

Risk-taking over the Life Cycle: Aggregate and Distributive Implications of Entrepreneurial Risk *

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January 2023

Abstract

We study the risk-taking behavior over the life cycle of entrepreneurs subject to partial insurance against idiosyncratic shocks. The model quantitatively accounts for the life-cycle profiles of consumption and risk-taking, the aggregate and idiosyncratic risk premia, and the patterns of wealth inequality observed in the data. A reform that relaxes the risk constraints leads to a reduction in the idiosyncratic risk premium and an investment boom. Consistent with a Kuznets curve, inequality increases in the short run and declines in the long run. The initial generation of entrepreneurs benefits from better insurance, but future generations of entrepreneurs are worse-off after the reform.

KEYWORDS: Entrepreneurship, risk taking, risk premium, insurance, inequality

JEL CLASSIFICATION: G11, G51, E44.

*We would like to thank Antonio Antunes, Francisco Buera, Pedro Brinca (discussant), Sylvain Catherine (discussant), David Lagakos, Juan Passadore, and seminar participants at Barcelona GSE Summer Forum, Econometric Society Meeting, FGV-EPGE, LACEA-LAMES, Lubramacro, SED, UFABC, WFA, and Yale for their helpful comments and suggestions.

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1 Introduction

Entrepreneurship is inherently a risky activity. Entrepreneurial risk is especially important in the context of developing economies, where the bulk of production takes place in privately-owned businesses and risk-sharing opportunities are particularly limited. Given that entrepreneurs hold under-diversified portfolios, the presence of such risks has far reaching implications.¹ At the individual level, entrepreneurial risk distorts investment and savings decisions. The importance of business income is highly heterogeneous and varies substantially over the life cycle, which shapes the patterns of wealth inequality. At the aggregate level, imperfect insurance leads to an inefficient risk premium which depresses the capital stock and hinders the economy's development. The pervasive effects of limited risk sharing highlight the importance of policy interventions that may alleviate the consequences of entrepreneurs' lack of diversification.

To assess the aggregate impact of such interventions, it is crucial to determine the quantitative importance of the limits to risk sharing. Identifying such limits, however, is challenging. For instance, despite holding concentrated portfolios, entrepreneurs may engage in self-insurance or participate in informal insurance arrangements, which attenuates the equilibrium impact of under-diversification. Such informal insurance arrangements are widespread in developing economies, as documented e.g. by [Kinnan and Townsend \(2012\)](#). Moreover, it is important to disentangle the effects of aggregate and idiosyncratic shocks. Lack of diversification is potentially less consequential if private businesses are mostly exposed to aggregate (non-diversifiable) risk. Therefore, the relevance of risk sharing frictions cannot be directly inferred from the degree of under-diversification or the volatility of business income.

In this paper, we study the aggregate and distributive implications of entrepreneurial risk in the context of a quantitative life-cycle model with limited idiosyncratic insurance. We discipline the model using a rich dataset on small business owners in Thailand, which includes information on the returns of entrepreneurial activity as well as entrepreneurs' risk-taking and consumption.² The model captures main features of the data, including the life-cycle profile of entrepreneurs' risk-taking and consumption, the patterns of inequality between- and within-age groups, as well as the risk premium on business returns. Moreover, the model implies that one can identify the limits to risk sharing, measured by the fraction of idiosyncratic risk entrepreneurs cannot diversify, using three potentially observable quantities: i) the exposure of entrepreneurs' wealth to the business (a measure of under-diversification); ii) the share of variance of business returns explained by idiosyncratic shocks (a measure of the im-

¹For evidence on the under-diversification of entrepreneurs see e.g. [Moskowitz and Vissing-Jørgensen \(2002\)](#) and [Herranz et al. \(2009\)](#) for a study focused on small businesses.

²A large literature studies the asset-pricing and macroeconomic implications of firm-level risk (see e.g. [Christiano et al. 2014](#), [Gârleanu et al. 2015](#), [Dou 2016](#), [Herskovic et al. 2016](#), [Di Tella 2017](#), [Herskovic et al. 2018](#)), but these studies usually rely on data of public companies to discipline their quantitative exercises.

portance of diversifiable risk); and iii) the equilibrium compensation for holding idiosyncratic risk, which we refer to as the *idiosyncratic risk premium*. By combining a theory of the idiosyncratic risk premium with detailed information on portfolios and business returns, one can assess the extent of limits to risk sharing faced by entrepreneurs and, ultimately, the impact of reforms relaxing these constraints.

We start by measuring the exposure of entrepreneurs' wealth to the business. Given these are risky activities, this measure effectively captures entrepreneurs' risk-taking behavior. We find that there is substantial variation in risk-taking over the life cycle. In particular, young entrepreneurs are 50% more exposed to the business than old entrepreneurs. These differences in risk-taking cannot be explained by differences in expected returns, suggesting that risk tolerance is heterogeneous across age groups. We consider next the risk and return of private businesses. In particular, it is important to disentangle how much of expected returns and volatility is explained by either aggregate or idiosyncratic shocks. Given information on a panel of realized returns, we can use standard asset-pricing techniques to perform this decomposition.³ We find that roughly half of expected returns consists of compensation for holding idiosyncratic risk. Moreover, idiosyncratic volatility is three times larger than aggregate volatility, so the Sharpe ratio for the idiosyncratic component is smaller than the one for the aggregate component. Differences in the compensation (per unit of risk) for holding aggregate and idiosyncratic risk is inconsistent with an autarky allocation, where only total volatility would matter. It is also inconsistent with perfect risk sharing, as the idiosyncratic risk premium would be zero, indicating that entrepreneurs partially insure these risks.

To capture these motivating facts, we propose a general equilibrium model with two main ingredients: limited idiosyncratic insurance and finite lives with imperfect altruism. The economy is populated by entrepreneurs and wage earners. Entrepreneurs are the only ones with access to a production technology, which is exposed to aggregate and idiosyncratic shocks. Aggregate shocks are public information, so there are no frictions in sharing aggregate risk. In contrast, idiosyncratic shocks are private information. Entrepreneurs face a moral hazard problem, as they can divert a fraction of capital every period. Under the optimal contract, entrepreneurs are subject to a skin-in-the-game constraint, so they must bear a fraction of the idiosyncratic risk in equilibrium. The moral hazard parameter controls the degree of partial insurance. We identify the moral hazard parameter by matching the decomposition of expected returns into aggregate and idiosyncratic components performed in the data. Intuitively, if entrepreneurs are able to diversify a large fraction of the risk, we would find that expected returns on the business would mostly reflect compensation for holding aggregate risk. The relative importance of the idiosyncratic risk premium provides information on the

³This decomposition extends the work of [Samphantharak and Townsend \(2018\)](#). It is computed by running [Fama and MacBeth \(1973\)](#) regressions, where average returns are regressed on a measure of aggregate risk (a "market" portfolio) and on idiosyncratic variance. See Section 2 for details.

quantitative importance of risk sharing frictions. We find that entrepreneurs are able to diversify a significant fraction of the risk, roughly one quarter of the idiosyncratic volatility, but they still have to absorb the majority of the idiosyncratic shocks. Therefore, entrepreneurs face significant frictions on their ability to diversify risks, which affects their decision of how much to invest in the business.

Importantly, we assume that entrepreneurial households also receive some labor income, presumably from members of the household working outside the business, consistent with what we observe in our data. An implication of this fact is that entrepreneurs' effective risk aversion depends on the ratio of human wealth, the present discounted value of future labor income, to financial wealth, the value of the investment in the business and other safe assets minus any liabilities. We find that the human-financial wealth ratio declines over the life cycle in the data. This mechanism endogenously creates heterogeneity in risk aversion, which is important to replicate the empirical life-cycle profiles.

The model matches quantitatively the decline in risk-taking over the life cycle by having young entrepreneurs being relatively risk tolerant, given a higher human-financial ratio early in life.⁴ The savings behavior of entrepreneurs also has an important life-cycle component. We find that the consumption-wealth ratio initially declines with age and then increases by the end of the life cycle. The model captures this pattern through the interplay of two different effects. First, the decline in the human-financial wealth ratio reduces the consumption-wealth ratio. Second, the marginal propensity to consume (MPC) increases with age due to imperfect altruism. This effect is particularly strong when entrepreneurs get older, which explains the increase in the consumption-wealth ratio late in life.

To focus on the impact of limits to risk sharing, we assume the moral hazard problem is the only friction in the baseline model. Importantly, our results do not hinge on abstracting from other commonly assumed financial frictions. In extensions, we consider the implications of adding two forms of borrowing constraints. First, we introduce limited pledgeability of physical capital by having collateral constraints. While this friction affects the allocation of capital between constrained and unconstrained entrepreneurs, it leaves the aggregate implications of our model essentially intact. Second, we introduce limited pledgeability of human wealth by having uninsurable labor income risk. This version of the model allow us to study the interactions between uninsurable labor income risk, as in [Aiyagari \(1994\)](#), and uninsurable investment risk, as in [Angeletos \(2007\)](#). We are able to obtain (approximate) analytical expressions, despite having aggregate shocks and incomplete markets. We find that the borrowing limit associated with uninsurable labor income risk has two effects on risk taking. On the one hand, it tends to reduce risk taking, as entrepreneurs have less resources available.

⁴This mechanism is reminiscent of the work on portfolio choice with labor income ([Bodie et al. 1992](#), [Heaton and Lucas 1997](#), [Koo 1998](#), [Viceira 2001](#)). See [Huggett and Kaplan \(2016\)](#) for a similar approach on valuing human wealth.

On the other hand, it endogenously reduces risk aversion, which pushes entrepreneurs to be more exposed to risk. Therefore, the borrowing limit affects mostly the dispersion in business investment and it has a more muted effect on its level. Moreover, the behavior of risk-taking over the life cycle is essentially the same as in the baseline model.

Entrepreneurial risk has a significant effect on inequality. On average, wealth increases initially with age and then declines later in life. A similar inverted-U shape holds for the standard deviation of wealth conditional on age, capturing the evolution of within-group inequality. The initial increase in inequality is explained by what we call a generalized " $r - g$ " effect. A positive difference between the expected return on wealth, which includes the aggregate and idiosyncratic risk premium, and the growth rate of the economy cause entrepreneurs to accumulate more wealth as they age. A countervailing force comes from the fact that the MPC increases with age, due to imperfect altruism. Even though this was not targeted during the calibration, the model roughly captures the patterns of between- and within-group inequality observed in the data.

At the aggregate level, the idiosyncratic risk premium and the capital stock are simultaneously determined. In equilibrium, the marginal product of capital (MPK) is pinned down by the expected return on the business. An increase in the idiosyncratic risk premium leads to a higher MPK and a smaller capital stock. In contrast, entrepreneurs demand a higher compensation for holding idiosyncratic risk when the capital stock is larger, as they are more exposed to risk. We find that an improvement in risk sharing leads to a decline in the idiosyncratic risk premium and an increase in the capital stock in the long run. The effect is quantitatively large, where a reform that reduces the risk premium by 100 basis points lead to an increase of the capital stock of 10%. This is consistent with the view that the level of financial development, captured here by the extent of idiosyncratic risk sharing, is an important determinant of the level of economic development.⁵

We consider as well the dynamic implications of relaxing risk constraints. Solving for the transitional dynamics is challenging, as one must compute the evolution over time of the distribution of different outcomes by age. We propose a new method to compute the transitional dynamics, based on a combination of perturbation and finite-difference methods, which is potentially useful in other applications involving heterogeneous agents. Improving idiosyncratic insurance leads to an investment boom that lasts for a decade, which is accompanied by a sharp increase in the value of private businesses in the short run. Inequality falls in the long run, as entrepreneurs are less exposed to risk after the reform. However, inequality increases in the short run, due to a revaluation effect. The increase in the value of the business benefits entrepreneurs with larger investments initially, which are the ones relatively richer before

⁵See [Levine \(2005\)](#) for a review of literature on the connections between financial and economic development.

the reform. It takes a long time for inequality to converge to its new long-run level, as the effects of the initial increase in inequality persists due to intergenerational links. This pattern is consistent with [Kuznets's \(1955\)](#) hypothesis over the relationship between inequality and economic development.⁶

The transitional dynamics is also important to assess the welfare effects of improvements in risk sharing. We find that entrepreneurs are actually worse-off in the long run after the reform, despite of the benefits of better diversification. The reason is that the reform reduces the expected return on entrepreneurial activity, so it becomes harder for entrepreneurs to accumulate wealth, affecting the bequests and welfare of future generations. In contrast, the welfare of the initial generation of entrepreneurs improves with the reform. They received their bequest before the intervention, and the value of their businesses increases substantially in the short run. Therefore, most of the gains of the reform are reaped by the initial generation of entrepreneurs and by wage earners, who receive higher wages given the larger capital stock. This demonstrates how the effect of risk constraints interacts in rich ways with the demographic structure.

Related literature. This paper is related to several strands of literature in macroeconomics and finance. First, the work in macro-finance studying how firm-level uncertainty affects asset prices and the real economy ([Gârleanu et al. 2015](#), [Di Tella 2017](#), [Dou 2016](#), [Herskovic et al. 2016](#), [Iachan et al. 2021](#)).⁷ While this literature focus mostly on business-cycle fluctuations, we study how firm-level uncertainty affects the economy in the long run. Second, the literature on how the lack of diversification of entrepreneurs' portfolios affects several firm outcomes, including real investment ([Panousi and Papanikolaou 2012](#)), capital structure ([Chen et al. 2010](#), [Herranz et al. 2015](#)), and risk-taking ([Chen and Strebulaev 2019](#)). This work is mostly in partial equilibrium and abstracts from the aggregate implications of entrepreneurial risk.

We also contribute to recent work on the importance of heterogeneous returns to understand the patterns of inequality. This literature documents substantial heterogeneity in portfolio returns ([Fagereng et al. 2019](#), [Fagereng et al. 2020](#), [Bach et al. 2020](#)), it finds that private business wealth is one the main sources of wealth at the top ([Smith et al. 2019](#), [Smith et al. 2020](#)), and that return heterogeneity is important to quantitatively match the observed levels of inequality ([Gomez 2017](#), [Hubmer et al. 2021](#)).⁸ We study how the presence of an idiosyncratic risk premium in private business affects inequality both between- and within-age

⁶[Greenwood and Jovanovic \(1990\)](#) provides an early treatment of the Kuznets dynamics and its connection with financial development.

⁷A related literature studies the asset-pricing implications of labor income risk in infinite-horizon ([Constantinides and Duffie 1996](#) and [Heaton and Lucas 1996](#)) and life-cycle models ([Constantinides et al. 2002](#) and [Storesletten et al. 2007](#)).

⁸See also [De Nardi and Fella \(2017\)](#) for a survey of the literature on quantitative Bewley models of wealth inequality.

groups, and how changes in this premium affect the dynamics of inequality. [Greenwald et al. \(2021\)](#) show that financial wealth inequality increases after a decline in interest rates. We find a similar effect for a reduction in the idiosyncratic risk premium in the short run, but inequality actually falls in the long run, showing that discount-rate effects may have very different implications over long horizons.

A related literature studies the extent households can partially insure *labor income* shocks ([Blundell et al. 2008](#) and [Kaplan and Violante 2010](#)). The work by e.g. [Krueger and Perri 2006](#), [Attanasio and Pavoni 2011](#), and [Heathcote et al. 2014](#) shows how the degree of partial insurance can be inferred using data on consumption and labor income. We focus instead on entrepreneurial risk and show that the degree of partial insurance can be identified using information on the idiosyncratic risk premium and the exposure of entrepreneurs' wealth to private businesses.

Our result that young entrepreneurs invest a larger fraction of their wealth on the business is closely related to the findings in the literature on portfolio choice over the life cycle ([Jagannathan et al. 1996](#), [Viceira 2001](#), [Cocco et al. 2005](#)), who found that young households should invest more in the stock market than old households.⁹ In particular, we show that variations in the human-financial wealth ratio accounts quantitatively for the pattern of entrepreneurial risk-taking over the life cycle. Related work uses business income as a source of background risk to explain stock market investment ([Heaton and Lucas 2000a](#), [Heaton and Lucas 2000b](#)). In contrast, we endogeneize the decision to invest in the business and show how non-business income plays an important role in accounting for cross-sectional differences in risk taking.

A large (micro) development literature studies risk sharing ([Townsend 1994](#), [Morduch 1995](#)) and the risk and return of production activities in developing economies ([Udry and Anagol 2006](#), [De Mel et al. 2008](#)). [Karlan et al. \(2014\)](#) conducted a randomized control trial extending credit and insurance to farmers and found that the the lack of insurance is the binding constraint to investment. Their results are consistent with our findings that relaxing risk constraints have large impacts on investment. The (macro) development literature studies the aggregate implications of credit constraints ([Buera and Shin 2013](#), [Midrigan and Xu 2014](#), [Moll 2014](#)). Our approach is complementary to theirs, as we focus instead on the role of risk constraints. Our work is closer to the original model of uninsurable investment risk by [Angeletos \(2007\)](#), which we extend to allow for partial idiosyncratic insurance, a rich demographic structure, and aggregate risk. These extensions are crucial to capture the patterns of consumption and risk-taking observed in the micro data as well as to derive the dynamics of inequality in response to a relaxation of risk constraints.

⁹A related literature studies the life-cycle patterns of consumption and labor income (see e.g. [Deaton and Paxson 1994](#), [Gourinchas and Parker 2002](#), and [Storesletten et al. 2004](#)).

2 Motivating evidence

In this section, we provide motivating evidence on entrepreneurial activity in the context of a developing country. First, we study the determinants of entrepreneurial returns. Second, we consider entrepreneurs' risk-taking and consumption decisions over the life cycle.

Data. We use data from the Townsend Thai Monthly Survey, an intensive monthly survey initiated in 1998 in four provinces of Thailand. Two provinces, Chachoengsao and Lopburi, are semi-urban in a more developed central region near the capital, Bangkok. The other two provinces are rural, Buriram and Srisaket, and are located in the less developed northeastern region by the border of Cambodia. In each of the four provinces, the survey is conducted in four villages, chosen at random within a given township. A detailed discussion of the Townsend Thai Monthly Survey can be found in [Samphantharak and Townsend \(2010\)](#).

Our sample covers 710 households and 14 years of monthly data, starting in January 1999. During this time, these economies were subject to all sorts of aggregate and idiosyncratic shocks. Rice cultivation is affected by seasonal variation in rainfall and temperature. Restrictions on exports to the EU affected shrimp ponds. The productivity of milk cows varies substantially both over time for a given animal and over the herd. The exposure to this different set of shocks highlights the importance of entrepreneurial risk, and it enable us to disentangle the role of aggregate and idiosyncratic shocks.

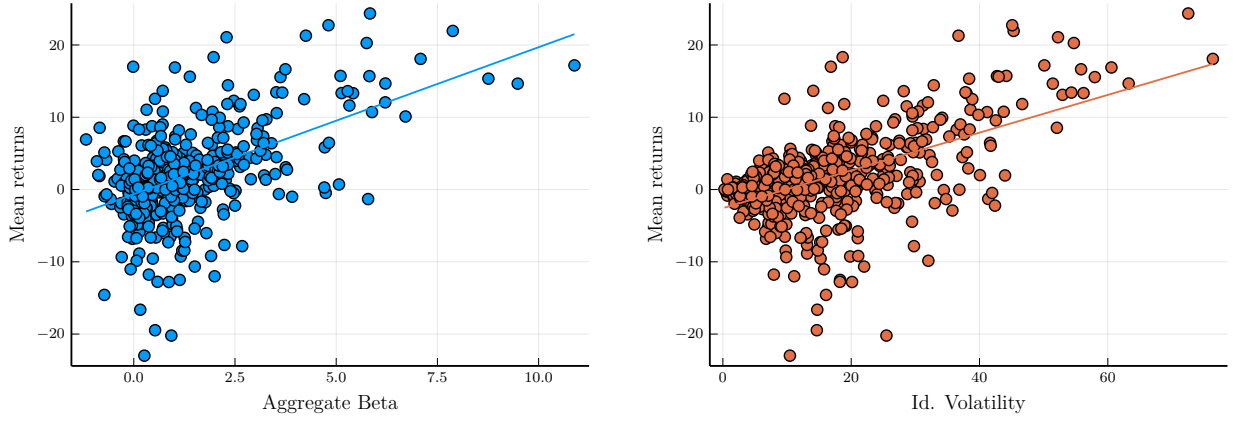
The data collected in the Townsend Thai Monthly Survey includes information on the net income generated by the business as well as total assets and liabilities of households. Importantly, the data are rich enough to allow us to construct a detailed balance sheet for these private businesses. We can then compute the return on assets (ROA), measured as net profits income over business assets. We also measure the fraction of entrepreneurs' wealth invested in the business and the fraction invested in safe (real or financial) assets, providing us with a measure of risk-taking for these entrepreneurs. Finally, the data also includes information on the household's labor income and consumption, which gives a characterization of the savings behavior and the importance of non-business income for entrepreneurial households. See Appendix [B](#) for a detailed description of these variables.¹⁰

2.1 The determinants of expected entrepreneurial returns

The decision of whether and how much to invest in a business depends on its risk/return trade-off. We study this trade-off by considering the determinants of expected entrepreneurial

¹⁰The household is the unit of measurement, even though households typically consist of multiple members doing separate or partially overlapping activities. We do treat the household as a whole as unitary (see e.g. [Doepke and Tertilt \(2016\)](#) for a discussion of unitary models of the household).

Figure 1: Average returns vs. aggregate and idiosyncratic risk



Note: The left (right) panel shows a scatter plot of aggregate beta (idiosyncratic volatility) against average time-series returns for each entrepreneur. Aggregate beta is measured as the slope of the time-series regression of individual returns on the cross-sectional average return in the entrepreneur's province. Idiosyncratic volatility is measured as the volatility of residuals from the same regression.

returns. The case of perfect risk sharing provides a natural benchmark. Under this assumption, differences in expected returns are entirely driven by differences in the exposure to aggregate risk, as idiosyncratic risk can be perfectly insured or diversified. In the other extreme, entrepreneurs can be in financial autarky, where they have no access to insurance. In this case, only the total amount of risk is relevant to entrepreneurs, causing expected returns to vary with both aggregate and idiosyncratic volatility. When idiosyncratic shocks can be partially insured or diversified, expected returns depend not only on the total variance, but also on the relative importance of each source of risk.¹¹

This discussion motivates the use of a two-factor model, based on the exposure to aggregate and idiosyncratic shocks, to explain the cross-section of expected returns of private businesses. This approach allow us to evaluate not only whether perfect risk sharing or autarky holds, but more importantly whether a risk-based explanation can quantitatively account for differences in entrepreneurial returns observed in the data. We measure the exposure to the two sources of risk following the approach of [Samphantharak and Townsend \(2018\)](#). Entrepreneurs' exposure to aggregate risk is measured by the slope of a time-series regression of returns on the cross-sectional average return across all entrepreneurs in a given province. This is analogous to the approach used to test the CAPM, see e.g. [Fama and MacBeth \(1973\)](#), where the average return in the province plays the role of the market return. Idiosyncratic volatility corresponds to the standard deviation of residuals for this regression. Figure 1 shows how the exposure to each source of risk is related to entrepreneurs' average (time-series) return. As

¹¹The different cases are captured by this expression for the risk premium: $p^{ag}\sigma_{ag} + p^{id}\sigma_{id}$, where (p^{ag}, p^{id}) is the return per unit of risk (also known as the price of risk) and $(\sigma_{ag}, \sigma_{id})$ the volatilities. Perfect risk sharing corresponds to $p^{id} = 0$. Autarky corresponds to $\frac{p^{ag}}{p^{id}} = \frac{\sigma_{ag}}{\sigma_{id}}$, so the risk premium is proportional to $\sigma_{ag}^2 + \sigma_{id}^2$.

Dependent Variable:	mean_return		
Model:	(1)	(2)	(3)
<i>Variables</i>			
(Intercept)	-0.1194 (0.5148)	-0.3102 (0.4330)	-1.034*** (0.3077)
beta	1.683*** (0.3614)		1.178*** (0.2091)
sigma2		0.0045*** (0.0007)	0.0036*** (0.0005)
<i>Fit statistics</i>			
Observations	24	24	24
R ²	0.49635	0.64057	0.85688
Adjusted R ²	0.47346	0.62423	0.84325

*Signif. Codes: ***: 0.01, **: 0.05, *: 0.1*

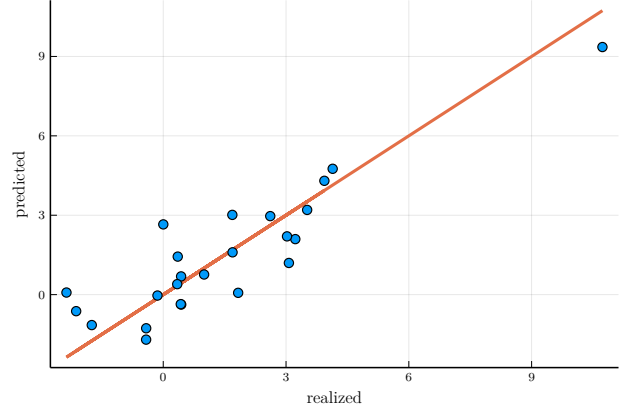


Table 1: Estimation of two-factor model

Figure 2: Realized vs. predicted returns

Note: The left panel shows a cross-sectional regression of average returns on either the aggregate beta or idiosyncratic variance (or both). The right panel shows the 45° degree line and a scatter plot of the predicted returns of the two-factor model and actual average returns.

expected, entrepreneurs who are more exposed to aggregate risk tend to have higher average returns. Similarly, we find a positive and significant relationship between average returns and idiosyncratic volatility, which is consistent with entrepreneurs' limited access to insurance.

One possible concern is that idiosyncratic volatility may be correlated with aggregate beta, e.g. riskier projects could be more exposed to both sources of risk. The positive association between idiosyncratic risk and expected returns could in principle still be consistent with perfect risk sharing. To address this point, we consider next a two-factor model where we control for both sources of risk. Another important issue is that our measures of risk exposure are noisy estimates of the true beta and idiosyncratic volatility, so measurement error may bias our results. To deal with this issue, we follow the standard procedure in empirical asset pricing (see e.g. [Black et al. 1972](#)) and group entrepreneurs into bins, or portfolios, according to a 5×5 double sort based on aggregate beta and idiosyncratic risk.¹²

Table 1 shows cross-sectional regressions of average portfolio returns on the exposure to the different sources of risk. The CAPM-inspired one-factor model explains a sizeable fraction of the variation in average portfolio returns, with an adjusted R^2 of 47%, perhaps surprisingly given its performance with public companies. Idiosyncratic variance by itself is able to explain an even larger fraction of the differences in returns, with an adjusted R^2 of 62%. In both cases, the coefficient on the risk exposure is highly significant. In the case of the two-factor model, the coefficients on the exposure to both aggregate and idiosyncratic risk are significant and the adjusted R^2 is 84%. Moreover, we can reject both one-factor models in favor of the two-factor model, indicating that both sources of risk are important to explain differences in

¹²As these are private businesses, the portfolios are not tradeable. Forming portfolios allow us to reduce the noise in the estimation and better capture the risk/return relationship. To avoid within-portfolio diversification, we assign to the portfolio the average idiosyncratic volatility, instead of the volatility of the average.

entrepreneurial returns. Figure 2 shows the fit of the two-factor model, by comparing predicted and realized average returns. The model successfully accounts for the large variation in expected returns observed in our data.

In the appendix, we explore the robustness of this result. We show that as we increase the number of portfolios, the two-factor still explains a substantial fraction of the variation. The adjusted R^2 is 69% in the case of a 8×8 sort in the two sources of risk and 53% in the case of a 10×10 sort. In both cases, we can reject the hypothesis of a single-factor model based on either source of risk.

To better grasp the role of each risk factor, we provide a decomposition of entrepreneurial returns into an aggregate and idiosyncratic component, following again Samphantharak and Townsend (2018). Table 2 decomposes the risk premium and variance of returns using the results from Table 1. We find that the majority of risk is idiosyncratic, where idiosyncratic shocks account for 93% of the variance, while idiosyncratic risk accounts for slightly more than half of the risk premium. This implies that the Sharpe ratio of the aggregate component is three times larger than the Sharpe ratio of the idiosyncratic component.

The results from Table 2 are inconsistent with both the perfect risk sharing and autarky benchmarks. The fact that the idiosyncratic risk premium explains more than half of total expected returns indicates that limits to diversification are substantial, which allow us to reject the full risk sharing benchmark. Financial autarky is inconsistent with the observed pattern in the Sharpe ratio. As shown in Appendix E.4, under autarky, the share of variance of the idiosyncratic component should coincide with its share of expected returns. Equivalently, the Sharpe ratio should be proportional to the volatility. This implies that the Sharpe ratio for the idiosyncratic component should be more than three times *larger* than the one for the aggregate component, but we observe the opposite pattern in the data.¹³ Therefore, these results suggest that partial insurance is important to explain the empirical patterns involving risk and return. We revisit this decomposition through the lenses of our structural model in Section 5.2.

2.2 Risk-taking and savings over the life cycle

We consider next the determinants of entrepreneurs' risk-taking and savings decisions. The decision of how much to invest in the business depends not only on the risk and return of the entrepreneurial activity, but also on households' attitude towards risk. Behavior towards risk is potentially shaped by households' characteristics, including e.g. age and household size. We study then how risk-taking and consumption decisions depend on expected returns, age, and a range of demographic controls.

¹³Financial autarky implies that the aggregate Sharpe ratio over the idiosyncratic Sharpe ratio should be $5.2/19.4 = 0.26$, while we observed a ratio of $0.35/0.11 = 3.18$, almost a twelvefold difference.

Table 2: Aggregate and idiosyncratic components of risk and return

This table presents the decomposition of total returns into an aggregate component and an idiosyncratic component. The aggregate component is measured as the coefficient on the market return on the cross-sectional regression times the market beta averaged across entrepreneurs. The idiosyncratic component is measured as the coefficient on idiosyncratic risk on the cross-sectional regression times the idiosyncratic variance averaged across entrepreneurs.

	Risk premium	% of returns	Volatility	% of variance	Sharpe ratio
Total returns	4.0%	100%	20.1%	100%	0.20
Aggregate component	1.9%	46.2%	5.2%	6.8%	0.35
Idiosyncratic component	2.2%	53.8%	19.4%	93.2%	0.11

Source: Townsend Thai Monthly Survey.

To ensure stationarity, we consider the behavior of the ratio of the value invested in the business and the entrepreneur’s financial wealth, which corresponds to our measure of risk-taking, and the ratio of consumption and financial wealth. By looking at ratios instead of levels, we limit the influence of aggregate shocks and focus on more stable relationships. We compute age effects controlling for year and demographic controls, following the methodology of [Kaplan \(2012\)](#). The age effects are computed for five age groups, ranging from 25 to 80 years old, where group 1 is the youngest and group 5 the oldest. The cutoffs for each group are chosen such that we have roughly the same number of households in each group. We cluster standard errors by household to allow shocks to be correlated over time. The details of the empirical methodology are described in [Appendix B.2](#).

Table 3 presents the results. We find that risk-taking has a strong life cycle pattern. Column 1, the case without any additional controls, shows that the amount invested in the business declines sharply with age, with the coefficient for the oldest group being roughly 30% smaller than the one for the youngest group. A similar pattern holds as we add more controls. Moreover, age effects account for a large fraction of the variation, with an R^2 of almost 70%. Column 2 adds the average return as a control. As expected, we find a positive and significant coefficient. However, this adds little to explain cross-section variation in risk taking, as the R^2 is virtually the same. Adding time fixed effects does not change the results, suggesting that aggregate shocks affect similarly the amount invested in the business and financial wealth, leaving their ratio roughly unchanged. Finally, we add demographic and geographic controls, such as household size, number of children, and the province the household is located, but we find again similar patterns.

The consumption-wealth ratio also has a strong life cycle pattern, where once again age effects account for a large fraction of the cross-sectional variation. However, instead of declining monotonically with age as our risk-taking measure, we find a U-shaped pattern:

Table 3: Risk-taking and consumption decisions over the life cycle

	Risk-taking				Consumption-wealth ratio			
	(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)
Age group: 1	0.278*** (0.012)	0.256*** (0.013)	0.231*** (0.014)	0.155*** (0.011)	0.112*** (0.005)	0.098*** (0.005)	0.100*** (0.005)	0.084*** (0.005)
Age group: 2	0.244*** (0.008)	0.229*** (0.008)	0.200*** (0.010)	0.124*** (0.010)	0.104*** (0.004)	0.096*** (0.004)	0.098*** (0.005)	0.082*** (0.005)
Age group: 3	0.242*** (0.009)	0.233*** (0.010)	0.202*** (0.011)	0.128*** (0.010)	0.092*** (0.004)	0.088*** (0.004)	0.090*** (0.005)	0.076*** (0.005)
Age group: 4	0.203*** (0.008)	0.191*** (0.009)	0.162*** (0.010)	0.092*** (0.010)	0.075*** (0.003)	0.069*** (0.003)	0.071*** (0.004)	0.056*** (0.005)
Age group: 5	0.192*** (0.008)	0.179*** (0.009)	0.147*** (0.010)	0.081*** (0.010)	0.081*** (0.004)	0.076*** (0.004)	0.077*** (0.005)	0.062*** (0.005)
Average return		0.340** (0.106)	0.338** (0.106)	0.346*** (0.034)		0.067 (0.042)	0.066 (0.043)	0.088*** (0.015)
Year FE			Yes	Yes			Yes	Yes
Demographic controls				Yes				Yes
N	7,542	5,932	5,932	5,776	7,542	6,019	6,019	5,838
R^2	0.692	0.698	0.703	0.719	0.621	0.622	0.623	0.633

consumption-wealth initially declines with age, but it increases again for the oldest group.

We derive three main conclusions from these results. First, the cross-sectional variation in both risk-taking and consumption is driven mostly by life-cycle considerations. This is indicative that entrepreneurs' risk appetite varies over the life cycle. Second, year fixed effects do not affect significantly the ratio of business assets and financial wealth or the consumption-wealth ratio. This suggests a relatively stationary environment, despite the presence of aggregate shocks. Third, expected returns affect positively risk-taking and consumption. These facts, in conjunction with the importance of partial insurance discussed in Section 2.1, motivates the ingredients of our theoretical model, which we consider next.

3 A life-cycle model of entrepreneurial risk taking

In this section, we consider a model of entrepreneurial activity with two main ingredients: i) imperfect idiosyncratic insurance and ii) finite lives with imperfect altruism. These two ingredients play a crucial role in capturing the empirical patterns described in Section 2.

3.1 Environment

Time is continuous and the economy is populated by two types of households: entrepreneurs and wage earners. Population grows at rate g , and the share of entrepreneurs in the population is constant and given by χ_e . The set of entrepreneurs and wage earners alive at period t are denoted by \mathcal{E}_t and \mathcal{W}_t , respectively. Entrepreneurs live for T periods and leave bequests to their offspring. For simplicity, we assume that wage earners have infinite horizon. While

all households receive labor income, only entrepreneurs have access to a production technology, which is exposed to both aggregate and idiosyncratic shocks. Households buy and sell aggregate insurance in a frictionless market, but entrepreneurs have access only to imperfect idiosyncratic insurance. Households borrow and lend at a riskless rate r_t . We now describe in detail the technology, preferences, and financial frictions faced by both types of households.

Technology. Entrepreneur i combines capital $k_{i,t}$ and hired labor $l_{i,t}$ to produce a final homogeneous good $\tilde{y}_{i,t}$, the numeraire in this economy, using the technology:

$$\tilde{y}_{i,t} = A_t k_{i,t}^\alpha l_{i,t}^{1-\alpha}, \quad (1)$$

and we denote scaled output by $y_t = \tilde{y}_t / A_t$.¹⁴

Productivity A_t is subject to aggregate shocks and follows a geometric Brownian motion:

$$\frac{dA_t}{A_t} = \mu_A dt + \sigma_A dZ_t, \quad (2)$$

where Z_t is a standard Brownian motion.

Entrepreneurs can adjust their capital stock by either investing in new capital or buying capital from another entrepreneur. The investment technology is risky and subject to adjustment costs. In particular, given a total investment of $\iota_{i,t} A_t k_{i,t}$, capital evolves according to

$$\frac{dk_{i,t}}{k_{i,t}} = (\Phi(\iota_{i,t}) - \delta) dt + \sigma_{id} dZ_{i,t}. \quad (3)$$

$Z_{i,t}$ is an idiosyncratic Brownian motion for entrepreneur i , which is independent across entrepreneurs. The investment function $\Phi(\cdot)$ satisfies $\Phi(0) = 0$, $\Phi'(\cdot) > 0$, and $\Phi''(\cdot) < 0$.

