# Online Appendix for "Doubly Robust Local Projections and Some Unpleasant VARithmetic"

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## Contents

| $p$ | $M$ | $\frac{M^2}{1+M^2}$ |
|---|---|---|
| 1 | 4.317 | 0.949 |
| 2 | 2.008 | 0.801 |
| 4 | 1.418 | 0.668 |
| 8 | 0.981 | 0.490 |
| 12 | 0.529 | 0.218 |
| 20 | 0.126 | 0.016 |
| 40 | 0.007 | 0.000 |

Table D.1: $M$ and $\frac{M^2}{1+M^2}$ as a function of $p$ in the structural model of Smets and Wouters (2007), with the researcher observing the monetary policy shock and output, and estimating a VAR($p$).

# Appendix D Further simulation results

We report results for two further sets of simulations from the structural model Smets and Wouters (2007): an observed monetary shock in Supplemental Appendix D.1, and recursive monetary shock identification in Supplemental Appendix D.2. This choice of shock of interest and shock identification schemes mimics much applied practice in macroeconometrics (e.g., see the review in Ramey, 2016). Finally, in Supplemental Appendix D.3, we present simulations with a larger sample size, which show that the headline simulation findings in Section 5.3 are consistent with our asymptotic results.

## D.1 Observed monetary shock

We again consider the model of Smets and Wouters. The econometrician now observes the monetary policy shock and total output, and the impulse response function of interest is that of output with respect to the monetary shock.

Results. We begin by quantifying the amount of misspecification, with Table D.1 showing the total degree of misspecification $M$ as well as the minimax MSE-optimal weight $M^2/(1+M^2)$ on LP in Corollary 4.2 as a function of the VAR lag length $p$. The value of $M$ is calculated for $T = 240$ and $\zeta = 1/2$ as in Section 5.3. As expected and as in our main exercise we see that larger $p$ give smaller $M$. Compared to our analysis in Section 5.3, $M$ now declines somewhat faster with the lag length $p$. However, for lag lengths typical in applied practice (for quarterly data), misspecification is still material, with $M \approx 1.42$ for a standard lag length of $p = 4$.

Next, Figure D.1 shows that, just as in our main exercise, VAR confidence intervals can

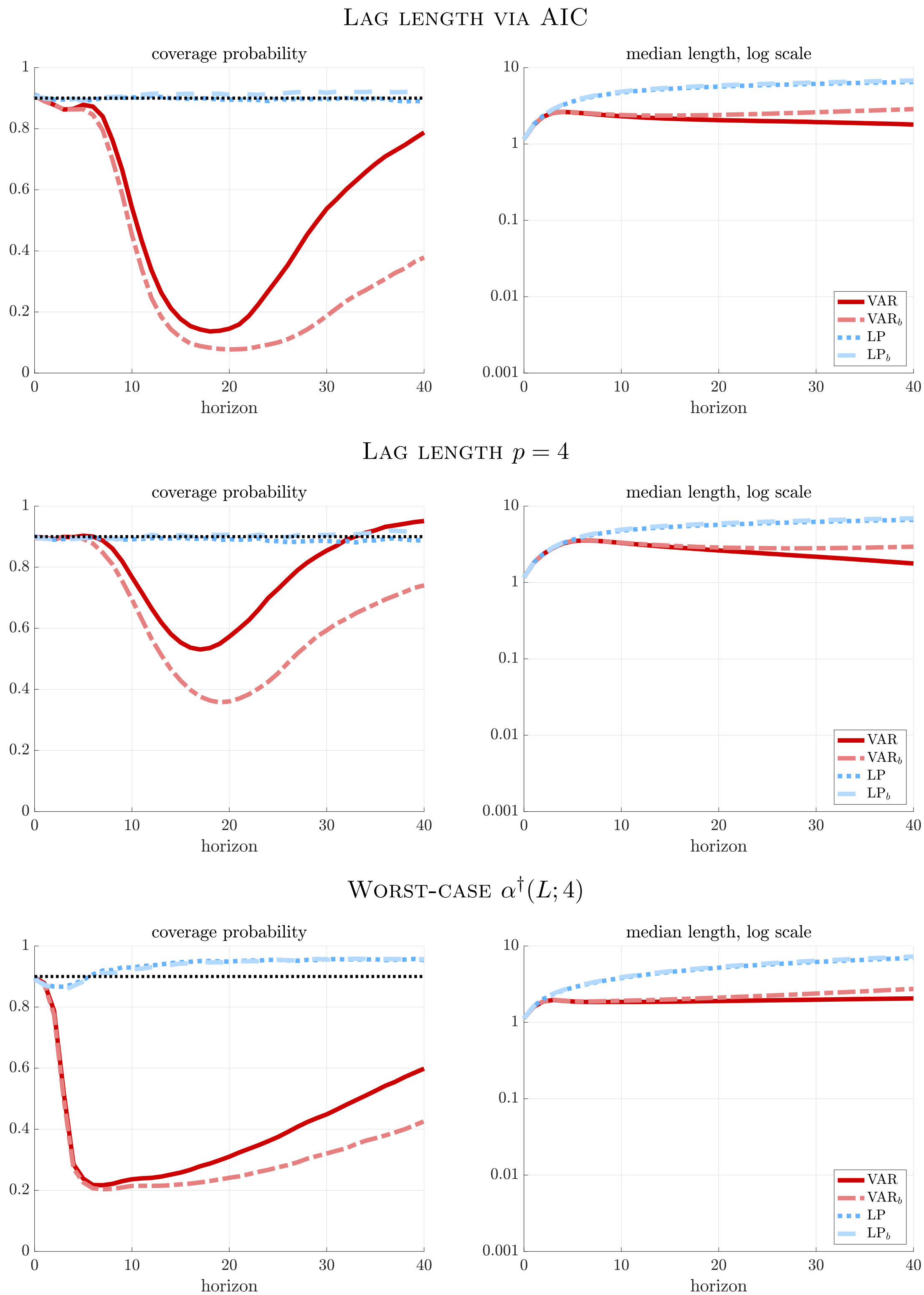


Figure D.1: See Figure 5.2. The DGP is the model of Smets and Wouters, and the researcher estimates the response of output to an observed monetary policy shock. Lag length $p$ is selected using the AIC for the top panel and set to $p = 4$ for the middle panel. The bottom panel changes the MA polynomial in the VARMA representation to the worst-case $\alpha^\dagger(L;4)$ at horizon $h = 4$.

severely undercover, while LP intervals remain robust. As in Section 5.3 we set $T = 240$, simulate 5,000 samples, and construct delta method and bootstrap confidence intervals. For the top panel lag length is selected using the AIC, giving a median selected lag length of $p = 2$. We see that VAR confidence intervals materially undercover, while LP attains close to the nominal coverage level, yet again consistent with our theoretical results. For the middle panel we instead set $p = 4$, again illustrating the "no free lunch" result: as the lag length is increased, VAR coverage gets closer to the nominal level for short and intermediate horizons, but at the same time confidence intervals become essentially as wide as for LP. Finally, in the bottom panel, we show what happens if the actual lag polynomial $\alpha(L)$ is replaced by the horizon-4 worst-case one, $\alpha^\dagger(L; 4)$.[D.1] VAR undercoverage is now severe even at shorter horizons. Overall, however, the magnitudes of undercoverage at medium and long horizons are broadly comparable with those obtained under the actual $\alpha(L)$ implied by the Smets and Wouters (2007) model, revealing that the least favorable MA polynomial $\alpha^\dagger(L, \bullet)$ is again not particularly pathological.

Taken together, the results presented here and in Section 5.3 reveal that, in a typical macroeconomic data-generating process, our theoretical results have bite for a menu of different (and widely studied) structural shocks.

## D.2 Recursively identified monetary shock

For our final exercise we consider an alternative shock identification scheme—identification of a monetary policy shock through a recursive ordering (plus the assumption of invertibility). The data-generating process is yet again the structural model of Smets and Wouters, and the researcher observes output, inflation, and the short-term nominal rate of interest. She identifies a monetary shock as the last innovation in that system under a recursive ordering, as in much of the traditional monetary policy shock literature (e.g., see Christiano, Eichenbaum, and Evans, 1999, and the references therein). We note that, while this identification scheme fails to exactly recover the model's true monetary shock, it does in population yield impulse responses that are qualitatively and quantitatively similar to the effects of a true monetary shock (see the discussion in Wolf, 2020).

[D.1] To be precise, we first set $p = 1$, derive the VARMA(1,$\infty$) as discussed in Section 5.1, and then switch out the implied lag polynomial $\alpha(L)$. The estimation lag length is selected by AIC.

| $p$ | $M$ | $\frac{M^2}{1+M^2}$ |
|---|---|---|
| 1 | 6.973 | 0.980 |
| 2 | 3.780 | 0.935 |
| 4 | 2.558 | 0.867 |
| 8 | 1.613 | 0.722 |
| 12 | 1.117 | 0.555 |
| 20 | 0.611 | 0.272 |
| 40 | 0.230 | 0.050 |

Table D.2: $M$ and $\frac{M^2}{1+M^2}$ as a function of $p$ in the structural model of Smets and Wouters (2007), with the researcher observing output, inflation, and interest rates, and estimating a VAR($p$).

