GLOBAL GAME SELECTIONS IN BINARY-ACTION SUPERMODULAR GAMES

STEPHEN MORRIS, DAISUKE OYAMA, AND SATORU TAKAHASHI

Abstract. We characterize global game selections in binary-action supermodular games in terms of *sequential obedience*.

1. Introduction

In *global games* (Carlsson and van Damme (1993), Frankel et al. (2003)), the payoffs are determined by the state of the world θ, while each player observes a noisy signal $x_i = \theta + \kappa\varepsilon_i$, where the noise terms ε_i are independent of the state θ, and $\kappa > 0$ is a scale parameter. Under supermodularity and state-monotonicity of payoffs and the existence of dominance regions, Frankel et al. (2003) showed for many-player many-action games that an essentially unique equilibrium survives iterative deletion of dominated strategies as $\kappa \to 0$ (limit uniqueness), while the limit equilibrium may depend on the distribution of the noise terms (noise dependence). An action profile a is a *global game selection* in a complete information game given by the payoffs at θ^* if there exists a noise distribution under which the limit equilibrium plays a at θ^*. In this note, we characterize global game selections in binary-action supermodular (BAS) games in terms of the condition of *sequential obedience* (and its reverse version) introduced by Morris et al. (2022) for a characterization of (smallest equilibrium or full) implementability by information design: we show that in BAS games, an action profile is a global game selection if and only if it satisfies sequential obedience and reverse sequential obedience.

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2. Binary-Action Supermodular Games

The finite set of players is denoted by $I = \{1, \dots, |I|\}$. Each player $i \in I$ has binary actions, $A_i = \{0, 1\}$. A set S of players in I (resp. players in $I \setminus \{i\}$) is identified with the action profile of all players (resp. opponents of player i) where player j chooses action 1 if and only if $j \in S$. Action profiles of all players and those of opponents of player i are partially ordered by set inclusion. A complete information binary-action game is represented by a profile $\mathbf{f} = (f_i)_{i \in I}$ of payoff gain functions $f_i \colon 2^{I \setminus \{i\}} \to \mathbb{R}$, where $f_i(S)$ is player i's payoff gain from playing action 1 over action 0 when the subset $S \subset I \setminus \{i\}$ of players play action 1. We assume *supermodular payoffs*: $f_i(S)$ is nondecreasing in S (i.e., $f_i(S) \leq f_i(S')$ whenever $S \subset S'$).

Let Γ be the set of all sequences of distinct players, and for $S \subset I$, let $\Pi(S) \subset \Gamma$ be the set of permutations of the players in S. For a sequence $\gamma \in \Gamma$ and for a player $i \in I$ that appears in γ, let $S(i, \gamma) \subset I \setminus \{i\}$ be the set of players who appear before i in γ (which represents the action profile of opponent players where these players play action 1 and the others play action 0), and let $S^0(i, \gamma) = (I \setminus \{i\}) \setminus S(i, \gamma)$ (which represents the action profile of opponent players where the players before i in γ play action 0 and the others play action 1). An action profile $S^* \subset I$ satisfies *sequential obedience* (resp. *strict sequential obedience*) in $\mathbf{f}$ if there exists $\rho \in \Delta(\Pi(S^*))$ such that

$$\sum_{\gamma \in \Pi(S^*)} \rho(\gamma) f_i(S(i, \gamma)) \geq \text{(resp. } > \text{) } 0 \tag{2.1}$$

for all $i \in S^*$; it satisfies *reverse sequential obedience* (resp. *strict reverse sequential obedience*) in $\mathbf{f}$ if there exists $\rho^0 \in \Delta(\Pi(I \setminus S^*))$ such that

$$\sum_{\gamma \in \Pi(I \setminus S^*)} \rho^0(\gamma) f_i(S^0(i, \gamma)) \leq \text{(resp. } < \text{) } 0 \tag{2.2}$$

for all $i \in I \setminus S^*$. We also say that ρ satisfies sequential obedience (resp. strict sequential obedience) in $\mathbf{f}$ if (2.1) holds and that ρ^0 satisfies reverse sequential obedience (resp. strict reverse sequential obedience) in $\mathbf{f}$ if (2.2) holds. Trivially by definition, $\emptyset$ satisfies strict sequential obedience, and I satisfies strict reverse sequential obedience. In *generic* BAS games, an action profile that satisfies sequential obedience (resp. reverse sequential obedience) also satisfies strict sequential obedience (resp. strict reverse sequential obedience).