The concavity of $\Phi(\cdot)$ captures the presence of adjustment costs. Importantly, investment is risky and subject to idiosyncratic shocks. Entrepreneurs can also adjust their capital stock by buying capital from other entrepreneurs at the price $\tilde{q}_t = q_t A_t$. The market value of the business is given by $\tilde{q}_t k_{i,t}$. In equilibrium, q_t will be non-stochastic, so the relative price of capital \tilde{q}_t moves proportionally with aggregate productivity shocks.

The evolution of the aggregate capital stock, $k_t = \int_{\mathcal{E}_t} k_{i,t} di$, is not affected by idiosyncratic shocks, as these shocks get diversified in the aggregate:

$$dk_t = \left[\int_{\mathcal{E}_t} (\Phi(\iota_{i,t}) - \delta) k_{i,t} di \right] dt. \quad (4)$$

The return of investing in the project can be written as the sum of the dividend yield, i.e.

¹⁴We adopt this convention throughout the paper: variables that grow with aggregate productivity A_t are denoted with a tilde and the corresponding scaled variable are denoted without a tilde.

profits net of investment expenses relative to the value of the business, and capital gains:

$$\begin{aligned} dR_{i,t} &= \frac{\tilde{y}_{i,t} - \tilde{w}_t l_{i,t} - l_{i,t} A_t k_{i,t}}{\tilde{q}_t k_{i,t}} dt + \frac{d(\tilde{q}_t k_{i,t})}{\tilde{q}_t k_{i,t}} \\ &= \mu_{i,t}^R dt + \sigma_A dZ_t + \sigma_{id} dZ_{i,t}, \end{aligned}$$

where $\tilde{w}_t = w_t A_t$ denotes the wage rate and $\mu_{i,t}^R$ is the expected return.

Using Ito's lemma to compute expected capital gains, the expected return on the project is

$$\mu_{i,t}^R \equiv \underbrace{\frac{y_{i,t} - w_t l_{i,t} - l_{i,t} k_{i,t}}{q_t k_{i,t}}}_{\text{dividend yield}} + \underbrace{\frac{\dot{q}_t}{q_t} + \mu_A + \Phi(l_{i,t}) - \delta}_{\text{expected capital gain}}. \quad (5)$$

Preferences and labor supply. Entrepreneurs live for T periods. They have isoelastic preferences over consumption $\tilde{c}_{i,t}$ with curvature parameter γ and derive utility of leaving bequests:

$$\mathbb{E}_{s_i} \left[\int_{s_i}^{s_i+T} e^{-\rho(t-s_i)} \frac{\tilde{c}_{i,t}^{1-\gamma}}{1-\gamma} dt + e^{-\rho(T-s_i)} (1-\psi)^\gamma V^* \frac{\tilde{n}_{i,s_i+T}^{1-\gamma}}{1-\gamma} \right], \quad (6)$$

where $\tilde{n}_{i,t}$ denotes financial wealth, and s_i denotes the birth date of entrepreneur i .¹⁵

The parameter ψ measures the strength of the bequest motive. If $\psi = 1$, entrepreneurs give no weight to their offspring, and there are no intergenerational linkages. If $\psi = 0$, the behavior of entrepreneurs coincides with the one of an agent with infinite horizon.¹⁶ The case $0 < \psi < 1$ captures a form of imperfect altruism.

In addition to business income, entrepreneurial households are allowed to receive labor income. This is consistent with the observation that households have multiple sources of income in our data. Labor is supplied inelastically, it is denoted by $\bar{l}_{i,t}$, and it can vary deterministically over the life cycle.¹⁷ As discussed in Section 3.3, there is significant variation in the importance of labor income over the life cycle for entrepreneurs.

Financial friction. Entrepreneurs face a moral hazard problem, similar to the one in [He and Krishnamurthy \(2012\)](#) and [Di Tella \(2017\)](#). We follow [Di Tella \(2017\)](#) and restrict attention to short-term contracts. Aggregate shocks are perfectly observable by all households, while idiosyncratic shocks are only observed by the entrepreneur. An entrepreneur can divert capital, but a fraction $1 - \phi$ of the diverted capital is lost in the process. The parameter $\phi \in (0, 1)$

¹⁵Financial wealth equals the entrepreneur's assets, including the value of the business, net of liabilities.

¹⁶The constant V^* , given in Appendix A.1, equals the value function coefficient for an infinite-horizon agent.

¹⁷Notice that $\bar{l}_{i,t}$ denotes the amount of labor supplied by the household, while $l_{i,t}$ is the amount of labor demanded by the entrepreneur to run her project.

controls the severity of the moral hazard problem.

Following the literature on dynamic moral hazard, we show in Appendix C.1 that the solution to the contracting problem can be implemented by a market structure where entrepreneurs have access to a riskless asset with return r_t and both aggregate and idiosyncratic insurance. There is no limit on aggregate insurance, as aggregate shocks are perfectly observable. The quantity of idiosyncratic insurance is limited for incentive purposes. Formally, entrepreneur i pays $p_t^{ag} \tilde{\theta}_{i,t}^{ag}$ to reduce aggregate volatility by $\tilde{\theta}_{i,t}^{ag}$, where p_t^{ag} denotes the price of aggregate insurance. In contrast, entrepreneur i can buy idiosyncratic insurance $\tilde{\theta}_{i,t}^{id}$ at zero cost in equilibrium, as providers of insurance can perfectly diversify across entrepreneurs. However, the amount of idiosyncratic insurance is limited by the *skin-in-the-game* constraint:

$$\tilde{\theta}_{i,t}^{id} \leq (1 - \phi) \tilde{q}_t k_{i,t} \sigma_{id}. \quad (7)$$

This particular market structure represents one possible implementation of the allocation under the optimal contract. For instance, instead of formal insurance contracts, this implementation may capture the presence of informal insurance arrangements. As documented by e.g. Kinnan and Townsend (2012), kinship networks play an important role in allowing households to partially share idiosyncratic risk. The insurance constraint can also be interpreted as an equity constraint, where entrepreneurs are unable to freely sell claims on their business to a diversified set of investors, as in e.g. Chen et al. (2010) or Panousi and Papanikolaou (2012).

The skin-in-the-game constraint is binding in equilibrium, causing entrepreneurs to be *insurance-constrained*. To focus on the implications of entrepreneurial risk, we abstract from borrowing constraints in our baseline model. We discuss the role of borrowing constraints in Section 3.4. Entrepreneurs can borrow freely against the present value of their future labor income, that is, their human wealth:

$$\tilde{n}_{i,t} \geq -\tilde{h}_{i,t}, \quad (8)$$

where $\tilde{n}_{i,t}$ denotes the entrepreneur's financial wealth, $\tilde{h}_{i,t} \equiv \mathbb{E}_t \left[\int_t^{s+T} \frac{\pi_z}{\pi_t} \tilde{w}_z \bar{l}_{i,z} dz \right]$ denotes human wealth, and π_t denotes a stochastic discount factor (SDF) for this economy.¹⁸ The SDF evolves according to $d\pi_t = -r_t \pi_t dt - p_t^{ag} \pi_t dZ_t$. Note that we can write the entrepreneur's financial wealth as $\tilde{n}_{i,t} = \tilde{q}_t k_{i,t} + \tilde{b}_{i,t}$, the sum of the value of the business and the amount invested in the riskless asset, denoted by $\tilde{b}_{i,t}$. Condition (8) can be written as $-\tilde{b}_{i,t} \leq \tilde{q}_t k_{i,t} + \tilde{h}_{i,t}$, so the entrepreneur can borrow freely against the value of the business and human wealth.

Entrepreneurs' problem. The problem of entrepreneur i with age a_i is to choose a vector of stochastic processes $(\tilde{c}_i, \tilde{\theta}_i^{ag}, \tilde{\theta}_i^{id}, k_i, l_i, \iota_i)$, taking the processes for prices $(\tilde{q}, \tilde{w}, r, p^{ag})$ as given,

¹⁸Even though markets are (endogenously) incomplete, households agree on the valuation of variables not exposed to idiosyncratic risk. Therefore, there is no ambiguity in defining the relevant SDF for aggregate payoffs.

to solve the following program:

$$\tilde{V}_t(\tilde{n}_i, a_i) = \max_{\tilde{c}_i, \tilde{\theta}_i^{ag}, \tilde{\theta}_i^{id}, k_i, l_i, t_i} \mathbb{E}_t \left[\int_0^{T-a_i} e^{-\rho z} \frac{\tilde{c}_{i,t+z}^{1-\gamma}}{1-\gamma} dz + e^{-\rho(T-a_i)} (1-\psi)^\gamma V^* \frac{\tilde{n}_{i,t+T-a_i}^{1-\gamma}}{1-\gamma} \right], \quad (9)$$

subject to (7), (8), non-negativity constraints $c_{i,t}, k_{i,t} \geq 0$, and the law of motion of $\tilde{n}_{i,t}$

$$d\tilde{n}_{i,t} = \left[(\tilde{n}_{i,t} - \tilde{q}_t k_{i,t}) r_t + \tilde{q}_t k_{i,t} \mu_{i,t}^R - p_t^{ag} \tilde{\theta}_{i,t}^{ag} + \tilde{w}_t \bar{l}_{i,t} - \tilde{c}_{i,t} \right] dt + \left(\tilde{q}_t k_{i,t} \sigma_A - \tilde{\theta}_{i,t}^{ag} \right) dZ_t + \left(\tilde{q}_t k_{i,t} \sigma_{id} - \tilde{\theta}_{i,t}^{id} \right) dZ_{i,t},$$

given initial financial wealth $\tilde{n}_{i,t} = \tilde{n}_i > -\tilde{h}_{i,t}$.

The term in brackets in the expression above represents the expected growth rate of financial wealth. The entrepreneur invests $\tilde{n}_{i,t} - \tilde{q}_t k_{i,t}$ in the riskless asset, which gives a return r_t , and she invests the amount $\tilde{q}_t k_{i,t}$ in the risky business technology, which gives an expected return $\mu_{i,t}^R$. The cost of aggregate insurance is $p_t^{ag} \tilde{\theta}_{i,t}^{ag}$. The entrepreneur receives labor income $\tilde{w}_t \bar{l}_{i,t}$ and consumes $\tilde{c}_{i,t}$. The last two terms represent the exposure to aggregate and idiosyncratic risk, which corresponds to the risk exposure through the business net of insurance.

Wage earners' problem. In contrast to entrepreneurs, wage earners do not have access to a production technology. To simplify exposition, we assume that wage earners have infinite horizon. Allowing for finite lives and a bequest motive does not change our main results, as shown in Appendix E.3. Wage earners and entrepreneurs share a common per-period isoelastic utility function with curvature parameter γ .¹⁹ As commonly assumed in models with heterogeneous rate of return, as e.g. Kiyotaki and Moore (1997), we allow for different discount rates between wage earners and entrepreneurs.

The problem of wage earner $j \in \mathcal{W}_t$ is given by

$$\tilde{V}_t^w(\tilde{n}_j) = \max_{\tilde{c}_j, \tilde{\theta}_j^{ag}} \mathbb{E}_t \left[\int_t^\infty e^{-\rho_w(z-t)} \frac{\tilde{c}_{j,z}^{1-\gamma}}{1-\gamma} dz \right], \quad (10)$$

subject to non-negativity constraint $\tilde{c}_{j,t} \geq 0$, $\tilde{n}_{j,t} \geq \tilde{h}_{j,t}$, where $\tilde{h}_{j,t}$ denotes wage earner j 's human wealth, and the law of motion of financial wealth $\tilde{n}_{j,t}$

$$d\tilde{n}_{j,t} = \left[\tilde{n}_{j,t} r_t - p_t^{ag} \tilde{\theta}_{j,t}^{ag} + \tilde{w}_t \bar{l}_{j,t} - \tilde{c}_{j,t} \right] dt - \tilde{\theta}_{j,t}^{ag} dZ_t,$$

given initial financial wealth $\tilde{n}_{j,t} = \tilde{n}_j > -\tilde{h}_{j,t}$. $\tilde{\theta}_{j,t}^{ag}$ can take positive or negative values, so wage earners can choose to either buy or provide aggregate insurance to entrepreneurs.²⁰

¹⁹ As shown by Swanson (2012), and consistent with Lemma 1 below, the coefficient of relative risk aversion in presence of labor is affected by, but is not equal to, γ .

²⁰ Wage earners can also provide idiosyncratic insurance. In our notation, we have already imposed that they

Equilibrium. We provide below a definition of the competitive equilibrium.

Definition 1. A competitive equilibrium is a set of aggregate stochastic processes: the aggregate capital stock k , the interest rate r , the wage rate \tilde{w} , the relative price of capital \tilde{q} , and the price of aggregate insurance p^{ag} ; and a set of stochastic processes for each entrepreneur $i \in \mathcal{E}_t$ and wage earner $j \in \mathcal{W}_t$: consumption \tilde{c}_i , financial wealth \tilde{n}_i , capital k_i , labor l_i , aggregate insurance $\tilde{\theta}_i^{ag}$, and idiosyncratic insurance $\tilde{\theta}_i^{id}$ for $i \in \mathcal{E}_t$; consumption \tilde{c}_j and aggregate insurance $\tilde{\theta}_j^{ag}$ for $j \in \mathcal{W}_t$ such that:

- (a) Aggregate capital stock satisfies the law of motion (4), given the initial capital stock k_0 .
- (b) $(\tilde{c}_i, \tilde{\theta}_i^{ag}, \tilde{\theta}_i^{id}, k_i, l_i, \iota_i)$ solve entrepreneurs' problem (9), given $(\tilde{q}, \tilde{w}, r, p^{ag})$.
- (c) $(\tilde{c}_j, \tilde{\theta}_j^{ag})$ solve wage earners' problem (10), given (\tilde{w}, r, p^{ag}) .
- (d) Markets clear for all $t \geq 0$:

i. Market for consumption goods

$$\int_{\mathcal{E}_t} \tilde{c}_{i,t} di + \int_{\mathcal{W}_t} \tilde{c}_{j,t} dj + \int_{\mathcal{E}_t} \iota_{i,t} A_t k_{i,t} di = \int_{\mathcal{E}_t} \tilde{y}_{i,t} di.$$

ii. Markets for capital and labor

$$\int_{\mathcal{E}_t} k_{i,t} di = k_t, \quad \int_{\mathcal{E}_t} l_{i,t} di = \int_{\mathcal{E}_t} \bar{l}_{i,t} di + \int_{\mathcal{W}_t} \bar{l}_{j,t} dj.$$

iii. Markets for aggregate insurance and riskless bonds

$$\int_{\mathcal{E}_t} \tilde{\theta}_{i,t}^{ag} di + \int_{\mathcal{W}_t} \tilde{\theta}_{j,t}^{ag} dj = 0, \quad \int_{\mathcal{E}_t} [\tilde{n}_{i,t} - \tilde{q}_t k_{i,t}] di + \int_{\mathcal{W}_t} \tilde{n}_{j,t} dj = 0.$$

3.2 Solution to entrepreneurs' problem

We describe next the solution to the entrepreneurs' problem. We focus on a stationary equilibrium, where scaled aggregate variables are constant, that is, $w_t = w$ and $q_t = q$. In Section 6, we consider a non-stationary environment where aggregate variables are time-varying.

Maximizing expected returns. As $(l_{i,t}, \iota_{i,t})$ enters the maximization problem in Equation (9) only through the expected return on the business, given in Equation (5), entrepreneurs choose these variables to maximize expected returns. Labor demand assumes the usual form:

$$w = (1 - \alpha) \left(\frac{k_{i,t}}{l_{i,t}} \right)^\alpha. \quad (11)$$

can diversify the exposure to idiosyncratic risk, so their financial wealth is only exposed to aggregate risk.

The capital-labor ratio is equalized across entrepreneurs and coincides with the aggregate capital-labor ratio $K_t \equiv k_t / \bar{l}_t$, where k_t denotes the aggregate capital stock and \bar{l}_t denotes the aggregate labor supply. In a stationary equilibrium, capital grows at the same rate as labor supply, which we assume grows with population at rate g .

The investment rate $\iota_{i,t}$ is given by

$$\Phi'(\iota_{i,t}) = \frac{1}{q} \Rightarrow \iota_{i,t} = (\Phi')^{-1} \left(\frac{1}{q} \right) \equiv \iota(q), \quad (12)$$

where $\iota(q)$ is increasing in q , given the concavity of $\Phi(\cdot)$. From Equation (5), an increase in the investment rate $\iota_{i,t}$ raises the expected capital gain by $\Phi'(\iota_{i,t})$, but it reduces the dividend yield by $1/q$. Expected returns are maximized when $\Phi'(\iota_{i,t}) = 1/q$, i.e., Equation (12) is satisfied.

Substituting Equations (11) and (12) into Equation (5), we obtain

$$\mu^R = \frac{\alpha K^{\alpha-1} - \iota(q)}{q} + \mu_A + \Phi(\iota(q)) - \delta, \quad (13)$$

where $\mu_{i,t}^R = \mu^R$ in a stationary equilibrium, so expected returns are equalized across entrepreneurs. Realized returns are, of course, still heterogeneous due to imperfect insurance.

Human and total wealth. The lemma below shows that the relevant notion of wealth is *total wealth*, $\omega_{i,t} \equiv n_{i,t} + h_{i,t}$, the sum of financial wealth and human wealth. In particular, the entrepreneur's value function depends on total wealth $\omega_{i,t}$ and on age $a_{i,t} = t - s_i$.

Lemma 1. *Suppose the economy is in a stationary equilibrium. Then,*

(a) *Human wealth evolves according to*

$$\frac{\partial h(a)}{\partial a} = (r + p^{ag} \sigma_A - \mu_A) h(a) - w\bar{l}(a), \quad (14)$$

given $h(T) = 0$. Human wealth is then given by

$$h(a) = \int_0^{T-a} e^{-(r+p^{ag}\sigma_A-\mu_A)z} w\bar{l}(a+z) dz. \quad (15)$$

(b) *The (scaled) value function is given by²¹*

$$V(n, a) = \zeta(a)^{-\frac{1}{\gamma}} \frac{(n + h(a))^{1-\gamma}}{1-\gamma}, \quad (16)$$

²¹Scaled value function is related to the original value function by the condition $\tilde{V}_t(\tilde{n}, a) = A_t^{1-\gamma} V\left(\frac{\tilde{n}}{A_t}, a\right)$.

where $\zeta(a)$ equals the ratio of consumption to total wealth. The entrepreneur's effective risk aversion is given by

$$-\frac{V_{nn}n}{V_n} = \frac{\gamma}{1 + \frac{h(a)}{n}}. \quad (17)$$

(c) Capital stock and the aggregate and idiosyncratic insurance solve the mean-variance problem:

$$\max_{k_i, \theta_i^{ag}, \theta_i^{id}} \frac{qk_{i,t}}{n_{i,t}} (\mu^R - r) - \frac{p^{ag}\theta_{i,t}^{ag}}{n_{i,t}} - \frac{1}{2} \frac{\gamma}{1 + \frac{h_{i,t}}{n_{i,t}}} \left[\left(\frac{qk_{i,t} + h_{i,t}}{n_{i,t}} \sigma_A - \frac{\theta_{i,t}^{ag}}{n_{i,t}} \right)^2 + \left(\frac{qk_{i,t}}{n_{i,t}} \sigma_{id} - \frac{\theta_{i,t}^{id}}{n_{i,t}} \right)^2 \right], \quad (18)$$

subject to (7).

Proof. See Appendix A.1. □

The first part of Lemma 1 shows that human wealth is the present discounted value of future labor income, where the discount rate incorporates the aggregate risk premium $p^{ag}\sigma_A$.²² This extra discount is required as wages move with aggregate productivity, so labor income is risky. Human wealth being a risky asset is consistent with Benzoni et al. (2007), who show that human wealth becomes highly correlated with stocks when aggregate output and labor income are cointegrated, as in our model. As human wealth depends only on the entrepreneur's age, we drop the dependence on the household, i.e. $h_{i,t} = h(a_{i,t})$.

The second part of Lemma 1 gives the value function, an age-dependent CRRA function of total wealth. Importantly, the entrepreneur's effective risk aversion depends on $h_{i,t}/n_{i,t}$, the *human-financial wealth* ratio, which varies substantially over the life cycle in the data.

The final part of the lemma shows that entrepreneurs' portfolio choice reduces to a mean-variance problem, given the effective risk aversion $\gamma/(1 + \frac{h_{i,t}}{n_{i,t}})$. This results from the continuous-time formulation, as Equation (18) comes from a rearrangement of the Hamilton-Jacobi-Bellman (HJB) equation of the entrepreneurs' problem. Problem (18) is subject to the skin-in-the-game constraint (7) and the Lagrange multiplier to this constraint, which we call the *shadow price of idiosyncratic insurance*, plays an important role in the entrepreneurs' risk-taking decision.

Policy functions. We consider next entrepreneurs' policy functions.

Proposition 1. Suppose the economy is in a stationary equilibrium. Then,

i. Demand for capital is given by

$$\frac{qk_{i,t}}{n_{i,t}} = \frac{1 + \frac{h_{i,t}}{n_{i,t}}}{\gamma} \frac{p^{id}}{\phi\sigma_{id}}, \quad (19)$$

²²We can write Equation (14) alternatively as $w\bar{l}(a) + \mu_A + \frac{1}{h(a)} \frac{\partial h(a)}{\partial a} - r = p^{ag}\sigma_A$, so the expected excess return on human wealth equals the aggregate risk premium $p^{ag}\sigma_A$.

where p^{id} is the shadow price of idiosyncratic insurance, given by

$$p^{id} = \frac{\mu^R - r - p^{ag}\sigma_A}{\phi\sigma_{id}}. \quad (20)$$

ii. The demand for aggregate insurance is given by

$$\frac{\theta_{i,t}^{ag}}{n_{i,t}} = \left(\frac{qk_{i,t}}{n_{i,t}} + \frac{h_{i,t}}{n_{i,t}} \right) \sigma_A - \frac{1 + \frac{h_{i,t}}{n_{i,t}}}{\gamma} p^{ag}. \quad (21)$$

iii. The consumption-wealth ratio is given by

$$\frac{c_{i,t}}{n_{i,t}} = \frac{\bar{r}}{1 - \psi e^{-\bar{r}(T-a_i)}} \left(1 + \frac{h_{i,t}}{n_{i,t}} \right), \quad (22)$$

where $\bar{r} \equiv \frac{1}{\gamma}\rho + \left(1 - \frac{1}{\gamma}\right)r^{MV}$ and $r^{MV} \equiv r + \frac{(p^{ag})^2 + (p^{id})^2}{2\gamma}$.

Proof. See Appendix A.1. □

The demand for capital has three components: entrepreneur's effective risk aversion, $\gamma/(1 + \frac{h_{i,t}}{n_{i,t}})$, the price of idiosyncratic insurance, p^{id} , and the quantity of non-diversified risk, $\phi\sigma_{id}$. The cross-sectional dispersion in risk-taking is captured by differences in effective risk aversion, which are driven by the human-financial wealth ratio. Entrepreneurial risk-taking then inherits the life-cycle patterns of the human-financial wealth ratio.

Equation (20) shows that the shadow price of idiosyncratic insurance, the Lagrange multiplier on the skin-in-the-game constraint, is equalized across entrepreneurs. Moreover, it equals the return per unit of risk (the Sharpe ratio) of an investor who fully insures the project against aggregate risk. In equilibrium, this Sharpe ratio is positive, so the skin-in-the-game constraint is always binding, that is, $\theta_{i,t}^{id} = (1 - \phi)q_t k_{i,t} \sigma_{id}$. Intuitively, entrepreneurs purchase as much idiosyncratic insurance as possible given it has zero cost.

Rearranging Equation (20), we can express expected excess returns as follows:

$$\mu^R - r = \underbrace{p^{ag}\sigma_A}_{\text{agg. risk premium}} + \underbrace{p^{id}\phi\sigma_{id}}_{\text{id. risk premium}}.$$

Hence, we obtain a two-factor model for expected returns, corresponding to the compensation for aggregate and idiosyncratic risk. In the absence of aggregate risk, growth, and adjustment costs, this expression simplifies to $\alpha K^{\alpha-1} - r = p^{id}\phi\sigma_{id}$, so the marginal product of capital (MPK) is not equal to the interest rate when $\phi > 0$, even though entrepreneurs face no borrowing constraints. Entrepreneurs do not expand the business, despite $\mu^R > r$, to limit their

exposure to risk. In equilibrium, the wedge $\mu^R - r$ depends on entrepreneurs' risk appetite and their ability to diversify these risks, as captured by ϕ .

Interestingly, the demand for capital depends on the price and quantity of *idiosyncratic* risk. Access to aggregate insurance leads to a separation between the choice of the scale of the business and how much aggregate risk entrepreneurs are willing to hold. The choice of how much to invest in the business is essentially a decision of how much idiosyncratic risk to bear.

The demand for aggregate insurance, given in Equation (21), is decreasing in the price of aggregate insurance with a slope given by the inverse of the effective risk aversion. Given that $p^{ag} = \gamma\sigma_A$, the demand for aggregate insurance simplifies to $\theta_{i,t}^{ag} = (qk_{i,t} - n_{i,t})\sigma_A$.²³ Poor entrepreneurs buy aggregate insurance, $\theta_{i,t}^{ag} > 0$, while rich entrepreneurs provide insurance, $\theta_{i,t}^{ag} < 0$. This arrangement can be implemented by having richer entrepreneurs sending transfers to poor entrepreneurs as an indemnity after a negative aggregate shock, with transfers in the opposite direction after a positive aggregate shock. Note that if entrepreneurs on average have positive savings on the risk-free asset, such that $\int_{\mathcal{E}_t} (qk_{i,t} - n_{i,t}) di < 0$, then they are net sellers of insurance. In this case, wage earners are borrowers in equilibrium. As they effectively hold a leveraged position on their risky human wealth, wage earners are disproportionately exposed to risk and demand aggregate insurance in equilibrium.

The consumption-wealth ratio is given in Equation (22). The first term $\bar{r}/(1 - \psi e^{-\bar{r}(T-a)})$ represents the marginal propensity to consume (MPC).²⁴ It is increasing in age, as it is typical in finite-horizon problems, and the bequest motive parameter ψ controls the strength of this effect. If $\psi = 0$, the MPC is constant, recovering a standard result in infinite-horizon problems. If $\psi = 1$, then the MPC gets arbitrarily large as the entrepreneur approaches the end of life, so the stock of wealth is fully consumed at the final age T , as in Merton (1969). When $\gamma > 1$, an increase in risk-adjusted returns raises the average MPC. Finally, the consumption-wealth ratio depends on the human-financial wealth ratio, which potentially varies over the life cycle.

3.3 Quantitative implications

We consider next the quantitative implications of the model for the life-cycle behavior of entrepreneurs. We start describing the calibration of the model and consider next the model performance accounting for the empirical life cycle patterns.

Technology, preferences, and demographics. We adopt the following calibration, which is summarized in Table 4. The capital share is set to $\alpha = 0.33$. The average growth rate of pro-

²³The result $p^{ag} = \gamma\sigma_A$ is obtained by combining Equation (21) with the market clearing condition for aggregate insurance, as shown in Appendix C.3.

²⁴The MPC is defined as the change in consumption in response to an increase in financial wealth, that is, the MPC is given by $\frac{\partial c_{i,t}}{\partial n_{i,t}} = \frac{\bar{r}}{1 - \psi e^{-\bar{r}(T-a_i)}}$.

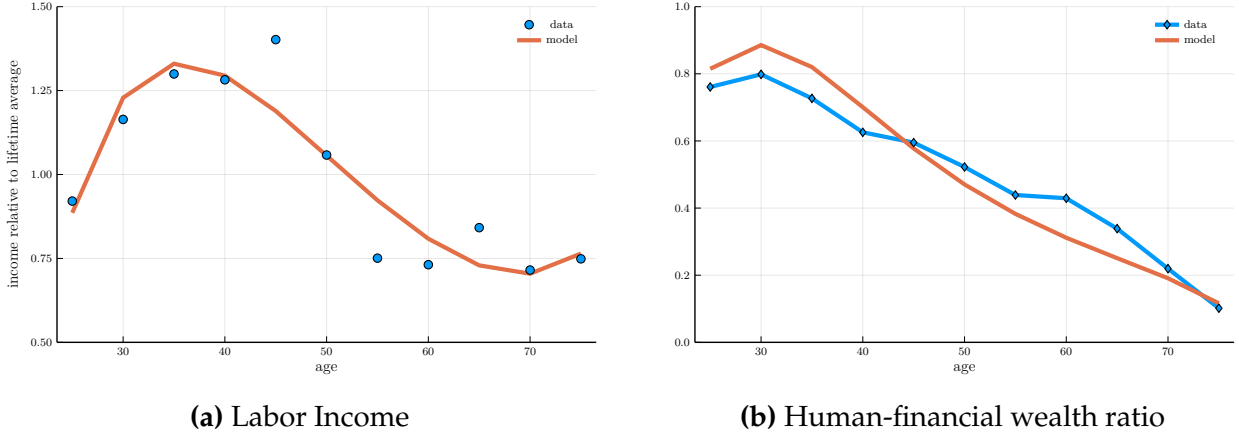
Table 4. Calibrated parameters

Parameter	Choice
<i>Preferences</i>	
ρ Entrepreneur's rate of time preference	0.096
ρ_w Wage earner's rate of time preference	0.089
ψ Bequest motive	0.394
γ Risk aversion	6.76
<i>Technology & financial friction</i>	
μ_A Average productivity growth rate	0.003
σ_A Aggregate volatility	0.052
σ_{id} Idiosyncratic volatility	0.194
ϕ Moral hazard parameter	0.73
Φ_0 Adjustment cost parameter (I)	0.89
Φ_1 Adjustment cost parameter (II)	1.05
α Capital share in production function	0.33
δ Depreciation rate	0.10
<i>Demographics</i>	
g Population growth	0.003
χ_e Share of entrepreneurs in the population	0.47
T Life span (adult life)	55

ductivity is set to $\mu = 0.003$, following the evidence provided by Jeong and Townsend (2007) for Thailand. The investment function assumes the functional form $\Phi(\iota) = \frac{\sqrt{\Phi_0^2 + 2\Phi_1\iota} - \Phi_0}{\Phi_1}$. This corresponds to the case of quadratic adjustment costs, as the investment rate required for capital to grow at rate g is $\iota = \Phi_0(g + \delta) + 0.5\Phi_1(g + \delta)^2$. The coefficients of the investment function are chosen to match an investment rate of roughly 20% and a long-run relative price of capital of one. The depreciation rate is set to $\delta = 0.10$. The discount rate of wage earners is chosen to match a risk-free rate of $r = 3.77\%$, consistent with the average real rate for Thailand over the last two decades. The discount rate of entrepreneurs and the bequest motive parameters are chosen to match the consumption-wealth ratio at the beginning and end of life. The life horizon is set to $T = 55$, so it covers the life span from 25 to 80 years old, and the population growth is set to $g = 0.3\%$, the most recent value for population growth in Thailand. The parameter χ_e is chosen to match the average ratio of the value of the business relative to the entrepreneurs' financial wealth.

Risk, return, and the moral hazard parameter. We choose the risk aversion coefficient, the aggregate and idiosyncratic volatility, and the moral hazard parameter to match the decomposition of risk and return provided in Table 2. The volatility parameters, σ_A and σ_{id} , are chosen to match the aggregate and idiosyncratic components of total volatility. The aggregate

Figure 3: Life Cycle Profiles: Labor Income and Human-Financial Ratio



risk premium is given by $p^{ag}\sigma_A = \gamma\sigma_A^2$, so we choose γ to match the aggregate risk premium. The idiosyncratic risk premium is informative about ϕ . If $\phi = 0$, we obtain the complete markets solution and there is no idiosyncratic risk premium. As we raise ϕ , the importance of the idiosyncratic risk premium increases. Using expression (19) to solve for p^{id} , we can write the idiosyncratic risk premium in terms of the moral hazard parameter ϕ and observable quantities, namely the level of idiosyncratic volatility and the exposure of entrepreneurs' total wealth to the private business.²⁵ Given these quantities, we can identify ϕ .

Measuring human wealth. It remains to specify the labor supply parameters. We follow [Gârleanu and Panageas \(2015\)](#) and assume that $\bar{l}_{i,t}$ is a function of the entrepreneur's age: $\bar{l}_{i,t} = \sum_{l=1}^L \Gamma_l e^{\phi_l a_i}$, where we normalize the average labor supply of entrepreneurs to one, that is, $\frac{\int_{\mathcal{E}_t} \bar{l}_{i,t} di}{\int_{\mathcal{E}_t} di} = 1$. This functional form is flexible enough to capture the empirical labor income dynamics while being analytically tractable. We estimate the parameters $(\Gamma_l, \phi_l)_{l=1}^L$ by non-linear least squares such that the distribution of labor income across age groups matches the one observed in the data. We set the number of exponential terms to $L = 3$. Wage earner's labor supply is constant and given by $\bar{l}_{j,t} = 1$. The left panel of Figure 3 shows that the functional form does a good job of approximating the empirical labor income profile.

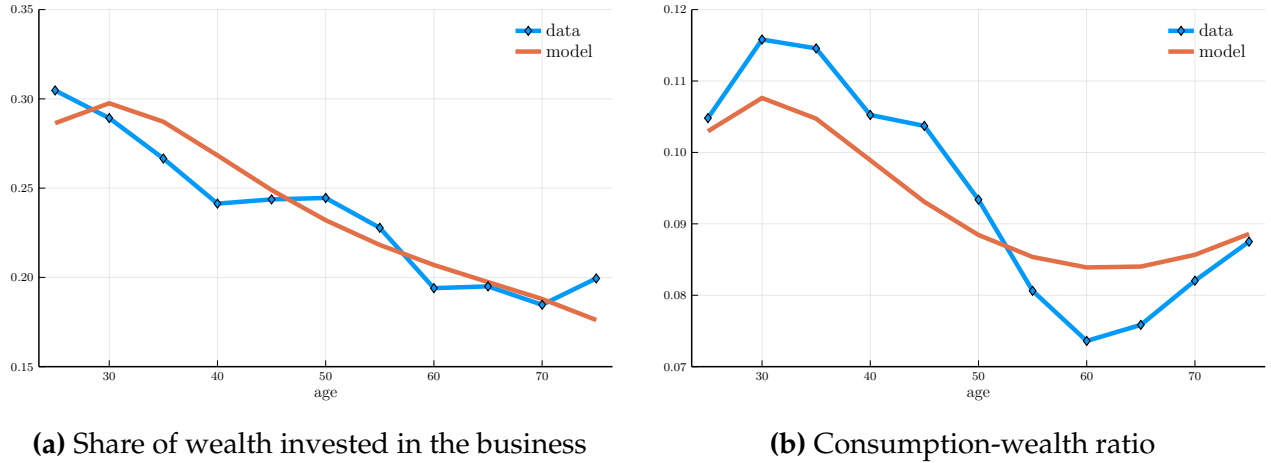
From Equation (15), human wealth can be computed as follows:

$$h_{i,t} = \int_0^{T-a_i} e^{-(r+p^{ag}\sigma_A-\mu_A)z} w \sum_{l=1}^L \Gamma_l e^{\phi_l(z+a_i)} dz.$$

Notice that if $T \rightarrow \infty$ and $\bar{l}_{i,t}$ was constant, then the expression above boils down to the Gordon growth formula applied to human wealth: $h_{i,t} = w/(r + p^{ag}\sigma_A - \mu_A)$.

²⁵See Section 5 for a discussion of the determination of p^{id} in equilibrium.

Figure 4: Life Cycle Profiles: Risk-taking and Consumption-Wealth Ratio



Given the discount rate and the labor income profile, we can compute the human-financial wealth $h_{i,t}/n_{i,t}$, both in the data and in the model. The right panel of Figure 3 shows that the human-financial wealth ratio declines over the life cycle. This is the result of labor income being relatively high at the beginning of the life cycle and the mechanical fact that households have less years of future income as time goes by. Human wealth is quantitatively as important as financial wealth at the beginning of the life. By the age of 60, human wealth is only half the financial wealth. Even though this was not a direct target of the calibration, the model captures reasonably well the evolution of the human-financial wealth over the life cycle.