Results. Our findings in this third application largely echo those of the previous two, so our discussion here will be brief. First, Table D.2 reveals that the degree of misspecification is again material for lag lengths typical in applied work (e.g., $M \approx 2.56$ for $p = 4$). Second, Figure D.2 shows that VAR undercoverage can yet again be material, while LP robustly achieves coverage close to the nominal level. In the top panel lag length is selected using the AIC (delivering a median lag length of $p = 2$), which as before results in VAR undercoverage. Finally, we in the bottom panel replace the model-implied lag polynomial $\alpha(L)$ by the worst-case one (with the same amount of overall misspecification), and now yet again find very material VAR undercoverage.

## D.3 Further results on the cost-push shock

To complement our simulation evidence in Section 5.3, we here repeat the cost-push shock exercise of that section for a larger sample size, now setting $T = 2{,}000$. We fix $p = 2$, in line with the median AIC lag length selection in our main exercise.

The results shown in Figure D.3 are similar both qualitatively and quantitatively to our main findings in the top panel of Figure 5.3, especially for the bootstrap confidence intervals. Hence, our results with $T = 240$ in Section 5.3 are not driven by small-sample phenomena. Figure D.3 also plots the theoretically predicted VAR coverage probability (orange dashed line), computed following Corollary 3.2. We see that this asymptotic coverage is very close to the actual one.

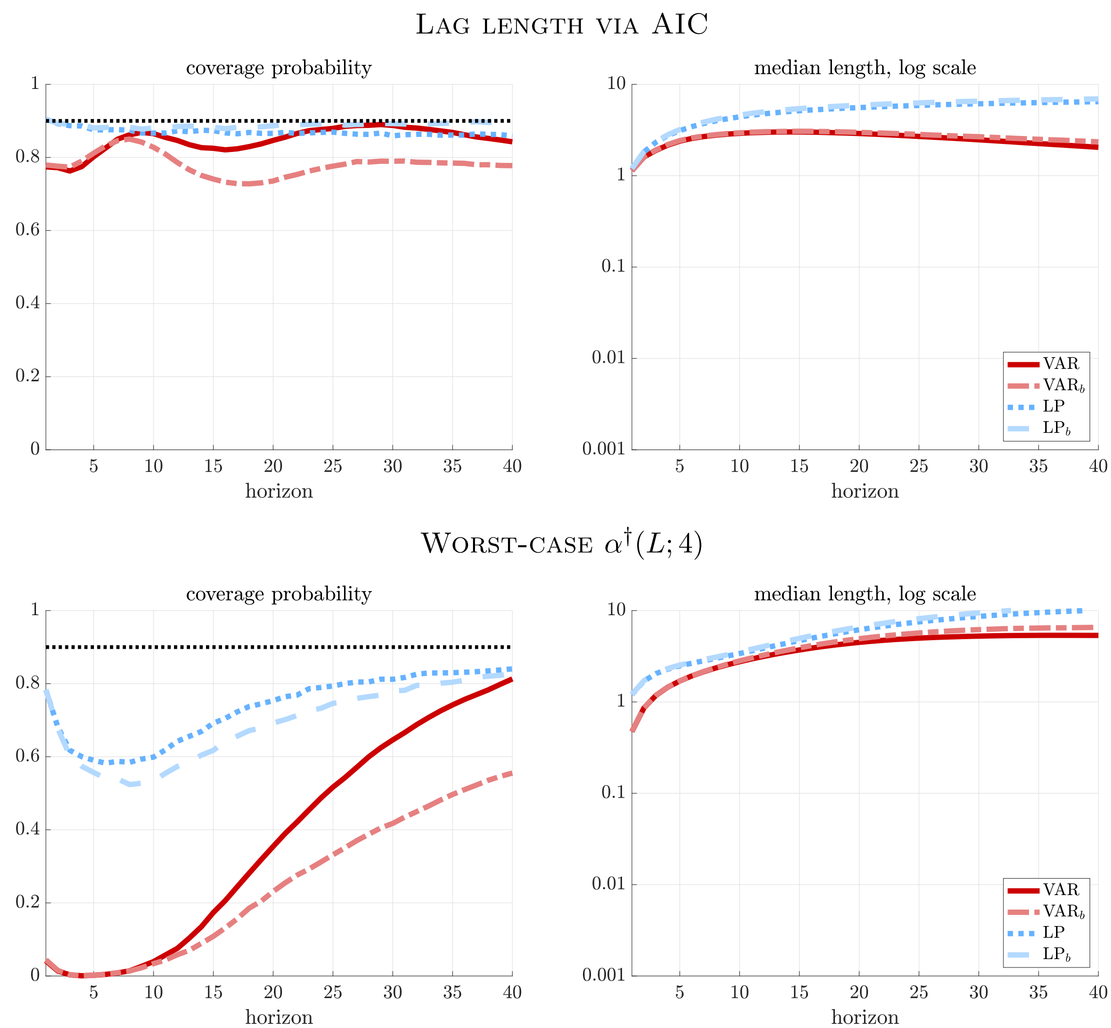


Figure D.2: See Figure 5.2. The DGP is the model of Smets and Wouters, and the researcher estimates the impulse response of output to a monetary policy shock identified through a recursive ordering in a trivariate system with output, inflation, and interest rates. Lag length $p$ is selected using the AIC for both panels. The bottom panel changes the MA polynomial in the VARMA representation to the worst-case $\alpha^\dagger(L;4)$ at horizon $h = 4$.

Lag length $p = 2$, larger sample

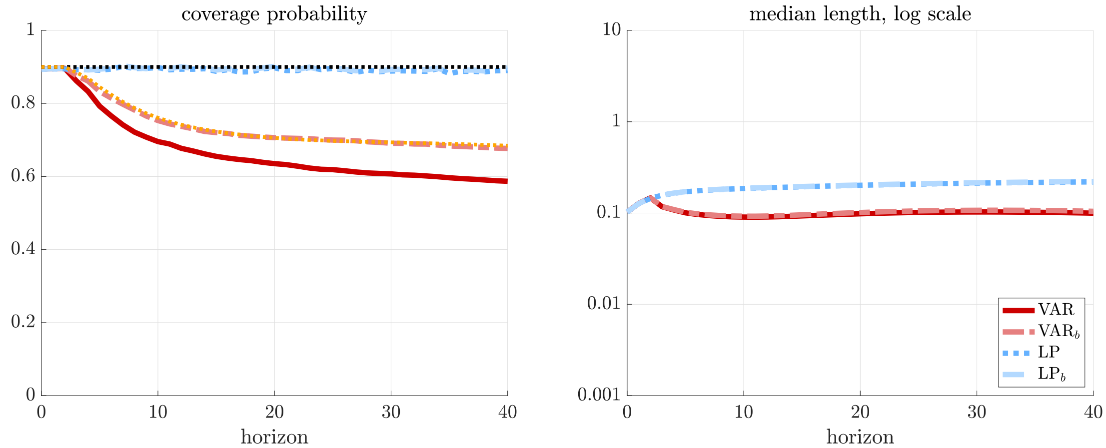


Figure D.3: See Figure 5.2. The DGP is the model of Smets and Wouters, and the researcher estimates the response of inflation to an observed cost-push shock. We set $T = 2{,}000$ and $p = 2$, in line with the AIC selection in Supplemental Appendix D.3. The orange dashed line indicates the asymptotic VAR coverage predicted by Corollary 3.2.

# Appendix E Proof details

We impose Assumption 3.1 throughout. Let $\|B\|$ denote the Frobenius norm of any matrix $B$. It is well known that this norm is sub-multiplicative: $\|BC\| \leq \|B\| \cdot \|C\|$. Let $I_n$ denote the $n \times n$ identity matrix, $0_{m\times n}$ the $m \times n$ matrix of zeros, and $e_{i,n}$ the $n$-dimensional unit vector with a 1 in the $i$-th position. Recall from Assumption 3.1 the definitions $D \equiv \text{Var}(\varepsilon_t) = \text{diag}(\sigma_1^2, \dots, \sigma_m^2)$, $\tilde{y}_t \equiv (I_n - AL)^{-1} H\varepsilon_t = \sum_{s=0}^{\infty} A^s H \varepsilon_{t-s}$, and $S \equiv \text{Var}(\tilde{y}_t)$.

## E.1 Main lemmas

**Lemma E.1.** *For any $i^* \in \{1, \dots, n\}$ and $j^* \in \{1, \dots, m\}$, we have*

$$y_{i^*,t+h} = \theta_{h,T}\varepsilon_{j^*,t} + \underline{B}'_{h,i^*,j^*}\underline{y}_{j^*,t} + B'_{h,i^*,j^*} y_{t-1} + \xi_{h,i^*,t} + T^{-\zeta}\Theta_h(L)\varepsilon_t,$$

*where*

$$\begin{aligned}
\theta_{h,T} &\equiv e'_{i^*,n}(A^h H + T^{-\zeta}\sum_{\ell=1}^{h} A^{h-\ell} H\alpha_\ell) e_{j^*,m}, \\
\underline{B}'_{h,i^*,j^*} &\equiv e'_{i^*,n} A^h \underline{H}_{j^*} H_{11}^{-1}, \\
B'_{h,i^*,j^*} &\equiv e'_{i^*,n}\left[A^{h+1} - A^h \underline{H}_{j^*} H_{11}^{-1} \underline{I}_{j^*} A\right], \\
\xi_{h,i^*,t} &\equiv e'_{i^*,n} A^h \overline{H}_{j^*} \overline{\varepsilon}_{j^*,t} + \sum_{\ell=1}^{h} e'_{i^*,n} A^{h-\ell} H \varepsilon_{t+\ell},
\end{aligned}$$

*and $\Theta_h(L) = \sum_{\ell=-\infty}^{\infty} \Theta_{h,\ell} L^\ell$ is an absolutely summable, $1 \times n$ two-sided lag polynomial with the $j^*$-th element of $\Theta_{h,0}$ equal to zero. Moreover,*

$$T^{-1}\sum_{t=1}^{T-h} (\Theta_h(L)\varepsilon_t)\varepsilon_{j^*,t} = O_p(T^{-1/2}).$$

*Proof.* Iteration on the model in Equation (3.1) yields

$$y_{t+h} = A^{h+1} y_{t-1} + \sum_{\ell=0}^{h} A^{h-\ell}(H\varepsilon_{t+\ell} + T^{-\zeta} H\alpha(L)\varepsilon_{t+\ell}). \tag{E.1}$$

As in Section 3.2, let $\underline{y}_{j^*,t} \equiv (y_{1,t}, \dots, y_{j^*-1,t})'$ denote the variables ordered before $y_{j^*,t}$ (if any). Analogously, let $\overline{y}_{j^*,t} \equiv (y_{j^*+1,t}, \dots y_{n,t})'$ denote the variables ordered after $y_{j^*,t}$.