If an action profile $S^* \subset I$ satisfies both sequential obedience and reverse sequential obedience (resp. strict sequential obedience and strict reverse sequential obedience) in $\mathbf{f}$, then by supermodularity, it is a Nash equilibrium (resp. strict Nash equilibrium) in $\mathbf{f}$. The converse does not hold in general. For example, in the case of two players where $f_i(\emptyset) < 0 < f_i(\{3-i\})$ for each $i \in I = \{1,2\}$, the Nash equilibrium I (which satisfies strict reverse sequential obedience trivially) satisfies sequential obedience (resp. strict sequential obedience) in $\mathbf{f}$ if and only if there exists $\rho \in \Delta(\Pi(I))$ (where $\Pi(I) = \{12, 21\}$) such that

$$\rho(12)f_1(\emptyset) + \rho(21)f_1(\{2\}) \geq \text{(resp. } >\text{) } 0,$$
$$\rho(21)f_2(\emptyset) + \rho(12)f_2(\{1\}) \geq \text{(resp. } >\text{) } 0,$$

which holds if and only if $f_1(\emptyset)f_2(\emptyset) \leq$ (resp. $<$) $f_1(\{2\})f_2(\{1\})$, that is, I is weakly risk dominant (resp. strictly risk dominant) in $\mathbf{f}$.

From Proposition B.2 in Morris et al. (2022), we have:

Proposition 1. (1) *In any BAS game, there exist (i) a largest action profile that satisfies sequential obedience, which also satisfies strict reverse sequential obedience, and (ii) a smallest action profile that satisfies reverse sequential obedience, which also satisfies strict sequential obedience.*

(2) *In any generic BAS game, there exist a largest and a smallest action profiles that satisfy strict sequential obedience and strict reverse sequential obedience.*

An action profile $S^* \subset I$ is a *strict monotone potential maximizer* (*strict MP-maximizer*) in $\mathbf{f}$ if there exist a function $v\colon 2^I \to \mathbb{R}$ and $\lambda = (\lambda_i)_{i \in I} \in \mathbb{R}^I_{++}$ such that

$$\lambda_i f_i(S) \geq v(S \cup \{i\}) - v(S) \tag{2.3}$$

for all $i \in S^*$ and $S \subset I \setminus \{i\}$,

$$\lambda_i f_i(S) \leq v(S \cup \{i\}) - v(S) \tag{2.4}$$

for all $i \in I \setminus S^*$ and $S \subset I \setminus \{i\}$, and $v(S^*) > v(S)$ for all $S \neq S^*$.[1] Such a function v is called a *strict monotone potential* of $\mathbf{f}$ for S^*.

From Lemmas 2 and A.1 in Oyama and Takahashi (2020) and Proposition 1(1) above, the following relationship holds between (reverse) sequential obedience and strict MP-maximization:

[1]In BAS games, this is equivalent to local potential maximizer of Frankel et al. (2003, Definition 1) and strict monotone potential maximizer and strict local potential maximizer of Oyama et al. (2008, Definitions 4.2 and 4.4) and is stronger than monotone potential maximizer and local potential maximizer of Morris and Ui (2005, Definitions 8 and 11). Oyama and Takahashi (2020) used this strict version, while referring to it simply as monotone potential maximizer (without the qualifier "strict").

Proposition 2. *For any BAS game* $\mathbf{f}$*, an action profile is a unique action profile that satisfies sequential obedience and reverse sequential obedience in* $\mathbf{f}$ *if and only if it is a strict MP-maximizer in* $\mathbf{f}$*.*

Note that by Proposition 1(1), if the condition in Proposition 2 holds, then that action profile in fact satisfies strict sequential obedience and strict reverse sequential obedience.

3. Global Game Selections

We define global games as in Frankel et al. (2003) (FMP, henceforth), but specializing to our binary-action case. In a global game with player set I and action sets $A_i = \{0, 1\}$, a state of the world θ is drawn from the real line according to a continuous density ϕ with connected support, and the payoffs are represented by a profile $\mathbf{d} = (d_i)_{i\in I}$ of payoff gain functions $d_i \colon 2^{I\setminus\{i\}} \times \mathbb{R} \to \mathbb{R}$, where $d_i(S, \theta)$ is player i's payoff gain from action 1 over action 0 when the subset $S \subset I \setminus \{i\}$ of players play action 1 and the state is $\theta \in \mathbb{R}$. Each player i observes a noisy signal $x_i = \theta + \kappa\varepsilon_i$, where $\kappa > 0$ is a scale parameter, and the noise profile $(\varepsilon_i)_{i\in I}$ is distributed independently of θ according to a continuous joint density ψ with support contained in $[-\frac{1}{2}, \frac{1}{2}]^I$.[2] The assumptions from FMP are imposed on the payoffs:

A1. Strategic complementarities: For all $i \in I$ and $\theta \in \mathbb{R}$, $d_i(S, \theta)$ is nondecreasing in S.