Implications for risk-taking and savings. We consider next the evolution of risk-taking and consumption over the life cycle. The left panel of Figure 4 shows that the share of wealth invested in the business declines with age, consistent with the evidence in Table 3. In a stationary environment, this pattern cannot be explained by differences in expected returns, which are assumed to be constant. The model generates this pattern by having the effective risk aversion to be decreasing in the human-financial wealth ratio. Given the presence of human wealth, young entrepreneurs are endogenously less risk averse than older entrepreneurs, so they invest a larger fraction of their wealth on the business. Notice that the ratio of the risk-taking measure at the beginning and at the end of life is entirely determined by the human-financial wealth ratio, which was calibrated independently of any information on the cross-section of entrepreneurs' risk-taking.

The consumption-wealth ratio is roughly U-shaped as a function of age. This non-monotonic behavior is the result of two forces. First, the human-financial wealth ratio declines with age, which induces households to reduce consumption. Second, the MPC increases with age, which induces households to consume more. The first effect dominates at the beginning of the life cycle, and the second effect dominates as the entrepreneur gets older.

3.4 The role of borrowing constraints and occupational choice

We consider next three extensions of our baseline model that show how our results are robust to introducing different forms of borrowing constraints or endogenous occupational choice.

Collateral constraints and heterogeneous productivities. In our baseline model, risk is the only factor preventing entrepreneurs from scaling up operations, as entrepreneurs can borrow freely against the value of their assets. In Appendix E.1, we consider the effects of limited pledgeability of physical capital by introducing collateral constraints. Moreover, we allow productivity and idiosyncratic volatility to vary across entrepreneurs, so the collateral constraints are binding for entrepreneurs who are more productive or have less risky projects. The Lagrange multiplier on the insurance constraint now varies across entrepreneurs. It depends not only on the risk-return trade-off, as in Equation (20), but also on the degree of pledgeability of physical capital. Importantly, we find that the demand for capital (19) holds *exactly* when we aggregate across entrepreneurs of a given age. Allowing for collateral constraints and heterogeneity changes the interpretation of p^{id} , which corresponds now to the average price of idiosyncratic risk, but does not affect the life cycle patterns discussed in this section. An important difference with the baseline model is that expected returns depend not only on the exposure to risk, but also on the net worth of constrained entrepreneurs.²⁶ One should expect entrepreneurs with lower amount of assets (and lower collateral) to have a higher expected return, after controlling for the exposure to idiosyncratic and aggregate risk. As shown in Appendix E.1, we find no evidence in favor of this hypothesis in our data. We find that i) the coefficient on log assets is not significant; ii) the coefficient has the opposite sign of the theoretical prediction; iii) log assets explains only a negligible fraction of the variation in average returns (with or without controls). This suggests that collateral constraints are not the main driver of expected entrepreneurial returns in our data.

Uninsurable labor income risk and borrowing constraints. We consider next the implications of limited pledgeability of human wealth. In Appendix E.2, we introduce uninsurable labor income risk, where labor (or non-business) income is subject to a Poisson shock. The magnitude of the shock, regardless of how likely it is, limits how much the entrepreneur can borrow against future income. Even though a closed-form solution is not available with both insurance and borrowing constraints, we are able to provide an analytical characterization using perturbation techniques.²⁷ Our main finding is that the inability to borrow against future

²⁶Another difference with the baseline model comes from the fact that heterogeneity in the idiosyncratic risk premium leads to dispersion in the MPK and misallocation of capital. This is analogous to the channel in David et al. (2020), but they focus on the dispersion of aggregate risk premium instead.

²⁷Our method extends the one used by Viceira (2001) and it is similar to risky steady state approximations, see e.g. Coeurdacier et al. 2011. Despite having aggregate risk and labor income risk, there is no need to approximate

income has two opposite effects on the demand for capital. First, as entrepreneurs have less resources available to invest with limited credit, this tends to reduce the scale of the business. This effect is intuitive and particularly important for entrepreneurs with relatively low financial wealth. Second, limited pledgeability of human wealth implies that entrepreneurs are effectively less risk averse, which tends to *increase* the scale of the business. As entrepreneurs cannot borrow against a fraction of human wealth, future income acts as a buffer making entrepreneurs more willing to take risks. This second effect is particularly relevant for richer entrepreneurs. In combination, these two effects imply that borrowing constraints increases the dispersion of capital holdings, but it has a more muted effect on average capital holdings. We also show that a declining human-financial wealth ratio over the life cycle causes entrepreneurs to reduce their exposure to the business with age. Therefore, uninsurable labor income does not substantially change our main results, while it adds a layer of complexity in the analysis.

Endogenous occupational choice. In our baseline model, households do not choose to be an entrepreneur or a wage earner, as the fraction of entrepreneurial households is set exogenously. In Appendix E.3, we introduce an endogenous occupational choice. We also assume that wage earners have finite lives and imperfect altruism, like entrepreneurs. To become an entrepreneur, the household must pay a fixed cost at the beginning of life. Households draw a cost parameter from a given distribution. The threshold to become an entrepreneur depends on financial wealth, such that households who receive larger bequests are more likely to become an entrepreneur, and depends as well as on the shadow price of idiosyncratic risk. We show in the appendix that our results carry through essentially unchanged to this setting, where the fraction of entrepreneurs in the economy is endogenous.

4 Distributive implications of entrepreneurial risk

In this section, we consider how entrepreneurial risk affects wealth inequality both between and within age groups. The presence of uninsurable idiosyncratic risk distorts the distribution of wealth in the economy in important ways. Imperfect insurance creates dispersion in wealth within age groups, and wealth across age groups depends on the magnitudes of risk and risk premia.

The main object of interest in this section is the joint distribution of (scaled) financial wealth and age, which we denote by $f_t(n, a)$. Let $f(a)$ be the (marginal) age distribution in the population. Given that population grows at rate g and that entrepreneurs live for T

the wealth distribution with finite moments as in Krusell and Smith (1998). See Appendix E.2 for details.

periods, $f(a)$ follows an exponential distribution truncated at age T . Given this joint distribution, one can compute the average wealth conditional on age a , $n_t(a)$, and the average wealth of all entrepreneurs, $n_{e,t}$:

$$n_t(a) = \int_{-h_t(a)}^{\infty} n f_t(n|a) dn, \quad n_{e,t} = \int_0^T n_t(a) f(a) da,$$

where $f_t(n|a) = f_t(n, a) / f(a)$.

We focus on the stationary distribution of financial wealth, such that $f_t(n, a) = f(n, a)$ for all t , which allow us to drop time subscripts, $n_t(a) = n(a)$ and $n_{e,t} = n_e$. We start by characterizing how $n(a)$ varies with age and the role of risk and demographics in shaping this distribution. We then characterize the distribution of wealth conditional on age and how inequality varies with age.

4.1 Between-group inequality

The next proposition provides a characterization of between-group inequality and entrepreneurs' average financial wealth.

Proposition 2. *Suppose the economy is in a stationary equilibrium.*

- i. **Between-group inequality:** *The share of wealth held by entrepreneurs of age a , $\frac{f(a)n(a)}{n_e}$, satisfies*

$$\log \frac{f(a)n(a)}{n_e} = \log \frac{f(0)n(0)}{n_e} + \underbrace{\log \left(\frac{1 + \frac{h(0)}{n(0)}}{1 + \frac{h(a)}{n(a)}} \right)}_{\text{human-to-financial wealth effect}} + \underbrace{\left[r + \frac{(p^{ag})^2}{\gamma} + \frac{(p^{id})^2}{\gamma} - (g + \mu_A) \right]}_{\text{generalized "r-g" effect}} a - \underbrace{\int_0^a \frac{\bar{r}}{1 - \psi e^{-\bar{r}(T-a')}} da'}_{\text{average MPC effect}}. \quad (23)$$

where

$$n(0) = \frac{e^{\left(r + \frac{(p^{ag})^2}{\gamma} + \frac{(p^{id})^2}{\gamma} - (g + \mu_A) - mpc_e \right) T}}{1 - e^{\left(r + \frac{(p^{ag})^2}{\gamma} + \frac{(p^{id})^2}{\gamma} - (g + \mu_A) - mpc_e \right) T}} h(0), \quad (24)$$

and $mpc_e = \frac{1}{T} \int_0^T \frac{\bar{r}}{1 - \psi e^{-\bar{r}(T-a)}} da$ is the average MPC across the life cycle.

- ii. **Average financial wealth:** *The average financial wealth of entrepreneurs is given by*

$$n_e = f(0)(n(0) + h(0)) \int_0^T e^{\left(r + \frac{(p^{ag})^2}{\gamma} + \frac{(p^{id})^2}{\gamma} - (g + \mu_A) \right) a} \frac{e^{-\bar{r}a} - \psi e^{-\bar{r}T}}{1 - \psi e^{-\bar{r}T}} da - h_e, \quad (25)$$

where $h_e = \int_0^T f(a)h(a)da$.

Proof. See Appendix A.2. □

The first part of Proposition 2 decomposes the distribution of wealth across age groups into three effects. First, a human-to-financial wealth effect. For older entrepreneurs, the human wealth has been mostly converted into financial wealth, that is, labor income accelerates the accumulation of financial wealth. The second term is a generalized “ $r - g$ ” effect.²⁸ The first component is the return on total wealth: $r + (p^{ag})^2/\gamma + (p^{id})^2/\gamma$. Hence, the correct notion of return in this context takes into account the aggregate and idiosyncratic risk premium. The second component is the growth rate of the economy, $g + \mu_A$, the sum of population and productivity growth. The generalized “ $r - g$ ” effect implies that the wealth share tends to increase with age if the return on total wealth exceeds the growth rate of the economy. The third term is the average MPC effect. It captures the fact that wealth accumulated at age a depends on the entrepreneur’s past consumption decisions. In particular, this term is small at the beginning of life and increases as the entrepreneur gets older. Therefore, entrepreneurs tend to accumulate wealth when young, but eventually consumption increases until wealth achieves a desired bequest level.

The wealth of newborn agents, that is the bequest they receive, depends on expected returns, consumption behavior, and the amount of human wealth. Notice that expression (24) can be written as

$$n(0) = e^{\left(r + \frac{(p^{ag})^2}{\gamma} + \frac{(p^{id})^2}{\gamma} - (g + \mu_A) - mpc_e\right)T} (n(0) + h(0)).$$

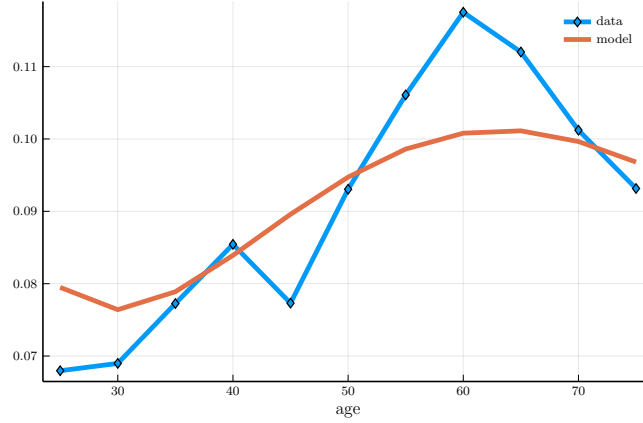
The exponential term captures the net accumulation rate of entrepreneurs’ wealth over their lifetime. In equilibrium, this number is less than one, as part of the total wealth is consumed and only the remaining part is transferred to the next generation. The amount of bequests is increasing in returns and decreasing in the growth rate of the economy and the average MPC.

The second part of Proposition 2 characterizes the average financial wealth of entrepreneurs. For a given level of the capital stock, this captures the wealth distribution between the two types of households, as $\frac{\chi_e n_e}{qK}$ equals the share of financial wealth held by entrepreneurs and $1 - \frac{\chi_e n_e}{qK}$ the share of financial wealth held by wage earners. As n_e is the average of the financial wealth conditional on age $n(a)$, the same effects that shape the distribution of wealth across age groups pin down the overall level of wealth held by entrepreneurs.

Figure 5 shows how financial wealth varies across age groups. Even though this was not part of the calibration targets, the model is able to capture the inverted-U pattern of financial

²⁸The importance of $r - g$ in determining wealth inequality has been emphasized by Piketty (2014). See Benhabib et al. (2011) for the implications to the tail of the wealth distribution and Jones (2015) for a review of the literature.

Figure 5: Financial wealth distribution across age groups



wealth. At the beginning of the life cycle, both the human-financial wealth effect and the “ $r - g$ ” effect dominate the average MPC effect, so the wealth share initially increases with age. The average MPC effect dominates later in the life cycle, bringing down the wealth share.

4.2 Within-group inequality

We turn next to the characterization of the wealth distribution conditional on age. Let $\mu_{n,t}(n, a)$ denote the expected change and $\sigma_{n,t}(n, a)$ the instantaneous volatility of financial wealth for an entrepreneur with financial wealth n and age a . The evolution of the distribution of wealth conditional on age can be characterized by the Kolmogorov Forward Equation, as shown in Lemma 2.

Lemma 2 (Kolmogorov Forward Equation). *The conditional distribution of financial wealth $f_t(n|a)$ satisfies the partial differential equation*

$$\frac{\partial f_t(n|a)}{\partial t} + \frac{\partial f_t(n|a)}{\partial a} = -\frac{\partial [f_t(n|a)\mu_{n,t}(n, a)]}{\partial n} + \frac{1}{2} \frac{\partial^2 [f_t(n|a)\sigma_{n,t}^2(n, a)]}{\partial n^2}, \quad (26)$$

and the boundary condition $f_t(e^{-gT}n|0) = f_t(n|T)$, given an initial condition $f_0(n|a)$.

Proof. See Appendix A.3. □

Despite the complexity created by the dependency on age of the expected change and volatility of wealth, it is possible to solve for the conditional distribution of financial wealth in closed-form for the special case where entrepreneurs leave no bequests, that is, $\psi = 1$. This allow us to characterize analytically the evolution of inequality over the life cycle.

Proposition 3 (Within-group inequality: no bequests). Suppose $\psi = 1$ and $r + \frac{(p^{as})^2}{\gamma} + \frac{(p^{id})^2}{\gamma} > \mu_A$.

- i. **Shifted log-normal distribution.** The distribution of financial wealth conditional on age is given by a shifted log-normal distribution with support $(-h(a), \infty)$, that is, the distribution of total wealth $n + h(a)$ is log-normal.
- ii. **Mean and variance by age.** The expected value and variance of n conditional on age are given by

$$\mathbb{E}[n|a] = h(0)e^{\left(r + \frac{(p^{as})^2}{\gamma} + \frac{(p^{id})^2}{\gamma} - \mu_A\right)a} \frac{e^{-\bar{r}a} - e^{-\bar{r}T}}{1 - e^{-\bar{r}T}} - h(a) \quad (27)$$

$$\mathbb{V}[n|a] = \left[e^{\left(\frac{p^{id}}{\gamma}\right)^2 a} - 1 \right] \left[h(0)e^{\left(r + \frac{(p^{as})^2}{\gamma} + \frac{(p^{id})^2}{\gamma} - \mu_A\right)a} \frac{e^{-\bar{r}a} - e^{-\bar{r}T}}{1 - e^{-\bar{r}T}} \right]^2. \quad (28)$$

- iii. **Inverted-U shape of inequality over the life cycle.** There exists $0 < \hat{a} < T$ such that $\mathbb{V}[n|a]$ is increasing in a for $a < \hat{a}$ and decreasing for $a > \hat{a}$.

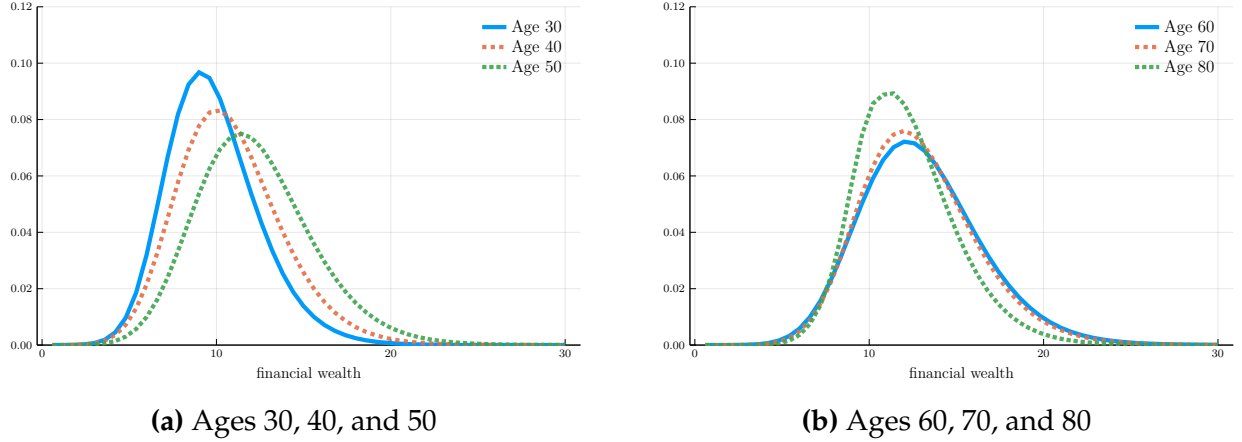
Proof. See Appendix A.4. □

Proposition 3 gives a complete characterization of the distribution of wealth conditional on age. Wealth has a *shifted log-normal* distribution, with an age-dependent shifter $-h(a)$. Since entrepreneurs are allowed to borrow, financial wealth clearly cannot be log-normally distributed, as n can take on negative values. However, financial wealth cannot go below the natural borrowing limit $-h(a)$, so total wealth assumes only positive values. Total wealth follows a log-normal distribution with mean and variance dependent on age.

The expression for the conditional mean $\mathbb{E}[n|a] = n(a)$ is essentially the same as the one in Equation (23), after rearranging and specializing to the case $\psi = 1$. As we have seen, average wealth tends to increase with age at the beginning of life, and it goes down by the end of the life cycle. This is the result of the increase in MPC balancing out the effect of wealth being accumulated over time.

Expression (28) shows how the variance of wealth evolves over the life cycle. In the case with no bequests, the variance is zero at ages $a = 0$ and $a = T$. The dispersion of wealth increases at the beginning of life, as some entrepreneurs receive a series of positive shocks, while others suffer a sequence of negative shocks. This force is magnified with the exposure to idiosyncratic risk, as measured by p^{id}/γ . The increase in MPC provides a countervailing force, as the impact of the proportional shocks is reduced as the level of wealth is brought down at the end of the life cycle.

Figure 6: Stationary distribution of financial wealth conditional on age



The results in Proposition 3 require the assumption of no bequests. In the general case, we must resort to numerical methods. In Appendix C.2, we adapt the methods in Achdou et al. (2017) to deal with the case of an economy with bequests, where the distribution at age 0 depends on the distribution at age T . Figure 6 shows the stationary distribution of financial wealth for selected ages. The mean and the dispersion of the distribution initially increases with age, then eventually both start to decline as entrepreneurs get older.

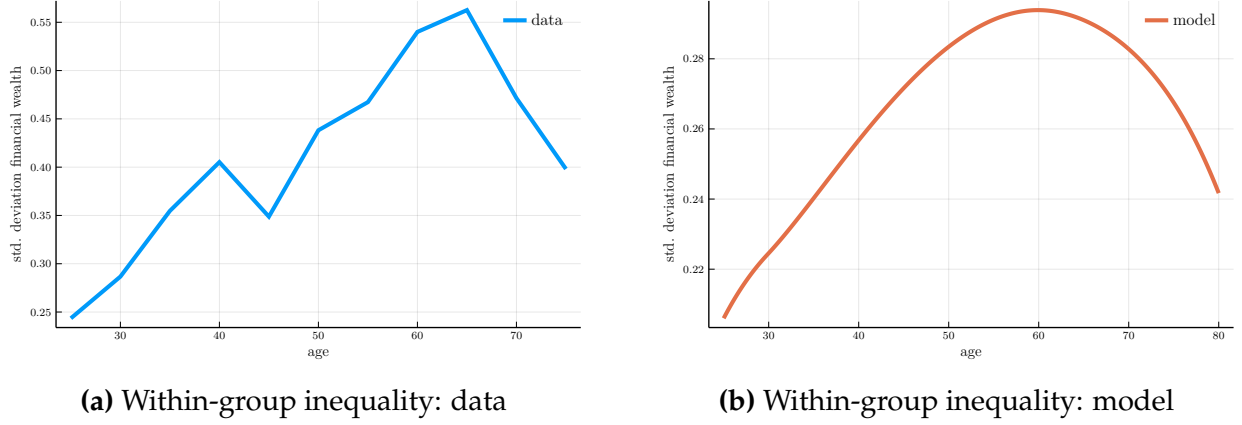
The same inverted-U pattern on wealth inequality can be found in the data. Figure 7 shows the evolution of within-group inequality over the life cycle in our data. To make the units easier to interpret, we divide financial wealth by the average wealth for all entrepreneurs. The figure shows that the standard deviation of n/n_e increases sharply until roughly age 60 and then declines until the end of the life cycle. Quantitatively, the model generates a substantial increase in inequality over the life cycle, over 50% from early in life until the peak. Notice that, even though idiosyncratic shocks to the business are the only source of heterogeneity across entrepreneurs, the model is able to capture a large fraction of the within-group inequality observed in the data. Heterogeneity in business returns can explain between 55% to 80% of the observed within-group inequality depending on the age group.

Wealth inequality and risk sharing. As shown in Appendix A.4, a corollary of Proposition 3 is that the within-group variance of log total wealth, and ultimately log consumption, increases with age:

$$\mathbb{V}[\log \omega_{i,t}|a] = \mathbb{V}[\log c_{i,t}|a] = \left(\frac{p^{id}}{\gamma} \right)^2 a.$$

The steepness of the variance life cycle profile depends on p^{id} , so it is a function of the degree of risk sharing in the economy. This is in line with the literature using the dispersion in consumption over the life cycle to infer how imperfect is risk sharing (see e.g. Storesletten

Figure 7: Standard deviation of financial wealth within age groups



Note: Panel (a) shows the standard deviation of financial wealth by age in the data. Panel (b) shows the standard deviation of financial wealth by age in the model economy. In both cases, we normalize the standard deviation by entrepreneurs' average wealth.

et al. 2004). The expression above shows that, in our setting with entrepreneurial risk, the shadow price of idiosyncratic insurance controls the level of consumption inequality in the economy.

5 Aggregate implications of entrepreneurial risk

In this section, we study the impact of entrepreneurial risk on the long-run level of the aggregate capital stock. We find that the degree of financial development, captured by the magnitude of the moral hazard parameter ϕ , is tightly linked to the level of economic development of the economy.

5.1 Equilibrium characterization

We present next the determination of aggregate variables in a stationary equilibrium. The detailed derivations are provided in Appendix C.3.

5.1.1 Aggregate risk premium, interest rate, and the relative price of capital

Equating supply and demand for aggregate insurance, we obtain the price of aggregate insurance,

$$p^{ag} = \gamma \sigma_A, \quad (29)$$

The interest rate is given by the condition

$$r = \rho_w + \gamma \mu_A - \frac{\gamma(\gamma + 1)}{2} \sigma_A^2. \quad (30)$$

The three terms in Equation (30) capture, respectively, the impact of impatience, intertemporal substitution, and precautionary savings. Note that the relevant impatience parameter is the one for the infinite-horizon agent.

In a stationary equilibrium, the capital-labor ratio is constant, so capital grows at the constant rate g . Given the investment function $\Phi(\iota) = \frac{\sqrt{\Phi_0^2 + 2\Phi_1\iota} - \Phi_0}{\Phi_1}$, the required investment rate to achieve a growth rate of capital g is given by $\iota = \Phi_0(g + \delta) + 0.5\Phi_1(g + \delta)^2$. Using the expression for optimal investment condition, Equation (12), we obtain the relative price of capital:

$$q = \Phi_0 + \Phi_1(g + \delta). \quad (31)$$

Note that p^{ag} , r , and q are independent of the moral hazard parameter ϕ , so they coincide with their corresponding value at the complete markets economy.

5.1.2 Aggregate capital stock and idiosyncratic risk premium

The aggregate capital stock and the idiosyncratic risk premium are jointly determined. From Equation (20), we obtain an expression for the expected return on the business

$$r + p^{ag}\sigma_A + p^{id}\phi\sigma_{id} = \frac{\alpha K^{\alpha-1} - \iota(q)}{q} + g + \mu_A. \quad (32)$$

The left-hand side captures the required rate of return to invest in the business, which includes a premium for holding aggregate and idiosyncratic risk. The right-hand side gives the actual expected return of investing in the business, a function of the marginal product of capital (MPK) net of adjustment costs. Notice that Equation (32) generalizes the standard textbook relation between MPK and interest rates to an environment with growth, risk, and adjustment costs. In the absence of these three elements, the expression above boils down to $r = \alpha K^{\alpha-1} - \delta$.

Expression (32) gives an inverse relation between the idiosyncratic risk premium $p^{id}\phi\sigma_{id}$ and the capital-labor ratio K . This downward-sloping relationship is represented by the solid blue line in the left panel of Figure 8, which we refer to as the MPK schedule.

We need another condition relating K and p^{id} . Aggregating the demand for capital (19) across all entrepreneurs, we obtain

$$p^{id} = \underbrace{\gamma}_{\text{risk aversion}} \underbrace{\phi\sigma_{id}}_{\text{effective risk}} \underbrace{\frac{qK}{\chi_e(n_e + h_e)}}_{\text{id. risk exposure}}, \quad (33)$$

where n_e and h_e denote the average financial and human wealth of entrepreneurs.

In the same way the price of aggregate risk depends on the product of risk aversion and

risk, $\gamma\sigma_A$, the price of idiosyncratic risk also depends on the product of the risk aversion γ and the idiosyncratic risk (net of insurance) $\phi\sigma_{id}$. However, the price of idiosyncratic risk depends on an additional term: the *idiosyncratic risk exposure*, that is, the ratio of physical assets to total wealth of entrepreneurs. This term captures the fact that entrepreneurs require a larger idiosyncratic risk premium when they have a larger fraction of their wealth invested in the business. From Equation (25), we obtain $n_e + h_e$ as a function of the capital stock and the price of idiosyncratic risk. After substituting $n_e + h_e$ into Equation (33), we obtain an implicit relationship between p^{id} and K . The left panel of Figure 8 plots this relationship as the solid upward-sloping curve, which we refer to as the *pricing schedule*.

The idiosyncratic risk premium and the capital stock in the economy are determined by the intersection of the MPK and pricing schedules. Hence, the degree of financial development in the economy and the amount of idiosyncratic risk are important determinants of the level of economic development.

5.2 The price of aggregate and idiosyncratic risk in the data

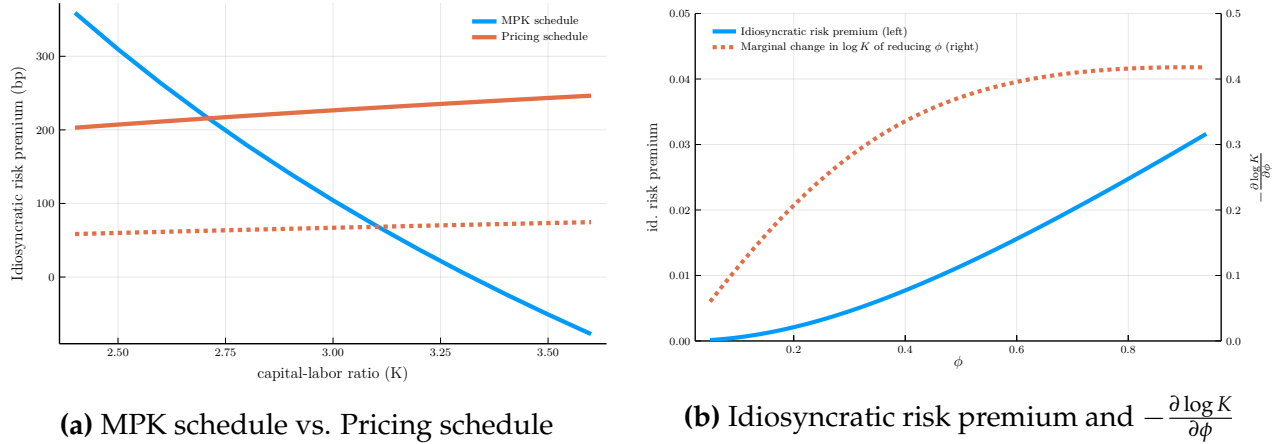
The model replicates the level of risk premium and volatility on aggregate and idiosyncratic risk observed in the data, as given in Table 2. A striking fact is that, despite idiosyncratic volatility being three times larger than aggregate volatility, the idiosyncratic risk premium is only slightly larger than the aggregate risk premium. This leads to a Sharpe ratio three times larger for aggregate risk. Equations (29) and (33) help to shed light on this pattern.

The Sharpe ratio of aggregate risk corresponds to p^{ag} and it is given by $\gamma\sigma_A$. The Sharpe ratio of idiosyncratic risk is given by $p^{id}\phi$, as the volatility computed in the data does not take into account any insurance available to entrepreneurs. If we were to naively price the idiosyncratic risk by analogy with the aggregate risk, the Sharpe ratio would be $\gamma\sigma_{id}$, that is, more than three times larger than the one for aggregate risk. Two factors explain why the price of idiosyncratic risk is actually three times *smaller* than the one for aggregate risk: idiosyncratic insurance and the risk exposure. The pricing equation for idiosyncratic risk shows the role of these two components:

$$p^{id}\phi = \gamma\sigma_{id}\phi^2 \frac{qK/\chi_e}{n_e(1 + h_e/n_e)}.$$

The parameter of the skin-in-the-game constraint is given by $\phi = 0.73$. This term by itself reduces the price of idiosyncratic risk by 45%. However, the bulk of the adjustment comes from the risk exposure factor, as $qK/\chi_e(n_e + h_e) \approx 0.15$. Intuitively, the reason for a much smaller price of idiosyncratic risk is that entrepreneurs are proportionally less exposed to idiosyncratic risk, either because of insurance mechanisms, or because only a fraction of their total wealth is exposed to this risk. Notice the importance of explicitly introducing human

Figure 8: Idiosyncratic Risk Premium and Capital Stock



Note: In the left panel, the solid (dashed) upward-sloping curve shows the pricing schedule in the initial (new) stationary equilibrium for $\phi_0 = 0.73$ ($\phi_1 = 0.365$). The right panel shows the equilibrium idiosyncratic risk premium and the marginal increase in the capital stock of reducing the moral hazard parameter for different values of ϕ .

wealth and heterogeneous agents into the model. In the absence of these elements, one would incorrectly attribute the low Sharpe ratio to a high degree of idiosyncratic insurance, and the economy would appear to have better insurance than it actually has.

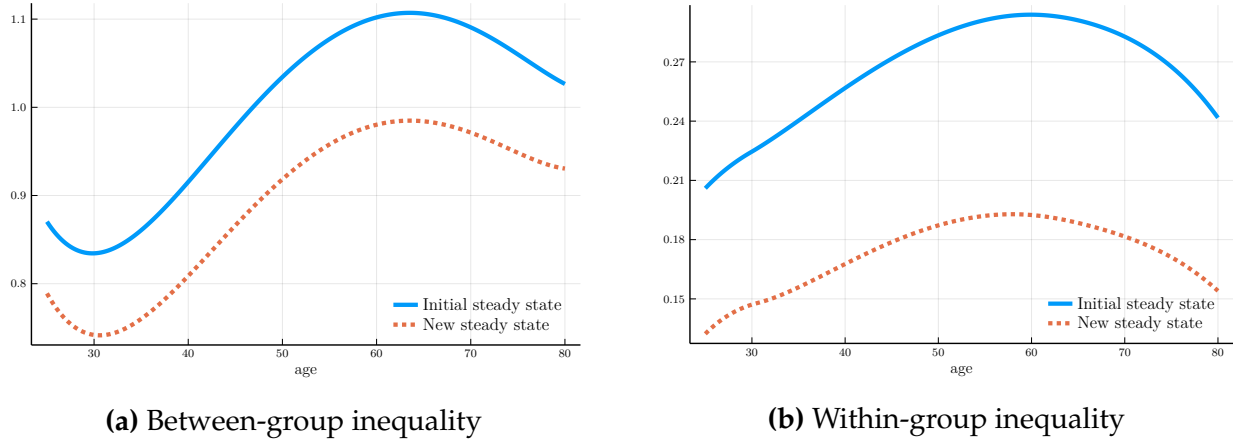
5.3 Long-run effects of relaxing insurance constraints

We consider next the aggregate implications of relaxing the insurance constraints that give rise to the idiosyncratic risk premium. The parameter ϕ measures the strength of contractual frictions in the economy. High values of ϕ are meant to capture situations where the access to insurance arrangements, formal or informal, is rather limited. In this case, entrepreneurs are forced to hold most of the idiosyncratic risk of their businesses, potentially limiting their choice of scale. As the institutional arrangements improve, in particular mechanisms to monitor entrepreneurs' activities, such frictions are expected to be reduced and entrepreneurs would bear a smaller fraction of the risk.²⁹ Hence, we see the process of financial development as leading to a reduction in the moral hazard parameter ϕ .

The left panel of Figure 8 shows the impact of an intervention or reform that reduces the parameter ϕ . The MPK schedule is unchanged with a reduction in ϕ , but the pricing schedule is shifted down. Entrepreneurs are now able to share a larger share of the risk of the project, so they demand a smaller compensation to hold this risk. In the long-run, the price of capital does not change, so the reduction in expected returns leads to a reduction in the MPK and to a higher capital stock. Hence, financial development and economic development

²⁹To focus on entrepreneurs, we assume that insurance providers can perfectly diversify idiosyncratic risk. If insurance providers also hold under-diversified portfolios, as in Gârleanu et al. (2015), a reduction in the risk premium can be caused by an improvement in their ability to diversify, see e.g. Khorrami (2019).

Figure 9: Financial development and inequality in the long-run



Note: Inequality in a stationary equilibrium for $\phi_0 = 0.73$ (initial steady state) and $\phi_1 = 0.365$ (new steady state). Panel (a) shows average financial wealth by age. Panel (b) shows the standard deviation of financial wealth by age. All variables are normalized by entrepreneurs' average wealth in the initial steady state.

go together in this economy. Reducing the moral hazard parameter in half, from $\phi_0 = 0.73$ to $\phi_1 = 0.365$, the idiosyncratic risk premium falls by about 150 basis point, leading to an increase of roughly 15% in the capital stock. Therefore, relaxing risk constraints have a quantitatively large impact on the aggregate capital stock in our calibrated economy.

The right panel of Figure 8 shows that the economy's response to changes in ϕ is highly *non-linear*. The idiosyncratic risk premium (solid line) is convex in ϕ , so changes in the moral hazard parameter lead to stronger risk premia effects in economies with low financial development. The dashed line shows the marginal change in (log) capital due to a reduction in the moral hazard parameter for different initial values of ϕ . The same reduction in ϕ can lead to an increase in the capital stock that is four times larger in an economy with low financial development (high ϕ) compared to an economy with better financial development (low ϕ).

Financial development also has important implications for wealth accumulation and inequality. The left panel on Figure 9 shows the average financial wealth for each age group relative to the average across all entrepreneurs on the *initial* stationary equilibrium. Financial wealth falls for all age groups in the new stationary equilibrium. From Equation (23), one can see that a reduction in the price of idiosyncratic insurance tends to reduce bequests and wealth accumulation over the life cycle. In the long-run, inequality falls in response to the reduction in ϕ , as indicated by the right panel on Figure 9. Equation (28) provides some intuition on the effects of p^{id} on wealth inequality. Both the reduction in the amount of risk entrepreneurs are exposed as well as a reduction on the level of financial wealth contributes to the reduction in the standard deviation of wealth.

6 Dynamic effects of relaxing risk constraints

In the previous section, we studied the long-run effects of relaxing risk constraints by comparing two stationary equilibria (or steady states). This approach ignores what happens during the transitional dynamics. To compute the welfare implications of relaxing risk constraints, it is important to explicitly take into account what happens during the transition. Moreover, some of the effects may take a long time to materialize. We then turn next to the dynamic effects of changes in ϕ .

6.1 Computing the transitional dynamics

We consider a "small open economy" version of the model, where the interest rate is kept at the level of the original stationary equilibrium.³⁰ This allow us to focus on the dynamic implications of fluctuations in the idiosyncratic risk premium on the capital stock. We show in Appendix D that the price of aggregate insurance is constant, and given by $p_t^{ag} = \gamma\sigma_A$, even during the transition. We still have to solve for the path of the capital stock and its relative price, K and q , the consumption-wealth ratio of entrepreneurs by age, $\zeta_t(a)$, and the level of total and human wealth, $\omega_t(a)$ and $h_t(a)$. The next proposition provides the conditions characterizing the evolution of these variables.