Using Assumption 3.1(iii), partition

$$H = (\underline{H}_{j^*}, H_{\bullet,j^*}, \overline{H}_{j^*}) = \begin{pmatrix} H_{11} & 0 & 0 \\ H_{21} & H_{22} & 0 \\ H_{31} & H_{32} & H_{33} \end{pmatrix}$$

conformably with the vector $y_t = (\underline{y}'_{j^*,t}, y_{j^*,t}, \overline{y}'_{j^*,t})'$. Let $\underline{I}_{j^*}$ denote the first $j^* - 1$ rows of the $n \times n$ identity matrix. Using the definition of $y_t$ in Equation (3.1),

$$\underline{y}_{j^*,t} = \underline{I}_{j^*} A y_{t-1} + H_{11} \underline{\varepsilon}_{j^*,t} + T^{-\zeta} H_{11} \underline{I}_{j^*} \alpha(L) \varepsilon_t,$$

where $\underline{\varepsilon}_{j^*,t} = \underline{I}_{j^*} \varepsilon_t$. Using the previous equation to solve for $\underline{\varepsilon}_{j^*,t}$ we get

$$\underline{\varepsilon}_{j^*,t} = H_{11}^{-1}(\underline{y}_{j^*,t} - \underline{I}_{j^*} A y_{t-1} - T^{-\zeta} H_{11} \underline{I}_{j^*} \alpha(L) \varepsilon_t). \tag{E.2}$$

Expanding the terms in (E.1) we get:

$$\begin{aligned} y_{t+h} &= A^{h+1} y_{t-1} + A^h H \varepsilon_t + T^{-\zeta} A^h H \alpha(L) \varepsilon_t + \sum_{\ell=1}^{h} A^{h-\ell} (H \varepsilon_{t+\ell} + T^{-\zeta} H \alpha(L) \varepsilon_{t+\ell}) \\ &= A^{h+1} y_{t-1} + \left( A^h \underline{H}_{j^*} \underline{\varepsilon}_{j^*,t} + A^h H_{\bullet,j^*} \varepsilon_{j^*,t} + A^h \overline{H}_{j^*} \overline{\varepsilon}_{j^*,t} \right) \\ &\quad + T^{-\zeta} A^h H \alpha(L) \varepsilon_t + \sum_{\ell=1}^{h} A^{h-\ell} (H \varepsilon_{t+\ell} + T^{-\zeta} H \alpha(L) \varepsilon_{t+\ell}) \\ &= A^{h+1} y_{t-1} + A^h \underline{H}_{j^*} H_{11}^{-1} (\underline{y}_{j^*,t} - \underline{I}_{j^*} A y_{t-1} - T^{-\zeta} H_{11} \underline{I}_{j^*} \alpha(L) \varepsilon_t) + A^h H_{\bullet,j^*} \varepsilon_{j^*,t} + A^h \overline{H}_{j^*} \overline{\varepsilon}_{j^*,t} \\ &\quad + T^{-\zeta} A^h H \alpha(L) \varepsilon_t + \sum_{\ell=1}^{h} A^{h-\ell} (H \varepsilon_{t+\ell} + T^{-\zeta} H \alpha(L) \varepsilon_{t+\ell}), \end{aligned}$$

where the last equality follows from substituting (E.2). Re-arranging terms we get

$$\begin{aligned} y_{i^*,t+h} &= \left( e'_{i^*,n} A^h H_{\bullet,j^*} \right) \varepsilon_{j^*,t} + \underbrace{\left( e'_{i^*,n} A^h \underline{H}_{j^*} H_{11}^{-1} \right)}_{\equiv \underline{B}'_{h,i^*,j^*}} \underline{y}_{j^*,t} + \underbrace{\left( e'_{i^*,n} \left[ A^{h+1} - A^h \underline{H}_{j^*} H_{11}^{-1} \underline{I}_{j^*} A \right] \right)}_{\equiv B'_{h,i^*,j^*}} y_{t-1} \\ &\quad + \underbrace{e'_{i^*,n} \left( A^h \overline{H}_{j^*} \overline{\varepsilon}_{j^*,t} + \sum_{\ell=1}^{h} A^{h-\ell} H \varepsilon_{t+\ell} \right)}_{=\xi_{h,i^*,t}} \\ &\quad + T^{-\zeta} e'_{i^*,n} \left( -A^h \underline{H}_{j^*} H_{11}^{-1} H_{11} \underline{I}_{j^*} \alpha(L) \varepsilon_t + \sum_{\ell=0}^{h} A^{h-\ell} H \alpha(L) \varepsilon_{t+\ell} \right), \end{aligned} \tag{E.3}$$

Using the definition of $\theta_{h,T} \equiv e'_{i^*,n}(A^h H + T^{-\zeta}\sum_{\ell=1}^{h} A^{h-\ell} H\alpha_\ell)e_{j^*,m}$ and adding and subtracting $e'_{i^*,n}\left(T^{-\zeta}\sum_{\ell=1}^{h} A^{h-\ell}H\alpha_\ell\right)e_{j^*,m}\varepsilon_{j^*,t}$ to (E.3), gives a representation of the form

$$y_{i^*,t+h} = \theta_{h,T}\varepsilon_{j^*,t} + \underline{B}'_{h,i^*,j^*}\underline{y}_{j^*,t} + B'_{h,i^*,j^*}y_{t-1} + \xi_{h,i^*,t} + T^{-\zeta}\tilde{u}_t, \tag{E.4}$$

where

$$\tilde{u}_t \equiv e'_{i^*,n}\left(-A^h\underline{H}_{j^*}\underline{I}_{j^*}\alpha(L)\varepsilon_t + \sum_{\ell=0}^{h} A^{h-\ell}H\alpha(L)\varepsilon_{t+\ell} - \left(\sum_{\ell=1}^{h} A^{h-\ell}H\alpha_\ell e_{j^*,m}e'_{j^*,m}\right)\varepsilon_t\right). \tag{E.5}$$

Algebra shows that $\tilde{u}_t$ can be written as a two-sided lag polynomial, $\Theta_h(L) = \sum_{\ell=-\infty}^{\infty}\Theta_{h,\ell}L^\ell$, with coefficients of dimension $1 \times n$ given by the following formulae:

1. For $\ell \geq 1$:
$$\Theta_{h,\ell} = -e'_{i^*,n}A^h\underline{H}_{j^*}\underline{I}_{j^*}\alpha_\ell + \sum_{s=0}^{h} e'_{i^*,n}A^{h-s}H\alpha_{\ell+s}.$$

2. For $\ell = 0$:
$$\Theta_{h,0} = \sum_{s=1}^{h} e'_{i^*,n}A^{h-s}H\alpha_s - \sum_{s=1}^{h} e'_{i^*,n}A^{h-s}H\alpha_s e_{j^*,m}e'_{j^*,m},$$
and, consequently, $\Theta_{h,0,j^*} \equiv \Theta_{h,0}e_{j^*,m} = 0$.