A2. Dominance regions: There exist $\underline{\theta} < \overline{\theta}$ in the interior of the support of ϕ such that for all $i \in I$, $d_i(I \setminus \{i\}, \theta) < 0$ if $\theta \le \underline{\theta}$ and $d_i(\emptyset, \theta) > 0$ if $\theta \ge \overline{\theta}$.

A3. State monotonicity: There exists $K_0 > 0$ such that for all $i \in I$ and all $S \subset I \setminus \{i\}$, $d_i(S, \theta) - d_i(S, \theta') \ge K_0(\theta - \theta')$ if $\theta \ge \theta'$, $\theta, \theta' \in [\underline{\theta}, \overline{\theta}]$.

A4. Payoff continuity: For all $i \in I$ and $S \subset I \setminus \{i\}$, $d_i(S, \theta)$ is continuous in θ.

We always consider $\kappa > 0$ small enough so that $[\underline{\theta} - \kappa, \overline{\theta} + \kappa]$ is contained in the interior of the support of ϕ. A strategy of player $i \in I$ is a measurable function $s_i \colon \mathbb{R} \to A_i$, where $s_i(x_i)$ is the action that the player plays when observing signal x_i. We denote this game by $G^\kappa(\mathbf{d}, \phi, \psi)$, and we refer to $\mathbf{d}$ as a base global game. FMP showed that in these games, an essentially unique equilibrium survives iterative deletion of dominated strategies as $\kappa \to 0$.

[2]We follow the original formulation of Carlsson and van Damme (1993) to allow correlation in noise terms among players. The results of FMP continue to hold even without their assumption of independence among noise terms.

To prove this result, FMP introduced *simplified global games.* In the simplified global game $G^{*\kappa}(\mathbf{d},\psi)$, the state θ is drawn from the uniform distribution over some large interval that contains $[\underline{\theta}-\kappa,\overline{\theta}+\kappa]$, and each player i's payoff gain is given by $d_i(S,x_i)$ depending directly on his signal x_i, rather than on θ. FMP showed that an essentially unique equilibrium survives iterative deletion of dominated strategies in the simplified game $G^{*\kappa}(\mathbf{d},\psi)$, and it converges as $\kappa\to 0$ to the same limit equilibrium as the original game $G^{\kappa}(\mathbf{d},\phi,\psi)$. Formally, applied to our binary-action setting, Theorem 1 and Lemmas A1, A3, and A4 of FMP imply:

- For each $\kappa>0$, there exists a nondecreasing strategy profile $(s_i^{*\kappa})_{i\in I}$, where we denote the cutoff of $s_i^{*\kappa}$ by $\xi_i^{*\kappa}$, such that if $(s_i)_{i\in I}$ is a strategy profile that survives iterated deletion of strictly dominated strategies in $G^{*\kappa}(\mathbf{d},\psi)$, then for each $i\in I$, s_i agrees with $s_i^{*\kappa}$ except perhaps at $\xi_i^{*\kappa}$.
- For each $i\in I$, $\xi_i^{*\kappa}$ converges to some ξ_i^* as $\kappa\to 0$.
- For any $\delta>0$, there exists $\bar{\kappa}>0$ such that for any $\kappa\in(0,\bar{\kappa}]$, if $(s_i)_{i\in I}$ is a strategy profile that survives iterated deletion of strictly dominated strategies in $G^{\kappa}(\mathbf{d},\phi,\psi)$, then for each $i\in I$, $s_i(x_i)=0$ for all $x_i<\xi_i^*-\delta$ and $s_i(x_i)=1$ for all $x_i>\xi_i^*+\delta$.

Note in particular that the cutoff profile $(\xi_i^*)_{i\in I}$ characterizes the limit equilibrium of $G^{\kappa}(\mathbf{d},\phi,\psi)$ independent of the prior distribution ϕ.