Proposition 4. *The evolution of $(q_t, K_t, \{\zeta_t(a), h_t(a), \omega_t(a)\})$ is characterized by a pair of ordinary differential equations (ODEs)*

$$\begin{aligned}\dot{K}_t &= [\Phi(\iota(q_t)) - \delta - g] K_t \\ \dot{q}_t &= \left[r + \gamma\sigma_A^2 + \gamma\phi^2\sigma_{id}^2 \frac{q_t K_t}{\chi_e \omega_{e,t}} + \delta - \mu_A - \Phi(\iota(q_t)) - \frac{\alpha K_t^{\alpha-1} - \iota(q_t)}{q_t} \right] q_t\end{aligned}$$

and three partial differential equations (PDEs)

$$\begin{aligned}\frac{\partial \zeta_t(a)}{\partial t} &= -\frac{\partial \zeta_t(a)}{\partial a} + \zeta_t^2(a) - \bar{r}_t \zeta_t(a) \\ \frac{\partial h_t(a)}{\partial t} &= -\frac{\partial h_t(a)}{\partial a} + (r + \gamma\sigma_A^2 - \mu_A) h_t(a) - (1 - \alpha) K_t^\alpha \bar{l}(a) \\ \frac{\partial \omega_t(a)}{\partial t} &= -\frac{\partial \omega_t(a)}{\partial a} + \left[r + \gamma\sigma_A^2 - \mu_A + \gamma\phi^2\sigma_{id}^2 \left(\frac{q_t K_t}{\chi_e \omega_{e,t}} \right)^2 - \zeta_t(a) \right] \omega_t(a),\end{aligned}$$

subject to the boundary conditions described in the appendix.

Proof. See Appendix A.5. □

³⁰We show in Appendix D that the interest rate is constant during the transitional dynamics when wage earners have Epstein-Zin utility with linear intertemporal preferences.

In the expressions above, we have eliminated the price of aggregate insurance using $p_t^{ag} = \gamma\sigma_A$ and the price of idiosyncratic insurance using $p_t^{id} = \gamma\phi\sigma_{id}\frac{q_t K_t}{\chi_e \omega_{e,t}}$, where $\omega_{e,t} = n_{e,t} + h_{e,t}$. We obtain the first ODE by aggregating (3) and the second one by rearranging the time-dependent version of (32). The first PDE comes from the HJB equation for entrepreneurs, the second one corresponds to the time-dependent version of (14), and the last PDE can be obtained by averaging the budget constraint of entrepreneurs.

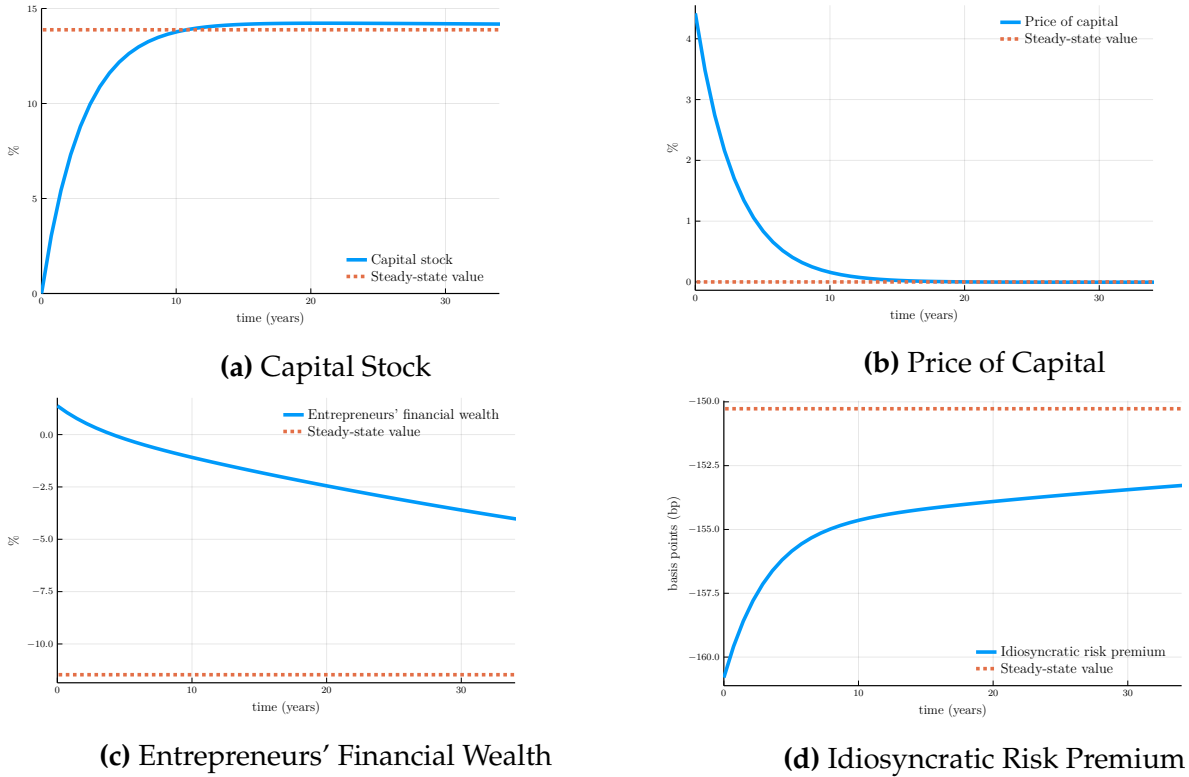
The transitional dynamics for heterogeneous agents models are often computed using a shooting algorithm, as in e.g. Guerrieri and Lorenzoni (2017) or Achdou et al. (2017). Given the large number of forward-looking variables, such algorithm is impractical in our setting. We propose an alternative form of solving for the transitional dynamics, by combining perturbation and finite-difference methods. First, we use finite differences to discretize the system of ODE/PDEs. We end up with a (large) non-linear boundary value problem, since $(K_t, \omega_t(a))$ have initial conditions, while $(q_t, \zeta_t(a), h_t(a))$ have terminal conditions. We then linearize the system around the new stationary equilibrium. In contrast to the approach in e.g. Ahn et al. (2018) that linearizes around the economy without aggregate shocks, we do not assume that aggregate or idiosyncratic shocks are small. By avoiding these small-risk approximations, we are able to capture time-varying risk premia and precautionary savings effects using this method.³¹ The final step consists of solving the resulting linear rational expectation model, which can be done by standard techniques, as the one proposed by Blanchard and Kahn (1980). See Appendix D for a detailed discussion of the method.

6.2 Short-run dynamics: the overshooting effect

Figure 10 shows the evolution of the capital stock, the relative price of capital, entrepreneurs' financial wealth, and the idiosyncratic risk premium, all as deviations from the initial steady state. We consider a reform that reduces the moral hazard parameter by half, from $\phi_0 = 0.73$ under our calibration to $\phi_1 = 0.365$. In response to a relaxation of risk constraints, we observe an investment boom and a sharp increase in the value of businesses that last for roughly a decade, as shown in Panels (a) and (b) of Figure 10. Importantly, the short-run response of the price of capital exceeds by a large margin its long-run level, i.e. there is an *overshooting* effect. The intuition for this result is the following. As entrepreneurs must bear less of the risk of the project, they require a smaller premium to invest in the business. In the long-run, this reduction in expected returns requires a smaller marginal product of capital and a larger capital stock. However, capital is fixed in the short-run, so the only way expected returns can go down is through expected capital losses. Since the price of capital must be expected to go down and its long-run level is unaffected by the financial friction, then it must jump

³¹See e.g. Winberry (2018) on the use of perturbation methods to solve heterogeneous agents models with aggregate risk.

Figure 10: Transitional dynamics: aggregate variables



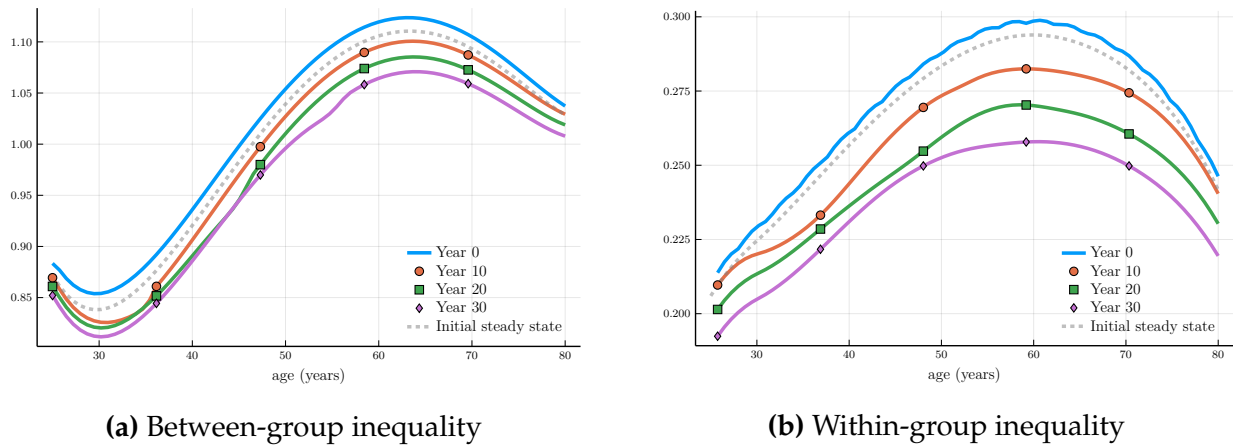
Note: Transitional dynamics from a stationary equilibrium with $\phi_0 = 0.73$ to a stationary equilibrium with $\phi_1 = 0.365$. Capital stock, relative price of capital, and entrepreneurs' financial wealth are expressed as percentage deviations from the initial steady state. The idiosyncratic risk premium is expressed as absolute deviation from the initial steady state in basis points.

up on impact. This increase in marginal q induces entrepreneurs to invest more, explaining the investment boom. This logic is reminiscent of [Dornbusch's \(1976\)](#) overshooting model, where exchange rates react more strongly to shocks in the short run to create expected capital losses to domestic investors.³² The overshooting effect has important implications for wealth accumulation and inequality.

As discussed in Subsection 5.3, the financial wealth of entrepreneurs goes down in the long run. In contrast, their financial wealth actually increases in the short run. The reason for the contrast between the short-run and long-run responses is a *revaluation* effect. Since entrepreneurs own the capital stock, their wealth jumps on impact as the relative price of capital goes up in the short-run. As expected returns on the business go down with the reduction in the idiosyncratic risk premium, entrepreneurs accumulate wealth at a slower pace and end up with a smaller level of wealth in the long run. This long-run effect takes a long time to materialize though, as shown in Panel (c) of Figure 10, given it is in part transmitted to fu-

³²The presence of adjustment costs is crucial for the overshooting result. In the absence of adjustment costs, capital jumps to the new steady-state level and there are no movement in prices. Similarly, there is no overshooting in Dornbusch's model when the price level can immediately jump to its steady-state level.

Figure 11: Transitional dynamics: inequality



Note: Transitional dynamics from a stationary equilibrium with $\phi_0 = 0.73$ to a stationary equilibrium with $\phi_1 = 0.365$. Panel (a) shows the dynamics of average financial wealth by age. Panel (b) shows the dynamics of the standard deviation of financial wealth by age. All variables are normalized by entrepreneurs' average wealth in the initial steady state.

ture generations through lower bequests. Even thirty years after the shock, the reduction in entrepreneurs' financial wealth is only 30% of the long-run effect.

The slow dynamics of wealth affects the behavior of the idiosyncratic risk premium. The short-run response of the idiosyncratic risk premium exceeds its long-run level by more than ten basis points, as shown in Panel (d) of Figure 10. After a decade, the difference of the risk premium to the new steady state level is cut by more than half, and then the risk premium increases slowly as entrepreneurs' wealth declines, in line with the risk exposure effect discussed in Subsection 5.2.

6.3 Kuznets dynamics

Relaxing the risk constraints has important implications for the dynamics of inequality. Figure 11 shows the evolution of between- and within-group wealth inequality. The left panel shows the evolution of average financial wealth for each age group in different points in time, normalized by average wealth across all entrepreneurs in the initial equilibrium. Because of the revaluation effect, financial wealth goes up on impact for all age groups, but the effect is stronger for younger entrepreneurs, as they are proportionally more exposed to the business. Over time the financial wealth goes down due to the reduction in expected returns.

The right panel on Figure 11 shows the standard deviation of financial wealth by age. Again the short-run and long-run responses are different. While wealth inequality goes down in the long run, wealth inequality goes up in the short run. Entrepreneurs with initially more wealth also hold more capital, so wealthier entrepreneurs benefit the most from the reform, which increase the dispersion in financial wealth in the economy.

The evolution of inequality interacts in interesting ways with the demographic structure. After the initial increase in inequality, wealth dispersion goes down for all age groups over time, but the cohort starting their professional life just after the reform are the ones mostly affected. For instance, ten years after the reform, the drop in inequality is three times larger for 35-year old entrepreneurs, who lived their entire professional life under the new regime, relative to 80-year old entrepreneurs, who lived most of their life under the old regime. Similarly, the drop in inequality is more pronounced to 35-year old entrepreneurs than to 25-year old entrepreneurs, who just inherited their wealth from old entrepreneurs. A similar argument explains why, twenty years after the reform, 45-year old entrepreneurs are the ones with the largest drop in wealth dispersion.

Taking stock. A reduction in the amount of risk that entrepreneurs must hold and the consequent compression of the idiosyncratic risk premium leads to an investment boom and an increase in wealth inequality. As the economy approaches its new level of output, inequality starts to recede and it reaches a lower level in the long-run. The initial increase in inequality as the economy enters in a high growth phase and eventual reduction in inequality as the economy reaches a higher level of development is consistent with a **Kuznets's (1955)** curve.³³

6.4 Welfare implications

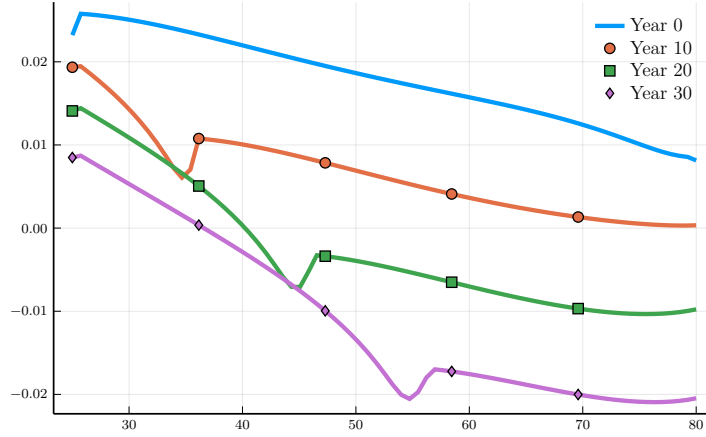
We turn next to the welfare implications of risk constraints. Remember that the value function of an entrepreneur of age a at time t is given by

$$V_t(n, a) = \zeta_t^{-\frac{1}{\gamma}}(a) \frac{(n + h_t(a))^{1-\gamma}}{1-\gamma}.$$

Hence, the welfare of an entrepreneur depends on financial wealth n , human wealth $h_t(a)$, and the consumption-wealth ratio $\zeta_t(a)$, which captures the path of expected future returns. We evaluate financial wealth at the average level of the age group, $n = n_t(a)$, but one can use the inequality results previously discussed to infer the dispersion in welfare within age groups. Finally, we take a monotonic transformation of the value function to measure welfare

³³**Moll (2012)** found a related Kuznets curve by showing that the steady-state level of the top wealth share is a hump-shaped function of financial development. In contrast, we show that inequality initially increases in response to a shock that reduces inequality in the long run, so we focus on the transitional dynamics instead of long-run comparisons.

Figure 12: Transitional dynamics: welfare



in units of consumption. Hence, our measure of welfare will be

$$\begin{aligned}\mathcal{W}_t(a) &= \log \left[u^{-1} (V_t(n_t(a), a)) \right] - \log \left[u^{-1} (V^*(n^*(a), a)) \right] \\ &= \frac{1}{\gamma(\gamma - 1)} \hat{\zeta}_t(a) + \hat{\omega}_t(a),\end{aligned}$$

where $u(c) = \frac{c^{1-\gamma}}{1-\gamma}$, and a hat denotes log deviations from the initial steady state.

Figure 12 shows the welfare gains for each age groups in different points in time. The generation that is alive at the moment of the reform benefits the most from the reform, with gains concentrated on younger entrepreneurs. This is the result of the revaluation effect, which raises not only financial wealth but also human wealth. However, the negative impact on wealth accumulation affects future generations of entrepreneurs. As they receive smaller bequests, and it becomes harder to accumulate wealth, their welfare is adversely impacted. Figure 12 shows how demographics affect the welfare gains. For instance, ten years after the intervention, the welfare gains for entrepreneurs who started their professional life after the reform, the ones with age 25 to 35 years old, have welfare gains that are smaller than the entrepreneurs of the same age at the time of the intervention. Thirty years after the reform, entrepreneurs with ages between 35 and 55 are worse off in comparison to an equilibrium without the reform, with larger welfare losses for the older entrepreneurs. Therefore, the initial generation of entrepreneurs reap most of the benefits of the reform, illustrating the rich interactions of the financial friction and the demographic structure.

7 Conclusion

In this paper, we provide a framework to analyze the aggregate and distributive implications of entrepreneurial risk. We propose a life-cycle model of entrepreneurship with aggregate and idiosyncratic risk under limited insurance. We show that entrepreneurial returns command a positive idiosyncratic risk premium, which accounts for a large fraction of total returns. The model captures quantitatively the empirical patterns of risk-taking and savings over the life cycle, the inverted-U shape of wealth inequality, and the level of aggregate and idiosyncratic risk premia.

We also study the impact of relaxing risk constraints. We show that an improvement in idiosyncratic insurance increases output and reduces inequality in the long-run and generates rich transitional dynamics. The price of capital overshoots in the short run, generating a large investment boom, and an increase in the value of the business. This overshooting leads to an initial increase in inequality. As the reduction in risk and expected returns have time to play out, inequality goes down in the long run with important intergenerational effects. Finally, we find that most of the welfare gains are concentrated in the generations that are alive at the time of the change and that future generations of entrepreneurs are actually worse-off.

An important direction of future research is the impact of risk constraints on misallocation and aggregate productivity. Given imperfect insurance leads to different incentives to self-insure than borrowing constraints, the aggregate consequences of financial frictions are potentially very different than the more conventional borrowing constraints explored in the literature.

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A Proofs

A.1 Proofs of Lemma 1 and Proposition 1

Proof. We start by showing part (b) of Lemma 1, that is, we solve for $h_t(a)$ and its dynamics. Then, we proceed to solve for the entrepreneurs' value function and policy functions, deriving items (a) and (c) of Lemma 1 as well as the results in Proposition 1.

Pricing human wealth. Define the stochastic discount factor (SDF) for this economy as the process π_t satisfying the law of motion

$$\frac{d\pi_t}{\pi_t} = -r_t dt - p_t^{ag} dZ_t. \quad (\text{A.1})$$

Without loss of generality, we assumed that the SDF is not exposed to idiosyncratic risk, as we only use the SDF to price human wealth which is not exposed to idiosyncratic risk. Integrating the process above, we obtain

$$\frac{\pi_z}{\pi_t} = \exp \left(- \int_t^z \left(r_u + \frac{(p_u^{ag})^2}{2} \right) du - \int_t^z p_u^{ag} dZ_u \right). \quad (\text{A.2})$$

Similarly, integrating the process for A_t

$$\frac{A_z}{A_t} = \exp \left(\int_t^z \left(\mu_A - \frac{\sigma_A^2}{2} \right) du + \int_t^z \sigma_A dZ_u \right). \quad (\text{A.3})$$

Hence, we can explicitly compute the following expectation

$$\begin{aligned} \mathbb{E}_t \left[\frac{\pi_z A_z}{\pi_t A_t} \right] &= \mathbb{E}_t \left[\exp \left(- \int_t^z \left(r_u - \mu_A + \frac{(p_u^{ag})^2 + \sigma_A^2}{2} \right) du - \int_t^z (p_u^{ag} - \sigma_A) dZ_u \right) \right] \\ &= \exp \left(- \int_t^z \left(r_u - \mu_A + \frac{(p_u^{ag})^2 + \sigma_A^2}{2} \right) du + \frac{1}{2} \int_t^z (p_u^{ag} - \sigma_A)^2 du \right) \\ &= \exp \left(- \int_t^z (r_u + p_u^{ag} \sigma_A - \mu_A) du \right), \end{aligned} \quad (\text{A.4})$$

where we used Ito's isometry in the second equality and the fact that p_t^{ag} is deterministic.

Human wealth is given by

$$\begin{aligned} h_t(a) &= \mathbb{E}_t \left[\int_t^{t+T-a} \frac{\pi_z A_z}{\pi_t A_t} w_z \bar{l}(a+z-t) dz \right] \\ &= \int_t^{t+T-a} e^{-\int_t^z (r_u + p_u^{ag} \sigma_A - \mu_A) du} w_z \bar{l}(a+z-t) dz. \end{aligned} \quad (\text{A.5})$$

Consider the human wealth for someone born at date s , so $a = t - s$:

$$h_t(t-s) = \int_t^{s+T} e^{-\int_t^z (r_u + p_u^{ag} \sigma_A - \mu_A) du} w_z \bar{l}(z-s) dz. \quad (\text{A.6})$$

Differentiating the expression above with respect to time yields

$$\frac{\partial h_t(a)}{\partial t} + \frac{\partial h_t(a)}{\partial a} = (r_t + p_t^{ag} \sigma_A - \mu_A) h_t(a) - w_t \bar{l}(a), \quad (\text{A.7})$$

which gives (14) in a stationary equilibrium.

The HJB equation. The HJB equation for problem (9) is given by

$$\rho \tilde{V}_t(\tilde{n}, t-s; A_t) = \max_{\tilde{c}_t, \tilde{\theta}_t^{ag}, \tilde{\theta}_t^{id}, k_t, l_t, \mu_t} \left\{ \frac{\tilde{c}_t^{1-\gamma}}{1-\gamma} + \frac{\mathbb{E}_t [d\tilde{V}_{i,t}]}{dt} \right\}, \quad (\text{A.8})$$

subject to (7) as well as the terminal and boundary conditions

$$\tilde{V}_t(\tilde{n}, T) = (1-\psi)^\gamma V^* \frac{\tilde{n}^{1-\gamma}}{1-\gamma}; \quad \lim_{\tilde{n} \rightarrow -\tilde{h}_t(a)} \tilde{V}_t(\tilde{n}, a) = \begin{cases} 0, & \text{if } \gamma < 1 \\ -\infty, & \text{if } \gamma \geq 1 \end{cases}, \quad (\text{A.9})$$

where the terminal condition captures the effect of bequests and the boundary condition captures the fact that consumption is zero if the entrepreneur hits the natural borrowing limit.

Using Ito's lemma, the HJB reduces to a partial differential equation for $\tilde{V}_t(\tilde{n}, a; A_t)$:

$$\rho \tilde{V}_t = \max_{\tilde{c}_t, \tilde{\theta}_t^{ag}, \tilde{\theta}_t^{id}, k_t, l_t, \mu_t} \left\{ \frac{\tilde{c}_t^{1-\gamma}}{1-\gamma} + \frac{\partial \tilde{V}_t}{\partial t} + \frac{\partial \tilde{V}_t}{\partial a} + \frac{\partial \tilde{V}_t}{\partial \tilde{n}} \mu_{\tilde{n},t} + \frac{\partial \tilde{V}_t}{\partial A_t} \mu_A A_t + \frac{1}{2} \frac{\partial^2 \tilde{V}_t}{\partial \tilde{n}^2} (\sigma_{ag,t}^2 + \sigma_{id,t}^2) + \frac{\partial^2 \tilde{V}_t}{\partial \tilde{n} \partial A_t} \sigma_{ag,t} \sigma_A A_t + \frac{1}{2} \frac{\partial^2 \tilde{V}_t}{\partial A_t^2} \sigma_A^2 A_t^2 \right\}, \quad (\text{A.10})$$

where $(\mu_{\tilde{n},t}, \sigma_{ag,t}, \sigma_{id,t})$ are the drift and diffusion terms for \tilde{n}_t , and the maximization is subject to (7).

First, we verify that the following guess for the value function solves the PDE

$$\tilde{V}_t(\tilde{n}, a; A_t) = \zeta_t(a)^{-\gamma} \frac{(\tilde{n} + A_t h_t(a))^{1-\gamma}}{1-\gamma}. \quad (\text{A.11})$$

Plugging the derivatives of the equation above into the HJB equation, we obtain

$$\begin{aligned} \frac{\rho}{1-\gamma} = & \max_{c_{i,t}, k_{i,t}, l_{i,t}, \iota_{i,t}, \theta_{i,t}^{ag}, \theta_{i,t}^{id}} \left\{ \frac{\zeta_t^\gamma(a)}{1-\gamma} \left(\frac{c_{i,t}}{\omega_{i,t}} \right)^{1-\gamma} - \frac{\gamma}{1-\gamma} \frac{1}{\zeta_t(a)} \left(\frac{\partial \zeta_t(a)}{\partial t} + \frac{\partial \zeta_t(a)}{\partial a} \right) + r_t + \frac{q_t k_{i,t}}{\omega_{i,t}} (\mu_{i,t}^R - r_t) \right. \\ & - \frac{p_t^{ag} \theta_{i,t}^{ag}}{\omega_{i,t}} + \frac{h_{i,t}}{\omega_{i,t}} \sigma_A p_t^{ag} - \frac{c_{i,t}}{\omega_{i,t}} - \frac{\gamma}{2} \left[\left(\frac{q_t k_{i,t} + h_{i,t}}{\omega_{i,t}} \sigma_A - \frac{\theta_{i,t}^{ag}}{\omega_{i,t}} \right)^2 + \left(\frac{q_t k_{i,t}}{\omega_{i,t}} \sigma_{id} - \frac{\theta_{i,t}^{id}}{\omega_{i,t}} \right)^2 \right] \\ & \left. + p_t^{id} \left[(1-\phi) \frac{q_t k_{i,t}}{\omega_{i,t}} \sigma_{id} - \frac{\theta_{i,t}^{id}}{\omega_{i,t}} \right] \right\}, \end{aligned} \quad (\text{A.12})$$

where $\omega_{i,t} = n_{i,t} + h_t(a)$ and $p_{i,t}^{id}$ denotes the Lagrange multiplier on the skin-in-the-game constraint.

From the expression above, it is immediate that the optimal value of $(l_{i,t}, \iota_{i,t})$ maximizes the expected return on the business. The first-order conditions for $(l_{i,t}, \iota_{i,t})$ are given in (11) and (12), respectively. The expected return on the business will be equalized, allowing us to write $\mu_{i,t}^R = \mu_t^R$.

The optimal quantity of capital, aggregate insurance, and idiosyncratic insurance solve the problem

$$\max_{k_{i,t}, \theta_{i,t}^{ag}, \theta_{i,t}^{id}} \left\{ \frac{q_t k_{i,t}}{\omega_{i,t}} (\mu_t^R - r) - \frac{p^{ag} \theta_{i,t}^{ag}}{\omega_{i,t}} + \frac{h_{i,t}}{\omega_{i,t}} p^{ag} \sigma_A - \frac{\gamma}{2} \left[\left(\frac{q_t k_{i,t} + h_{i,t}}{\omega_{i,t}} \sigma_A - \frac{\theta_{i,t}^{ag}}{\omega_{i,t}} \right)^2 + \left(\frac{q_t k_{i,t}}{\omega_{i,t}} \sigma_{id} - \frac{\theta_{i,t}^{id}}{\omega_{i,t}} \right)^2 \right] \right\}, \quad (\text{A.13})$$

subject to (7).

Multiplying the expression above by $\omega_{i,t}/n_{i,t}$ gives (18).

Policy functions. The first order condition for $\theta_{i,t}^{id}$ is given by

$$\gamma \left[\frac{q_t k_{i,t}}{\omega_{i,t}} \sigma_{id} - \frac{\theta_{i,t}^{id}}{\omega_{i,t}} \right] = p_{i,t}^{id}. \quad (\text{A.14})$$

The equation above implies that the skin-in-the-game constraint is always binding, so $p_{i,t}^{id} > 0$ and $\theta_{i,t}^{id} = (1-\phi)q_t k_{i,t} \sigma_{id}$. If this was not the case, i.e. $p_{i,t}^{id} = 0$, then we would have $\theta_{i,t}^{id} = q_t k_{i,t} \sigma_{id}$, which violates the skin-in-the-game constraint.

The first-order conditions for capital and aggregate insurance are given by

$$\begin{aligned}\mu_t^R - r + p_{i,t}^{id}(1 - \phi)\sigma_{id} &= \gamma \left[\left(\frac{q_t k_{i,t} + h_{i,t}}{\omega_{i,t}} \sigma_A - \frac{\theta_{i,t}^{ag}}{\omega_{i,t}} \right) \sigma_A + \left(\frac{q_t k_{i,t}}{\omega_{i,t}} \sigma_{id} - \frac{\theta_{i,t}^{id}}{\omega_{i,t}} \right) \sigma_{id} \right] \\ p_t^{ag} &= \gamma \left(\frac{q_t k_{i,t} + h_{i,t}}{\omega_{i,t}} \sigma_A - \frac{\theta_{i,t}^{ag}}{\omega_{i,t}} \right).\end{aligned}\tag{A.15}$$

Combining the expressions above, we obtain

$$p_{i,t}^{id} = \frac{\mu_t^R - r_t - p_t^{ag} \sigma_A}{\phi \sigma_{id}},\tag{A.16}$$

which coincides with expression (20) after we write $p_{i,t}^{id} = p_t^{id}$.

The demand for capital can be written as

$$\frac{q_t k_{i,t}}{\omega_{i,t}} = \frac{p_t^{id}}{\gamma \phi \sigma_{id}}.\tag{A.17}$$

Multiplying by $\omega_{i,t}/n_{i,t}$, we obtain expression (19). Solving for $\theta_{i,t}^{ag}$ in the optimality condition for aggregate insurance we obtain (21).

The first-order condition for consumption gives

$$\frac{c_{i,t}}{\omega_{i,t}} = \zeta_t(a).\tag{A.18}$$

Plugging the expressions above back into the HJB, we obtain a PDE for $\zeta_t(a)$

$$\frac{\partial \zeta_t(a)}{\partial t} + \frac{\partial \zeta_t(a)}{\partial a} = \zeta_t^2(a) - \bar{r}_t \zeta_t(a),\tag{A.19}$$

where

$$\bar{r}_t \equiv \frac{1}{\gamma} \rho + \left(1 - \frac{1}{\gamma}\right) \left[r + \frac{(p_t^{id})^2 + (p_t^{ag})^2}{2\gamma} \right].\tag{A.20}$$

Define $z_{s,t} \equiv \zeta_t^{-1}(t - s)$ as the wealth-consumption ratio for an entrepreneur born at date s . Differentiating with respect to t , we obtain

$$\dot{z}_{s,t} = -\frac{1}{\zeta_t^2(a)} \left[\frac{\partial \zeta_t}{\partial t} + \frac{\partial \zeta_t}{\partial a} \right] = \bar{r}_t z_{s,t} - 1.\tag{A.21}$$

Solving the above differential equation, we get

$$z_{s,t} = \int_t^{s+T} e^{-\int_t^u \bar{r}_z dz} du + e^{-\int_t^{s+T} \bar{r}_z dz} z_{s,s+T}, \quad (\text{A.22})$$

or in terms of $\zeta_t(a)$, we have

$$\zeta_t(a) = \frac{1}{\int_t^{t+T-a} e^{-\int_t^u \bar{r}_z dz} du + e^{-\int_t^{t+T-a} \bar{r}_z dz} (1 - \psi)(V^*)^{\frac{1}{\gamma}}}, \quad (\text{A.23})$$

where we used the boundary condition $\zeta_t^{-1}(T) = (1 - \psi)(V^*)^{\frac{1}{\gamma}}$.

Assuming $(V^*)^{\frac{1}{\gamma}} = \frac{1}{\bar{r}}$ and a stationary equilibrium, where $\bar{r}_t = \bar{r}$, we obtain

$$\zeta_t(a) = \frac{\bar{r}}{1 - \psi e^{-\bar{r}(T-a)}}, \quad (\text{A.24})$$

which coincides with (22).

Notice that the assumption $(V^*)^{\frac{1}{\gamma}} = \frac{1}{\bar{r}}$ guarantees that the consumption-wealth ratio for $\psi = 0$ is the same as in the infinite horizon economy.

□

A.2 Proof of Proposition 2

Proof. We start by deriving the law of motion of financial wealth for an entrepreneur of a given age. Using the value of capital, $k_{i,t}$, aggregate and idiosyncratic insurance, $(\theta_{i,t}^{ag}, \theta_{i,t}^{id})$, and the definition of the price of idiosyncratic risk, p_t^{id} , given in Proposition 1, we can write the law of motion of financial wealth as follows

$$d\tilde{n}_{i,t} = \left[r_t \tilde{\omega}_{i,t} + \frac{(p_t^{id})^2}{\gamma} \tilde{\omega}_{i,t} + \frac{(p_t^{ag})^2}{\gamma} \tilde{\omega}_{i,t} - \tilde{n}_{i,t}(r_t + p_t^{ag} \sigma_A) + \tilde{\omega}_{i,t} \bar{l}_{i,t} - \tilde{c}_{i,t} \right] dt + \left(\tilde{\omega}_{i,t} \frac{p_t^{ag}}{\gamma} - \tilde{n}_{i,t} \sigma_A \right) dZ_t + \frac{p_t^{id}}{\gamma} \tilde{\omega}_{i,t} dZ_{i,t}, \quad (\text{A.25})$$

where $\tilde{\omega}_{i,t} = \tilde{n}_{i,t} + \tilde{h}_{i,t}$.

Using the fact that $p_t^{ag} = \gamma \sigma_A$ in equilibrium, we find that the aggregate risk exposure of entrepreneurs is given $\tilde{n}_{i,t} \sigma_A$. Hence, scaled financial wealth, $n_{i,t} = \tilde{n}_{i,t} / A_t$, does not respond to aggregate shocks. The evolution of $n_{i,t}$ can then be written as

$$dn_{i,t} = \mu_{n,t}(n_{i,t}, a) dt + \sigma_{n,t}(n_{i,t}, a) dZ_{i,t}, \quad (\text{A.26})$$

where

$$\mu_{n,t}(n, a) = \left[r_t + \frac{(p_t^{id})^2}{\gamma} + \frac{(p_t^{ag})^2}{\gamma} - \mu_A - \zeta_t(a) \right] (n + h_t(a)) - \mu_{h,t}(a) \quad (\text{A.27})$$

$$\sigma_{n,t}(n, a) = \frac{p_t^{id}}{\gamma} (n + h_t(a)), \quad (\text{A.28})$$

and $\mu_{h,t}(a)$ is the drift of $h_t(a)$.

Derivation of Equation (23). Notice that total wealth evolves according to

$$\frac{d\omega_{i,t}}{\omega_{i,t}} = \left[r_t + \frac{(p_t^{ag})^2}{\gamma} + \frac{(p_t^{id})^2}{\gamma} - \mu_A - \zeta_t(t - s_i) \right] dt + \frac{p_t^{id}}{\gamma} dZ_{i,t}, \quad (\text{A.29})$$

where s_i is the birthdate of entrepreneur i .

Let $\bar{\omega}_{s,t} \equiv \frac{\int_{\mathcal{E}_t} \mathbb{1}_{s_i=s} \omega_{i,t} di}{\int_{\mathcal{E}_t} \mathbb{1}_{s_i=s} di}$ denote the average total wealth of entrepreneurs born at date s . The law of motion of $\bar{\omega}_{s,t}$ is given by

$$d\bar{\omega}_{s,t} = \left[r_t + \frac{(p_t^{ag})^2}{\gamma} + \frac{(p_t^{id})^2}{\gamma} - \mu_A - \zeta_t(t - s) \right] \bar{\omega}_{s,t} dt \quad (\text{A.30})$$

where the idiosyncratic risk is diversified by averaging out across entrepreneurs of a given cohort.