3. For $\ell \in \{-(h-1),\ldots,-1\}$:
$$\sum_{s=1}^{h+\ell} e'_{i^*,n}A^{h-s+\ell}H\alpha_s.$$

4. For $\ell \leq -h$, $\Theta_{h,\ell} = 0_{1\times n}$.

We next show that $\Theta_h(L)$ is absolutely summable, that is

$$\sum_{\ell=-\infty}^{\infty} \|\Theta_{h,l}\| < \infty.$$

To do this, it suffices to show that

$$\sum_{\ell=1}^{\infty} \|\Theta_{h,l}\| < \infty,$$

since all the coefficients with index $\ell \leq -h$ are 0. Note that, by definition, for any $\ell \geq 1$:

$$\|\Theta_{h,\ell}\| \leq \|A^h\|\|\underline{H}_{j^*}\underline{I}_{j^*}\|\|\alpha_\ell\| + \sum_{s=0}^{h}\|A^{h-s}\|\|H\|\|\alpha_{\ell+s}\|.$$

Thus,

$$\sum_{\ell=1}^{\infty} \|\Theta_{h,\ell}\| \leq \|A^h\| \|\underline{H}_{j^*} \underline{I}_{j^*}\| \sum_{\ell=1}^{\infty} \|\alpha_\ell\| + \|H\| \sum_{\ell=1}^{\infty} \sum_{s=0}^{h} \|A^{h-s}\| \|\alpha_{\ell+s}\|.$$

Let $\lambda \in [0, 1)$ and $C > 0$ be chosen such that $\|A^\ell\| \leq C\lambda^\ell$ for all $\ell \geq 0$ (such constants exists by Assumption 3.1(ii)). Then

$$\begin{aligned}
\sum_{\ell=1}^{\infty} \sum_{s=0}^{h} \|A^{h-s}\| \|\alpha_{\ell+s}\| &\leq C \sum_{\ell=1}^{\infty} \sum_{s=1}^{h} \lambda^{h-s} \|\alpha_{\ell+s}\| \\
&\leq C \sum_{\ell=1}^{\infty} \sum_{s=1}^{h} \|\alpha_{\ell+s}\| \\
&\leq Ch \sum_{\ell=1}^{\infty} \|\alpha_\ell\| \\
&< \infty,
\end{aligned}$$

where the last inequality holds because the coefficients of $\alpha(L)$ are summable. We thus conclude that

$$y_{i^*,t+h} = \theta_{h,T} \varepsilon_{j^*,t} + \underline{B}'_{h,y} \underline{y}_{j^*,t} + B'_{h,y} y_{t-1} + \xi_{h,i^*,t} + T^{-\zeta} \Theta_h(L) \varepsilon_t,$$

where $\Theta_h(L)$ is a two-sided lag-polynomial with summable coefficients.

Finally, we show that

$$T^{-1} \sum_{t=1}^{T-h} (\Theta_h(L) \varepsilon_t) \varepsilon_{j^*,t} = O_p(T^{-1/2}).$$

To do this, we write

$$\Theta_h(L) \varepsilon_t = \sum_{\ell=1}^{\infty} \Theta_{h,\ell} \varepsilon_{t-\ell} + \Theta_{h,0} \varepsilon_t + \sum_{\ell=-(h-1)}^{-1} \Theta_{h,\ell} \varepsilon_{t-\ell}.$$

1. Note first that the process

$$\left\{ \left( \sum_{\ell=1}^{\infty} \Theta_{h,\ell} \varepsilon_{t-\ell} \right) \varepsilon_{j^*,t} \right\}_{t=1}^{\infty}$$

is white noise (mean-zero and serially uncorrelated components). The summability of coefficients of $\Theta_h(L)$ further implies that

$$\text{Var}\left(\frac{1}{\sqrt{T}}\sum_{t=1}^{T-h}\left(\sum_{\ell=1}^{\infty}\Theta_{h,\ell}\varepsilon_{t-\ell}\right)\varepsilon_{j^*,t}\right) = \frac{T-h}{T}\text{Var}\left(\left(\sum_{\ell=1}^{\infty}\Theta_{h,\ell}\varepsilon_{t-\ell}\right)\varepsilon_{j^*,t}\right) < \infty.$$

Thus, by Markov's inequality, we have that

$$\frac{1}{T}\sum_{t=1}^{T-h}\left(\sum_{\ell=1}^{\infty}\Theta_{h,\ell}\varepsilon_{t-\ell}\right)\varepsilon_{j^*,t} = O_p(T^{-1/2}).$$

2. Note second that the process

$$\{(\Theta_{h,0}\varepsilon_t)\,\varepsilon_{j^*,t}\}_{t=1}^{\infty}$$

is i.i.d. with mean zero (since $\varepsilon_t$ has independent components and $\Theta_{0,\ell,j^*} = 0$). Since the process has finite variance, we conclude that

$$\frac{1}{T}\sum_{t=1}^{T-h}\left(\sum_{\ell=1}^{\infty}\Theta_{h,0}\varepsilon_{t}\right)\varepsilon_{j^*,t} = O_p(T^{-1/2}).$$

3. Finally, note that the process

$$\left\{\left(\sum_{\ell=-(h-1)}^{-1}\Theta_{h,\ell}\varepsilon_{t-\ell}\right)\varepsilon_{j^*,t}\right\}_{t=1}^{\infty}$$

is white noise (mean-zero and serially uncorrelated components). Therefore,

$$\text{Var}\left(\frac{1}{\sqrt{T}}\sum_{t=1}^{T-h}\left(\sum_{\ell=-(h-1)}^{-1}\Theta_{h,\ell}\varepsilon_{t-\ell}\right)\varepsilon_{j^*,t}\right) = \frac{T-h}{T}\text{Var}\left(\left(\sum_{\ell=-(h-1)}^{-1}\Theta_{h,\ell}\varepsilon_{t-\ell}\right)\varepsilon_{j^*,t}\right) < \infty.$$

We conclude that

$$\frac{1}{T}\sum_{t=1}^{T-h}\left(\sum_{\ell=-(h-1)}^{-1}\Theta_{h,\ell}\varepsilon_{t-\ell}\right)\varepsilon_{j^*,t} = O_p(T^{-1/2}).$$

Consequently,

$$T^{-1}\sum_{t=1}^{T-h}(\Theta_h(L)\varepsilon_t)\varepsilon_{j^*,t} = O_p(T^{-1/2}). \qquad \square$$

**Lemma E.2.**

$$\hat{A} - A = T^{-\zeta} H \sum_{\ell=1}^{\infty} \alpha_\ell D H' (A')^{\ell-1} S^{-1} + T^{-1} \sum_{t=1}^{T} H \varepsilon_t \tilde{y}'_{t-1} S^{-1} + o_p(T^{-\zeta}).$$

*In particular,* $\hat{A} - A = O_p(T^{-\zeta} + T^{-1/2})$.

*Proof.* Since,

$$\hat{A} - A = \left( T^{-1} \sum_{t=1}^{T-h} u_t y'_{t-1} \right) \left( T^{-1} \sum_{t=1}^{T-h} y_{t-1} y'_{t-1} \right)^{-1},$$

the result follows from Lemmas E.7 and E.8. □

**Lemma E.3.**

$$\hat{\nu} - H_{\bullet, j^*} = \frac{1}{\sigma_{j^*}^2} T^{-1} \sum_{t=1}^{T} \xi_{0,t} \varepsilon_{j^*,t} + o_p(T^{-1/2}).$$

*Proof.* By Lemma E.5, $\hat{\nu} = (0_{1\times(j^*-1)}, 1, \hat{\overline{\nu}}')$, where the $j$-th element of $\hat{\overline{\nu}}$ equals the on-impact local projection of $y_{i^*+j,t}$ on $y_{j^*,t}$, controlling for $\underline{y}_{j^*,t}$ and $y_{t-1}$. The statement of the lemma is therefore a direct consequence of Proposition 3.1 and the fact that (by definition) $\xi_{0,i,t} = 0$ for $i \le j^*$. □

**Lemma E.4.** *Fix* $h \ge 0$*. Consider the regression of* $y_{j^*,t}$ *on* $q_{j^*,t} \equiv (\underline{y}'_{j^*,t}, y'_{t-1})'$*, using the observations* $t = 1, 2, \dots, T-h$*:*

$$y_{j^*,t} = \hat{\vartheta}'_h q_{j^*,t} + \hat{x}_{h,t}.$$

*Note that the residuals* $\hat{x}_{h,t}$ *are consistent with the earlier definition in the proof of Proposition 3.1. Let* $\underline{\lambda}'_{j^*}$ *be the row vector containing the first* $j^*-1$ *elements of the last row of* $-\tilde{H}^{-1}$ *(where* $\tilde{H}$ *is defined in Assumption 3.1(iii)). Let* $\lambda'_{j^*} \equiv (-\underline{\lambda}'_{j^*}, 1, 0_{1\times(n-j^*)})$ *and* $\vartheta \equiv (\underline{\lambda}'_{j^*}, (\lambda'_{j^*} A))'$*. Then:*

*i)* $\hat{\vartheta}_h - \vartheta = O_p(T^{-\zeta} + T^{-1/2})$.

*ii)* $T^{-1} \sum_{t=1}^{T-h} (\hat{x}_{h,t} - \varepsilon_{j^*,t}) \varepsilon_{j^*,t} = o_p(T^{-1/2})$.

*iii) For* $\ell \ge 1$*,* $T^{-1} \sum_{t=1}^{T-h} (\hat{x}_{h,t} - \varepsilon_{j^*,t}) \varepsilon_{t+\ell} = o_p(T^{-1/2})$.

*iv)* $T^{-1} \sum_{t=1}^{T-h} (\hat{x}_{h,t} - \varepsilon_{j^*,t}) \hat{x}_{h,t} = o_p(T^{-1/2})$.

*v)* $T^{-1} \sum_{t=1}^{T-h} \hat{x}_{h,t}^2 \xrightarrow{p} \sigma_{j^*}^2$.

*vi) For any absolutely summable two-sided lag polynomial* $B(L)$, $T^{-1}\sum_{t=1}^{T-h}(\hat{x}_{h,t}-\varepsilon_{j^*,t})B(L)\varepsilon_t = O_p(T^{-\zeta}+T^{-1/2})$.

*Proof.* By Equation (3.1), the outcome variables in the model satisfy

$$y_t = Ay_{t-1} + H[I_m + T^{-\zeta}\alpha(L)]\varepsilon_t,\ t = 1, 2, \dots, T.$$

By Assumption 3.1(iii), the first $j^*$ rows of the matrix $H$ above are of the form $(\tilde{H}, 0_{j^*\times(j^*-m)})$, where $m$ is the number of shocks and $\tilde{H}$ is a $j^*\times j^*$ lower triangular matrix with 1's on the diagonal.