To define global game selections in complete information BAS game $\mathbf{f}$, fix any state $\theta^*\in(\underline{\theta},\overline{\theta})$, where we let $\theta^*=0$ without loss of generality. Base global game $\mathbf{d}$ *embeds* $\mathbf{f}$ (at $\theta^*=0$) if $d_i(\cdot,0)=f_i(\cdot)$ for all $i\in I$. Let $\mathbf{d}$ embed $\mathbf{f}$, and let $(\xi_i^*)_{i\in I}$ be the (common) limit cutoff profile for $G^{\kappa}(\mathbf{d},\phi,\psi)$ and $G^{*\kappa}(\mathbf{d},\psi)$ as above. Denote $\underline{S}(\mathbf{f},\mathbf{d},\psi)=\{i\in I\mid \xi_i^*<0\}$ and $\overline{S}(\mathbf{f},\mathbf{d},\psi)=\{i\in I\mid \xi_i^*\leq 0\}$, which are the action profiles played at $\theta^*=0$ by the left- and the right-continuous cutoff strategy profiles defined by $(\xi_i^*)_{i\in I}$, respectively. In fact, as shown by Basteck et al. (2013), $\underline{S}(\mathbf{f},\mathbf{d},\psi)$ and $\overline{S}(\mathbf{f},\mathbf{d},\psi)$ do not depend on the choice of the base global game $\mathbf{d}$ that embeds $\mathbf{f}$; thus we denote these by $\underline{S}(\mathbf{f},\psi)$ and $\overline{S}(\mathbf{f},\psi)$. An action profile $S^*\subset I$ is a *global game selection* in $\mathbf{f}$ if $S^*=\underline{S}(\mathbf{f},\psi)=\overline{S}(\mathbf{f},\psi)$ for some noise distribution ψ; S^* is a *noise-independent global game selection* in $\mathbf{f}$ if it is a ψ-global game selection in $\mathbf{f}$ for all noise distributions ψ.

4. Results

Our main result is:

Theorem 1. *For any BAS game* $\mathbf{f}$*, an action profile is a global game selection in* $\mathbf{f}$ *if and only if it satisfies strict sequential obedience and strict reverse sequential obedience in* $\mathbf{f}$*.*

The theorem follows from Lemmas 1 and 2 to be stated and proved in Section 6.

By Theorem 1 and Proposition 1(2), we have:

Corollary 1. *In any generic BAS game, there exist a largest and a smallest global game selections.*

By Lemmas 1 and 2 and Propositions 1(1) and 2, we also have:

Theorem 2. *For any BAS game* $\mathbf{f}$*, an action profile is a noise-independent global game selection in* $\mathbf{f}$ *if and only if it is a strict MP-maximizer in* $\mathbf{f}$*.*

FMP showed that a local potential maximizer is a noise-independent global game selection in many-action supermodular games that satisfy own-action concavity.[3] Thus, the "if" part of Theorem 2 follows also as a special case of that result. Oyama and Takahashi (2020) proved the "only if" part of Theorem 2 for BAS games under a genericity assumption. Note that our results, Theorems 1 and 2, apply to all (whether generic or nongeneric) BAS games. Our Theorem 1 refines these existing results for BAS games with no strict MP-maximizer, characterizing the set of all global game selections in terms of the condition of (reverse) sequential obedience.

In Morris et al. (2022), we used (reverse) sequential obedience to characterize implementability by information design in incomplete information BAS games (with discrete states). In particular, we showed that (1) if an outcome (a joint distribution over actions and states) is induced by a unique equilibrium for some information structure, then it satisfies (an incomplete information version of) sequential obedience and reverse sequential obedience, and (2) the converse also holds under a dominance states assumption. In this context, Theorem 1 in the present paper may be viewed as a characterization result on information design by global games (with continuous states and under state monotonicity).

[3] See also Oyama and Takahashi (2009, Remark 1).

5. Example

As an illustration, we consider a three-player BAS game with cyclically symmetric interactions, studied in Oyama and Takahashi (2019, Example 8). Let $I = \{1, 2, 3\}$, and let $\mathbf{f} = (f_i)_{i \in I}$ be given by

$$f_i(S) = r_i(S) - c_i \tag{5.1a}$$

with

$$r_i(S) = \begin{cases} 0 & \text{if } S = \emptyset, \\ b_0 & \text{if } S = \{i-1\}, \\ b_1 & \text{if } S = \{i+1\}, \\ b_2 & \text{if } S = \{i-1, i+1\}, \end{cases} \tag{5.1b}$$

where $0 < b_0 < b_1 < b_2$ and $0 \leq c_i \leq b_2$, and $i-1$ and $i+1$ are understood modulo 3. Action profiles I and $\emptyset$ are the (pure action) Nash equilibria of this game. We want to identify conditions for I and $\emptyset$ to be global game selections, or equivalently, satisfy strict sequential obedience and strict reverse sequential obedience.