It is convenient to express total wealth as a function of age instead of the entrepreneurs' birthdate. Let $\omega_t(a)$ denote the average total wealth of investors with age a at period t . Using the fact that $\bar{\omega}_{s,t} = \omega_t(t - s)$, we obtain the following PDE for $\omega_t(a)$:

$$\frac{\partial \omega_t(a)}{\partial t} + \frac{\partial \omega_t(a)}{\partial a} = \left[r_t + \frac{(p_t^{ag})^2}{\gamma} + \frac{(p_t^{id})^2}{\gamma} - \mu_A - \zeta_t(a) \right] \omega_t(a). \quad (\text{A.31})$$

In a stationary equilibrium, $\omega_t(a)$ does not depend on calendar time t , which allow us to write

$$\frac{d \log \omega(a)}{da} = r + \frac{(p^{ag})^2}{\gamma} + \frac{(p^{id})^2}{\gamma} - \mu_A - \zeta(a). \quad (\text{A.32})$$

Integrating the expression above, we obtain

$$\log \omega(a) = \log \omega(0) + \left[r + \frac{(p^{ag})^2}{\gamma} + \frac{(p^{id})^2}{\gamma} - \mu_A \right] a - \int_0^a \zeta(u) du. \quad (\text{A.33})$$

Using the fact $\log \frac{\omega(a)}{\omega(0)} = \log \frac{f(a)\omega(a)}{f(0)\omega(0)} + ga$ and the identity $\omega(a) = n(a) \left(1 + \frac{h(a)}{n(a)} \right)$, we

obtain expression (23) after some rearrangement.

Derivation of Equation (24). The expression for $\omega(a)$ in levels can be written as

$$\omega(a) = \omega(0)e^{\left(r + \frac{(p^a g)^2}{\gamma} + \frac{(p^{id})^2}{\gamma} - \mu_A\right)a} \frac{e^{-\bar{r}a} - \psi e^{-\bar{r}T}}{1 - \psi e^{-\bar{r}T}}. \quad (\text{A.34})$$

Evaluating at $a = T$ gives

$$\omega(T) = \omega(0)e^{\left(r + \frac{(p^a g)^2}{\gamma} + \frac{(p^{id})^2}{\gamma} - \mu_A - mpc_e\right)T}, \quad (\text{A.35})$$

where $mpc_e = \frac{1}{T} \int_0^T \zeta(a) da$.

The boundary condition at age T implies $\omega(0) = e^{-gT} \omega(T) + h(0)$, then

$$\omega(0) = \frac{h(0)}{1 - e^{\left(r + \frac{(p^a g)^2}{\gamma} + \frac{(p^{id})^2}{\gamma} - \mu_A - g - mpc_e\right)T}}. \quad (\text{A.36})$$

Using $\omega(0) = n(0) + h(0)$ and rearranging the resulting expression, we obtain (24).

Derivation of Equation (25). Multiplying Equation (A.34) by $f(a)$, integrating over age, and using the fact that $f(a) = e^{-ga} f(0)$, we obtain

$$n_e + h_e = f(0)\omega(0) \int_0^T e^{\left(r + \frac{(p^a g)^2}{\gamma} + \frac{(p^{id})^2}{\gamma} - (g + \mu_A)\right)a} \frac{e^{-\bar{r}a} - \psi e^{-\bar{r}T}}{1 - \psi e^{-\bar{r}T}} da, \quad (\text{A.37})$$

which gives Equation (25) after some rearrangement .

□

A.3 Proof of Lemma 2

Proof. We derive the Kolmogorov Forward Equation as the limit of a discrete-time economy. The discrete-time approximation goes as follows. Time takes values on the discrete set $\{t^1, \dots, t^L\}$, where $\Delta t = t^{l+1} - t^l$ is the constant time step. Scaled financial wealth $n_{i,t}$ takes values on a discrete grid, $n_{i,t} \in \{n^1, n^2, \dots, n^J\}$ with a constant step size $\Delta n = n^{j+1} - n^j$. Age is also assumed to take values in a discrete grid $\{a^1, \dots, a^K\}$, where $\Delta a = a^{k+1} - a^k$, $a^1 = 0$, and $a^K = T$. For simplicity, assume $\Delta a = \Delta t$. The probability of moving up, down, or staying

at the same point of the grid are chosen to approximate (A.26) and are given, respectively, by

$$p_u(n^j, a^k) = \frac{1}{2} \left[\frac{\sigma_n(n^j, a^k)^2}{\bar{\sigma}^2} + \frac{\mu_n(n^j, a^k)}{\bar{\sigma}^2} \Delta n \right] \quad (\text{A.38})$$

$$p_d(n^j, a^k) = \frac{1}{2} \left[\frac{\sigma_n(n^j, a^k)^2}{\bar{\sigma}^2} - \frac{\mu_n(n^j, a^k)}{\bar{\sigma}^2} \Delta n \right] \quad (\text{A.39})$$

$$p_s(n^j, a^k) = 1 - \frac{\sigma_n(n^j, a^k)^2}{\bar{\sigma}^2}. \quad (\text{A.40})$$

where $\bar{\sigma} = \max_{1 \leq j \leq J, 1 \leq k \leq K} \sigma_n(n^j, a^k)$, $\Delta n = \bar{\sigma} \sqrt{\Delta t}$, and $\Delta a = \Delta t$.

Notice that the expected change in $n_{i,t}$, where $n_{i,t} = n^j$ and $a_i = a^k$, is given by

$$\mathbb{E} [n_{i,t+1} - n_{i,t}] = p_u(n^j, a^k) \Delta n + p_d(n^j, a^k) (-\Delta n) = \mu_n(n^j, a^k) \Delta t, \quad (\text{A.41})$$

and

$$\mathbb{E} [(n_{i,t+1} - n_{i,t})^2] = p_u(n^j, a^k) \Delta n^2 + p_d(n^j, a^k) (-\Delta n)^2 = \sigma_n(n^j, a^k)^2 \Delta t. \quad (\text{A.42})$$

Let $m(n^j, a^k, t^l)$ denote the mass of agents with financial wealth n^j , age a^k , at period t^l . Summing over n^j , we obtain the mass of agents with age a^k , $M_{k,l} \equiv \sum_{j=1}^J m(n^j, a^k, t^l) = e^{g(t^l - (k-1)\Delta t)}$. Summing over (n^j, a^k) , we obtain the total population $M_l = \sum_{k=1}^K M_{k,l}$, so $M_{l+1} = e^{g\Delta t} M_l$. The law of motion of m , for $k > 1$ and $1 < j < J$, is given by

$$\begin{aligned} m(n^j, a^k, t^l + \Delta t) &= p_u(n^j - \Delta n, a^k - \Delta a) m(n^j - \Delta n, a^k - \Delta a, t^l) + p_s(n^j, a^k - \Delta a) m(n^j, a^k - \Delta a, t^l) \\ &\quad + p_d(n^j + \Delta n, a^k - \Delta a) m(n^j + \Delta n, a^k - \Delta a, t^l). \end{aligned} \quad (\text{A.43})$$

The boundary conditions are defined as follows. For $j = 1$ and $j = J$, we will assume a reflecting boundary, that is, if n moves up from n_J or down from n_1 , it is immediately reflected back to its initial position

$$\begin{aligned} m(n^J, a^k, t^l + \Delta t) &= p_u(n^J - \Delta n, a^k - \Delta a) m(n^J - \Delta n, a^k - \Delta a, t^l) + p_s(n^J, a^k - \Delta a) m(n^J, a^k - \Delta a, t^l) \\ &\quad + p_u(n^j, a^k - \Delta a) m(n^J, a^k - \Delta a, t^l), \end{aligned} \quad (\text{A.44})$$

and analogously for $j = 1$.

Finally, for $k = 1$, we have

$$\begin{aligned} m(e^{-gT} n^j, a^1, t^l + \Delta t) &= e^{gT} \left[p_u(n^j - \Delta n, a^K) m(n^j - \Delta n, a^K, t^l) + p_s(n^j, a^K) m(n^j, a^K, t^l) \right. \\ &\quad \left. + p_d(n^j + \Delta n, a^K) m(n^j + \Delta n, a^K, t^l) \right]. \end{aligned} \quad (\text{A.45})$$

since each one of the e^{gT} heirs inherit $e^{-gT}n^j$, where we assumed $e^{-gT}n^j$ belongs to the grid.

Let $f(n^j, a^k, t^l) \equiv \frac{m(n^j, a^k, t^l)}{M_l}$ denote the share of agents in state (n^j, a^k) in period t^l . Dividing both sides of (A.43) by M_l and taking a Taylor expansion, we obtain

$$\begin{aligned} (1 + g\Delta t)(f + f_t\Delta t) &= \frac{1}{2} \left(\frac{\sigma_n^2 - (\sigma_n^2)_n\Delta n + 0.5(\sigma_n^2)_{nn}\Delta n^2 - (\sigma_n^2)_a\Delta t}{\bar{\sigma}^2} + \frac{\mu_n - (\mu_n)_n\Delta n}{\bar{\sigma}^2} \Delta n \right) (f - f_a\Delta t - f_n\Delta n + 0.5f_{nn}\Delta n^2) \\ &\quad + \frac{1}{2} \left(\frac{\sigma_n^2 + (\sigma_n^2)_n\Delta n + 0.5(\sigma_n^2)_{nn}\Delta n^2 - (\sigma_n^2)_a\Delta t}{\bar{\sigma}^2} - \frac{\mu_n + (\mu_n)_n\Delta n}{\bar{\sigma}^2} \Delta n \right) (f - f_a\Delta t + f_n\Delta n + 0.5f_{nn}\Delta n^2) \\ &\quad + \left(1 - \frac{\sigma_n^2 - (\sigma_n^2)_a\Delta t}{\bar{\sigma}^2} \right) (f - f_a\Delta t) + o(\Delta t). \end{aligned} \quad (\text{A.46})$$

Simplifying the expression above and taking the limit $\Delta t \rightarrow 0$, we obtain

$$f_t + f_a + gf = \frac{1}{2}(\sigma_n^2)_{nn}f - (\mu_n)_nf + (\sigma_n^2)_nf_n - \mu_nf_n + \frac{1}{2}\sigma_n^2f_{nn}, \quad (\text{A.47})$$

or, more explicitly, we can write the expression as follows

$$\frac{\partial f(n, a, t)}{\partial t} + \frac{\partial f(n, a, t)}{\partial a} + gf(n, a, t) = -\frac{\partial [f(n, a, t)\mu_n(n, a)]}{\partial n} + \frac{1}{2} \frac{\partial [f(n, a, t)\sigma_n^2(n, a)]}{\partial n^2}. \quad (\text{A.48})$$

Let $f_t(n|a)$ denote the conditional density at date t , so $f_t(n, a) = f_t(n|a)f(a)$. We can write the Kolmogorov Forward Equation in terms of the conditional density:

$$f(a) \frac{\partial f_t(n|a)}{\partial t} + f(a) \frac{\partial f_t(n|a)}{\partial a} + f_t(n|a)f'(a) = -f(a) \frac{\partial [f_t(n|a)\mu_n(n, a)]}{\partial n} + f(a) \frac{1}{2} \frac{\partial [f_t(n|a)\sigma_n^2(n, a)]}{\partial n^2} - gf_t(n, a). \quad (\text{A.49})$$

Dividing by $f(a)$ and using the fact that $f'(a) = -gf(a)$, we obtain

$$\frac{\partial f_t(n|a)}{\partial t} + \frac{\partial f_t(n|a)}{\partial a} = -\frac{\partial [f_t(n|a)\mu_n(n, a)]}{\partial n} + \frac{1}{2} \frac{\partial [f_t(n|a)\sigma_n^2(n, a)]}{\partial n^2}. \quad (\text{A.50})$$

In a stationary equilibrium, we can ignore the dependence on calendar time to obtain

$$\frac{\partial f(n|a)}{\partial a} = -\frac{\partial [f(n|a)\mu_n(n, a)]}{\partial n} + \frac{1}{2} \frac{\partial [f(n|a)\sigma_n^2(n, a)]}{\partial n^2}. \quad (\text{A.51})$$

□

A.4 Proof of Proposition 3

Proof. The law of motion of (log) total wealth is

$$d \log \omega_{i,t} = \left[r + \frac{(p^{ag})^2}{\gamma} + \frac{(p^{id})^2}{\gamma} - \frac{\bar{r}}{1 - e^{-\bar{r}(T-(t-s))}} - \mu_A - \frac{1}{2} \left(\frac{p^{id}}{\gamma} \right)^2 \right] dt + \frac{p^{id}}{\gamma} dZ_{i,t}, \quad (\text{A.52})$$

where s denotes the birth date of entrepreneur i .

Integrating the expression above, we obtain

$$\log \omega_{i,t} = \log \omega_{i,s} + \int_s^t \left[r + \frac{(p^{ag})^2}{\gamma} + \frac{(p^{id})^2}{\gamma} - \frac{\bar{r}}{1 - e^{-\bar{r}(T-(t'-s))}} - \mu_A - \frac{1}{2} \left(\frac{p^{id}}{\gamma} \right)^2 \right] dt' + \frac{p^{id}}{\gamma} (Z_{i,t} - Z_{i,s}), \quad (\text{A.53})$$

where $Z_{i,t} - Z_{i,s} \sim \mathcal{N}(0, a)$ and $a = t - s$.

Hence, $\log \omega_{i,t} \sim \mathcal{N}(m(a), v(a))$, where the mean and variance are given by

$$m(a) = \log h(0) + \left[r + \frac{(p^{ag})^2}{\gamma} + \frac{(p^{id})^2}{\gamma} - \mu_A - \frac{1}{2} \left(\frac{p^{id}}{\gamma} \right)^2 - \bar{r} \right] a + \log \frac{1 - e^{-\bar{r}(T-a)}}{1 - e^{-\bar{r}T}} \quad (\text{A.54})$$

$$v(a) = \left(\frac{p^{id}}{\gamma} \right)^2 a, \quad (\text{A.55})$$

using the fact that $\omega_{i,s_i} = h(0)$ when $\psi = 1$.

Note that, as the ratio of consumption to total wealth is the same for all entrepreneurs with the same age, the variance of log consumption is then given by

$$\mathbb{V}[\log c_{i,t} | a] = \mathbb{V}[\log \omega_{i,t} | a] = \left(\frac{p^{id}}{\gamma} \right)^2 a. \quad (\text{A.56})$$

Normalized financial wealth $n_{i,t} = \omega_{i,t} - h_{i,t}$ has a shifted log-normal distribution conditional on $s_i = s$, with support $(-h(a), \infty)$. The expected value and variance of $n_{i,t}$ is given by

$$\mathbb{E}[n|a] = h(0) e^{\left(r + \frac{(p^{ag})^2}{\gamma} + \frac{(p^{id})^2}{\gamma} - \mu_A - \bar{r} \right) a} \frac{1 - e^{-\bar{r}(T-a)}}{1 - e^{-\bar{r}T}} - h(a) \quad (\text{A.57})$$

$$\mathbb{V}[n|a] = \left[e^{\left(\frac{p^{id}}{\gamma} \right)^2 a} - 1 \right] \left[h(0) e^{\left(r + \frac{(p^{ag})^2}{\gamma} + \frac{(p^{id})^2}{\gamma} - \mu_A \right) a} \frac{e^{-\bar{r}a} - e^{-\bar{r}T}}{1 - e^{-\bar{r}T}} \right]^2. \quad (\text{A.58})$$

We show next that $\mathbb{V}[n|a]$ has an inverted U shape. Define the following functions:

$$v_1(a) = \left[e^{\left(\frac{p^{id}}{\gamma} \right)^2 a} - 1 \right]^{\frac{1}{2}} e^{\left(r + \frac{(p^{ag})^2}{\gamma} + \frac{(p^{id})^2}{\gamma} - \mu_A \right) a}; \quad v_2(a) = \frac{e^{-\bar{r}a} - e^{-\bar{r}T}}{1 - e^{-\bar{r}T}}. \quad (\text{A.59})$$

The derivative of the product of $v_1(a)$ and $v_2(a)$ will be positive if

$$v_1'(a)v_2(a) + v_1(a)v_2'(a) > 0 \iff \frac{v_1'(a)}{v_1(a)} > -\frac{v_2'(a)}{v_2(a)}, \quad (\text{A.60})$$

for $a \neq 0$ and $a \neq T$.

Notice that $-v'_2(a)/v_2(a)$ is positive, monotonically increasing, and approaches ∞ as a approaches T :

$$-\frac{v'_2(a)}{v_2(a)} = \bar{r} \frac{1}{1 - e^{-\bar{r}(T-a)}}. \quad (\text{A.61})$$

The term $v'_1(a)/v_1(a)$ is positive, monotonically decreasing, and approaches $+\infty$ as $a \rightarrow 0$:

$$\frac{v'_1(a)}{v_1(a)} = \frac{1}{2} \left(\frac{p^{id}}{\gamma} \right)^2 \frac{e^{\left(\frac{p^{id}}{\gamma} \right)^2 a}}{e^{\left(\frac{p^{id}}{\gamma} \right)^2 a} - 1} + r + \frac{(p^{ag})^2}{\gamma} + \frac{(p^{id})^2}{\gamma} - \mu_A. \quad (\text{A.62})$$

Hence, there exists a unique $0 < \hat{a} < T$ such that $v'_1(a)v_2(a) + v_1(a)v'_2(a) > 0$ for all $a < \hat{a}$ and $v'_1(a)v_2(a) + v_1(a)v'_2(a) < 0$ for all $a > \hat{a}$. Hence, $\mathbb{V}[n|a]$ is equal to zero at $a = 0$, it increases monotonically for $a < \hat{a}$, where it achieves the maximum, and it decreases towards zero for $\hat{a} < a \leq T$. □

A.5 Proof of Proposition 4

Proof. Aggregating Equation (3) and using the fact that labor supply grows at rate g , we obtain

$$\dot{K}_t = [\Phi(\iota(q_t)) - \delta - g] K_t, \quad (\text{A.63})$$

given the initial condition $K_0 = K^*$.

From (A.16) and (5), we obtain the expression

$$r + p_t^{ag} \sigma_A + p_t^{id} \phi \sigma_{id} = \frac{\alpha K_t^{\alpha-1} - \iota(q_t)}{q_t} + \frac{\dot{q}_t}{q_t} + \Phi(\iota(q_t)) - \delta + \mu_A. \quad (\text{A.64})$$

Using $p_t^{ag} = \gamma \sigma_A$ and $p_t^{id} = \gamma \phi \sigma_{id} \frac{q_t K_t}{\chi_e \omega_{e,t}}$ and solving for \dot{q}_t , we obtain

$$\dot{q}_t = \left[r + \gamma \sigma_A^2 + \gamma \phi^2 \sigma_{id}^2 \frac{q_t K_t}{\chi_e \omega_{e,t}} + \delta - \mu_A - \Phi(\iota(q_t)) - \frac{\alpha K_t^{\alpha-1} - \iota(q_t)}{q_t} \right] q_t, \quad (\text{A.65})$$

where $\omega_{e,t} = n_{e,t} + h_{e,t}$.

The ODE above is subject to the terminal condition

$$\lim_{t \rightarrow \infty} q_t = q, \quad (\text{A.66})$$

where q is the value of q_t in the new stationary equilibrium.

The first PDE was derived in the proof of Lemma 1 and it was given in (A.19). The boundary conditions are $\zeta_t(T) = (1 - \psi)^{-1} (V^*)^{-\frac{1}{\gamma}}$ and

$$\lim_{t \rightarrow \infty} \zeta_t(a) = \zeta(a), \quad (\text{A.67})$$

where $\zeta(a)$ is the value in the new stationary equilibrium.

The PDE for the human wealth was given in (A.7). The boundary conditions are $h_t(T) = 0$ and

$$\lim_{t \rightarrow \infty} h_t(a) = h(a), \quad (\text{A.68})$$

where $h(a)$ is the value in the new stationary equilibrium.

The PDE for total wealth was derived in the proof of Proposition 2 and it was given in (A.31). The first boundary condition is $\omega_t(0) = e^{-\delta T} \omega_t(T) + h_t(0)$. The initial condition for $\omega_t(a)$ is given by

$$\omega_0(a) = n^*(a) + (q_0 - q^*)k^*(a) + h_0(a), \quad (\text{A.69})$$

where variables with an asterisk denote values before the change in ϕ .

The initial condition captures two types of revaluation effects. First, the value of the capital stock changes, since the price of capital jumps on impact. Second, the value of human wealth also changes since expected future wages respond on impact.

□

A.6 Proof of Lemma 3

Proof. With a slight abuse of notation, we denote the entrepreneur's value function as a function of total wealth, age, and aggregate productivity: $\tilde{V}_t(\tilde{\omega}_i, a_i) = \tilde{V}(\tilde{\omega}_i, a_i; A_t)$. The HJB for the entrepreneur's problem is given by

$$\rho \tilde{V}_t = \max_{\tilde{c}_i, \tilde{\theta}_i^{ag}, \tilde{\theta}_i^{id}, k_i, l_i, t_i} \frac{\tilde{c}_t^{1-\gamma}}{1-\gamma} + \frac{\mathbb{E}_t[d\tilde{V}_t]}{dt} \quad (\text{A.70})$$

subject to the insurance and borrowing constraints (E.10).

We guess and verify that the value function can be written as

$$\tilde{V}(\tilde{\omega}, a; A_t) = A_t^{1-\gamma} V\left(\frac{\tilde{\omega}}{A_t}, a\right), \quad \tilde{V}^d(\tilde{\omega}, a; A_t) = A_t^{1-\gamma} V^d\left(\frac{\tilde{\omega}}{A_t}, a\right), \quad (\text{A.71})$$

where $V(\omega, a)$ and $V^d(\omega, a)$ are independent of A_t .

Let $\omega_{i,t} \equiv \tilde{\omega}_{i,t}/A_t$ denote scaled total wealth. Using an argument analogous to the one used in Lemma 1, we can derive the law of motion $h_{i,t} = \tilde{h}_{i,t}/A_t$, which then gives the law of motion of $\omega_{i,t}$:

$$d\omega_{i,t} = \mu_{\omega_{i,t}} dt + \sigma_{i,t}^{ag} dZ_t + \sigma_{i,t}^{id} dZ_{i,t}, \quad (\text{A.72})$$

where $\sigma_{i,t}^{ag} \equiv (qk_{i,t} + (1 - \xi_d)h_{i,t} - \omega_{i,t})\sigma_A - \theta_{i,t}^{ag}$, $\sigma_{i,t}^{id} \equiv qk_{i,t}\sigma_{id} - \theta_{i,t}^{id}$, and

$$\mu_{\omega_{i,t}} \equiv (r + p^{ag}\sigma_A - \mu_A)\omega_{i,t} + qk_{i,t}(\mu_{i,t}^R - r - p^{ag}\sigma_A) + (p^{ag} - \sigma_A)\sigma_{i,t}^{ag} + \xi_d w_t \bar{l}_{i,t} - c_{i,t}. \quad (\text{A.73})$$

The HJB equation for the scaled value function can be written as

$$\begin{aligned} \hat{\rho}V = & \max_{c_i, \theta_i^{ag}, \theta_i^{id}, k_i, l_i, \iota_i} \frac{c_t^{1-\gamma}}{1-\gamma} + V_a + V_\omega \left[\hat{r}\omega_{i,t} + qk_{i,t}(\mu_{i,t}^R - r - p^{ag}\sigma_A) + \hat{p}^{ag}\sigma_{i,t}^{ag} + \xi_d w_t \bar{l}_{i,t} - c_{i,t} \right] + \\ & + \frac{1}{2}V_{\omega\omega} \left((\sigma_{i,t}^{ag})^2 + (\sigma_{i,t}^{id})^2 \right) + \lambda_d (V^d - V) \end{aligned} \quad (\text{A.74})$$

subject to $\theta_{i,t}^{id} \leq (1 - \phi)qk_{i,t}\sigma_{id}$ and $\omega_{i,t} \geq 0$, where

$$\hat{\rho} \equiv \rho - (1 - \gamma) \left(\mu_A - \frac{\gamma\sigma_A^2}{2} \right), \quad \hat{r} \equiv r + p^{ag}\sigma_A - \mu_A, \quad \hat{p}^{ag} \equiv p^{ag} - \gamma\sigma_A. \quad (\text{A.75})$$

As the HJB equation above is independent of A_t , we conclude that $V(\omega, a)$ is also independent of A_t , confirming our initial guess.

As $l_{i,t}$ and $\iota_{i,t}$ only enter the problem through $\mu_{i,t}^R$, it is optimal to choose them to maximize expected returns. Expected returns is then constant and equalized across entrepreneurs, so we drop the dependence on entrepreneur and time: $\mu_{i,t}^R = \mu^R$. The first-order condition with respect to $\theta_{i,t}^{ag}$ is given by

$$\hat{p}^{ag} = -\frac{V_{\omega\omega}}{V_\omega}\sigma_{i,t}^{ag} \Rightarrow \theta_{i,t}^{ag} = (qk_{i,t} + (1 - \xi_d)h_{i,t} - \omega_{i,t})\sigma_A + \frac{V_\omega}{V_{\omega\omega}}\hat{p}^{ag}. \quad (\text{A.76})$$

An argument analogous to the one used for the model without labor income risk establishes that $p^{ag} = \gamma\sigma_A$ in a stationary equilibrium. This implies that $\hat{p}^{ag} = 0$ and $\sigma_{i,t}^{ag} = 0$, so entrepreneurs choose the same exposure to aggregate risk in equilibrium.

Let $V_\omega p^{id}$ denote the Lagrange multiplier on the insurance constraint. The first-order conditions with respect to $k_{i,t}$ and $\theta_{i,t}^{id}$ are given by

$$\mu_{i,t}^R - r - \gamma\sigma_A^2 + p^{id}(1 - \phi)\sigma_{id} = -\frac{V_{\omega\omega}}{V_\omega} \left[\sigma_{i,t}^{ag}\sigma_A + \sigma_{i,t}^{id}\sigma_{id} \right], \quad p^{id} = -\frac{V_{\omega\omega}}{V_\omega}\sigma_{i,t}^{id}. \quad (\text{A.77})$$

Given that $\sigma_{i,t}^{id} > 0$ by the insurance constraint and given the concavity of the value function, $V_{\omega\omega} < 0$, we have that $p^{id} > 0$. Therefore, the insurance constraint is always binding, that is, $\theta_{i,t}^{id} = (1 - \phi)qk_{i,t}\sigma_{id}$. Rearranging the expressions above, we obtain

$$qk_{i,t} = -\frac{V_\omega}{V_{\omega\omega}} \frac{p^{id}}{\phi\sigma_{id}}, \quad p^{id} = \frac{\mu^R - r - p^{ag}\sigma_A}{\phi\sigma_{id}}. \quad (\text{A.78})$$

Finally, the optimality condition for consumption is given by

$$c_{i,t}^{-\gamma} = V_\omega \Rightarrow c(\omega_{i,t}, a_{i,t}) = V_\omega^{-\frac{1}{\gamma}}(\omega_{i,t}, a_{i,t}). \quad (\text{A.79})$$

□

A.7 Proof of Proposition 5

Proof. We provide next a complete characterization of the first-order approximation of the entrepreneurs' problem. We proceed in four steps. First, we derive the law of motion of $\hat{\omega}_{i,t}$. Second, we solve for the demand for capital. Third, we will solve for the consumption function. Fourth, we derive the conditions that determine the approximation point $\bar{\omega}$.

Law of motion of the state. The log of total wealth for entrepreneur i evolves according to

$$d \log \omega_{i,t} = \left[\hat{r} + \frac{qk_{i,t}}{\omega_{i,t}} p^{id} \phi \sigma_{id} - \frac{1}{2} \left(\frac{qk_{i,t}}{\omega_{i,t}} \phi \sigma_{id} \right)^2 + \xi_d \frac{w\bar{l}_{i,t}}{\omega_{i,t}} - \frac{c_{i,t}}{\omega_{i,t}} \right] dt + \frac{qk_{i,t}}{\omega_{i,t}} \phi \sigma_{id} dZ_{i,t}. \quad (\text{A.80})$$

Log-linearizing the law of motion of $\omega_{i,t}$, we obtain

$$\begin{aligned} d \log \omega_{i,t} = & \left[\hat{r} + \frac{q\bar{k}}{\bar{\omega}} p^{id} \phi \sigma_{id} (1 + \hat{k}_{i,t} - \hat{\omega}_{i,t}) - \frac{1}{2} \left(\frac{q\bar{k}}{\bar{\omega}} \phi \sigma_{id} \right)^2 (1 + 2(\hat{k}_{i,t} - \hat{\omega}_{i,t})) \right. \\ & \left. + \xi_d \frac{w\bar{l}(\bar{a})}{\bar{\omega}} \left(1 + \frac{\bar{l}'(\bar{a})}{\bar{l}(\bar{a})} \hat{a}_{i,t} - \hat{\omega}_{i,t} \right) - \frac{\bar{c}}{\bar{\omega}} (1 + \hat{c}_{i,t} - \hat{\omega}_{i,t}) \right] dt + \frac{q\bar{k}}{\bar{\omega}} \phi \sigma_{id} (1 + \hat{k}_{i,t} - \hat{\omega}_{i,t}) dZ_{i,t}. \end{aligned} \quad (\text{A.81})$$

Rearranging the expression above, we get

$$\begin{aligned} d\hat{\omega}_{i,t} = & \left[\left(\frac{q\bar{k}}{\bar{\omega}} p^{id} \phi \sigma_{id} - \left(\frac{q\bar{k}}{\bar{\omega}} \phi \sigma_{id} \right)^2 \right) (\hat{k}_{i,t} - \hat{\omega}_{i,t}) + \xi_d \frac{w\bar{l}(\bar{a})}{\bar{\omega}} \left(\frac{\bar{l}'(\bar{a})}{\bar{l}(\bar{a})} \hat{a}_{i,t} - \hat{\omega}_{i,t} \right) - \frac{\bar{c}}{\bar{\omega}} (\hat{c}_{i,t} - \hat{\omega}_{i,t}) \right] dt \\ & + \mathbb{E}[d \log \omega_{i,t} | a_{i,t} = \bar{a}] + \frac{q\bar{k}}{\bar{\omega}} \phi \sigma_{id} (1 + \hat{k}_{i,t} - \hat{\omega}_{i,t}) dZ_{i,t}. \end{aligned} \quad (\text{A.82})$$

We can write the expression above in more compact form:

$$d\hat{\omega}_{i,t} = [\psi_{\omega,0} + \psi_{\omega,a} \hat{a}_{i,t} + \psi_{\omega,\omega} \hat{\omega}_{i,t}] dt + [\psi_{\omega,0}^\sigma + \psi_{\omega,a}^\sigma \hat{a}_{i,t} + \psi_{\omega,\omega}^\sigma \hat{\omega}_{i,t}] dZ_{i,t}, \quad (\text{A.83})$$

where

$$\psi_{\omega,0}^\sigma \equiv \frac{q\bar{k}}{\bar{\omega}} \phi \sigma_{id}, \quad \psi_{\omega,\omega}^\sigma \equiv \psi_{\omega,0}^\sigma (\psi_{k,\omega} - 1), \quad \psi_{\omega,a}^\sigma \equiv \psi_{\omega,0}^\sigma \psi_{k,a}. \quad (\text{A.84})$$

and

$$\psi_{\omega,0} = \hat{r} + \frac{q\bar{k}}{\bar{\omega}} p^{id} \phi \sigma_{id} - \frac{1}{2} \left(\frac{q\bar{k}}{\bar{\omega}} \phi \sigma_{id} \right)^2 + \xi_d \frac{w\bar{l}(\bar{a})}{\bar{\omega}} - \frac{\bar{c}}{\bar{\omega}} \quad (\text{A.85})$$

$$\psi_{\omega,\omega} = \left(\frac{q\bar{k}}{\bar{\omega}} p^{id} \phi \sigma_{id} - \left(\frac{q\bar{k}}{\bar{\omega}} \phi \sigma_{id} \right)^2 \right) (\psi_{k,\omega} - 1) - \xi_d \frac{w\bar{l}(\bar{a})}{\bar{\omega}} - \frac{\bar{c}}{\bar{\omega}} (\psi_{c,\omega} - 1), \quad (\text{A.86})$$

$$\psi_{\omega,a} = \left(\frac{q\bar{k}}{\bar{\omega}} p^{id} \phi \sigma_{id} - \left(\frac{q\bar{k}}{\bar{\omega}} \phi \sigma_{id} \right)^2 \right) \psi_{k,a} + \xi_d \frac{w\bar{l}'(\bar{a})}{\bar{\omega}} - \frac{\bar{c}}{\bar{\omega}} \psi_{c,a}, \quad (\text{A.87})$$

Demand for capital. The first-order condition for capital and consumption are given by

$$-V_{\omega\omega} \sigma_{i,t}^{id} = V_\omega p^{id}, \quad V_\omega = c_{i,t}^{-\gamma}. \quad (\text{A.88})$$

Using the fact that $dV_\omega dZ_{i,t} = V_{\omega\omega} \sigma_{i,t}^{id} dt$ and the expressions above, we can express the optimality condition for capital as follows:

$$p^{id} dt = \gamma \frac{dc_{i,t}}{c_{i,t}} dZ_{i,t}. \quad (\text{A.89})$$

Up to first order, we have that $\frac{dc_{i,t}}{c_{i,t}} dZ_{i,t} = d\hat{c}_{i,t} dZ_{i,t} = \psi_{c,\omega} d\hat{\omega}_{i,t} dZ_{i,t}$. We can then write the expression above as follows:

$$p^{id} = \gamma \psi_{c,\omega} [\psi_{\omega,0}^\sigma + \psi_{\omega,a}^\sigma \hat{a}_{i,t} + \psi_{\omega,\omega}^\sigma \hat{\omega}_{i,t}]. \quad (\text{A.90})$$

As the expression above must hold for all \hat{a} and $\hat{\omega}_{i,t}$, then we must have $\psi_{\omega,\omega}^\sigma = \psi_{\omega,a}^\sigma = 0$. This implies that coefficients in the expansion for capital are given by

$$\psi_{k,\omega} = 1, \quad \psi_{k,a} = 0. \quad (\text{A.91})$$

The exposure to the business relative to total wealth is then the same for all entrepreneurs, in line with the results in Section 3:

$$\frac{qk_{i,t}}{\omega_{i,t}} = \frac{q\bar{k}}{\bar{\omega}}. \quad (\text{A.92})$$

Using the fact that $V_{\omega\omega} = -\gamma c^{-\gamma-1} c_\omega$ and evaluating the first-order condition for capital

at $(\bar{\omega}, \bar{a})$, we obtain

$$\gamma \frac{c_\omega(\bar{\omega}, \bar{a})}{c(\bar{\omega}, \bar{a})} qk(\bar{\omega}, \bar{a}) \phi \sigma_{id} = p^{id} \Rightarrow \frac{qk(\bar{\omega}, \bar{a})}{\bar{\omega}} = \frac{1}{\gamma \psi_{c,\omega}} \frac{p^{id}}{\phi \sigma_{id}}, \quad (\text{A.93})$$

where we used that $\psi_{c,\omega} = \frac{c_\omega(\bar{\omega}, \bar{a}) \bar{\omega}}{c(\bar{\omega}, \bar{a})}$.