$\tilde{H}$ is invertible, which means we can premultiply the first $j^*$ equations of (3.1) by $\tilde{H}^{-1}$ to obtain:

$$[\tilde{H}^{-1}, 0_{j^*\times(n-j^*)}]y_t = [\tilde{H}^{-1}, 0_{j^*\times(n-j^*)}]Ay_{t-1} + [I_{j^*}, 0_{j^*\times(m-j^*)}][I_m + T^{-\zeta}\alpha(L)]\varepsilon_t.$$

By definition, $-\underline{\lambda}'_{j^*}$ is the row vector containing the first $j^*-1$ elements of the last row of $\tilde{H}^{-1}$ and $\lambda'_{j^*} \equiv (-\underline{\lambda}'_{j^*}, 1, 0_{1\times(n-j^*)})$. Thus, we can re-write the $j^*$-th equation above as

$$[-\underline{\lambda}'_{j^*}, 1, 0_{j^*\times(n-j^*)}]y_t = \lambda'_{j^*}Ay_{t-1} + \varepsilon_{j^*,t} + T^{-\zeta}\alpha_{j^*}(L)\varepsilon_t,$$

where $\alpha_{j^*}(L)$ is the $j^*$-th row of $\alpha(L)$. Re-arranging terms we get

$$y_{j^*,t} = \vartheta' q_{j^*,t} + \varepsilon_{j^*,t} + T^{-\zeta}\alpha_{j^*}(L)\varepsilon_t,$$

where $\vartheta \equiv (\underline{\lambda}'_{j^*}, (\lambda'_{j^*}A))'$ and $q_{j^*,t} \equiv (\underline{y}'_{j^*,t}, y'_{t-1})'$. In a slight abuse of notation, and for notational simplicity, we henceforth replace $q_{j^*,t}$ by $q_t$.

Statement (i) follows from standard OLS algebra if we can show that a) $T^{-1}\sum_{t=1}^{T-h} q_t\varepsilon_{j^*,t} = O_p(T^{-\zeta}+T^{-1/2})$, b) $(T^{-1}\sum_{t=1}^{T-h} q_t q_t')^{-1} = O_p(1)$, and c) $T^{-\zeta-1}\sum_{t=1}^{T-h} q_t(\alpha_{j^*}(L)\varepsilon_t) = O_p(T^{-\zeta})$. Lemma E.9 establishes these results.

Statements (ii)–(iii) are proved in Lemma E.10 below.

For statement (iv), note that

$$T^{-1}\sum_{t=1}^{T-h}(\hat{x}_{h,t}-\varepsilon_{j^*,t})\hat{x}_{h,t} = T^{-1}\sum_{t=1}^{T-h}(\hat{x}_{h,t}-\varepsilon_{j^*,t})^2 + T^{-1}\sum_{t=1}^{T-h}(\hat{x}_{h,t}-\varepsilon_{j^*,t})\varepsilon_{j^*,t}.$$

Lemma E.11 shows that $T^{-1}\sum_{t=1}^{T-h}(\hat{x}_{h,t}-\varepsilon_{j^*,t})^2 = o_p(T^{-1/2})$. This result, combined with (ii), implies that statement (iv) holds.

For statement (v), note that

$$T^{-1}\sum_{t=1}^{T-h}(\hat{x}_{h,t})^2 = T^{-1}\sum_{t=1}^{T-h}(\hat{x}_{h,t}-\varepsilon_{j^*,t}+\varepsilon_{j^*,t})^2$$
$$= T^{-1}\sum_{t=1}^{T-h}(\hat{x}_{h,t}-\varepsilon_{j^*,t})^2 - 2T^{-1}\sum_{t=1}^{T-h}(\hat{x}_{h,t}-\varepsilon_{j^*,t})\varepsilon_{j^*,t} + T^{-1}\sum_{t=1}^{T-h}\varepsilon_{j^*,t}^2.$$

Lemma E.11 and statement (ii) imply that the first two terms converge in probability to zero. Since $T^{-1}\sum_{t=1}^{T-h}\varepsilon_{j^*,t}^2 \xrightarrow{p} \sigma_{j^*}^2$ (by the Law of Large Numbers), statement (v) holds.

Finally, statement (vi) obtains by decomposing

$$T^{-1}\sum_{t=1}^{T-h}B(L)\varepsilon_t(\hat{x}_{h,t}-\varepsilon_{j^*,t}) = T^{-1}\sum_{t=1}^{T-h}B(L)\varepsilon_t q_t'(\vartheta-\hat{\vartheta}_h) + T^{-\zeta}T^{-1}\sum_{t=1}^{T-h}B(L)\varepsilon_t[\alpha_{j^*}(L)\varepsilon_t]'$$
$$= O_p(1)\times O_p(T^{-\zeta}+T^{-1/2}) + T^{-\zeta}\times O_p(1),$$

where the last line follows from statement (i), Lemma E.6, and moment calculations. □

## E.2 Auxiliary numerical lemma

**Lemma E.5.** *Define $\overline{y}_{i,t} \equiv (y_{i+1,t}, y_{i+2,t}, \dots, y_{nt})'$ to be the (possibly empty) vector of variables that are ordered after $y_{i,t}$ in $y_t$. Partition*

$$\hat{\Sigma} = \begin{pmatrix} \hat{\Sigma}_{11} & \hat{\Sigma}_{12} & \hat{\Sigma}_{13} \\ \hat{\Sigma}_{21} & \hat{\Sigma}_{22} & \hat{\Sigma}_{23} \\ \hat{\Sigma}_{31} & \hat{\Sigma}_{32} & \hat{\Sigma}_{33} \end{pmatrix}, \quad \hat{C} = \begin{pmatrix} \hat{C}_{11} & 0 & 0 \\ \hat{C}_{21} & \hat{C}_{22} & 0 \\ \hat{C}_{31} & \hat{C}_{32} & \hat{C}_{33} \end{pmatrix},$$

*conformably with $y_t = (\underline{y}_{j^*,t}', y_{j^*,t}, \overline{y}_{j^*,t}')'$, where $\hat{\Sigma} = \hat{C}\hat{C}'$ (in particular, $\hat{C}_{22} = \hat{C}_{j^*,j^*}$). Then*

$$(\hat{\Sigma}_{31}, \hat{\Sigma}_{32})\begin{pmatrix} \hat{\Sigma}_{11} & \hat{\Sigma}_{12} \\ \hat{\Sigma}_{21} & \hat{\Sigma}_{22} \end{pmatrix}^{-1} e_{j^*,j^*} = \hat{C}_{22}^{-1}\hat{C}_{32}. \tag{E.6}$$

Note that the lemma implies $\hat{\beta}_0 = \hat{\delta}_0$: If $i^* < j^*$ or $i^* = j^*$, then both estimators equal 0 or 1 (by definition), respectively; if $i^* > j^*$, then $\hat{\beta}_0$ is defined as the $i^* - j^*$ element of the left-hand side of (E.6) (by Frisch-Waugh), while $\hat{\delta}_0$ is defined as the $i^* - j^*$ element of the right-hand side of (E.6).

*Proof.* From the relationship $\hat{\Sigma} = \hat{C}\hat{C}'$, we get

$$\begin{pmatrix} \hat{\Sigma}_{11} & \hat{\Sigma}_{12} \\ \hat{\Sigma}_{21} & \hat{\Sigma}_{22} \\ \hat{\Sigma}_{31} & \hat{\Sigma}_{32} \end{pmatrix} = \begin{pmatrix} \hat{C}_{11}\hat{C}_{11}' & \hat{C}_{11}\hat{C}_{21}' \\ \hat{C}_{21}\hat{C}_{11}' & \hat{C}_{21}\hat{C}_{21}' + \hat{C}_{22}^2 \\ \hat{C}_{31}\hat{C}_{11}' & \hat{C}_{31}\hat{C}_{21}' + \hat{C}_{32}\hat{C}_{22} \end{pmatrix}.$$

The partitioned inverse formula implies

$$\begin{aligned} \begin{pmatrix} \hat{\Sigma}_{11} & \hat{\Sigma}_{12} \\ \hat{\Sigma}_{21} & \hat{\Sigma}_{22} \end{pmatrix}^{-1} e_{j^*,j^*} &= \frac{1}{\hat{C}_{21}\hat{C}_{21}' + \hat{C}_{22}^2 - \hat{C}_{21}\hat{C}_{11}'(\hat{C}_{11}\hat{C}_{11}')^{-1}\hat{C}_{11}\hat{C}_{21}'} \begin{pmatrix} -(\hat{C}_{11}\hat{C}_{11}')^{-1}\hat{C}_{11}\hat{C}_{21}' \\ 1 \end{pmatrix} \\ &= \frac{1}{\hat{C}_{22}^2} \begin{pmatrix} -\hat{C}_{11}^{-1\prime}\hat{C}_{21}' \\ 1 \end{pmatrix}, \end{aligned}$$

so

$$(\hat{\Sigma}_{31}, \hat{\Sigma}_{32}) \begin{pmatrix} \hat{\Sigma}_{11} & \hat{\Sigma}_{12} \\ \hat{\Sigma}_{21} & \hat{\Sigma}_{22} \end{pmatrix}^{-1} e_{j^*,j^*} = \frac{1}{\hat{C}_{22}^2} \left( -\hat{C}_{31}\hat{C}_{11}'\hat{C}_{11}^{-1\prime}\hat{C}_{21}' + \hat{C}_{31}\hat{C}_{21}' + \hat{C}_{32}\hat{C}_{22} \right) = \frac{1}{\hat{C}_{22}}\hat{C}_{32}. \quad \square$$