For $\gamma \in \Pi(I)$, let $\alpha^\gamma = (\alpha_i^\gamma)_{i \in I} \in \mathbb{R}^I$ be the vector defined by $\alpha_i^\gamma = r_i(S(i, \gamma))$. For example, $\alpha^{123} = (0, b_0, b_2)$ and $\alpha^{132} = (0, b_2, b_1)$. Define the sets

$$\begin{aligned} C_I &= \{x \in [0, b_2]^I \mid x \leq y \text{ for some } y \in \mathrm{conv}\{\alpha^\gamma \mid \gamma \in \Pi(I)\}\}, \\ C_I^\circ &= \{x \in [0, b_2]^I \mid x \ll y \text{ for some } y \in \mathrm{conv}\{\alpha^\gamma \mid \gamma \in \Pi(I)\}\}, \\ C_\emptyset &= \{x \in [0, b_2]^I \mid x \geq y \text{ for some } y \in \mathrm{conv}\{\alpha^\gamma \mid \gamma \in \Pi(I)\}\}, \\ C_\emptyset^\circ &= \{x \in [0, b_2]^I \mid x \gg y \text{ for some } y \in \mathrm{conv}\{\alpha^\gamma \mid \gamma \in \Pi(I)\}\}. \end{aligned}$$

The set C_I is depicted in Figure 1. Immediately from the definition of (reverse) sequential obedience, I satisfies sequential obedience (resp. strict sequential obedience) if and only if $(c_1, c_2, c_3) \in C_I$ (resp. $(c_1, c_2, c_3) \in C_I^\circ$), while $\emptyset$ satisfies reverse sequential obedience (resp. strict reverse sequential obedience) if and only if $(c_1, c_2, c_3) \in C_\emptyset$ (resp. $(c_1, c_2, c_3) \in C_\emptyset^\circ$). Therefore, by Theorems 1 2, we have:

Proposition 3. *In the BAS game* $\mathbf{f}$ *given by* (5.1),
(1) *I is a global game selection if and only if* $(c_1, c_2, c_3) \in C_I^\circ$;
(2) *I is a noise-independent global game selection if and only if* $(c_1, c_2, c_3) \in C_I^\circ \setminus C_\emptyset$;
(3) *$\emptyset$ is a global game selection if and only if* $(c_1, c_2, c_3) \in C_\emptyset^\circ$; *and*
(4) *$\emptyset$ is a noise-independent global game selection if and only if* $(c_1, c_2, c_3) \in C_\emptyset^\circ \setminus C_I$.

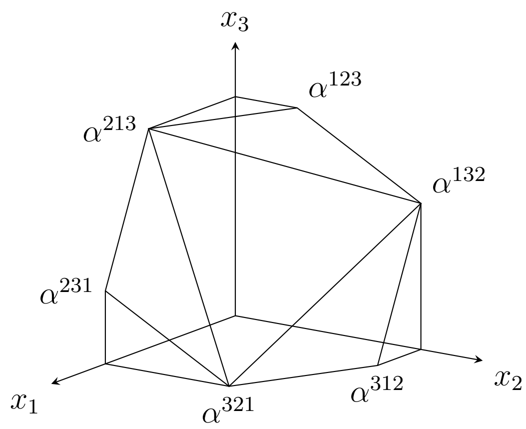

FIGURE 1. The set C_I

In the symmetric case where $c_1 = c_2 = c_3 = c$ (the case considered in Oyama and Takahashi (2019, Example 8)), if $c < \frac{b_1+b_2}{3}$, then I is a global game selection; if $c < \frac{b_0+b_2}{3}$, then I is a noise-independent global game selection; if $c > \frac{b_0+b_2}{3}$, then $\emptyset$ is a global game selection; and if $c > \frac{b_1+b_2}{3}$, then $\emptyset$ is a noise-independent global game selection.

6. Proof of Theorem 1

We prove the "only if" part and the "if" part of Theorem 1 in Subsections 6.1 and 6.2, respectively.