The demand for capital can then be written as

$$\frac{qk_{i,t}}{n_{i,t}} = \frac{1 + (1 - \xi_d) \frac{h_{i,t}}{n_{i,t}}}{\gamma \psi_{c,\omega}} \frac{p^{id}}{\phi \sigma_{id}}. \quad (\text{A.94})$$

Consumption. The envelope condition with respect to ω for problem (E.11) is given by

$$\hat{\rho} V_\omega = V_\omega \hat{r} + \frac{\mathbb{E}[dV_\omega]}{dt}. \quad (\text{A.95})$$

Using Ito's lemma, we can write the expression above as follows

$$r = \rho + \gamma \left(\mu_A + \frac{1}{dt} \mathbb{E}[d\hat{c}_{i,t}] \right) - \frac{\gamma(\gamma+1)}{2} \sigma_A^2 - \frac{\gamma^2}{2} \mathbb{E}[d\hat{c}_{i,t}^2] - \lambda_d \left(\left(\frac{c_{i,t}^d}{c_{i,t}} \right)^{-\gamma} - 1 \right). \quad (\text{A.96})$$

Up to first order, the expression above can be written as

$$\gamma \psi_{c,\omega} (\psi_{\omega,\omega} \hat{\omega}_{i,t} + \psi_{\omega,a} \hat{a}_{i,t}) + \gamma \lambda_d \left(\frac{\zeta(\bar{a}) \bar{\omega}}{\bar{c}} \right)^{-\gamma} \left(\hat{\omega}_{i,t} (1 - \psi_{c,\omega}) + \left(\frac{\zeta'(\bar{a})}{\zeta(\bar{a})} - \psi_{c,a} \right) \hat{a} \right) = \text{constant}. \quad (\text{A.97})$$

As the expression above must hold for all values of $\hat{\omega}_{i,t}$ and $\hat{a}_{i,t}$, we obtain $\psi_{\omega,\omega} = \psi_{\omega,a} = 0$. This implies the following conditions must hold:

$$\psi_{c,\omega} \psi_{\omega,\omega} + \lambda_d \left(\frac{\zeta(\bar{a}) \bar{\omega}}{\bar{c}} \right)^{-\gamma} (1 - \psi_{c,\omega}) = 0, \quad \psi_{c,\omega} \psi_{\omega,a} + \lambda_d \left(\frac{\zeta(\bar{a}) \bar{\omega}}{\bar{c}} \right)^{-\gamma} \left(\frac{\zeta'(\bar{a})}{\zeta(\bar{a})} - \psi_{c,a} \right) = 0. \quad (\text{A.98})$$

Using the expression for $\psi_{\omega,\omega}$, we obtain a quadratic equation for $\psi_{c,\omega}$:

$$\psi_{c,\omega}^2 - \psi_{c,\omega} \left[1 - \lambda_d \frac{\bar{\omega}}{\bar{c}} \left(\frac{\zeta(\bar{a}) \bar{\omega}}{\bar{c}} \right)^{-\gamma} - \xi_d \frac{w \bar{l}(\bar{a})}{\bar{c}} \right] - \lambda_d \frac{\bar{\omega}}{\bar{c}} \left(\frac{\zeta(\bar{a}) \bar{\omega}}{\bar{c}} \right)^{-\gamma} = 0. \quad (\text{A.99})$$

The equation above has a positive and a negative solution, where the economically rele-

vant solution is the positive one:

$$\psi_{c,\omega} = \frac{1}{2} \left[1 - \lambda_d \frac{\bar{\omega}}{\bar{c}} \left(\frac{\zeta(\bar{a})\bar{\omega}}{\bar{c}} \right)^{-\gamma} - \xi_d \frac{w\bar{l}(\bar{a})}{\bar{c}} + \sqrt{\left(1 - \lambda_d \frac{\bar{\omega}}{\bar{c}} \left(\frac{\zeta(\bar{a})\bar{\omega}}{\bar{c}} \right)^{-\gamma} - \xi_d \frac{w\bar{l}(\bar{a})}{\bar{c}} \right)^2 + 4\lambda_d \frac{\bar{\omega}}{\bar{c}} \left(\frac{\zeta(\bar{a})\bar{\omega}}{\bar{c}} \right)^{-\gamma}} \right] \quad (\text{A.100})$$

We will show next that $\psi_{c,\omega} < 1$. Assuming that $\psi_{c,\omega} > 1$, the expression above implies that

$$\sqrt{\left(1 - \lambda_d \frac{\bar{\omega}}{\bar{c}} \left(\frac{\zeta(\bar{a})\bar{\omega}}{\bar{c}} \right)^{-\gamma} - \xi_d \frac{w\bar{l}(\bar{a})}{\bar{c}} \right)^2 + 4\lambda_d \frac{\bar{\omega}}{\bar{c}} \left(\frac{\zeta(\bar{a})\bar{\omega}}{\bar{c}} \right)^{-\gamma}} > 1 + \lambda_d \frac{\bar{\omega}}{\bar{c}} \left(\frac{\zeta(\bar{a})\bar{\omega}}{\bar{c}} \right)^{-\gamma} + \xi_d \frac{w\bar{l}(\bar{a})}{\bar{c}}. \quad (\text{A.101})$$

Squaring both sides of the inequality above, we obtain

$$\left(1 - \lambda_d \frac{\bar{\omega}}{\bar{c}} \left(\frac{\zeta(\bar{a})\bar{\omega}}{\bar{c}} \right)^{-\gamma} - \xi_d \frac{w\bar{l}(\bar{a})}{\bar{c}} \right)^2 + 4\lambda_d \frac{\bar{\omega}}{\bar{c}} \left(\frac{\zeta(\bar{a})\bar{\omega}}{\bar{c}} \right)^{-\gamma} > \left(1 + \lambda_d \frac{\bar{\omega}}{\bar{c}} \left(\frac{\zeta(\bar{a})\bar{\omega}}{\bar{c}} \right)^{-\gamma} + \xi_d \frac{w\bar{l}(\bar{a})}{\bar{c}} \right)^2. \quad (\text{A.102})$$

Rearranging the expression above, we obtain

$$4\lambda_d \frac{\bar{\omega}}{\bar{c}} \left(\frac{\zeta(\bar{a})\bar{\omega}}{\bar{c}} \right)^{-\gamma} > 4 \left(\lambda_d \frac{\bar{\omega}}{\bar{c}} \left(\frac{\zeta(\bar{a})\bar{\omega}}{\bar{c}} \right)^{-\gamma} + \xi_d \frac{w\bar{l}(\bar{a})}{\bar{c}} \right), \quad (\text{A.103})$$

which is a contradiction. Therefore, we must have $0 < \psi_{c,\omega} < 1$.

We solve next for $\psi_{c,a}$. The coefficient $\psi_{c,a}$ satisfies the equation:

$$\psi_{c,\omega} \left(\xi_d \frac{w\bar{l}'(\bar{a})}{\bar{\omega}} - \frac{\bar{c}}{\bar{\omega}} \psi_{c,a} \right) + \lambda_d \left(\frac{\zeta(\bar{a})\bar{\omega}}{\bar{c}} \right)^{-\gamma} \left(\frac{\zeta'(\bar{a})}{\zeta(\bar{a})} - \psi_{c,a} \right) = 0. \quad (\text{A.104})$$

Rearranging the expression above, we obtain

$$\psi_{c,a} = \frac{\psi_{c,\omega} \frac{\bar{c}}{\bar{\omega}} \xi_d \frac{w\bar{l}'(\bar{a})}{\bar{c}} + \lambda_d \left(\frac{\zeta(\bar{a})\bar{\omega}}{\bar{c}} \right)^{-\gamma} \frac{\zeta'(\bar{a})}{\zeta(\bar{a})}}{\psi_{c,\omega} \frac{\bar{c}}{\bar{\omega}} + \lambda_d \left(\frac{\zeta(\bar{a})\bar{\omega}}{\bar{c}} \right)^{-\gamma}}. \quad (\text{A.105})$$

For ξ_d sufficiently small, the expression above is positive, as $\zeta'(a)$, the consumption-wealth ratio in the absence of labor income risk, is positive.

We can then write consumption as follows

$$c_{i,t} = c(\bar{\omega}, \bar{a}) e^{\psi_{c,a}(a_{i,t} - \bar{a})} \left(\frac{\omega_{i,t}}{\bar{\omega}} \right)^{\psi_{c,\omega}} \equiv f_c(a_{i,t}) \omega_{i,t}^{\psi_{c,\omega}}, \quad (\text{A.106})$$

where the second equality defines the age-dependent function $f_c(a)$. Note that $f_c(a)$ is increasing in a for ξ_d sufficiently small.

It remains to solve for the consumption-wealth ratio at $(\bar{\omega}, \bar{a})$. From the envelope condition, we obtain

$$r = \rho + \gamma(\mu_A + \psi_{c,\omega}\psi_{\omega,0} + \psi_{c,a}) - \frac{\gamma(\gamma+1)}{2}\sigma_A^2 - \frac{\gamma^2}{2}(\psi_{c,\omega}\psi_{\omega,0}^\sigma)^2 - \lambda_d \left(\left(\frac{\zeta(\bar{a})\bar{\omega}}{\bar{c}} \right)^{-\gamma} - 1 \right). \quad (\text{A.107})$$

where $\psi_{\omega,0} = r + \gamma\sigma_A^2 - \mu_A + \frac{q\bar{k}}{\bar{\omega}}p^{id}\phi\sigma_{id} - \frac{1}{2}\left(\frac{q\bar{k}}{\bar{\omega}}\phi\sigma_{id}\right)^2 + \xi_d\frac{w\bar{l}(\bar{a})}{\bar{\omega}} - \frac{\bar{c}}{\bar{\omega}}$.

Rearranging the expression above, we obtain

$$\psi_{c,a} = \psi_{c,\omega} \left(\frac{\bar{c}}{\bar{\omega}} - \xi_d \frac{w\bar{l}(\bar{a})}{\bar{\omega}} \right) - \bar{r}_d - (1 - \psi_{c,\omega}) \left(\mu_A - \frac{\gamma\sigma_A^2}{2} \right) + \frac{\lambda_d}{\gamma} \left(\left(\frac{\zeta(\bar{a})\bar{\omega}}{\bar{c}} \right)^{-\gamma} - 1 \right), \quad (\text{A.108})$$

where

$$\bar{r}_d \equiv \frac{1}{\gamma}\rho + \left(\psi_{c,\omega} - \frac{1}{\gamma} \right) \left(r + \frac{(p^{ag})^2}{2\gamma} + \frac{(p^{id})^2}{2\gamma\psi_{c,\omega}} \right). \quad (\text{A.109})$$

Using the expressions for $\psi_{c,\omega}$ and $\psi_{c,a}$, we obtain a non-linear equation for \bar{c} given $\bar{\omega}$. Note that if $\xi_d = \lambda_d = 0$, we recover a linearized version of the equation determining $\frac{\zeta'(a)}{\zeta(a)}$, Equation (A.19).

Wealth dynamics and the approximation point. The law of motion of $\hat{\omega}_{i,t}$ can be written as

$$d\hat{\omega}_{i,t} = [\psi_{\omega,0} + \psi_{\omega,a}\hat{a}_{i,t} + \psi_{\omega,\omega}\hat{\omega}_{i,t}]dt + \frac{p^{id}}{\gamma\psi_{c,\omega}}dZ_{i,t}, \quad (\text{A.110})$$

where

$$\psi_{\omega,\omega} = \frac{\bar{c}}{\bar{\omega}} \left[1 - \psi_{c,\omega} - \xi_d \frac{w\bar{l}(\bar{a})}{\bar{c}} \right], \quad \psi_{\omega,a} = \lambda_d \left(\frac{\zeta(\bar{a})\bar{\omega}}{\bar{c}} \right)^{-\gamma} \frac{\bar{c}}{\bar{\omega}} \frac{\xi_d \frac{w\bar{l}'(\bar{a})}{\bar{c}} - \frac{\zeta'(\bar{a})}{\zeta(\bar{a})}}{\psi_{c,\omega} \frac{\bar{c}}{\bar{\omega}} + \lambda_d \left(\frac{\zeta(\bar{a})\bar{\omega}}{\bar{c}} \right)^{-\gamma}}. \quad (\text{A.111})$$

We will show next that $\psi_{\omega,\omega} < 0$. For the sake of reaching a contradiction, assume that $\psi_{\omega,\omega} \geq 0$. This implies that the following condition is satisfied:

$$1 + \lambda_d \frac{\bar{\omega}}{\bar{c}} \left(\frac{\zeta(\bar{a})\bar{\omega}}{\bar{c}} \right)^{-\gamma} - \xi_d \frac{w\bar{l}(\bar{a})}{\bar{c}} \geq \sqrt{\left(1 - \lambda_d \frac{\bar{\omega}}{\bar{c}} \left(\frac{\zeta(\bar{a})\bar{\omega}}{\bar{c}} \right)^{-\gamma} - \xi_d \frac{w\bar{l}(\bar{a})}{\bar{c}} \right)^2 + 4\lambda_d \frac{\bar{\omega}}{\bar{c}} \left(\frac{\zeta(\bar{a})\bar{\omega}}{\bar{c}} \right)^{-\gamma}} \quad (\text{A.112})$$

Squaring both sides of the inequality and rearranging, we obtain

$$4\lambda_d \frac{\bar{\omega}}{\bar{c}} \left(\frac{\zeta(\bar{a})\bar{\omega}}{\bar{c}} \right)^{-\gamma} \left(1 - \zeta_d \frac{w\bar{l}(\bar{a})}{\bar{c}} \right) \geq 4\lambda_d \frac{\bar{\omega}}{\bar{c}} \left(\frac{\zeta(\bar{a})\bar{\omega}}{\bar{c}} \right)^{-\gamma}, \quad (\text{A.113})$$

which is a contradiction. Thus, we must have $\psi_{\omega,\omega} < 0$.

We solve next for $\bar{\omega}$. Let $\bar{\omega}(a) \equiv \exp(\mathbb{E}[\log \omega_{i,t} | a_{i,t} = a])$, then

$$\frac{d \log \bar{\omega}(a)}{da} = \psi_{\omega,0} + \psi_{\omega,a}(a - \bar{a}) + \psi_{\omega,\omega}(\log \bar{\omega}(a) - \log \bar{\omega}). \quad (\text{A.114})$$

Solving the differential equation above, we obtain

$$\log \bar{\omega}(a) = e^{\psi_{\omega,\omega}a} \log \bar{\omega}(0) + \int_0^a [\tilde{\psi}_{\omega,0} + \psi_{\omega,a}a'] e^{\psi_{\omega,\omega}(a-a')} da'. \quad (\text{A.115})$$

Using the fact that the wealth of entrepreneurs at date T is left as a bequest to the next generation, we can pin down $\bar{\omega}(0)$. Evaluating the expression above at $a = \bar{a}$, for a given reference age \bar{a} , we obtain an equation for $\bar{\omega}$. \square

B Data

In this appendix, we discuss the construction of our empirical measures in more detail. For an extensive discussion of the Townsend Thai Monthly Survey and the derivation of entrepreneurs' balance sheet information from the survey questionnaire, see [Samphantharak and Townsend \(2010\)](#).

B.1 Sample selection and variable definition

The dataset includes information on both economic and demographic variables. Economic variables include households' asset and liabilities, financial wealth, business and labor income, and consumption. Demographic and geographic variables consists of the age of the households' head, year, the number of members of the household, the number of children in the household, and the province the household is located. We focus on a sample of households from age 25 to 80. We drop observations for households without information on age or financial wealth. We end up with an unbalanced panel of 796 households over a time period of 14 years, from 1999 to 2012.

Business exposure. The Townsend Thai Monthly Survey contains detailed information on the assets held by entrepreneurs, including fixed assets, inventories, and financial assets. We

classify these assets into business assets and non-business assets, which we treat as safe. The value of the business includes: inventories, livestock, agricultural assets, business assets, and household assets. We follow [Samphantharak and Townsend \(2018\)](#) and include the value of household assets (cars, pick-up trucks, fishing boat, and so on) as part of the business. The motivation for this choice is that many of these assets are also used by households in their production activities. The value of safe assets includes: cash in hand, account receivables, deposit at financial institutions, ROSCA, other lending, prepaid insurance, and land. Given this breakdown, we can compute the fraction of financial wealth (total asset net of liabilities) invested in the business, which we use as our measure of risk-taking.

Return on business activity. Given the estimates for the value of entrepreneur’s business as well the flow of business income generated in a given period, we can compute the return on assets (ROA) as the ratio of business income over the value of the business. ROA is a common accounting measure used to capture the profitability of business activities.

Human wealth. We compute our empirical measure of human wealth in a way analogous to its counterpart in the model, as the present discounted value of future expected labor income. This requires us to specify the discount rate and the expected value of future labor. We use the same discount rate used in the model, as shown in Equation (15). The expected value of future labor income is age-dependent and computed as the average labor income for households of that age. As in the construction of life-cycle profiles discussed below, we use trimmed means to limit the influence of outliers. Consistent with the assumption in the model, there is no idiosyncratic component in the human wealth, as it corresponds to the present discounted value of future labor income across different households conditional on age.

B.2 Life-cycle profiles

When computing life-cycle profiles for a given variable, we aggregate households into 15 age groups. The thresholds that determine each group are chosen such that groups have roughly the same number of households. To limit the influence of outliers, we compute trimmed means with a trimming parameter of 7.5% in each side. We trim the data in a similar manner before running the regressions.

The life-cycle profiles presented in the main text are computed without any controls, which we denote by *raw* moments. We show next that controlling for year fixed effects or demographics variables maintain our results essentially unchanged.

Let $z_{i,k,t}$ denote variable z for household i in age group k at year t . Consider the following process for $z_{i,k,t}$:

$$z_{i,k,t} = \alpha_t + age_k + \delta' x_{i,k,t} + u_{i,k,t}, \quad (\text{B.1})$$

where α_t represents the year fixed-effect, age_k is the age-group effect, and $x_{i,k,t}$ is a vector of demographic and geographic controls, which includes the size of the household, the number of children in the household, and a set of province dummies.

The raw age-group effect is given by

$$age_k^{raw} = \mathbb{E} [z_{i,k',t} | k' = k] . \quad (B.2)$$

The raw age-group effect can be estimated by taking averages by age or by regressing $z_{i,k,t}$ on a set of dummies for age groups.

We follow [Kaplan \(2012\)](#) and define the age-group effect controlling for year fixed effects as follows:

$$age_k^{year-FE} = \frac{1}{\bar{T}} \sum_{t=1}^{\bar{T}} \mathbb{E} [z_{i,k',t'} | k' = k, t' = t] . \quad (B.3)$$

We can estimate $age_k^{year-FE}$ by running a regression on a set of dummy of age-group and year fixed effects and computing the predicted value of the regression evaluated at age k and at an "average" year.

Similarly, we define the age-group effect controlling for year fixed effects and demographic/geographic controls as follows:

$$age_k^{year-FE+dem} = \frac{1}{\bar{T}} \sum_{t=1}^{\bar{T}} \mathbb{E} [z_{i,k',t'} | k' = k, t' = t, x_{i,k',t'} = \bar{x}_{k,t}] , \quad (B.4)$$

where $\bar{x}_{k,t}$ denotes the average value of the controls $x_{i,k,t}$ conditional on age group k and year t . We can estimate $age_k^{year-FE+dem}$ from the full regression with year and age-group fixed-effects as well as demographic controls.

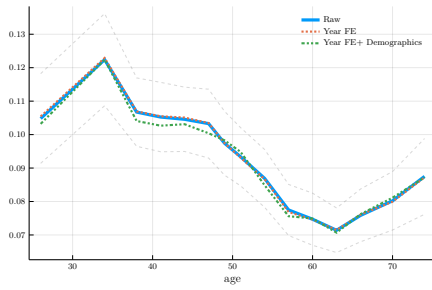
Figure [B.1](#) shows our estimates of the different age-group effects for the variables of interest. The grey dashed lines show the 95% confidence interval for the raw estimates. We cluster standard errors by household to allow shocks to be correlated over time. The life-cycle patterns obtained under the raw measure are essentially identical to the ones obtained after controlling for year fixed-effects and demographic variables.

B.3 Life-cycle profiles and average business returns

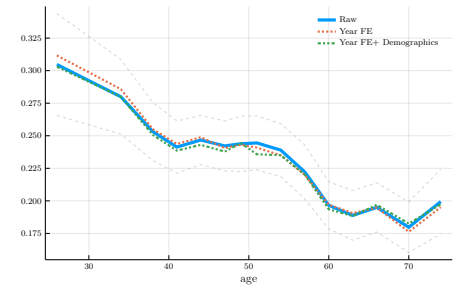
We have seen that an entrepreneur's business exposure, or the fraction of financial wealth invested in the business, varies substantially over the life cycle. We consider next the effect of average return on the business on risk-taking and savings decisions.

Table [3](#) shows regressions of business exposure and the consumption-wealth ratio on a set of age dummies and the average return on the business. We also consider specifications where

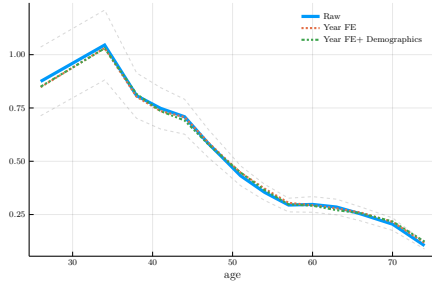
Figure B.1: Life-cycle profiles: raw vs. fixed effects



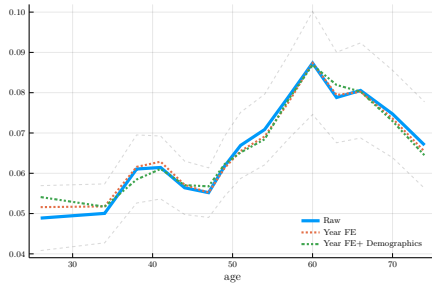
(a) Consumption-wealth ratio



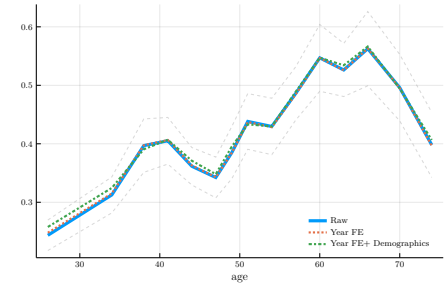
(b) Business exposure



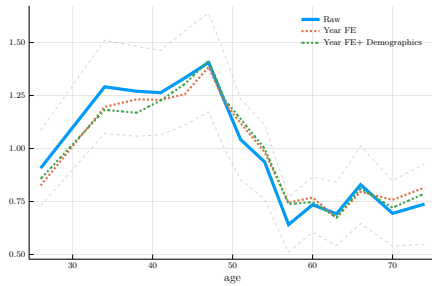
(c) Human-financial wealth ratio



(d) Mean of financial wealth



(e) Standard-deviation of wealth



(f) Labor income

we control for year fixed effects and demographic controls. Standard-errors are clustered at the household level as before. The table shows that the age effects are strongly significant and account for a large fraction of the variation in both risk-taking and savings decisions, with the regression R^2 ranging from 60% to 70%.

The effect of the average return on risk-taking is positive and significant. Note that adding the average return on the business affects only marginally the regression R^2 . This is consistent

with the approach in the model where cross-sectional differences are generated by variables with a strong life-cycle component instead of differences in expected returns.

The effect of the expected return on the consumption-wealth ratio is positive and strongly significant in the case with year fixed effects and demographic controls and marginally insignificant in the other cases. The positive effect is consistent with the income effect dominating the substitution effect in the savings decision, in line with the calibrated model.

C Derivations

C.1 The Optimal Contract

In this appendix, we consider the contracting problem in more detail. In particular, we show that the market structure assumed in Section 3, where entrepreneurs have access to a riskless bond and both aggregate and idiosyncratic insurance, corresponds to a specific implementation of the optimal contract allocation. The derivation follows closely the work of Di Tella (2017) and it is provided for completeness.

C.1.1 Moral hazard

We assume that the aggregate productivity shock Z_t and the individual cumulative return $R_{i,t}$ are publicly observable, but the idiosyncratic investment shock $Z_{i,t}$ is privately observed by entrepreneur i . Moreover, the entrepreneur may secretly divert capital at rate $\varsigma_{i,t}$. The return on the business is then given by

$$dR_{i,t} = \left[\frac{y_{i,t} - w_t l_{i,t} - \iota_{i,t} k_{i,t}}{q_t k_{i,t}} + \frac{\dot{q}_t}{q_t} + \mu_A + \Phi(\iota_{i,t}) - \delta - \varsigma_{i,t} \right] dt + \sigma_A dZ_t + \sigma_{id} dZ_{i,t}. \quad (\text{C.1})$$

Because $Z_{i,t}$ and $\varsigma_{i,t}$ are not publicly observable, a principal contracting with the entrepreneur cannot determine whether a low return is the result of a negative investment shock or positive stealing. The optimal contract ensures that it is incentive-compatible for the entrepreneur to choose $\varsigma_{i,t} = 0$ at all times.³⁴ Note that the expected return coincides with the one in condition (5) in the case of no stealing.

Diverted capital can be sold in the market, but a fraction $1 - \phi$ is lost in the process. The proceedings of the sale is invested in a hidden account, which is remunerated at the risk-free rate r_t . The entrepreneur's hidden savings $S_{i,t}$ evolve as follows:

$$dS_{i,t} = r_t S_{i,t} dt + \phi \tilde{q}_t k_{i,t} \varsigma_{i,t} dt. \quad (\text{C.2})$$

³⁴This is typical of cash-flow diversion models, see e.g. DeMarzo and Sannikov (2006) and DeMarzo and Fishman (2007).

C.1.2 The optimal contract problem

Consumption, investment, and factor demands are contractible. A contract between a principal and an entrepreneur is then given by $(\tilde{c}_i, \iota_i, k_i, l_i, \tilde{F}_i)$, where all variables are adapted to the filtration generated by (Z, R_i) , and $\tilde{F}_{i,t}$ denotes the transfer to the principal. Entrepreneurs cannot commit to long term contracts. At any point in time, an entrepreneur can settle her promises with the principal, transfer the funds from the hidden account to her bank account, and offer a new contract to the principal. Therefore, contracts are effectively short-term and the contract can be redefined at every period.

The continuation value to the principal is given by

$$\tilde{J}_{i,t} = \mathbb{E}_t \left[\int_t^{s_i+T} \frac{\pi_z}{\pi_t} \tilde{F}_{i,z} dz \right], \quad (\text{C.3})$$

where the expectation is taken under no stealing, $\varsigma_i = 0$, and π_t corresponds to the principal's SDF, which evolves according to $d\pi_t = -r_t \pi_t dt - p_t^{ag} \pi_t dZ_t$, given the processes for r_t and p_t^{ag} .³⁵

To compute the law of motion of $\tilde{J}_{i,t}$, let $G_{i,t}$ denote a martingale defined as follows

$$G_{i,t} = \int_{s_i}^t \pi_z d\tilde{F}_{i,z} + \mathbb{E}_t \left[\int_t^{s_i+T} \pi_z d\tilde{F}_{i,z} \right], \quad (\text{C.4})$$

where $G_{i,s_i} = \mathbb{E}_{s_i}[G_{i,t}]$. By the martingale representation theorem, there exists $\sigma_{G_{i,t}}^Z$ and $\sigma_{G_{i,t}}^R$ such that

$$\pi_t \tilde{F}_{i,t} dt + d(\pi_t \tilde{J}_{i,t}) = \pi_t \sigma_{G_{i,t}}^Z dZ_t + \pi_t \sigma_{G_{i,t}}^R (dR_{i,t} - \mathbb{E}_t[dR_{i,t}]). \quad (\text{C.5})$$

Applying Ito's lemma on $\pi_t \tilde{J}_{i,t}$ and combining with the expression above, we obtain

$$d\tilde{J}_{i,t} = \left[r_t \tilde{J}_{i,t} + p_t^{ag} (\sigma_{\tilde{J}_{i,t}}^Z + \sigma_{\tilde{J}_{i,t}}^R \sigma_A) - \tilde{F}_{i,t} \right] dt + \sigma_{\tilde{J}_{i,t}}^Z dZ_t + \sigma_{\tilde{J}_{i,t}}^R (dR_{i,t} - \mathbb{E}_t[dR_{i,t}]), \quad (\text{C.6})$$

where $\sigma_{\tilde{J}_{i,t}}^Z = \sigma_{G_{i,t}}^Z + p_t^{ag} \tilde{J}_{i,t}$ and $\sigma_{\tilde{J}_{i,t}}^R = \sigma_{G_{i,t}}^R$.

The financial wealth of entrepreneur i is defined as $\tilde{n}_{i,t} \equiv \tilde{b}_{i,t} + \tilde{q}_t k_{i,t} - \tilde{J}_{i,t}$, which corresponds to the sum of holdings of the risk-free asset $\tilde{b}_{i,t}$ and the value of the business $\tilde{q}_t k_{i,t}$, net of the promised payments to the principal $\tilde{J}_{i,t}$. The law of motion of financial wealth is given

³⁵When contracting with a wage earner, the relevant SDF is $\pi_t = e^{-\rho w t} \tilde{c}_{j,t}^{-\gamma}$. In a stationary equilibrium, consumption follows the process $d\tilde{c}_{j,t} = \mu_A \tilde{c}_{j,t} dt + \sigma_A \tilde{c}_{j,t} dZ_t$, then $d\pi_t = - \left[\rho_w + \gamma \mu_A - \gamma(\gamma+1) \frac{\sigma_A^2}{2} \right] dt - \gamma \sigma_A dZ_t$, where the drift corresponds to the interest rate r_t and the diffusion term corresponds to the the price of aggregate risk p_t^{ag} .

by

$$d\tilde{n}_{i,t} = \left[r_t \tilde{b}_{i,t} + \tilde{w}_t \bar{l}_{i,t} - \tilde{F}_{i,t} - \tilde{c}_{i,t} \right] dt + \tilde{q}_t k_{i,t} dR_{i,t} - d\tilde{J}_{i,t}. \quad (\text{C.7})$$

Combining (C.6) and (C.7), and assuming $\varsigma_{i,t} = 0$, we obtain

$$\begin{aligned} d\tilde{n}_{i,t} = & \left[r_t (\tilde{n}_{i,t} - \tilde{q}_t k_{i,t}) + \tilde{q}_t k_{i,t} \mu_{i,t}^R - p_t^{ag} (\sigma_{j,t}^Z + \sigma_{j,t}^R \sigma_A) + \tilde{w}_t \bar{l}_{i,t} - \tilde{c}_{i,t} + \left(-\tilde{q}_t k_{i,t} + \phi \tilde{q}_t k_{i,t} + \frac{\tilde{\theta}_t^{id}}{\sigma_{id}} \right) \varsigma_{i,t} \right] dt \\ & + \left(\tilde{q}_t k_{i,t} \sigma_A - \left(\sigma_{j,t}^Z + \sigma_{j,t}^R \sigma_A \right) \right) dZ_t + \left(\tilde{q}_t k_{i,t} \sigma_{id} - \sigma_{j,t}^R \sigma_{id} \right) dZ_{i,t}. \end{aligned} \quad (\text{C.8})$$

By imposing $\varsigma_{i,t} = 0$ and defining $\tilde{\theta}_{i,t}^{ag} \equiv \sigma_{j,t}^Z + \sigma_{j,t}^R \sigma_A$ and $\tilde{\theta}_{i,t}^{id} \equiv \sigma_{j,t}^R \sigma_{id}$, we obtain the law of motion of financial wealth presented in Section 3. Note that the transfers to the principal $\tilde{F}_{i,t}$ only affect the law of motion of $\tilde{n}_{i,t}$ through the diffusion terms of the principal's continuation value \tilde{J}_i . Therefore, we can write the contract in terms of $(\tilde{\theta}_{i,t}^{ag}, \tilde{\theta}_{i,t}^{id})$ instead of $\tilde{F}_{i,t}$.³⁶ The entrepreneur's problem can then be written as

$$\rho \tilde{V}_t = \max_{\tilde{c}_t, \tilde{\theta}_t^{ag}, \tilde{\theta}_t^{id}, k_t, l_t, \mu_t} \frac{c_{i,t}^{1-\gamma}}{1-\gamma} + \mathbb{E}_t [dV_t], \quad (\text{C.9})$$

subject to the law of motion of financial wealth, $\tilde{n}_{i,t} \geq -\tilde{h}_{i,t}$, and the incentive-compatibility (IC) constraint

$$0 \in \arg \max_{\varsigma_{i,t} \geq 0} \left\{ \frac{c_{i,t}^{1-\gamma}}{1-\gamma} + \mathbb{E}_t [dV_t] \right\}. \quad (\text{C.10})$$

Applying Ito's lemma to the value function, we can write the IC constraint as follows

$$-\tilde{q}_t k_{i,t} + \phi \tilde{q}_t k_{i,t} + \frac{\tilde{\theta}_t^{id}}{\sigma_{id}} \leq 0 \Rightarrow \tilde{\theta}_{i,t}^{id} \leq (1 - \phi) \tilde{q}_t k_{i,t} \sigma_{id}, \quad (\text{C.11})$$

where we used the fact that $V_{n,t}$ is positive.

Therefore, the optimal contract problem, where entrepreneurs choose transfers to a principal, is equivalent to problem (9), where entrepreneurs have access to aggregate and idiosyncratic insurance subject to the skin-in-the-game constraint (7).

C.2 Numerical solution of the KFE

We compute the solution of the KFE using a finite-difference scheme. We consider first the case of a stationary equilibrium and then discuss the solution in the case of a time-dependent KFE.

³⁶This reformulation also avoids the issue that the path of transfers $\tilde{F}_{i,t}$ is not uniquely determined, as an entrepreneur can, for instance, borrow from the principal and invest in the risk-free asset without affecting her utility.

C.2.1 Stationary KFE.

Consider the case of a stationary solution to the KFE. We solve the PDE (26) using a finite-differences method. As in the proof of Lemma 2, we assume that n takes values in the grid $\{n_1, n_2, \dots, n_I\}$ and age a takes values in the grid $\{a_1, a_2, \dots, a_K\}$, with $a_1 = 0$ and $a_K = T$. We adopt the following (upwind) difference scheme for $1 < i < I$ and $k < K$

$$\begin{aligned} \frac{f^{i,k+1} - f^{i,k}}{\Delta a} = & -\frac{(\mu_n^{i,k})^+ f^{i,k} - (\mu_n^{i-1,k})^+ f^{i-1,k}}{\Delta n} - \frac{(\mu_n^{i+1,k})^- f^{i+1,k} - (\mu_n^{i,k})^- f^{i,k}}{\Delta n} \\ & + \frac{(\sigma_n^{i+1})^2 f^{i+1,k} - 2(\sigma_n^i)^2 f^{i,k} + (\sigma_n^{i-1})^2 f^{i-1,k}}{2\Delta n^2}, \end{aligned} \quad (\text{C.12})$$

where $f^{i,k} \equiv f(n_i, a_k)$, $\mu_n^{i,k} \equiv \mu_n(n_i, a_k)$, $\sigma_n^i \equiv \sigma_n(n_i)$, $(x)^+ = \max\{x, 0\}$, and $(x)^- = \min\{x, 0\}$.

Rearranging the expression above and collecting terms, we obtain

$$f^{i,k+1} = \pi_u^{i-1,k} f^{i-1,k} + \pi_s^{i,k} f^{i,k} + \pi_d^{i+1,k} f^{i+1,k}, \quad (\text{C.13})$$

where

$$\pi_u^{i,k} = (\mu_n^{i,k})^+ \frac{\Delta a}{\Delta n} + \frac{(\sigma_n^i)^2}{2} \frac{\Delta a}{\Delta n^2}, \quad \pi_d^{i,k} = -(\mu_n^{i,k})^- \frac{\Delta a}{\Delta n} + \frac{(\sigma_n^i)^2}{2} \frac{\Delta a}{\Delta n^2}, \quad \pi_s^{i,k} = 1 - \left(|\mu_n^{i,k}| \frac{\Delta a}{\Delta n} + (\sigma_n^i)^2 \frac{\Delta a}{\Delta n^2} \right).$$

The above scheme converges if, for all (i, k) , the following variant of the Courant-Friedrichs-Lewy (CFL) condition holds

$$|\mu_n^{i,k}| \frac{\Delta a}{\Delta n} + (\sigma_n^i)^2 \frac{\Delta a}{\Delta n^2} \leq 1. \quad (\text{C.14})$$

We adopt a reflecting boundary at n_1 and n_I

$$f^{1,k+1} = (\pi_d^{1,k} + \pi_s^{1,k}) f^{1,k} + \pi_d^{2,k} f^{2,k} \quad (\text{C.15})$$

$$f^{I,k+1} = \pi_u^{I,k} f^{I-1,k} + (\pi_s^{I,k} + \pi_u^{I,k}) f^{I,k}. \quad (\text{C.16})$$

In matrix form, we can write

$$\underbrace{f^{k+1} = \Pi^k f^k}_{\text{explicit method}}, \quad \underbrace{f^{k+1} = [2I - \Pi^k]^{-1} f^k}_{\text{implicit method}}, \quad (\text{C.17})$$

where

$$\Pi^k = \begin{bmatrix} \pi_d^{1,k} + \pi_s^{1,k} & \pi_d^{2,k} & 0 & 0 & \cdots & 0 & 0 & 0 \\ \pi_u^{1,k} & \pi_s^{2,k} & \pi_d^{3,k} & 0 & \cdots & 0 & 0 & 0 \\ 0 & \pi_u^{2,k} & \pi_s^{2,k} & \pi_d^{3,k} & \cdots & 0 & 0 & 0 \\ \vdots & \ddots & \ddots & \ddots & \cdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & \pi_s^{I-2,k} & \pi_d^{I-1,k} & 0 \\ 0 & 0 & 0 & 0 & \cdots & \pi_u^{I-2,k} & \pi_s^{I-1,k} & \pi_d^{I,k} \\ 0 & 0 & 0 & 0 & \cdots & 0 & \pi_u^{I-1,k} & \pi_s^{I,k} + \pi_u^{I,k} \end{bmatrix}. \quad (\text{C.18})$$

Notice that $(\Pi^k)'$ is a stochastic matrix, so we can interpret the coefficients as probabilities. It remains to specify the boundary condition in the age dimension. An entrepreneur with age T and financial wealth n leaves as bequest $e^{-gT}n$ for each children. The quantity $e^{-gT}n$ may not be in the grid, so we linearly interpolate between the points in the grid. For any point n_i in the grid, there exists coefficients (i', b_i) such that

$$e^{-gT}n_i = n_{i'} + b_i \Delta n = (1 - b_i)n_{i'} + b_i n_{i'+1}, \quad (\text{C.19})$$

where $0 \leq b_i < 1$.