## E.3 Auxiliary asymptotic lemmas

**Lemma E.6.** *$T^{-1}\sum_{t=1}^{T} \|y_t - \tilde{y}_t\|^2 = O_p(T^{-2\zeta})$ and $T^{-1}\sum_{t=1}^{T} u_t(y_{t-1} - \tilde{y}_{t-1})' = O_p(T^{-2\zeta} + T^{-\zeta-1/2})$, where $u_t \equiv y_t - Ay_{t-1}$.*

*Proof.* Using Equation (3.1), write $y_t$ as

$$\begin{aligned} y_t &= \sum_{s=0}^{\infty} A^s H(I_m + T^{-\zeta}\alpha(L))\varepsilon_{t-s} \\ &= \underbrace{\sum_{s=0}^{\infty} A^s H\varepsilon_{t-s}}_{\equiv \tilde{y}_t} + T^{-\zeta}\sum_{s=0}^{\infty} A^s H\alpha(L)\varepsilon_{t-s}. \end{aligned}$$

Thus, the definition of $\tilde{y}_t$ implies

$$y_t - \tilde{y}_t = T^{-\zeta}\sum_{s=0}^{\infty} A^s H\alpha(L)\varepsilon_{t-s}.$$

Lemma E.12 below shows that, under Assumption 3.1, $T^{-1}\sum_{t=1}^{T} E\left[\|y_t - \tilde{y}_t\|^2\right] = O(T^{-2\zeta})$. Consequently, the first part of Lemma E.6 follows from Markov's inequality.

In order to establish the second part of Lemma E.6, note that

$$u_t\left(y_{t-1}-\tilde{y}_{t-1}\right)' = H[I_m + T^{-\zeta}\alpha(L)]\varepsilon_t\left(y_{t-1}-\tilde{y}_{t-1}\right)'.$$

Lemma E.13 below implies that

$$\frac{1}{T}\sum_{t=1}^{T}\varepsilon_t(y_{t-1}-\tilde{y}_{t-1})' = O_p\left(T^{-\zeta-1/2}\right). \tag{E.7}$$

Finally, Lemma E.14 below implies that

$$\frac{1}{T}\sum_{t=1}^{T}\alpha(L)\varepsilon_t(y_{t-1}-\tilde{y}_{t-1})' = O_p(T^{-\zeta}). \tag{E.8}$$

Equations (E.7) and (E.8) imply

$$\frac{1}{T}\sum_{t=1}^{T}u_t(y_{t-1}-\tilde{y}_{t-1})' = O_p(T^{-2\zeta}+T^{-\zeta-1/2}). \qquad \square$$

**Lemma E.7.**

$$T^{-1}\sum_{t=1}^{T}u_t y_{t-1}' = T^{-\zeta}H\sum_{\ell=1}^{\infty}\alpha_\ell DH'(A')^{\ell-1} + T^{-1}\sum_{t=1}^{T}H\varepsilon_t\tilde{y}_{t-1}' + o_p(T^{-\zeta}).$$

*Proof.*

$$\begin{aligned}
T^{-1}\sum_{t=1}^{T}u_t y_{t-1}' &= T^{-1}\sum_{t=1}^{T}u_t\tilde{y}_{t-1}' + \underbrace{T^{-1}\sum_{t=1}^{T}u_t(y_{t-1}-\tilde{y}_{t-1})'}_{=o_p(T^{-\zeta})\text{ by Lemma E.6}}\\
&= T^{-1}\sum_{t=1}^{T}H\varepsilon_t\tilde{y}_{t-1}' + T^{-\zeta-1}\sum_{t=1}^{T}H\alpha(L)\varepsilon_t\tilde{y}_{t-1}' + o_p(T^{-\zeta})\\
&= T^{-1}\sum_{t=1}^{T}H\varepsilon_t\tilde{y}_{t-1}' + T^{-\zeta}H\left(T^{-1}\sum_{t=1}^{T}E[\alpha(L)\varepsilon_t\tilde{y}_{t-1}'] + o_p(1)\right) + o_p(T^{-\zeta}),
\end{aligned}$$

where the last equality follows from Lemma E.15 below. Finally, note that

$$E[\alpha(L)\varepsilon_t\tilde{y}_{t-1}'] = \sum_{\ell=1}^{\infty}\sum_{s=0}^{\infty}\alpha_\ell E[\varepsilon_{t-\ell}\varepsilon_{t-s-1}']H'(A')^s = \sum_{\ell=1}^{\infty}\alpha_\ell DH'(A')^{\ell-1}. \qquad \square$$

**Lemma E.8.** $T^{-1}\sum_{t=1}^{T}y_{t-1}y_{t-1}' \xrightarrow{p} S$.

*Proof.* By Lemma E.6 and Cauchy-Schwarz, $T^{-1}\sum_{t=1}^{T} y_{t-1}y_{t-1}' = T^{-1}\sum_{t=1}^{T} \tilde{y}_{t-1}\tilde{y}_{t-1}' + o_p(1)$. The rest of the proof is standard. □

**Lemma E.9.** *Fix $h \geq 0$ and $j^* \in \{1,\ldots,n\}$. In a slight abuse of notation, let $q_t \equiv (\underline{y}_{j^*,t}', y_{t-1}')'$. Then*

*i)* $T^{-1}\sum_{t=1}^{T-h} q_t\varepsilon_{j^*,t} = O_p(T^{-\zeta} + T^{-1/2})$,

*ii)* $(T^{-1}\sum_{t=1}^{T-h} q_t q_t')^{-1} = O_p(1)$,

*iii)* $T^{-1}\sum_{t=1}^{T-h} q_t(\alpha_{j^*}(L)\varepsilon_t) = O_p(1)$,

*where $\alpha_{j^*}(L)$ is the $j^*$-th row of $\alpha(L)$.*

*Proof.* Let $\tilde{q}_t \equiv (\underline{\tilde{y}}_{j^*,t}', \tilde{y}_{t-1}')'$ and $\Delta_t \equiv q_t - \tilde{q}_t$. Note that

$$T^{-1}\sum_{t=1}^{T-h} q_t\varepsilon_{j^*,t} = T^{-1}\sum_{t=1}^{T-h} \Delta_t\varepsilon_{j^*,t} + T^{-1}\sum_{t=1}^{T-h} \tilde{q}_t\varepsilon_{j^*,t}. \tag{E.9}$$

Cauchy-Schwarz implies

$$\left\| T^{-1}\sum_{t=1}^{T-h} \Delta_t\varepsilon_{j^*,t} \right\| \leq \left( \frac{1}{T}\sum_{t=1}^{T-h} \|\Delta_t\|^2 \right)^{1/2} \left( \frac{1}{T}\sum_{t=1}^{T-h} \varepsilon_{j^*,t}^2 \right)^{1/2}.$$

Lemma E.6 implies the first term to the right of the inequality is $O_p(T^{-\zeta})$. Assumption 3.1(i) implies that the second term to the right of the inequality is $O_p(1)$. Thus, from (E.9) we have

$$T^{-1}\sum_{t=1}^{T-h} q_t\varepsilon_{j^*,t} = O_p\left(T^{-\zeta}\right) + T^{-1}\sum_{t=1}^{T-h} \tilde{q}_t\varepsilon_{j^*,t}.$$

Direct second-moment calculations imply that the last term is $O_p\left(T^{-1/2}\right)$. This establishes part (i) of the lemma.

For part (ii) of the lemma, note that

$$\frac{1}{T}\sum_{t=1}^{T-h} q_t q_t' = \frac{1}{T}\sum_{t=1}^{T-h} \Delta_t\Delta_t' + \frac{1}{T}\sum_{t=1}^{T-h} \tilde{q}_t\Delta_t' + \frac{1}{T}\sum_{t=1}^{T-h} \Delta_t\tilde{q}_t' + \frac{1}{T}\sum_{t=1}^{T-h} \tilde{q}_t\tilde{q}_t'. \tag{E.10}$$

Lemma E.6 implies that the first term is $O_p\left(T^{-2\zeta}\right)$. Cauchy-Schwarz, along with Assumption 3.1 and Lemma E.6, imply that the second and third terms are $O_p(T^{-\zeta})$. The last term converges in probability to $\text{Var}(\tilde{q}_t)$. This matrix is non-singular, since $\tilde{q}_t = (\underline{\tilde{y}}_{j^*,t}', \tilde{y}_{t-1}')'$, where $\mathrm{Var}(\tilde{y}_{t-1}) = S$ is non-singular by Assumption 3.1(iv), and Assumption 3.1(iii) implies that $\tilde{\underline{y}}_{j^*,t}$ equals a linear transformation of $\tilde{y}_{t-1}$ plus a non-singular independent noise term.

