{p}$.*

Proof. This follows from $\hat{f}(p) = f(p)$ and $\sum_{\tilde{p}\geq p'} \hat{f}(p,\tilde{p}) = \sum_{\tilde{p}\geq p'} f(p,\tilde{p})$ for all $p' < \bar{p}$. ■

Together, Lemmas 8–11 imply that all incentive constraints are satisfied. This completes the proof.

A.7 Proof of Proposition 8

First, suppose that $n \geq 3$. Note that π as a function of x is larger under (optimal) almost-uniform bids than under bid rotation, because the static bid distribution under bid rotation is almost-uniform (as at most one firm bids below 1). It thus suffices to show that x as a function of π is also larger under almost-uniform bids: that is, that

$$\begin{aligned} \frac{\delta}{(1-\delta)n} &\geq \frac{\delta^{n-1}}{1-\delta^n}, \qquad \text{or equivalently} \\ 1+\delta+\ldots+\delta^{n-3}+\delta^{n-1} &\geq (n-1)\,\delta^{n-2}. \end{aligned}$$

This holds whenever $n \geq 3$, because

$$1+\delta+\ldots+\delta^{n-3}+\delta^{n-1} \geq (n-3)\,\delta^{n-2}+\delta^{n-3}+\delta^{n-1} \geq (n-1)\,\delta^{n-2},$$

where the last line holds because $\delta^2+1 \geq 2\delta$ for all $\delta \in [0,1]$.

Now suppose that $n = 2$. Substituting for π in the system for (π, x) under bid rotation, we see that profit under bid rotation equals

$$\frac{1-\delta^2}{\delta}\exp\left(-\frac{1-\delta-\delta^2}{\delta}\right) = \frac{1-\delta}{\delta}\exp\left(-\frac{1-\delta}{\delta}\right)\exp\left(\delta\log\left(1+\delta\right)\right). \tag{36}$$

Substituting for x in the system for (π, x) under almost-uniform bids, we have

$$\begin{aligned}
\frac{2(1-\delta)}{\delta} &= \frac{1-2\log(2x)}{1-x}, \qquad \text{or equivalently} \\
x &= \frac{1}{2}\exp\left(-\frac{1-\delta}{\delta}\right)\exp\left(\frac{1}{2}\right)\exp\left(\frac{1-\delta}{\delta}x\right).
\end{aligned}$$

Hence, we have

$$\begin{aligned}
\pi &= \frac{2(1-\delta)}{\delta}x \\
&= \frac{1-\delta}{\delta}\exp\left(-\frac{1-\delta}{\delta}\right)\exp\left(\frac{1}{2}\right)\exp\left(\frac{1-\delta}{\delta}x\right).
\end{aligned} \tag{37}$$

Since $x > 0$, we see that (37) is greater than (36) whenever

$$\frac{1}{2} \geq \delta \log(1+\delta).$$

This inequality holds for all $\delta < \frac{1}{2}$, which completes the proof.

B Asymmetric Equilibrium without Mediation

Let $n = 3$ and $x < \frac{1}{4}$. Firms 1 and 2 mix independently, with

$$p_i = \begin{cases} x & \text{with prob } \frac{2}{7} \\ \frac{3}{2}x & \text{with prob } \frac{2}{7} \\ 4x & \text{with prob } \frac{3}{7} \end{cases}$$

Firm 3 stays out (or, equivalently, bids above $4x$). There are no on-path punishments. Any off-path bid is punished by x.

The winning bid distribution is given by

$$\min\{p_1, p_2\} = \begin{cases} x & \text{with prob } \frac{24}{49} \\ \frac{3}{2}x & \text{with prob } \frac{16}{49} \\ 4x & \text{with prob } \frac{9}{49} \end{cases}$$

Hence, industry profit equals $\frac{24}{49}(x)+\frac{16}{49}\left(\frac{3}{2}x\right)+\frac{9}{49}(4x)=\frac{12}{7}x$. This is greater than $\frac{3}{2}x$, which by Proposition 1, is the maximum cartel profit when all firms win with positive probability.

We check that this is an equilibrium. For firm 3, bidding just below x gives $x-x=0$, bidding just below $\frac{3}{2}x$ gives $\frac{25}{49}\left(\frac{3}{2}x\right)-x<0$, and bidding just below $4x$ gives $\frac{9}{49}(4x)-x<0$, so it is optimal for firm 3 to stay out. For firm 1 or 2, bidding at x, $\frac{3}{2}x$, or $4x$ all give an expected payoff of $\frac{6}{7}x$, because

$$\underbrace{\frac{6}{7}}_{\text{prob win if } p_i=x}(x)=\underbrace{\frac{4}{7}}_{\text{prob win if } p_i=\frac{3}{2}x}\left(\frac{3}{2}x\right)=\underbrace{\frac{3}{14}}_{\text{prob win if } p_i=4x}(4x).$$

Finally, for firm 1 or 2, any off-path bid gives a strictly lower payoff, because bidding just below x gives $x-x=0$, bidding just below $\frac{3}{2}x$ gives $\frac{5}{7}\left(\frac{3}{2}x\right)-x=\frac{1}{14}x$, and bidding just below $4x$ gives $\frac{3}{7}(4x)-x=\frac{5}{7}x$.