We can interpret b_i as the probability of receiving a bequest of size $n_{i'+1}$ and $1 - b_i$ as the probability of receiving a bequest of size $n_{i'}$. The boundary condition can be written as follows:

$$f(n_{i'}, 0) = (1 - b_i)f(n_i, T), \quad f(n_{i'+1}, 0) = b_i f(n_i, T). \quad (\text{C.20})$$

Collecting the coefficients in a matrix B , we obtain the following condition in matrix form

$$f^1 = B f^K. \quad (\text{C.21})$$

Combining the expressions above with the difference scheme for the f , we obtain

$$f^K = \bar{\Pi} B f^K \iff [I - \bar{\Pi} B] f^K = 0, \quad (\text{C.22})$$

where

$$\bar{\Pi} \equiv \Pi^1 \times \Pi^2 \times \cdots \times \Pi^{K-1}. \quad (\text{C.23})$$

Since B' and $(\Pi^k)'$ are stochastic matrices, we have that $B'(\bar{\Pi})'$ is also a stochastic matrix. Hence, the matrix has a unit eigenvalue, and a solution to the system of equations above exists.

C.2.2 Time-dependent KFE.

The discretized time-dependent KFE can be written as

$$f_{t+1}^{i,k} = \frac{\Delta t}{\Delta a} f_t^{i,k-1} + \pi_{u,t}^{i-1,k} f_t^{i-1,k} + \pi_{s,t}^{i,k} f_t^{i,k} + \pi_{d,t}^{i+1,k} f_t^{i+1,k}, \quad (\text{C.24})$$

for $1 < i \leq I$ and $1 < k \leq K$, where

$$\pi_{u,t}^{i,k} = (\mu_n^{i,k})^+ \frac{\Delta t}{\Delta n} + \frac{(\sigma_{n,t}^i)^2}{2} \frac{\Delta t}{\Delta n^2} \quad (\text{C.25})$$

$$\pi_{d,t}^{i,k} = -(\mu_n^{i,k})^- \frac{\Delta t}{\Delta n} + \frac{(\sigma_{n,t}^i)^2}{2} \frac{\Delta t}{\Delta n^2} \quad (\text{C.26})$$

$$\pi_{s,t}^{i,k} = 1 - \frac{\Delta t}{\Delta a} - \left(|\mu_{n,t}^{i,k}| \frac{\Delta t}{\Delta n} + (\sigma_{n,t}^i)^2 \frac{\Delta t}{\Delta n^2} \right). \quad (\text{C.27})$$

Note that the difference equation above corresponds to the implicit scheme for the stationary KFE if $f_{t+1}^{i,k} = f_t^{i,k}$. The boundary conditions are given by

$$f_{t+1}^{i,1} = (1 - b_i) f_{t+1}^{i',K} + b_i f_{t+1}^{i'+1,K} \quad (\text{C.28})$$

$$f_{t+1}^{1,k} = \frac{\Delta t}{\Delta a} f_t^{1,k-1} + (\pi_{d,t}^{1,k} + \pi_{s,t}^{1,k}) f_t^{1,k} + \pi_{d,t}^{2,k} f_t^{2,k}, \quad (\text{C.29})$$

$$f_{t+1}^{I,k} = \frac{\Delta t}{\Delta a} f_t^{I,k-1} + \pi_{u,t}^{I-1,k} f_t^{I-1,k} + (\pi_{s,t}^{I,k} + \pi_{u,t}^{I,k}) f_t^{I,k}. \quad (\text{C.30})$$

Let $f_t^k \equiv [f_t^{1,k}, f_t^{2,k}, \dots, f_t^{I,k}]'$ and $f_t \equiv [f_t^{2'}, f_t^{3'}, \dots, f_t^{I'}]'$. The recursion for f_t can be written as

$$\begin{bmatrix} f_{t+1}^2 \\ f_{t+1}^3 \\ \vdots \\ f_{t+1}^{K-1} \\ f_{t+1}^K \end{bmatrix} = \begin{bmatrix} \Pi_t^2 & 0 & \cdots & 0 & \frac{\Delta t}{\Delta a} B \\ \frac{\Delta t}{\Delta a} I_I & \Pi_t^3 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & \Pi_t^{K-1} & 0 \\ 0 & 0 & \cdots & \frac{\Delta t}{\Delta a} I_I & \Pi_t^K \end{bmatrix} \begin{bmatrix} f_t^2 \\ f_t^3 \\ \vdots \\ f_t^{K-1} \\ f_t^K \end{bmatrix}. \quad (\text{C.31})$$

In matrix form, we can write the recursion for both the explicit scheme above and an implicit scheme:

$$f_{t+1} = A_t f_t, \quad f_{t+1} = [2I - A_t]^{-1} f_t. \quad (\text{C.32})$$

C.3 Equilibrium prices and capital stock in a stationary equilibrium

We derive next the equilibrium prices and the capital stock in a stationary equilibrium.

C.3.1 Price of aggregate insurance, interest rate, and the relative price of capital

Price of aggregate insurance. The demand for aggregate insurance for wage earners is given by

$$\theta_{j,t}^{ag} = h_{j,t}\sigma_A - (n_{j,t} + h_{j,t})\frac{p^{ag}}{\gamma}, \quad (\text{C.33})$$

which is analogous to the expression for entrepreneurs (21).

Combining the demand for aggregate insurance for entrepreneurs and wage earners with the corresponding market clearing condition, we obtain

$$\int_{\mathcal{E}_t} \left[(qk_{i,t} + h_{i,t})\sigma_A - (n_{i,t} + h_{i,t})\frac{p^{ag}}{\gamma} \right] di + \int_{\mathcal{W}_t} \left[h_{j,t}\sigma_A - (n_{j,t} + h_{j,t})\frac{p^{ag}}{\gamma} \right] dj = 0. \quad (\text{C.34})$$

Rearranging the expression above, we can solve for the price of aggregate insurance p^{ag}

$$p^{ag} = \frac{\int_{\mathcal{E}_t} (qk_{i,t} + h_{i,t})di + \int_{\mathcal{W}_t} h_{j,t}dj}{\int_{\mathcal{E}_t} (n_{i,t} + h_{i,t})di + \int_{\mathcal{W}_t} (n_{j,t} + h_{j,t})dj} \gamma\sigma_A = \gamma\sigma_A, \quad (\text{C.35})$$

using the fact that $\int_{\mathcal{E}_t} n_{i,t}di + \int_{\mathcal{W}_t} n_{j,t}dj = \int_{\mathcal{E}_t} qk_{i,t}di$. Notice that this result does not rely on the assumption of a stationary equilibrium.

Interest rate. The financial wealth of wage earners evolve according to

$$d\tilde{n}_{j,t} = \left[(r + \gamma\sigma_A^2)\tilde{n}_{j,t} + \tilde{w}_t\bar{l}_{j,t} - \tilde{c}_{j,t} \right] dt + \tilde{n}_{j,t}\sigma_A dZ_t. \quad (\text{C.36})$$

using the fact that the demand for aggregate insurance is given by $\theta_{j,t} = -n_{j,t}\sigma_A$ in equilibrium.

Combining the expression above with the law of motion for human wealth, we obtain the law of motion of total wealth:

$$\frac{d\tilde{\omega}_{i,t}}{\tilde{\omega}_{i,t}} = \left[r + \gamma\sigma_A^2 - \frac{c_{j,t}}{n_{j,t} + h_{j,t}} \right] dt + \sigma_A dZ_t. \quad (\text{C.37})$$

In a stationary equilibrium, wage earners' scaled total wealth, $\omega_{j,t} = \tilde{\omega}_{j,t}/A_t$, is constant. Therefore, the drift of $\omega_{j,t}$ must be zero. We can compute the drift of $\omega_{j,t}$ using Ito's lemma:

$$\frac{d\omega_{i,t}}{\omega_{i,t}} = \frac{d\tilde{\omega}_{i,t}}{\tilde{\omega}_{i,t}} - \frac{dA_t}{A_t} + \left(\frac{dA_t}{A_t} \right)^2 - \frac{dA_t}{A_t} \frac{d\tilde{\omega}_{i,t}}{\tilde{\omega}_{i,t}} \quad (\text{C.38})$$

$$= \left[r + \gamma\sigma_A^2 - \frac{c_{j,t}}{n_{j,t} + h_{j,t}} - \mu_A \right] dt. \quad (\text{C.39})$$

The interest rate must then satisfy the condition

$$r + \gamma\sigma_A^2 - \left[\frac{1}{\gamma}\rho_w + \left(1 - \frac{1}{\gamma}\right) \left(r + \frac{\gamma\sigma_A^2}{2}\right) \right] - \mu_A = 0, \quad (\text{C.40})$$

using the fact that the consumption-wealth is given by

$$\frac{c_{j,t}}{n_{j,t} + h_{j,t}} = \frac{1}{\gamma}\rho_w + \left(1 - \frac{1}{\gamma}\right) \left(r + \frac{(p^{ag})^2}{2\gamma}\right), \quad (\text{C.41})$$

which is a special case of (22), as we set $p_t^{id} = 0$ and $T \rightarrow \infty$.

Rearranging expression (C.40), we obtain

$$r = \rho_w + \gamma\mu_A - \gamma(\gamma + 1)\frac{\sigma_A^2}{2}. \quad (\text{C.42})$$

Relative price of capital. Plugging the expression for $\Phi'(\iota)$ into the first-order condition for ι in equation (12), we obtain

$$\frac{1}{\sqrt{\Phi_0^2 + 2\Phi_1\iota}} = \frac{1}{q} \Rightarrow \iota = \frac{q^2 - \Phi_0^2}{2\Phi_1}. \quad (\text{C.43})$$

In a stationary equilibrium, the capital-labor ratio is constant. Thus, capital grows at the population rate g , which gives us the condition

$$\Phi(\iota) - \delta = g \Rightarrow q = \Phi_0 + \Phi_1(g + \delta), \quad (\text{C.44})$$

where we used the fact that $\Phi(\iota) = \frac{q - \Phi_0}{\Phi_1}$, obtained by plugging the expression for ι into the functional form for $\Phi(\iota)$.

C.3.2 Entrepreneurs' human and total wealth

The value of human wealth for an entrepreneur with age a is given by

$$\begin{aligned} h(a) &= \int_0^{T-a} e^{-(r+p^{ag}\sigma_A-\mu_A)z} \sum_{l=1}^L \Gamma_l e^{\varphi_l(z+a)} dz w \bar{l}_e \\ &= (1-\alpha)K^\alpha \bar{l}_e \sum_{l=1}^L \Gamma_l e^{\varphi_l a} \frac{1 - e^{-(r+p^{ag}\sigma_A-\mu_A-\varphi_l)(T-a)}}{r + p^{ag}\sigma_A - \mu_A - \varphi_l}. \end{aligned} \quad (\text{C.45})$$

Average human wealth for entrepreneurs is given by

$$h_e = \int_0^T \frac{ge^{-ga}}{1 - e^{-gT}} h(a) da = (1 - \alpha) K^\alpha \bar{l}_e \sum_{l=1}^L \Gamma_l \frac{ge^{(\varphi_l - g)a}}{1 - e^{-gT}} \frac{1 - e^{-(r + p^{ag}\sigma_A - \mu_A - \varphi_l)(T-a)}}{r + p^{ag}\sigma_A - \mu_A - \varphi_l}. \quad (\text{C.46})$$

Expressing condition (23) in levels, and after some rearrangement, we obtain entrepreneur's total wealth by age

$$\omega(a) = \omega(0) e^{\left(r + \frac{(p^{ag})^2}{\gamma} + \frac{(p^{id})^2}{\gamma} - \mu_A\right)a} \frac{e^{-\bar{r}a} - \psi e^{-\bar{r}T}}{1 - \psi e^{-\bar{r}T}}, \quad (\text{C.47})$$

where $\omega(0)$ is given by

$$\omega(0) = \frac{h(0)}{1 - e^{\left(r + \frac{(p^{ag})^2}{\gamma} + \frac{(p^{id})^2}{\gamma} - g - \mu_A - \bar{r}\right)T} \frac{1 - \psi}{1 - \psi e^{-\bar{r}T}}}. \quad (\text{C.48})$$

Average total wealth for entrepreneurs is given by

$$\omega_e = \int_0^T \frac{ge^{-ga}}{1 - e^{-gT}} \omega(a) da = \omega(0) f(0) \int_0^T \frac{e^{\hat{r}_1 a} - \psi e^{-\bar{r}T} e^{\hat{r}_2 a}}{1 - \psi e^{-\bar{r}T}} da, \quad (\text{C.49})$$

where $\hat{r}_1 = r + \gamma\sigma_A^2 + \frac{(p^{id})^2}{\gamma} - \mu_A - g - \bar{r}$ and $\hat{r}_2 = \hat{r}_1 + \bar{r}$.

C.3.3 Capital stock and idiosyncratic risk premium

Rearranging the expression for the shadow price of idiosyncratic risk (20), and using the definition of expected returns (13), we obtain the MPK schedule, as discussed in Section 5:

$$r + p^{ag}\sigma_A + p^{id}\phi\sigma_{id} = \frac{\alpha K^{\alpha-1} - \iota(q)}{q} + \mu_A + \Phi(\iota(q)) - \delta. \quad (\text{C.50})$$

where r , p^{ag} , and q are functions of parameters, as derived above.

Integrating condition (19) across all entrepreneurs, we obtain

$$p^{id} = \gamma\phi\sigma_{id} \frac{qK}{\chi_e \omega_e}, \quad (\text{C.51})$$

where χ_e is the fraction of entrepreneurs in the population.

Note that ω_e is a function of p^{id} and K . Plugging the expression for (C.49) into the equation above, we obtain p^{id} implicitly as a function of K . This relationship between p^{id} and K corresponds to the pricing schedule discussed in Section 5. Therefore, we obtain the equilib-

rium capital stock and equilibrium price of idiosyncratic risk by finding the pair (K, p^{id}) that simultaneously satisfies the MPK schedule and the pricing schedule.

D Transitional dynamics

In this section, we describe the computation of the transitional dynamics. We focus on the case where the interest rate is fixed during the transition. This simplifies the numerical solution while allow us to focus on the response of the idiosyncratic risk premium. To ensure that the interest rate is constant, we assume that wage earners have Epstein-Zin preferences and take the limit as the elasticity of intertemporal substitution goes to infinity, that is, workers have linear intertemporal preferences.³⁷

D.1 Wage earners with Epstein-Zin preferences

Wage earners have the continuous-time analog of Epstein-Zin preferences with EIS ψ_w and risk aversion γ . The wage earner's problem is given by

$$\tilde{V}_t^w(\tilde{n}_j) = \max_{\tilde{c}_j, \tilde{\theta}_j^{ag}} \mathbb{E}_t \left[\int_t^\infty f_w(\tilde{c}_{j,z}, \tilde{V}_z) dz \right], \quad (\text{D.1})$$

subject to $\tilde{n}_{j,t} \geq \tilde{h}_{j,t}$, where $\tilde{h}_{j,t}$ denotes wage earner j 's human wealth, non-negativity constraint $\tilde{c}_{j,t} \geq 0$, and the law of motion of financial wealth $\tilde{n}_{j,t}$

$$d\tilde{n}_{j,t} = \left[\tilde{n}_{j,t} r_t - p_t^{ag} \tilde{\theta}_{j,t}^{ag} + \tilde{w}_t \bar{l}_{j,t} - \tilde{c}_{j,t} \right] dt - \tilde{\theta}_{j,t}^{ag} dZ_t,$$

where $f_w(\tilde{c}, V)$ is the aggregator given by

$$f_w(\tilde{c}, \tilde{V}) = \rho_w \frac{(1-\gamma)\tilde{V}}{1-\psi_w^{-1}} \left\{ \left(\frac{\tilde{c}}{((1-\gamma)\tilde{V})^{\frac{1}{1-\gamma}}} \right)^{1-\psi_w^{-1}} - 1 \right\}. \quad (\text{D.2})$$

It is convenient to work with the scaled value function $V_t^w(n)$, which satisfies the condition $\tilde{V}_t^w(\tilde{n}) = A_t^{1-\gamma} V_t^w\left(\frac{\tilde{n}_t}{A_t}\right)$, where $V_t^w(\cdot)$ is independent of A_t . The HJB equation in terms of the

³⁷This assumption is meant to capture, in a extreme form, the essence of the macro-finance literature which assumes a high EIS (see e.g. [Bansal and Yaron 2004](#) and [Barro 2009](#)). In these models, the high EIS dampens movements in interest rates, so risk premia accounts for most of the variation in discount rates.

scaled value function is given by

$$\begin{aligned} \tilde{\rho}_w \frac{(1-\gamma)V^w}{1-\psi_w^{-1}} &= \max_{\tilde{c}, \theta^{ag}} \rho_w \frac{(1-\gamma)V^w}{1-\psi_w^{-1}} \left(\frac{c}{((1-\gamma)V^w)^{\frac{1}{1-\gamma}}} \right)^{1-\psi_w^{-1}} + V_t^w + V_n^w \left[\tilde{r}n + (\gamma\sigma_A - p^{ag})\theta^{ag} + w\bar{l} - c \right] \\ &\quad + \frac{1}{2} V_{nn}^w (\theta^{ag} + n\sigma_A)^2. \end{aligned} \quad (\text{D.3})$$

where $\tilde{\rho}_w \equiv \rho_w - (1 - \psi_w^{-1}) \left(\mu_A - \frac{\gamma\sigma_A^2}{2} \right)$ and $\tilde{r}_t \equiv r_t + \gamma\sigma_A^2 - \mu_A$.

Policy functions. The first-order conditions for this problem are given by

$$\rho_w ((1-\gamma)V^w)^{\frac{\psi_w^{-1}-\gamma}{1-\gamma}} c^{-\psi_w^{-1}} = V_n^w, \quad \gamma\sigma_A - p^{ag} = -\frac{V_{nn}^w}{V_n^w} (\theta^{ag} + n\sigma_A). \quad (\text{D.4})$$

We will guess and verify that the scaled value function takes the form

$$V_t^w(n_{j,t}) = \left(\frac{\zeta_{w,t}}{\rho_w^{\psi_w}} \right)^{\frac{1-\gamma}{1-\psi_w}} \frac{(n_{j,t} + h_{j,t})^{1-\gamma}}{1-\gamma}, \quad (\text{D.5})$$

where $\zeta_{w,t}$ and $h_{j,t}$ are potentially time-varying, but they are non-stochastic.

Using the expression for the value function above, we obtain the policy functions:

$$\frac{c_{j,t}}{n_{j,t} + h_{j,t}} = \zeta_{w,t}, \quad \theta_{j,t}^{ag} = \sigma_A h_{j,t} - \frac{p_t^{ag}}{\gamma} (n_{j,t} + h_{j,t}). \quad (\text{D.6})$$

Inserting the policy functions derived above back into the HJB equation, we obtain

$$\frac{\tilde{\rho}_w}{1-\psi_w^{-1}} = \frac{\psi_w^{-1}}{1-\psi_w^{-1}} \zeta_{w,t} + \frac{1}{1-\psi_w} \frac{\dot{\zeta}_{w,t}}{\zeta_{w,t}} + \tilde{r}_t. \quad (\text{D.7})$$

Rearranging the expression above for the case of a stationary equilibrium, so $\dot{\zeta}_{w,t} = 0$, we obtain the consumption-wealth ratio:

$$\zeta_{w,t} = \psi_w \rho_w + (1 - \psi_w) \left(r + \frac{\gamma\sigma_A^2}{2} \right), \quad (\text{D.8})$$

which coincides with the expression for the consumption-wealth ratio for wage earners given in (C.41) in the special case of CRRA preferences, i.e. $\psi_w^{-1} = \gamma$.

Price of aggregate risk and interest rate. The demand for aggregate insurance derived above coincides with the expression for aggregate insurance in the CRRA case (see Equation

C.33). Therefore, the same argument used in Section **C.3** to solve for the price of aggregate risk can be applied in the case with Epstein-Zin preferences in a non-stationary setting. This implies that the price of aggregate risk is constant and given by $p_t^{ag} = \gamma\sigma_A$ during the transitional dynamics.

We consider next the behavior of the interest rate. A derivation analogous to the one in Section **C.3** shows that the interest rate in a stationary equilibrium is given by

$$r = \rho_w + \psi_w^{-1}\mu_A - (1 + \psi_w^{-1})\frac{\gamma\sigma_A^2}{2}, \quad (\text{D.9})$$

which coincides with (30) when $\psi_w^{-1} = \gamma$.

Taking the limit of (D.7) as $\psi_w \rightarrow \infty$, we obtain r_t in the case of a non-stationary equilibrium:

$$r_t + \gamma\sigma_A^2 - \mu_A = \rho_w + \frac{\gamma\sigma_A^2}{2} - \mu_A \Rightarrow r_t = \rho_w - \frac{\gamma\sigma_A^2}{2}. \quad (\text{D.10})$$

Therefore, the interest rate is constant when wage earners have linear intertemporal preferences. Moreover, the expression above coincides with the one for the interest rate in the stationary equilibrium (D.9) when specialized to $\psi_w^{-1} = 0$.

D.2 Computation of the transition dynamics

Let $\hat{x}_t \equiv \log \frac{x_t}{x}$ for any variable x_t , where variables without a time subscript indicates the value in the new stationary equilibrium. The system of differential equations can then be written as

$$\dot{\hat{K}}_t = \frac{q}{\Phi_1}(e^{\hat{q}_t} - 1) \quad (\text{D.11})$$

$$\dot{\hat{q}}_t = \gamma\phi^2\sigma_{id}^2\frac{qK}{\chi_e\omega_e}\left(e^{\hat{q}_t+\hat{K}_t-\hat{\omega}_{e,t}} - 1\right) - \frac{q}{\Phi_1}(e^{\hat{q}_t} - 1) - \left[\frac{\alpha K^{\alpha-1}e^{(\alpha-1)\hat{K}_t} - \iota(qe^{\hat{q}_t})}{qe^{\hat{q}_t}} - \frac{\alpha K^{\alpha-1} - \iota(q)}{q}\right] \quad (\text{D.12})$$

$$\frac{\partial \hat{h}_t(a)}{\partial t} = -\frac{\partial \hat{h}_t(a)}{\partial a} - (1 - \alpha)\frac{K^\alpha \bar{l}(a)}{h(a)}\left(e^{\alpha\hat{K}_t - \hat{h}_t(a)} - 1\right) \quad (\text{D.13})$$

$$\frac{\partial \hat{\omega}_t(a)}{\partial t} = -\frac{\partial \hat{\omega}_t(a)}{\partial a} + \gamma(\phi\sigma_{id})^2\left(\frac{qK}{\chi_e\omega_e}\right)^2\left(e^{2(\hat{q}_t+\hat{K}_t-\hat{\omega}_{e,t})} - 1\right) - \zeta(a)\left(e^{\hat{\zeta}_t(a)} - 1\right) \quad (\text{D.14})$$

$$\frac{\partial \hat{\zeta}_t(a)}{\partial t} = -\frac{\partial \hat{\zeta}_t(a)}{\partial a} + \zeta(a)\left(e^{\hat{\zeta}_t(a)} - 1\right) - (\gamma - 1)\frac{(\phi\sigma_{id})^2}{2}\left(\frac{qK}{\chi_e\omega_e}\right)^2\left(e^{2(\hat{q}_t+\hat{K}_t-\hat{\omega}_{e,t})} - 1\right). \quad (\text{D.15})$$

Linearizing the system above, we obtain

$$\dot{\hat{K}}_t = \frac{q}{\Phi_1} \hat{q}_t \quad (\text{D.16})$$

$$\dot{\hat{q}}_t = \gamma \phi^2 \sigma_{id}^2 \frac{qK}{\chi_e \omega_e} (\hat{q}_t + \hat{K}_t - \hat{\omega}_{e,t}) + \frac{\alpha K^{\alpha-1} - \iota(q)}{q} \hat{q}_t + (1-\alpha) \frac{\alpha K^{\alpha-1}}{q} \hat{K}_t \quad (\text{D.17})$$

$$\frac{\partial \hat{h}_t(a)}{\partial t} = -\frac{\partial \hat{h}_t(a)}{\partial a} - (1-\alpha) \frac{K^\alpha \bar{l}(a)}{h(a)} (\alpha \hat{K}_t - \hat{h}_t(a)) \quad (\text{D.18})$$

$$\frac{\partial \hat{\omega}_t(a)}{\partial t} = -\frac{\partial \hat{\omega}_t(a)}{\partial a} + 2\gamma(\phi \sigma_{id})^2 \left(\frac{qK}{\chi_e \omega_e} \right)^2 (\hat{q}_t + \hat{K}_t - \hat{\omega}_{e,t}) - \zeta(a) \hat{\zeta}_t(a) \quad (\text{D.19})$$

$$\frac{\partial \hat{\zeta}_t(a)}{\partial t} = -\frac{\partial \hat{\zeta}_t(a)}{\partial a} + \zeta(a) \hat{\zeta}_t(a) - (\gamma-1)(\phi \sigma_{id})^2 \left(\frac{qK}{\chi_e \omega_e} \right)^2 (\hat{q}_t + \hat{K}_t - \hat{\omega}_{e,t}), \quad (\text{D.20})$$

where

$$\hat{\omega}_{e,t} = \int_0^T \frac{\omega(a)f(a)}{\omega_e} \hat{\omega}_t(a) da. \quad (\text{D.21})$$

We now discretize the system using a finite-differences method. The time variable t and age a will take values in the equally spaced grid $\{t_1, t_2, \dots, t_N\}$ and $\{a_1, a_2, \dots, a_K\}$, respectively. We adopt the following notation: $\zeta_n^k = \zeta_{t_n}(a_k)$ denotes the consumption-wealth ratio at time t_n and age a_k , and an analogous notation holds for the remaining variables. The time and age steps are denoted by $\Delta t = t_{n+1} - t_n$ and $\Delta a = a_{k+1} - a_k$. The discretized version of the ODEs are given by

$$\frac{\hat{K}_{n+1} - \hat{K}_n}{\Delta t} = \frac{q}{\Phi_1} \hat{q}_n \quad (\text{D.22})$$

$$\frac{\hat{q}_{n+1} - \hat{q}_n}{\Delta t} = \gamma \phi^2 \sigma_{id}^2 \frac{qK}{\omega_e} (\hat{q}_n + \hat{K}_n - \hat{\omega}_{e,n}) + \frac{\alpha K^{\alpha-1} - \iota(q)}{q} \hat{q}_n + \alpha(1-\alpha) \frac{K^{\alpha-1}}{q} \hat{K}_n. \quad (\text{D.23})$$

Discretizing the PDEs, we obtain at the interior points

$$\frac{\hat{h}_{n+1}^k - \hat{h}_n^k}{\Delta t} = -\frac{\hat{h}_n^{k+1} - \hat{h}_n^k}{\Delta a} - (1-\alpha) \frac{K^\alpha \bar{l}^k}{h^k} (\alpha \hat{K}_n - \hat{h}_n^k) \quad (\text{D.24})$$

$$\frac{\hat{\omega}_{n+1}^k - \hat{\omega}_n^k}{\Delta t} = -\frac{\hat{\omega}_n^k - \hat{\omega}_n^{k-1}}{\Delta a} + 2\gamma(\phi \sigma_{id})^2 \left(\frac{qK}{\omega_e} \right)^2 (\hat{q}_n + \hat{K}_n - \hat{\omega}_{e,n}) - \zeta^k \hat{\zeta}_n^k \quad (\text{D.25})$$

$$\frac{\hat{\zeta}_{n+1}^k - \hat{\zeta}_n^k}{\Delta t} = -\frac{\hat{\zeta}_n^{k+1} - \hat{\zeta}_n^k}{\Delta a} + \zeta^k \hat{\zeta}_n^k - (\gamma-1)(\phi \sigma_{id})^2 \left(\frac{qK}{\omega_e} \right)^2 (\hat{q}_n + \hat{K}_n - \hat{\omega}_{e,n}), \quad (\text{D.26})$$

where, using the Trapezoidal rule, $\hat{\omega}_{e,n}$ is given by

$$\hat{\omega}_{e,n} = \left[\sum_{k=2}^{K-1} \hat{\omega}_n^k \frac{f(a_k)\omega(a_k)}{\omega_e} + \hat{\omega}_n^1 \frac{f(a_1)\omega(a_1)}{2\omega_e} + \hat{\omega}_n^K \frac{f(a_K)\omega(a_K)}{2\omega_e} \right] \Delta a. \quad (\text{D.27})$$

It remains to specify the boundary conditions. We have initial conditions for state variables and terminal conditions for jump variables. For the first two equations, we have that capital starts at the old steady state and the relative price of capital converges to the new one.

$$\hat{k}_1 = \hat{k}^*; \quad \lim_{n \rightarrow \infty} \hat{q}_n = 0, \quad (\text{D.28})$$

where \hat{k}^* is the log-deviation of the capital stock at the old steady state relative to the new one.

The boundary conditions associated with $\hat{\zeta}_t(a)$ and $\hat{h}_t(a)$ are the following

$$\hat{\zeta}_n^K = 0; \quad \lim_{n \rightarrow \infty} \hat{\zeta}_n^k = 0, \forall k \quad (\text{D.29})$$

$$\hat{h}_n^K = 0; \quad \lim_{n \rightarrow \infty} \hat{h}_n^k = 0, \forall k. \quad (\text{D.30})$$

The consumption-wealth ratio and human wealth are forward-looking variables, so they have terminal conditions instead of initial conditions. Notice that $h_t(T) = 0$, while $\zeta_t(T)$ is determined by the bequest motive, so deviations from the new steady state are equal to zero. We then only need to solve for the vectors $\hat{\zeta}_n = [\hat{\zeta}_n^1, \hat{\zeta}_n^2, \dots, \hat{\zeta}_n^{K-1}]$ and $\hat{h}_n = [\hat{h}_n^1, \hat{h}_n^2, \dots, \hat{h}_n^{K-1}]$, since the value at the final age is pinned down by the boundary condition.

The boundary condition for total wealth is given by

$$\hat{\omega}_1^k = \hat{\omega}_1^{k,*}; \quad \hat{\omega}_n^1 = \frac{f^K \omega^K}{f^1 \omega^1} \hat{\omega}_n^K + \frac{h^1}{\omega^1} \hat{h}_n^1. \quad (\text{D.31})$$

Note that the value of $\hat{\omega}_n^1$ is pinned down by the boundary condition, given $\hat{\omega}_n^K$ and \hat{h}_n^1 . In this case, we have to solve for $\hat{\omega}_n = [\hat{\omega}_n^2, \dots, \hat{\omega}_n^K]'$, a $K - 1$ -dimensional vector. The determination of $\hat{\omega}_1^{k,*}$ will be discussed below.

Given the boundary conditions we can assemble the system in matrix form. First, we can write the difference equations for the state variables in matrix form

$$\begin{bmatrix} \hat{K}_{n+1} \\ \hat{\omega}_{n+1} \end{bmatrix} = \begin{bmatrix} 1 & \mathbf{0}'_{K-1} & \frac{q}{\Phi_1} \Delta t & \mathbf{0}_{K-1} & \mathbf{0}'_{K-1} \\ a_\omega \mathbf{1}_{K-1} & A_{\omega\omega} & a_{\omega} \mathbf{1}_{K-1} & A_{\omega h} & A_{\omega \zeta} \end{bmatrix} \begin{bmatrix} \hat{K}_n \\ \hat{\omega}_n \\ \hat{q}_n \\ \hat{h}_n \\ \hat{\zeta}_n \end{bmatrix}, \quad (\text{D.32})$$

where $a_\omega = 2\gamma(\phi\sigma_{id})^2 \left(\frac{qK}{\chi_e \omega_e}\right)^2 \Delta t$, and $A_{\omega\omega}$, $A_{\omega h}$, and $A_{\omega \zeta}$ are $K - 1 \times K - 1$ matrices given

by

$$A_{\omega\omega} = \begin{bmatrix} 1 - a_\omega \frac{f^2 \omega^2}{\omega_e} \Delta a - \frac{\Delta t}{\Delta a} & -a_\omega \frac{f^3 \omega^3}{\omega_e} \Delta a & \cdots & -a_\omega \frac{f^{K-1} \omega^{K-1}}{\omega_e} \Delta a & -a_\omega \frac{f^K \omega^K}{\omega_e} \Delta a + \frac{f^K \omega^K}{f^1 \omega^1} \frac{\Delta t}{\Delta a} \\ -a_\omega \frac{f^2 \omega^2}{\omega_e} \Delta a + \frac{\Delta t}{\Delta a} & 1 - a_\omega \frac{f^3 \omega^3}{\omega_e} \Delta a - \frac{\Delta t}{\Delta a} & \cdots & -a_\omega \frac{f^{K-1} \omega^{K-1}}{\omega_e} \Delta a & -a_\omega \frac{f^K \omega^K}{\omega_e} \Delta a \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ -a_\omega \frac{f^2 \omega^2}{\omega_e} \Delta a & -a_\omega \frac{f^3 \omega^3}{\omega_e} \Delta a & \cdots & 1 - a_\omega \frac{f^{K-1} \omega^{K-1}}{\omega_e} \Delta a - \frac{\Delta t}{\Delta a} & -a_\omega \frac{f^K \omega^K}{\omega_e} \Delta a \\ -a_\omega \frac{f^2 \omega^2}{\omega_e} \Delta a & -a_\omega \frac{f^3 \omega^3}{\omega_e} \Delta a & \cdots & -a_\omega \frac{f^{K-1} \omega^{K-1}}{\omega_e} \Delta a + \frac{\Delta t}{\Delta a} & 1 - a_\omega \frac{f^K \omega^K}{\omega_e} \Delta a - \frac{\Delta t}{\Delta a} \end{bmatrix}, \quad (\text{D.33})$$

$$A_{\omega h} = \begin{bmatrix} -a_\omega \frac{f^1 h^1}{2\omega_e} \Delta a + \frac{h^1}{\omega^1} \frac{\Delta t}{\Delta a} & 0 & \cdots & 0 & 0 \\ -a_\omega \frac{f^1 h^1}{2\omega_e} \Delta a & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ -a_\omega \frac{f^1 h^1}{2\omega_e} \Delta a & 0 & \cdots & 0 & 0 \\ -a_\omega \frac{f^1 h^1}{2\omega_e} \Delta a & 0 & \cdots & 0 & 0 \end{bmatrix}, \quad A_{\omega\zeta} = \begin{bmatrix} 0 & -\zeta^2 \Delta t & 0 & \cdots & 0 & 0 \\ 0 & 0 & -\zeta^3 \Delta t & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & -\zeta^{K-1} \Delta t \\ 0 & 0 & 0 & \cdots & 0 & 0 \end{bmatrix}. \quad (\text{D.34})$$

The difference equations for the jump variables can be written as

$$\begin{bmatrix} \hat{q}_{n+1} \\ \hat{h}_{n+1} \\ \hat{\zeta}_{n+1} \end{bmatrix} = \begin{bmatrix} A_{qK} & A_{q\omega} \bar{\omega}' & A_{qq} & A_{qh} & \mathbf{0}'_{K-1} \\ A_{hK} & \mathbf{0}_{K-1,K-1} & \mathbf{0}_{K-1} & A_{hh} & \mathbf{0}_{K-1,K-1} \\ a_\zeta \mathbf{1}_{K-1} & -a_\zeta \mathbf{1}_{K-1} \bar{\omega}' & a_\zeta \mathbf{1}_{K-1} & A_{\zeta h} & A_{\zeta\zeta} \end{bmatrix} \begin{bmatrix} \hat{K}_n \\ \hat{\omega}_n \\ \hat{q}_n \\ \hat{h}_n \\ \hat{\zeta}_n \end{bmatrix}, \quad (\text{D.35})$$

where

$$A_{qK} = \left[\gamma \phi^2 \sigma_{id}^2 \frac{qK}{\chi_e \omega_e} + \alpha(1-\alpha) \frac{K^{\alpha-1}}{q} \right] \Delta t, \quad A_{q\omega} = -\gamma \phi^2 \sigma_{id}^2 \frac{qK}{\omega_e} \Delta t, \quad \bar{\omega} = \left[\frac{f^2 \omega^2}{\omega_e}, \frac{f^3 \omega^3}{\omega_e}, \dots, \frac{f^K \omega^K}{\omega_e} \right]' \Delta a, \quad (\text{D.36})$$

$$A_{qq} = 1 + \left[\gamma \phi^2 \sigma_{id}^2 \frac{qK}{\chi_e \omega_e} + \frac{\alpha K^{\alpha-1} - \iota(q)}{q} \right] \Delta t, \quad A_{qh} = -\gamma \phi^2 \sigma_{id}^2 \frac{qK}{\chi_e \omega_e} \Delta t \frac{f^1 h^1}{2\omega_e} \Delta a \mathbf{e}'_{1,K-1}, \quad (\text{D.37})$$

$$A_{hK} = -(1-\alpha) \alpha K^\alpha \left[\frac{\bar{l}(a^1)}{h(a^1)}, \dots, \frac{\bar{l}(a^{K-1})}{h(a^{K-1})} \right]' \Delta t, \quad a_\zeta = -(\gamma-1)(\phi \sigma_{id})^2 \left(\frac{qK}{\chi_e \omega_e} \right)^2 \Delta t, \quad (\text{D.38})$$

$$A_{\zeta h} = -a_\zeta \frac{f^1 h^1}{2\omega_e} \Delta a \mathbf{1}_{K-1} \mathbf{e}'_{1,K-1} \quad (\text{D.39})$$

$$A_{hh} = \begin{bmatrix} 1 + w \frac{\bar{l}(a^1)}{h(a^1)} \Delta t + \frac{\Delta t}{\Delta a} & -\frac{\Delta t}{\Delta a} & 0 & \dots & 0 & 0 \\ 0 & 1 + w \frac{\bar{l}(a^2)}{h(a^2)} \Delta t + \frac{\Delta t}{\Delta a} & -\frac{\Delta t}{\Delta a} & \dots & 0 & 0 \\ \vdots & \ddots & \ddots & \dots & \ddots & \ddots \\ 0 & 0 & 0 & \dots & 1 + w \frac{\bar{l}(a^{K-2})}{h(a^{K-2})} \Delta t + \frac{\Delta t}{\Delta a} & -\frac{\Delta t}{\Delta a} \\ 0 & 0 & 0 & \dots & 0 & 1 + w \frac{\bar{l}(a^{K-1})}{h(a^{K-1})} \Delta t + \frac{\Delta t}{\Delta a} \end{bmatrix} \quad (D.40)$$

$$A_{\zeta\zeta} = \begin{bmatrix} 1 + \hat{\zeta}^1 \Delta t + \frac{\Delta t}{\Delta a} & -\frac{\Delta t}{\Delta a} & 0 & \dots & 0 & 0 \\ 0 & 1 + \hat{\zeta}^2 \Delta t + \frac{\Delta t}{\Delta a} & -\frac{\Delta t}{\Delta a} & \dots & 0 & 0 \\ \vdots & \ddots & \ddots & \dots & \ddots & \ddots \\ 0 & 0 & 0 & \dots & 1 + \hat{\zeta}^{K-2} \Delta t + \frac{\Delta t}{\Delta a} & -\frac{\Delta t}{\Delta a} \\ 0 & 0 & 0 & \dots & 0 & 1 + \hat{\zeta}^{K-1} \Delta t + \frac{\Delta t}{\Delta a} \end{bmatrix}. \quad (D.41)$$

Let $x_n = [\hat{K}_n, \hat{\omega}'_n]'$ denote the vector of predetermined variables, $y_n = [\hat{q}_n, \hat{\zeta}'_n, \hat{h}'_n]'$ the vector of jump variables, and $z_n = [x'_n, y'_n]'$ a vector containing the state and jump variables. We can then write the system in matrix form

$$z_{n+1} = Az_n, \quad (D.42)$$

for a matrix of coefficients A .