For part (iii) of the lemma, note that

$$\frac{1}{T}\sum_{t=1}^{T-h} q_t(\alpha_{j^*}(L)\varepsilon_t) = \frac{1}{T}\sum_{t=1}^{T-h} \Delta_t(\alpha_{j^*}(L)\varepsilon_t) + \frac{1}{T}\sum_{t=1}^{T-h} \tilde{q}_t(\alpha_{j^*}(L)\varepsilon_t). \tag{E.11}$$

Assumption 3.1(i) and (v) and Lemma E.6 imply that the first term is $O_p(T^{-\zeta})$. Markov's inequality and a moment calculation imply that the last term is $O_p(1)$. □

**Lemma E.10.** *Fix $h \geq 0$ and $j^* \in \{1, \dots, n\}$. Then*

$$T^{-1}\sum_{t=1}^{T-h}(\hat{x}_{h,t} - \varepsilon_{j^*,t})\varepsilon_{j^*,t} = o_p(T^{-1/2}). \tag{E.12}$$

*Moreover, for $\ell \geq 1$,*

$$T^{-1}\sum_{t=1}^{T-h}(\hat{x}_{h,t} - \varepsilon_{j^*,t})\varepsilon_{t+\ell} = o_p(T^{-1/2}). \tag{E.13}$$

*Proof.* In a slight abuse of notation, let $q_t \equiv (\underline{y}'_{j^*,t}, y'_{t-1})'$. We first establish (E.13). By definition of $\hat{x}_{h,t}$, we have $\hat{x}_{h,t} - \varepsilon_{j^*,t} = (\vartheta - \hat{\vartheta}_h)' q_t + T^{-\zeta}\alpha_{j^*}(L)\varepsilon_t$. As in Lemma E.6 define $\tilde{y}_t = \sum_{s=0}^{\infty} A^s H \varepsilon_{t-s}$. Let $\tilde{q}_t \equiv (\tilde{\underline{y}}'_{j^*,t}, \tilde{y}'_{t-1})'$ and $\Delta_t \equiv q_t - \tilde{q}_t$. Thus,

$$\begin{aligned} T^{-1}\sum_{t=1}^{T-h}(\hat{x}_{h,t} - \varepsilon_{j^*,t})\varepsilon_{t+\ell} &= (\vartheta - \hat{\vartheta}_h)'\left(\frac{1}{T}\sum_{t=1}^{T-h}\Delta_t\varepsilon_{t+\ell}\right) & \text{(E.14)} \\ &+ (\vartheta - \hat{\vartheta}_h)'\left(\frac{1}{T}\sum_{t=1}^{T-h}\tilde{q}_t\varepsilon_{t+\ell}\right) & \text{(E.15)} \\ &+ \frac{1}{T^\zeta}\left(\frac{1}{T}\sum_{t=1}^{T-h}(\alpha_{j^*}(L)\varepsilon_t)\,\varepsilon_{t+\ell}\right). & \text{(E.16)} \end{aligned}$$

By Lemma E.9, $(\vartheta - \hat{\vartheta}_h) = O_p(T^{-\zeta} + T^{-1/2})$. Direct second-moment calculations can be used to show that the terms in (E.15)–(E.16) are of order

$$O_p(T^{-\zeta} + T^{-1/2})O_p(T^{-1/2}) \text{ and } O_p(T^{-\zeta-1/2}),$$

respectively. This implies that both terms are $o_p(T^{-1/2})$.

Finally, note that Lemma E.6 and Assumption 3.1(i) imply that the sum in (E.14) is

$O_p\left(T^{-\zeta}\right)$. Thus, (E.14) is of order

$$O_p(T^{-\zeta}+T^{-1/2})O_p(T^{-\zeta}) = o_p\left(T^{-1/2}\right),$$

using $\zeta > 1/4$. Since we have shown that (E.14)–(E.16) are $o_p(T^{-1/2})$, then for $\ell \geq 1$,

$$T^{-1}\sum_{t=1}^{T-h}(\hat{x}_{h,t}-\varepsilon_{j^*,t})\varepsilon_{t+\ell} = o_p(T^{-1/2}).$$

The proof of (E.12) is entirely analogous. □

**Lemma E.11.** *Fix $h \geq 0$ and $j^* \in \{1,\ldots,n\}$. In a slight abuse of notation, let $q_t \equiv (\underline{y}'_{j^*,t}, y'_{t-1})'$ and*

$$\hat{x}_{h,t} \equiv (\vartheta-\hat{\vartheta}_h)'q_t + \varepsilon_{j^*,t} + T^{-\zeta}\alpha_{j^*}(L)\varepsilon_t,$$

*where $\alpha_{j^*}(L)$ is the $j^*$-th row of $\alpha(L)$. Then*

$$T^{-1}\sum_{t=1}^{T-h}(\hat{x}_{h,t}-\varepsilon_{j^*,t})^2 = o_p(T^{-1/2}). \tag{E.17}$$

*Proof.* Let $\tilde{q}_t \equiv (\underline{\tilde{y}}'_{j^*,t}, \tilde{y}'_{t-1})'$ and $\Delta_t \equiv q_t - \tilde{q}_t$. Then

$$T^{-1}\sum_{t=1}^{T-h}(\hat{x}_{h,t}-\varepsilon_{j^*,t})^2 = T^{-1}\sum_{t=1}^{T-h}\left((\vartheta-\hat{\vartheta}_h)'\Delta_t + (\vartheta-\hat{\vartheta}_h)'\tilde{q}_t + T^{-\zeta}\alpha_{j^*}(L)\varepsilon_t\right)^2.$$

To establish (E.17), it suffices by the $c_r$-inequality to show that

a) $T^{-1}\sum_{t=1}^{T-h}\left((\vartheta-\hat{\vartheta}_h)'\Delta_t\right)^2 = o_p\left(T^{-1/2}\right)$,

b) $T^{-1}\sum_{t=1}^{T-h}\left((\vartheta-\hat{\vartheta}_h)'\tilde{q}_t\right)^2 = o_p\left(T^{-1/2}\right)$,

c) $T^{-1}\sum_{t=1}^{T-h}\left(\alpha_{j^*}(L)\varepsilon_t\right)^2 = O_p(1)$.

To establish (a), note first that Cauchy-Schwarz implies

$$\frac{1}{T}\sum_{t=1}^{T-h}\left((\vartheta-\hat{\vartheta}_h)'\Delta_t\right)^2 \leq \left\|\vartheta-\hat{\vartheta}_h\right\|^2\left(\frac{1}{T}\sum_{t=1}^{T-h}\|\Delta_t\|^2\right).$$

Lemma E.6 implies that the term inside the parenthesis is $O_p(T^{-2\zeta})$. Lemma E.9 implies $(\vartheta-\hat{\vartheta}_h) = O_p\left(T^{-\zeta}+T^{-1/2}\right)$. Since $\zeta > 1/4$, statement (a) follows.

To establish (b), we apply Cauchy-Schwarz to obtain

$$\frac{1}{T}\sum_{t=1}^{T-h}\left((\vartheta-\hat{\vartheta}_h)'\tilde{q}_t\right)^2 \leq \left\|\vartheta-\hat{\vartheta}_h\right\|^2\left(\frac{1}{T}\sum_{t=1}^{T-h}\tilde{q}_t^2\right).$$

Assumption 3.1 implies that the term inside the parenthesis is $O_p(1)$. As in the previous paragraph, $\|\vartheta-\hat{\vartheta}_h\|^2 = O_p\left((T^{-\zeta}+T^{-1/2})^2\right)$. Since $\zeta > 1/4$, statement (b) follows.

Finally, statement (c) follows from Assumption 3.1(i) and (v). □

## E.4 Auxiliary lemmas to the auxiliary lemmas

**Lemma E.12.** *There exists a constant* $\tilde{C} \in (0,\infty)$ *such that*

$$E\left[\|y_t-\tilde{y}_t\|^2\right] \leq \tilde{C}T^{-2\zeta}. \tag{E.18}$$

*Proof.* The definition of $\tilde{y}_t$ implies

$$y_t-\tilde{y}_t = T^{-\zeta}\sum_{s=0}^{\infty}A^sH\alpha(L)\varepsilon_{t-s}.$$

Expanding $\alpha(L)=\sum_{\ell=1}^{\infty}\alpha_\ell L^\ell$, we obtain

$$y_t-\tilde{y}_t = T^{-\zeta}\sum_{s=1}^{\infty}B_s\varepsilon_{t-s}, \quad \text{where} \quad B_s \equiv \sum_{\ell=1}^{s}A^{s-\ell}H\alpha_\ell. \tag{E.19}$$

By the independence assumption on $\varepsilon_t$ in Assumption 3.1(i),

$$E\left[\|y_t-\tilde{y}_t\|^2\right] = T^{-2\zeta}\sum_{s=1}^{\infty}\text{trace}\left(B_sDB_s'\right).$$

Expanding $B_s$ and changing the summation indices shows that $E\left[\|y_t-\tilde{y}_t\|^2\right]$ equals

$$T^{-2\zeta}\sum_{s=1}^{\infty}\sum_{\ell_1=1}^{s}\sum_{\ell_2=1}^{s}\text{trace}\left(A^{s-\ell_1}H\alpha_{\ell_1}D\alpha_{\ell_2}'H'(A')^{s-\ell_2}\right).$$

Moreover, since for any two matrices $M_1, M_2$ of conformable dimensions $\text{trace}(M_1M_2) \leq \|M_1\|\|M_2\|$, then

$$\text{trace}\left(A^{s-\ell_1}H\alpha_{\ell_1}D\alpha_{\ell_2}'H'(A')^{s-\ell_2}\right) \leq \|H\|^2\cdot\|D\|\cdot\|A^{s-\ell_1}\|\cdot\|(A')^{s-\ell_2}\|\cdot\|\alpha_{\ell_1}\|\cdot\|\alpha_{\ell_2}\|.$$