C Optimal Bid Distributions with Only Downward Incentive Constraints

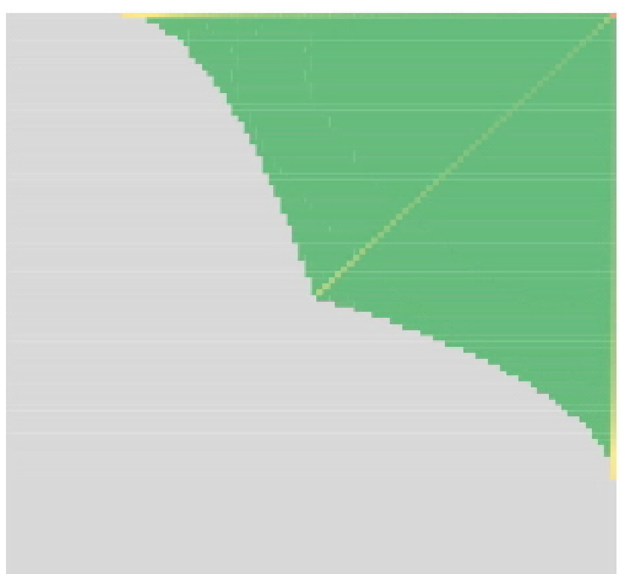

Figure 7: The optimal bid distribution when $x=.1$, with only downward incentive constraints. The color scheme in Figures 7–9 is the same as in Figure 1. Bid pair $p_1=p_2=1$ is recommended with probability .06.

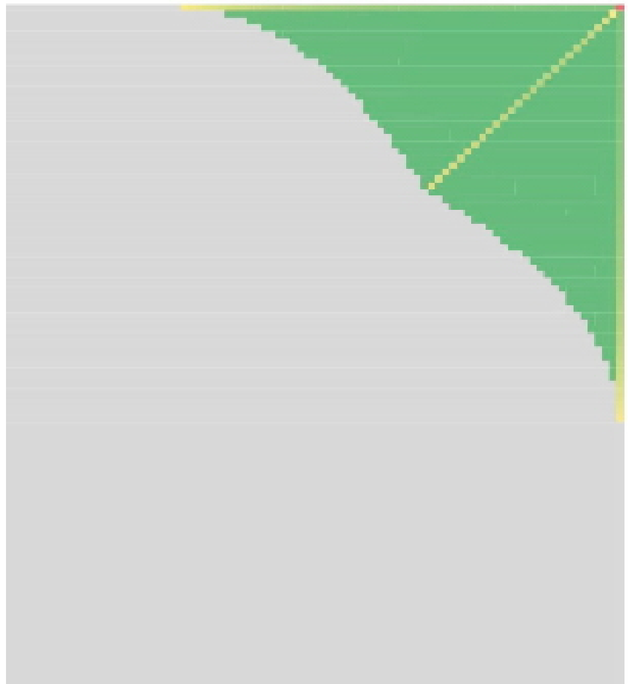

Figure 8: The optimal bid distribution when $x = .2$, with only downward incentive constraints. Bid pair $p_1 = p_2 = 1$ is recommended with probability .16.

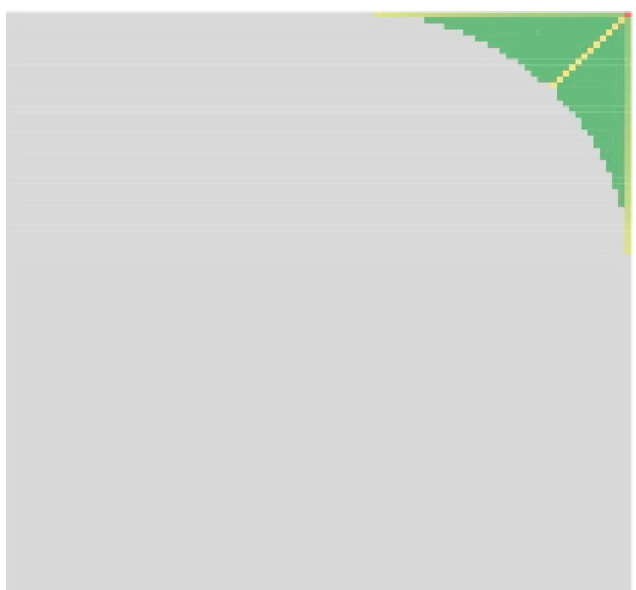

Figure 9: The optimal bid distribution when $x = .3$, with only downward incentive constraints. Bid pair $p_1 = p_2 = 1$ is recommended with probability .31.

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