Provided the **Blanchard and Kahn (1980)** conditions are satisfied, there exists a unique pair of matrices (P, H) such that

$$x_{n+1} = Px_n; \quad y_n = Hx_n. \quad (D.43)$$

The initial value of entrepreneurs' total wealth by age is given by

$$\omega_0(a) = \omega^*(a) + \underbrace{(q_0 - q^*)k^*(a) + h_0(a) - h^*(a)}_{\text{revaluation effect}}, \quad (D.44)$$

where $\omega^*(a)$ and $h^*(a)$ denote total wealth and human wealth at the initial steady state, respectively, and $k^*(a)$ denotes the amount of capital held by entrepreneurs of age a in the old steady state, which is given by $k^*(a) = \frac{\omega^*(a)}{\omega_e} K^*$.

Log-linearizing the expression above around the new steady state, we obtain

$$\hat{\omega}_0(a) = \hat{\omega}^*(a) + \frac{qK}{\omega_e} (\hat{q}_0 - \hat{q}^*) + \frac{h(a)}{\omega(a)} (\hat{h}_0(a) - \hat{h}^*(a)). \quad (D.45)$$

We can write the initial condition for x_1 as follows

$$x_1 = x^* + G(y_1 - y^*) \Rightarrow x_1 = [I - GH]^{-1}(x^* - Gy^*), \quad (D.46)$$

where $x^* = [\hat{k}^*, (\hat{\omega}^*)']'$, $y^* = [\hat{q}^*, (\hat{h}^*)', (\hat{\zeta}^*)']'$, and G is a $K \times 2K - 1$ matrix given by

$$G = \begin{bmatrix} 0 & \mathbf{0}'_{K-1} & \mathbf{0}'_{K-1} \\ \frac{qK}{\omega_e} \mathbf{1}_{K-1} & G_{\omega h} & \mathbf{0}_{K-1, K-1} \end{bmatrix}, \quad (\text{D.47})$$

where $G_{\omega h}$ has entries $(k, k+1)$ equal to $\frac{h(a_{k+1})}{\omega(a_{k+1})}$ for $k = 1, \dots, K-1$, and zero otherwise.

E Extensions

E.1 Heterogeneous productivity and limited pledgeability

In this subsection, we extend the basic environment in three dimensions: i) heterogeneous productivity, ii) heterogeneous idiosyncratic volatility, and iii) limited pledgeability of physical assets. We focus on the case of a stationary equilibrium.

E.1.1 Heterogeneous productivity and volatility

Entrepreneurs draw at birth an idiosyncratic productivity B_i which is fixed for the entrepreneur's lifetime, it has mean one in the cross-section $\mathbb{E}_i[B_i] = 1$, and the draw is independent of the previous generation productivity. We also allow entrepreneurs to have different levels of idiosyncratic volatility $\sigma_{i,id}$. The production technology for entrepreneur i is given by

$$\tilde{y}_{i,t} = A_t B_i k_{i,t}^\alpha l_{i,t}^{1-\alpha}. \quad (\text{E.1})$$

As before, labor is chosen to maximize returns, so the labor demand is given by

$$w = (1 - \alpha) B_i \left(\frac{k_{i,t}}{l_{i,t}} \right)^\alpha. \quad (\text{E.2})$$

Expected return for entrepreneur i is given by

$$\mu_i = \frac{\alpha B_i^{\frac{1}{\alpha}} \left(\frac{1-\alpha}{w} \right)^{\frac{(1-\alpha)}{\alpha}} - \iota(q)}{q} + \mu_A + \Phi(\iota(q)) - \delta. \quad (\text{E.3})$$

In contrast to the baseline model, expected returns on the business varies across entrepreneurs, as it depends on the idiosyncratic productivity level B_i .

E.1.2 Limited pledgeability and business exposure

Let $b_{i,t} \equiv n_{i,t} - q_t k_t$ denote the amount of safe assets held by entrepreneur i . The natural borrowing limit can be written as

$$-b_{i,t} \leq h_{i,t} + q k_{i,t}. \quad (\text{E.4})$$

This allows the entrepreneur to borrow freely against physical assets or human wealth. Let's now assume that there is limited pledgeability of physical assets, that is, entrepreneurs can only borrow a fraction of $1 - \lambda^{-1}$ of the value of physical assets, a form of collateral constraint:

$$-b_{i,t} \geq h_{i,t} + (1 - \lambda^{-1})q k_{i,t} \Rightarrow q k_{i,t} \leq \lambda \omega_{i,t}, \quad (\text{E.5})$$

where $\lambda \geq 1$. Hence, the entrepreneur faces a portfolio problem subject to leverage constraints. Given that the leverage constraint is linear in total wealth, we are able to obtain a closed-form solution for the portfolio problem with leverage constraints.³⁸ The HJB for an entrepreneur can be written as

$$\begin{aligned} \frac{\rho}{1-\gamma} = \max_{c_{i,t}, k_{i,t}, l_{i,t}, \theta_{i,t}^{ag}, \theta_{i,t}^{id}} & \left\{ \frac{\zeta_t^\gamma(a)}{1-\gamma} \left(\frac{c_{i,t}}{\omega_{i,t}} \right)^{1-\gamma} - \frac{\gamma}{1-\gamma} \frac{1}{\zeta_t(a)} \left(\frac{\partial \zeta_t(a)}{\partial t} + \frac{\partial \zeta_t(a)}{\partial a} \right) + r_t + \frac{q_t k_{i,t}}{\omega_{i,t}} (\mu_{i,t}^R - r_t) \right. \\ & \left. - \frac{p_t^{ag} \theta_{i,t}^{ag}}{\omega_{i,t}} + \frac{h_{i,t}}{\omega_{i,t}} \sigma_A p_t^{ag} - \frac{c_{i,t}}{\omega_{i,t}} - \frac{\gamma}{2} \left[\left(\frac{q_t k_{i,t} + h_{i,t}}{\omega_{i,t}} \sigma_A - \frac{\theta_{i,t}^{ag}}{\omega_{i,t}} \right)^2 + \left(\frac{q_t k_{i,t}}{\omega_{i,t}} \sigma_{i,id} - \frac{\theta_{i,t}^{id}}{\omega_{i,t}} \right)^2 \right] \right\}, \end{aligned}$$

subject to leverage and insurance constraints: $q_t k_{i,t} \leq \lambda \omega_{i,t}$ and $\theta_{i,t}^{id} \leq (1 - \phi) q_t k_{i,t} \sigma_{i,id}$.

The optimal capital demand is given by

$$\frac{q k_{i,t}}{n_{i,t}} = \frac{p_i^{id}}{\gamma \phi \sigma_{i,id}} \left(1 + \frac{h_{i,t}}{n_{i,t}} \right), \quad (\text{E.6})$$

where the price of idiosyncratic risk p_i^{id} is given by

$$p_i^{id} = \min \left\{ \frac{\mu_i^R - r - p^{ag} \sigma_A}{\phi \sigma_{i,id}}, \gamma \phi \sigma_{i,id} \lambda \right\}.$$

The price of idiosyncratic risk is now heterogeneous across entrepreneurs. In particular, the price of risk, and ultimately the demand for capital, is larger for entrepreneurs with higher expected return or smaller idiosyncratic risk. For entrepreneurs with more profitable or less risky projects, the leverage constraint may be binding. In this case, the price of idiosyncratic

³⁸A similar result on the optimal portfolio share with leverage constraints can be found for investors without labor income in, for instance, [Grossman and Vila \(1992\)](#) and [Detemple and Murthy \(1997\)](#).

risk is given $p^{id} = \gamma\phi\sigma_{i,id}\lambda$, which implies that $q_t k_{i,t} = \lambda\omega_{i,t}$ when p^{id} is inserted in Equation (E.6). Note that, in the absence of (ex-ante) heterogeneity, we would have that either all entrepreneurs would be constrained or all entrepreneurs would be unconstrained at all periods, regardless of their financial wealth.

To aggregate Equation (E.6) across types, it is convenient to first take logs, average for a given age group, and then convert the expression back to levels. Define $k(a) = \exp(\mathbb{E}_i[\log k_{i,t}|a_i = a])$ and $\omega(a) = \exp(\mathbb{E}_i[\log(n_{i,t} + h_{i,t})|a_i = a])$ as the relevant cross-sectional average of capital and total wealth conditional on age. The aggregate exposure to the business for entrepreneurs of age a is then given by

$$\frac{qk(a)}{n(a)} = \frac{1 + \frac{h(a)}{n(a)}}{\gamma} \frac{p^{id}}{\phi\sigma_{id}}, \quad (\text{E.7})$$

where $n(a) = \omega(a) - h(a)$ and

$$\sigma_{id} = \exp(\mathbb{E}_i[\log \sigma_{i,id}]), \quad p^{id} = \exp(\mathbb{E}_i[\log p_i^{id}]). \quad (\text{E.8})$$

Therefore, we obtain the same expression for the business exposure after aggregation as in the baseline model, showing that our results extend to the case with limited pledgeability and heterogeneity in productivity and risk. An analogous derivation shows that our results for the consumption-wealth ratio extend to this case as well.

E.2 Uninsurable labor income risk and borrowing constraints

In this subsection, we introduce uninsurable labor income risk into the entrepreneur's problem. This will enable us to study the implications of both insurance and borrowing constraints on entrepreneurial behavior. In particular, our focus is on how the inability to borrow against future income affects the entrepreneur's risk-taking decision.

E.2.1 The entrepreneurs' problem with labor income risk

Entrepreneurs receive labor income $\bar{l}_{i,t}$. With Poisson intensity λ_d , entrepreneurs suffer a "disability" shock that reduces their labor income by a factor $1 - \xi_d$, that is, labor income is given by $\bar{l}_{i,t} = \bar{l}(a_i)$ in the no-disability state and $\bar{l}_{i,t} = (1 - \xi_d)\bar{l}(a_i)$ in the disability state. The disability shock happens only once in the entrepreneurs' lifetime and it is permanent. When either $\lambda_d = 0$ or $\xi_d = 0$, we recover the model with no labor income risk discussed in Section 3.

We assume that households are subject to a natural borrowing limit, given by $\tilde{n}_{i,t} \geq -(1 - \xi_d)\tilde{h}_{i,t}$.³⁹ As discussed in Aiyagari (1994), the natural borrowing limit under incom-

³⁹Kaplan and Violante (2010) argue that the standard incomplete markets model with a natural borrowing

plete markets corresponds to the worst-case scenario of the realization of idiosyncratic shocks, which in our setting corresponds to the disability shock happening immediately. Therefore, under the natural borrowing limit, entrepreneurs can borrow at most a fraction $1 - \xi_d$ of the human wealth $\tilde{h}_{i,t}$. Note that, even for an arbitrarily small value of λ_d , the borrowing limit is tighter in the presence of uninsurable income risk. In this case, the parameter ξ_d effectively controls the pledgeability of human wealth.

The entrepreneur's problem in the no-disability state is given by

$$\tilde{V}_t(\tilde{n}_i, a_i) = \max_{\tilde{c}_i, \tilde{\theta}_i^{as}, \tilde{\theta}_i^{id}, k_i, l_i} \mathbb{E}_t \left[\int_t^{t+T_d} e^{-\rho(z-t)} \frac{\tilde{c}_i^{1-\gamma}}{1-\gamma} dz + e^{-\rho T_d} \tilde{V}_{t+T_d}^d(\tilde{n}_{i,t+T_d}, a_i + T_d) \right], \quad (\text{E.9})$$

subject to non-negativity constraints $c_{i,t}, k_{i,t} \geq 0$, the law of motion of $\tilde{n}_{i,t}$

$$d\tilde{n}_{i,t} = \left[\tilde{n}_{i,t}r + \tilde{q}_t k_{i,t}(\mu_{i,t}^R - r) - p^{as} \tilde{\theta}_{i,t}^{as} + \tilde{w}_t \bar{l}_{i,t} - \tilde{c}_{i,t} \right] dt + \left(\tilde{q}_t k_{i,t} \sigma_A - \tilde{\theta}_{i,t}^{as} \right) dZ_t + \left(\tilde{q}_t k_{i,t} \sigma_{id} - \tilde{\theta}_{i,t}^{id} \right) dZ_{i,t},$$

and insurance and borrowing constraints

$$\tilde{\theta}_{i,t}^{id} \leq (1 - \phi) \tilde{q}_t k_{i,t} \sigma_{id}, \quad \tilde{n}_{i,t} \geq -(1 - \xi_d) \tilde{h}_{i,t}, \quad (\text{E.10})$$

where $\tilde{h}_{i,t} \equiv \mathbb{E}_t \left[\int_t^{t+T-a_i} \frac{\pi_z}{\pi_t} \tilde{w}_z \bar{l}_i dz \right]$, T_d is the minimum of the (random) arrival time of a Poisson process with intensity $\lambda_d > 0$ and the entrepreneur's life horizon $T - a_i$. If $T_d < T - a_i$, then $\tilde{V}^d(\tilde{n}_i, a_i)$ corresponds to the value function in the disability state. As there is no labor income risk in the disability state, this is equal to the value function derived in Section 3. The value function evaluated at $a_i = T$, $\tilde{V}_t^d(\tilde{n}_i, T)$, is given by the bequest function.

E.2.2 The stationary problem

It is convenient to adopt a change of variables and to detrend the problem. First, we follow [Aiyagari \(1994\)](#) and define total (pledgeable) wealth $\tilde{\omega}_{i,t} = \tilde{n}_{i,t} + (1 - \xi_d) \tilde{h}_{i,t}$. This corresponds to the total amount of funds available to an entrepreneur, that is, the sum of the financial wealth and the borrowing limit. In the special case where $\xi_d = 0$, we recover the definition of total wealth given in Section 3. With a slight abuse of notation, we denote the entrepreneur's value function as $\tilde{V}_t(\tilde{\omega}_{i,t}, a_{i,t}) = \tilde{V}(\tilde{\omega}_{i,t}, a_{i,t}; A_t)$. Therefore, the entrepreneur's problem depends on the level of aggregate productivity. The next lemma shows that, despite the presence of uninsurable labor income risk and aggregate shocks, it is possible to detrend the problem and to define a stationary equilibrium in a way analogous to Lemma 1.

Lemma 3. *Suppose the economy is in a stationary equilibrium. Then,*

limit better captures the degree of partial insurance observed in the data than versions of the model with tighter borrowing limits.

- i. Scaled variables are independent of aggregate productivity, that is, the scaled value function $V(\omega_t, a_t) = \frac{\tilde{V}_t(\omega_t A_t, a_t)}{A_t^{1-\gamma}}$ and the scaled policy functions $c_{i,t} = \frac{\tilde{c}_{i,t}}{A_t}$, $\theta_{i,t}^{id} = \frac{\tilde{\theta}_{i,t}^{id}}{A_t}$, and $\theta_{i,t}^{ag} = \frac{\tilde{\theta}_{i,t}^{ag}}{A_t}$ do not depend on A_t .
- ii. The optimal value of $l_{i,t}$ and $\iota_{i,t}$ are given by (11) and (12), respectively. The insurance constraint is binding and the shadow price of idiosyncratic insurance is given by (20). The price of aggregate insurance is given by (29) and the demand for aggregate insurance is given by $\theta_{i,t}^{ag} = (qk_{i,t} - n_{i,t}) \sigma_A$.
- iii. The scaled value function satisfies the HJB equation

$$\hat{\rho}V = \max_{c_i, \sigma_{i,t}^{id}} \frac{c_t^{1-\gamma}}{1-\gamma} + V_a + V_\omega \left[\hat{r}\omega_{i,t} + p^{id}\sigma_{i,t}^{id} + \xi_d w_t \bar{l}_{i,t} - c_{i,t} \right] + \frac{1}{2} V_{\omega\omega} (\sigma_{i,t}^{id})^2 + \lambda_d (V^d - V), \quad (\text{E.11})$$

where $\sigma_{i,t}^{id} = qk_{i,t}\phi\sigma_{id}$, $\hat{\rho} \equiv \rho - (1-\gamma) \left(\mu_A - \frac{\gamma\sigma_A^2}{2} \right)$, and $\hat{r} \equiv r + \gamma\sigma_A^2 - \mu_A$.

Proof. See Appendix A.6. □

In the first part of Lemma 3, we show that scaled variables are independent of aggregate productivity. This implies that a stationary distribution of scaled wealth exists and there is no need to approximate the aggregate wealth distribution by a finite number of moments as in Krusell and Smith (1998), despite the presence of aggregate risk and incomplete markets. Two assumptions are important to obtain this result: i) a borrowing limit that is proportional to aggregate income; ii) the probability of switching to the disability state is independent of aggregate shocks. Note that the natural borrowing limit in our setting satisfies the first condition. This result echoes the one obtained by Krueger and Lustig (2010), who derive conditions under which uninsurable labor income risk has no effect for the price of aggregate risk. Consistent with their findings, we show that the price of aggregate insurance is $p^{ag} = \gamma\sigma_A$, the same we would obtain in a complete-markets economy.

The second part of the lemma shows that the solution to entrepreneur's problem share many of the features we derived in the absence of idiosyncratic labor income risk. Labor demand and investment rates are chosen to maximize expected returns, insurance constraints are always binding, and entrepreneurs with low financial wealth have a positive demand for aggregate insurance.

The third part of Lemma 3 shows how the entrepreneur's problem ultimately reduce to a choice of (scaled) consumption and the exposure to idiosyncratic risk, or equivalently the amount of capital to be employed in the business. In contrast to the problem in Section 3, a closed-form solution to this problem is not available, as the consumption function is not a linear function of total wealth anymore.

E.2.3 The approximate solution

We consider next an approximate solution to problem (E.11). Despite the well-known lack of closed-form solutions, we are able to provide an analytical characterization using perturbation techniques. In particular, we extend the methods used in Viceira (2001) to a general equilibrium life-cycle model. This approximate solution will allow us to show analytically how borrowing constraints affect entrepreneurs' risk-taking decisions, which is challenging to obtain from problem (E.11) otherwise.

Let $\omega(a)$ denote the average total wealth of entrepreneurs conditional on age a . We consider a log-linear approximation of the consumption and capital functions around the point $(\bar{\omega}, \bar{a})$:

$$\log c(\omega_{i,t}, a_{i,t}) = \log c(\bar{\omega}, \bar{a}) + \psi_{c,\omega} \hat{\omega}_i + \psi_{c,a} \hat{a}_i + \mathcal{O}(\hat{\omega}_i^2, \hat{a}_i^2) \quad (\text{E.12})$$

$$\log k(\omega_{i,t}, a_{i,t}) = \log k(\bar{\omega}, \bar{a}) + \psi_{k,\omega} \hat{\omega}_i + \psi_{k,a} \hat{a}_i + \mathcal{O}(\hat{\omega}_i^2, \hat{a}_i^2) \quad (\text{E.13})$$

where $\hat{\omega}_i \equiv \log \omega_i - \log \bar{\omega}$ and $\bar{\omega} \equiv \exp \mathbb{E}[\log \omega_{i,t} | a_{i,t} = \bar{a}]$, given $0 < \bar{a} < T$.

We can write the expressions above in more compact form:

$$\hat{c}_{i,t} = \psi_{c,\omega} \hat{\omega}_{i,t} + \psi_{c,a} \hat{a}_{i,t}, \quad \hat{k}_{i,t} = \psi_{k,\omega} \hat{\omega}_{i,t} + \psi_{k,a} \hat{a}_{i,t}, \quad (\text{E.14})$$

up to first order in $(\hat{\omega}_{i,t}, \hat{a}_{i,t})$, where $\hat{z}_{i,t} \equiv \log z(\omega_{i,t}, a_{i,t}) - \log z(\bar{\omega}, \bar{a})$ for $z \in \{c, k\}$.

Note that this log-linear approximation is only feasible when total wealth is always positive, that is, $\omega_{i,t} > 0$ for all i and t . This condition holds in the solution to the entrepreneurs' problem given the combination of the natural borrowing limit with an unbounded marginal utility as $c_{i,t}$ approaches zero (see e.g. the discussion in Chamberlain and Wilson 2000). If an entrepreneur were to borrow the maximum amount, such that $\omega_{i,t} = 0$, then consumption would be zero in the advent of a disability shock, a state with infinite marginal utility. An entrepreneur is better off by reducing borrowing and avoiding this possibility.

The next proposition provides a characterization of the first-order approximation of the entrepreneurs' problem.

Proposition 5. *Suppose the economy is in a stationary equilibrium and consider a first-order approximation of the policy functions and wealth dynamics around $(\bar{\omega}, \bar{a})$. Then,*

- i. *Consumption is an age-dependent concave function of total wealth given by*

$$c_{i,t} = f_c(a_{i,t}) \omega_{i,t}^{\psi_{c,\omega}}, \quad (\text{E.15})$$

where $0 < \psi_{c,\omega} < 1$ and $f'_c(a_{i,t}) > 0$ for ξ_d sufficiently small.

ii. The demand for capital is given by

$$\frac{qk_{i,t}}{n_{i,t}} = \frac{1 + (1 - \xi_d) \frac{h_{i,t}}{n_{i,t}}}{\gamma \psi_{c,\omega}} \frac{p^{id}}{\phi \sigma_{id}}. \quad (\text{E.16})$$

iii. Log total wealth evolves according to

$$d \log \omega_{i,t} = [\tilde{\psi}_{\omega,0} + \psi_{\omega,a} a_{i,t} + \psi_{\omega,\omega} \log \omega_{i,t}] dt + \frac{p^{id}}{\gamma \psi_{c,\omega}} dZ_{i,t}, \quad (\text{E.17})$$

where $\psi_{\omega,\omega} < 0$ and $\psi_{\omega,a} < 0$ for ξ_d sufficiently small.

Proof. See Appendix A.7. □

The first part of Proposition 5 describes the consumption function. Consumption is a strictly concave function of total wealth, as it is typical of problems with uninsurable labor income risk, where $\psi_{c,\omega} \in (0, 1)$ represents the elasticity of consumption with respect to total wealth. This is in contrast to the consumption function derived in Proposition 1, where consumption is a linear function of $\omega_{i,t}$, that is, $\psi_{c,\omega} = 1$. For ξ_d sufficiently small, consumption is an increasing function of age given $\omega_{i,t}$, reflecting the impact of entrepreneurs' finite horizon.⁴⁰

The second part of Proposition 5 gives the demand for capital and it corresponds to the main result of this section. The fact that entrepreneurs cannot borrow against future labor income has two (opposite) effects on the demand for capital. First, entrepreneurs have less resources available to them when $\xi_d > 0$, which tends to reduce the scale of the business. This effect is particularly more pronounced for entrepreneurs who are close to the borrowing limit, that is, $-n_{i,t}$ is close to $(1 - \xi_d)h_{i,t}$. Second, the limited pledgeability of human wealth *reduces* the effective risk aversion of entrepreneurs, as $\gamma \psi_{c,\omega} < \gamma$. The fraction of human wealth that cannot be used to fund an investment in the business acts as a buffer against future shocks, which makes the entrepreneur less concerned with taking investment risk.

It can be shown that uninsurable labor income risk reduces the scale of the business relative to an economy with $\xi_d = 0$ if and only if

$$n_{i,t} < \left[\frac{\xi_d}{1 - \psi_{c,\omega}} - 1 \right] h_{i,t}. \quad (\text{E.18})$$

Tighter borrowing constraints, captured by $\xi_d > 0$, reduces investment in the business for poor entrepreneurs, but it increases investment in the business for rich entrepreneurs. Given

⁴⁰The condition on ξ_d is necessary, as with uninsurable labor income risk the slope of the labor income profile also plays a role. The effect of a finite horizon on consumption is attenuated when labor income declines with age, and this effect is amplified when labor income increases with age.

that these two forces move in opposite directions, the aggregate effect of borrowing constraints tend to be muted. Importantly, for all values of ξ_d , a declining human-financial wealth ratio over the life-cycle causes entrepreneurs' exposure to the business to decline with age, as in the baseline model.

Entrepreneurs have an age-dependent target for wealth: $\frac{\tilde{\psi}_{\omega,0} + \psi_{\omega,a}a}{|\psi_{\omega,\omega}|}$. Entrepreneurs build up wealth when $\omega_{i,t}$ is below target and they decumulate wealth when $\omega_{i,t}$ is above target. When $\psi_{\omega,a} < 0$, the target on total wealth drifts down with age, again an implication of the entrepreneurs' finite horizon.

Note that the ratio of consumption to financial wealth can be written as

$$\frac{c_{i,t}}{n_{i,t}} = f_c(a_{i,t})\omega_{i,t}^{-(1-\psi_{c,\omega})} \left(1 + \frac{h_{i,t}}{n_{i,t}}\right). \quad (\text{E.19})$$

As typically $f_c(\cdot)$ is increasing with age and $\omega_{i,t}$ is decreasing with age on average, we obtain that the first two terms in the expression above increase with age. Figure 3 shows that the human-financial wealth ratio is declining with age. In line with our discussion in Section 3, the consumption-financial wealth ratio then depends on two forces that move in opposite directions with age.

Therefore, we conclude that introducing uninsurable labor income risk and borrowing constraints does not change substantially our results, while it adds a significant layer of complexity to the analysis.

E.3 Endogenous occupational choice

In this subsection, we introduce an occupational choice into the households' problem. Moreover, we assume that wage earners have finite horizon and imperfect altruism in the same way as entrepreneurs. For simplicity, we abstract from limited pledgeability and ex-ante heterogeneity on entrepreneurs, and once again focus on a stationary equilibrium.

E.3.1 The occupational choice

At the beginning of life, a household can choose to become an entrepreneur or a wage earner. To become an entrepreneur, household i must pay a fixed cost $\varphi_i \tilde{y}_{i,t}$, where φ_i is a cost parameter draw from a distribution $F_\varphi(\cdot)$ with support $[\underline{\varphi}, \overline{\varphi}]$. Let $\tilde{V}_t(\tilde{n}_i, a)$ denote the value function of a household who chose to become an entrepreneur and $\tilde{V}_t^w(\tilde{n}_i, a)$ the value function of a household who chose to become a wage earner. In contrast to the model from Section 3, a wage earner lives for T periods and derive the same utility of bequests as entrepreneurs.

A household who inherits financial wealth \tilde{n}_i will choose to become an entrepreneur if

$$\tilde{V}_t(\tilde{n}_i - \varphi_i \tilde{y}_t, 0) > \tilde{V}_t^w(\tilde{n}_i, 0). \quad (\text{E.20})$$

The value function of an entrepreneur can be written, after normalization, as $V(n, a) = \zeta(a)^{-\frac{1}{\gamma}} \frac{(n+h(a))^{1-\gamma}}{1-\gamma}$. Similarly, the value function of a wage earner can be written as $V^w(n, a) = \zeta_w(a)^{-\frac{1}{\gamma}} \frac{(n+h_w(a))^{1-\gamma}}{1-\gamma}$.

The condition for becoming an entrepreneur can then be written as

$$\zeta(0)^{\frac{1}{\gamma(\gamma-1)}} (n_i + h(0) - \varphi_i y) > \zeta_w(0)^{\frac{1}{\gamma(\gamma-1)}} (n_i + h_w(0)). \quad (\text{E.21})$$

Rearranging the expression above, we obtain that a household becomes an entrepreneur if $\varphi_i < \varphi^*(n_i)$, where the threshold $\varphi^*(n_i)$ is given by

$$\varphi^*(n_i) \equiv \frac{1}{y} \left[\left(\left(\frac{\zeta(0)}{\zeta_w(0)} \right)^{\frac{1}{\gamma(\gamma-1)}} - 1 \right) n_i + \left(\frac{\zeta(0)}{\zeta_w(0)} \right)^{\frac{1}{\gamma(\gamma-1)}} h(0) - h_w(0) \right]. \quad (\text{E.22})$$

It can be shown that $\zeta(0) > \zeta_w(0)$ for $\gamma > 1$, so households who received larger bequests are more likely to become entrepreneurs. The difference between $\zeta(0)$ and $\zeta_w(0)$ is increasing in p^{id} , the shadow price of idiosyncratic risk.

As the cost parameter is draw independently of the bequest a household receives, then the mass of entrepreneurs in a stationary equilibrium is given by

$$\chi_e = F_\varphi(\varphi^*(n(0))), \quad (\text{E.23})$$

where $n(0)$ is the average financial wealth of newborn entrepreneurs.

In a stationary equilibrium, the mass of entrepreneurs is constant. As $\zeta(0)$, $\zeta_w(0)$, and $n(0)$ depend on the interest rate and the aggregate and idiosyncratic risk premia, then the share of entrepreneurs in the economy depend on equilibrium expected returns.

E.3.2 Wage earners' problem and equilibrium determination

The optimal consumption-wealth ratio and demand for insurance for wage earners are now given by

$$\frac{c_{j,t}}{\omega_{j,t}} = \zeta_w(a) = \frac{\bar{r}_w}{1 - \psi e^{-\bar{r}_w(T-a_j)}}, \quad \theta_{j,t}^{ag} = h_{j,t} \sigma_A - \frac{p^{ag}}{\gamma} \omega_{j,t}, \quad (\text{E.24})$$

where

$$\bar{r}_w = \frac{1}{\gamma} \rho_w + \left(1 - \frac{1}{\gamma} \right) \left(r + \frac{(p^{ag})^2}{2\gamma} \right). \quad (\text{E.25})$$

The price of aggregate insurance, wages, and the relative price of capital are the same as in the baseline model:

$$p^{ag} = \gamma\sigma_A, \quad w = (1 - \alpha)K^\alpha, \quad q = \Phi_0 + \Phi_1(g + \delta). \quad (\text{E.26})$$

Finite lives for wage earners changes the determination of the interest rate. The interest rate is now jointly determined with the capital-labor ratio and the price of idiosyncratic risk by conditions (32), (33), and the market clearing condition for consumption

$$\int_0^T \frac{\bar{r}\omega(a)}{1 - \psi e^{-\bar{r}(T-a)}} f(a) da + \int_0^T \frac{\bar{r}_w \omega_w(a)}{1 - \psi e^{-\bar{r}_w(T-a)}} f(a) da = \alpha K^\alpha - \iota K, \quad (\text{E.27})$$

where

$$\omega(a) = \omega(0) e^{\left(r + \gamma\sigma_A^2 + \frac{(p^{id})^2}{\gamma} - \mu_A - \bar{r}\right)a} \frac{1 - \psi e^{-\bar{r}(T-a)}}{1 - \psi e^{-\bar{r}T}}, \quad \omega_w(a) = \omega_w(0) e^{\left(r + \gamma\sigma_A^2 - \mu_A - \bar{r}_w\right)a} \frac{1 - \psi e^{-\bar{r}_w(T-a)}}{1 - \psi e^{-\bar{r}_w T}}.$$

Assuming finite lives for wage earners would change the calibration of ρ_w but otherwise would not affect our main results.

E.4 Financial autarky

We consider next the case of financial autarky, where entrepreneurs have no access to either aggregate or idiosyncratic insurance. To shut down idiosyncratic insurance, we just have to set $\phi = 1$. To capture the absence of aggregate insurance, we will focus on the special case where there is no demand for aggregate insurance, so the solution would coincide to the case where entrepreneurs have no access to insurance.

Suppose that $h_{i,t} = 0$, so entrepreneurs have no labor income, and assume that $n_{j,t} = 0$ for $j \in \mathcal{W}_t$, so wage earners have no financial wealth. The first assumption implies that $\frac{qk_{i,t}}{n_{i,t}}$ is equalized across entrepreneurs and the second assumption implies that $qk_{i,t} = n_{i,t}$. As the demand for aggregate insurance is given by $\theta_{i,t}^{ag} = (qk_{i,t} - n_{i,t})\sigma_A$, we obtain that entrepreneurs do not demand aggregate insurance. The solution will then coincide with the case where aggregate insurance is not available.

Under these assumptions, the price of idiosyncratic risk, given in Equation (33), specializes to the following expression:

$$p^{id} = \gamma\sigma_{id}. \quad (\text{E.28})$$

The risk premium is then given by

$$p^{ag}\sigma_A + p^{id}\sigma_{id} = \gamma \left[\sigma_A^2 + \sigma_{id}^2 \right], \quad (\text{E.29})$$

and the aggregate Sharpe ratio relative to the idiosyncratic Sharpe ratio is given by $p^{ag}/p^{id} = \sigma_A/\sigma_{id}$.