Let $\lambda \in [0, 1)$ and $C > 0$ be chosen such that $\|A^\ell\| \le C\lambda^\ell$ for all $\ell \ge 0$ (such constants exists by Assumption 3.1(ii)). Then

$$\begin{aligned} E\left[\|y_t - \tilde{y}_t\|^2\right] &\le T^{-2\zeta} C^2 \|H\|^2 \|D\| \left(\sum_{\tau=0}^{\infty} \lambda^{2\tau}\right) \left(\sum_{\ell_1=1}^{\infty} \|\alpha_{\ell_1}\|\right) \left(\sum_{\ell_2=1}^{\infty} \|\alpha_{\ell_2}\|\right), \\ &\le T^{-2\zeta} C^2 \|H\|^2 \|D\| \left(\sum_{\tau=0}^{\infty} \lambda^{2\tau}\right) \left(\sum_{\ell=1}^{\infty} \|\alpha_\ell\|\right)^2, \\ &= T^{-2\zeta} \frac{C^2 \|H\|^2 \|D\|}{1-\lambda^2} \left(\sum_{\ell=1}^{\infty} \|\alpha_\ell\|\right)^2. \end{aligned}$$

□

**Lemma E.13.**

$$\frac{1}{\sqrt{T}} \sum_{t=1}^{T} \varepsilon_t \left(y_{t-1} - \tilde{y}_{t-1}\right)' = O_p\left(T^{-\zeta}\right).$$

*Proof.* By Markov's inequality, we need to show that the following expression is bounded:

$$T^{2\zeta} E\left[\left\|\frac{1}{\sqrt{T}} \sum_{t=1}^{T} \varepsilon_t \left(y_{t-1} - \tilde{y}_{t-1}\right)'\right\|^2\right].$$

Equation (E.19) in the proof of Lemma E.12 and Assumption 3.1(i) imply that the summands are serially uncorrelated, so the above expression equals

$$\begin{aligned} & T^{2\zeta} \frac{1}{T} \sum_{t=1}^{T} E\left[\|\varepsilon_t \left(y_{t-1} - \tilde{y}_{t-1}\right)'\|^2\right] \\ & \le T^{2\zeta} \frac{1}{T} \sum_{t=1}^{T} E\left[\|\varepsilon_t\|^2 \|y_{t-1} - \tilde{y}_{t-1}\|^2\right], \\ & = T^{2\zeta} \frac{1}{T} \sum_{t=1}^{T} E\left[\|\varepsilon_t\|^2\right] E\left[\|y_{t-1} - \tilde{y}_{t-1}\|^2\right], \\ & = T^{2\zeta} \operatorname{trace}(D)\, E\left[\|y_{t-1} - \tilde{y}_{t-1}\|^2\right]. \end{aligned}$$

The third line follows from Assumption 3.1(i), while the last line follows from stationarity. Lemma E.12 implies that the final expression is bounded. □

**Lemma E.14.**

$$\frac{1}{T} \sum_{t=1}^{T} \alpha(L) \varepsilon_t (y_{t-1} - \tilde{y}_{t-1})' = O_p(T^{-\zeta}).$$

*Proof.* By Markov's inequality, we need to show that

$$T^{\zeta} E\left[\left\|\frac{1}{T}\sum_{t=1}^{T}\alpha(L)\varepsilon_t\left(y_{t-1}-\tilde{y}_{t-1}\right)'\right\|\right]$$

is bounded. By stationarity and Cauchy-Schwarz, the expression is bounded above by

$$\begin{aligned}&T^{\zeta} E\left[\|\alpha(L)\varepsilon_t\|\,\|y_{t-1}-\tilde{y}_{t-1}\|\right]\\ &\leq T^{\zeta}\left(E\left[\|\alpha(L)\varepsilon_t\|^2\right]\right)^{1/2}\left(E\left[\|(y_t-\tilde{y}_{t-1})\|^2\right]\right)^{1/2}.\end{aligned}$$

The first expectation on the right-hand side is bounded due to Assumption 3.1(v). Hence, Lemma E.12 implies that the entire final expression is bounded. □

**Lemma E.15.**

$$T^{-1}\sum_{t=1}^{T}\left(\alpha(L)\varepsilon_t\tilde{y}'_{t-1}-E[\alpha(L)\varepsilon_t\tilde{y}'_{t-1}]\right)=o_p(1).$$

*Proof.* For an arbitrary $i\in\{1,\ldots,n\}$ and $s\geq 1$, define

$$\begin{aligned}\Gamma_s &\equiv \mathrm{Cov}(\alpha(L)\varepsilon_t\tilde{y}_{i,t-1},\alpha(L)\varepsilon_{t-s}\tilde{y}_{i,t-s-1})\\ &=\mathrm{Cov}\left(\sum_{\ell_1=1}^{\infty}\alpha_{\ell_1}\varepsilon_{t-\ell_1}\tilde{y}_{i,t-1},\sum_{\ell_2=1}^{\infty}\alpha_{\ell_2}\varepsilon_{t-s-\ell_2}\tilde{y}_{i,t-s-1}\right)\\ &=\sum_{\ell_1=1}^{\infty}\sum_{\ell_2=1}^{\infty}\alpha_{\ell_1}\,\mathrm{Cov}(\varepsilon_{t-\ell_1}\tilde{y}_{i,t-1},\varepsilon_{t-s-\ell_2}\tilde{y}_{i,t-s-1})\alpha'_{\ell_2}.\end{aligned}$$

By Theorem 7.1.1 in Brockwell and Davis (1991), the statement of the lemma follows if we can show that $\Gamma_s\to 0$ as $s\to\infty$.

Decompose

$$\tilde{y}_{i,t-1}=\underbrace{E[\tilde{y}_{i,t-1}\mid\{\varepsilon_{t-s}\}_{s=1}^{\ell_1-1}]}_{\equiv\tilde{y}^{(-)}_{i,t-1}}+\underbrace{E[\tilde{y}_{i,t-1}\mid\varepsilon_{t-\ell_1}]}_{\equiv\tilde{y}^{(0)}_{i,t-1}}+\underbrace{E[\tilde{y}_{i,t-1}\mid\{\varepsilon_{t-s}\}_{s=\ell_1+1}^{\infty}]}_{\equiv\tilde{y}^{(+)}_{i,t-1}}.$$

For $\ell_1\leq s$, the serial independence of $\varepsilon_t$ implies that

$$\mathrm{Cov}(\varepsilon_{t-\ell_1}\tilde{y}^{(-)}_{i,t-1},\varepsilon_{t-s-\ell_2}\tilde{y}_{i,t-s-1})=E[\tilde{y}^{(-)}_{i,t-1}]E[\varepsilon_{t-\ell_1}\varepsilon'_{t-s-\ell_2}\tilde{y}_{i,t-s-1}]=0,$$

$$\mathrm{Cov}(\varepsilon_{t-\ell_1}\tilde{y}^{(0)}_{i,t-1},\varepsilon_{t-s-\ell_2}\tilde{y}_{i,t-s-1})=0,$$

$$\mathrm{Cov}(\varepsilon_{t-\ell_1}\tilde{y}^{(+)}_{i,t-1},\varepsilon_{t-s-\ell_2}\tilde{y}_{i,t-s-1})=E[\varepsilon_{t-\ell_1}]E[\tilde{y}^{(+)}_{i,t-1}\varepsilon'_{t-s-\ell_2}\tilde{y}_{i,t-s-1}]=0,$$

and therefore

$$\operatorname{Cov}(\varepsilon_{t-\ell_1}\tilde{y}_{i,t-1}, \varepsilon_{t-s-\ell_2}\tilde{y}_{i,t-s-1}) = 0.$$

Inserting this result back into the earlier expression for $\Gamma_s$, we get

$$\begin{aligned}
|\Gamma_s| &= \left| \sum_{\ell_1=s+1}^{\infty} \sum_{\ell_2=1}^{\infty} \alpha_{\ell_1} \operatorname{Cov}(\varepsilon_{t-\ell_1}\tilde{y}_{i,t-1}, \varepsilon_{t-s-\ell_2}\tilde{y}_{i,t-s-1}) \alpha'_{\ell_2} \right| \\
&\leq \sum_{\ell_1=s+1}^{\infty} \sum_{\ell_2=1}^{\infty} \|\alpha_{\ell_1}\| \cdot \|\alpha_{\ell_2}\| \cdot \| \operatorname{Cov}(\varepsilon_{t-\ell_1}\tilde{y}_{i,t-1}, \varepsilon_{t-s-\ell_2}\tilde{y}_{i,t-s-1})\| \\
&\leq \sum_{\ell_1=s+1}^{\infty} \sum_{\ell_2=1}^{\infty} \|\alpha_{\ell_1}\| \cdot \|\alpha_{\ell_2}\| \cdot \sup_{\ell \geq 1} \| \operatorname{Var}(\varepsilon_{t-\ell}\tilde{y}_{i,t-1})\| \\
&\leq \underbrace{\left( E[\|\varepsilon_t^4\|] \cdot E[\tilde{y}_{i,t}^4] \right)^{1/2} \left( \sum_{\ell_2=1}^{\infty} \|\alpha_{\ell_2}\| \right)}_{<\infty} \left( \sum_{\ell_1=s+1}^{\infty} \|\alpha_{\ell_1}\| \right) \\
&\to 0 \quad \text{as } s \to \infty,
\end{aligned}$$

where the last line uses absolute summability of $\alpha(L)$. □

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