6.1. **Proof of the "Only If" Part of Theorem 1.**

The "only if" part of Theorem 1 immediately follows from the following:

Lemma 1. *For any BAS game* $\mathbf{f}$ *and for any noise distribution* ψ*,*
(1) $\overline{S}(\mathbf{f},\psi)$ *satisfies sequential obedience and strict reverse sequential obedience in* $\mathbf{f}$*, and*
(2) $\underline{S}(\mathbf{f},\psi)$ *satisfies strict sequential obedience and reverse sequential obedience in* $\mathbf{f}$*.*

Proof. We only prove part (1); the proof of part (2) is symmetric.

Given any BAS game $\mathbf{f}$, take any base global game $\mathbf{d}$ that embeds $\mathbf{f}$ and any noise distribution ψ. Let $(\xi_i^{*\kappa})_{i\in I}$ be the equilibrium cutoff profile of the simplified global game $G^{*\kappa}(\mathbf{d},\psi)$, and let $\xi_i^* = \lim_{\kappa\to 0}\xi_i^{*\kappa}$. Denote $S^* = \overline{S}(\mathbf{f},\psi)$ $(= \{i\in I \mid \xi_i^* \le 0\})$. Let $\alpha > 0$ be such that $\xi_i^* \ge \alpha$ for all $i \in I\setminus S^*$. For each $\delta \in (0, \frac{\alpha}{3}]$, let $\kappa(\delta)\in(0,\delta]$ be such that $\xi_i^{*\kappa(\delta)} < \delta$ for all $i\in S^*$ and $\xi_i^{*\kappa(\delta)} > \alpha-\delta$ for all $i\in I\setminus S^*$. Then define $\rho^\delta \in \Delta(\Pi(S^*))$ by

$$\rho^\delta(\gamma) = \mathbb{P}\left(\kappa(\delta)\varepsilon_{i_1} - \xi_{i_1}^{*\kappa(\delta)} \ge \cdots \ge \kappa(\delta)\varepsilon_{i_{|S^*|}} - \xi_{i_{|S^*|}}^{*\kappa(\delta)}\right)$$

for $\gamma = (i_1, \ldots, i_{|S^*|}) \in \Pi(S^*)$, and $\rho^{0,\delta} \in \Delta(\Pi(I \setminus S^*))$ by

$$\rho^{0,\delta}(\gamma) = \mathbb{P}\left(\kappa(\delta)\varepsilon_{i_1} - \xi_{i_1}^{*\kappa(\delta)} \leq \cdots \leq \kappa(\delta)\varepsilon_{i_{|I\setminus S^*|}} - \xi_{i_{|I\setminus S^*|}}^{*\kappa(\delta)}\right)$$

for $\gamma = (i_1, \ldots, i_{|I\setminus S^*|}) \in \Pi(I \setminus S^*)$.

For $\delta \in (0, \frac{\alpha}{3}]$, let $\kappa = \kappa(\delta)$ $(\leq \delta)$, and consider the game $G^{*\kappa}(\mathbf{d}, \psi)$. By construction, if a player observes a signal equal to δ (resp. $\alpha - \delta$), then he knows that every player $j \in I \setminus S^*$ (resp. $j \in S^*$) receives a signal $x_j \leq \alpha - \delta < \xi_j^{*\kappa}$ (resp. $x_j \geq \delta > \xi_j^{*\kappa}$). Therefore, for each player $i \in S^*$, if he observes a signal $x_i = \delta$ $(> \xi_i^{*\kappa})$, we have

$$\begin{aligned}
0 &\leq \mathbb{E}[d_i(\{j \in I \setminus \{i\} \mid x_j \geq \xi_j^{*\kappa}\}, x_i)|x_i = \delta] \\
&= \mathbb{E}[d_i(\{j \in S^* \setminus \{i\} \mid \kappa\varepsilon_j - \xi_j^{*\kappa} \geq \kappa\varepsilon_i - \xi_i^{*\kappa}\}, x_i)|x_i = \delta] \\
&= \mathbb{E}[d_i(\{j \in S^* \setminus \{i\} \mid \kappa\varepsilon_j - \xi_j^{*\kappa} \geq \kappa\varepsilon_i - \xi_i^{*\kappa}\}, \delta)] \\
&= \sum_{\gamma \in \Pi(S^*)} \rho^\delta(\gamma) d_i(S(i, \gamma), \delta),
\end{aligned}$$

where the second equality holds due to the assumption of the uniform prior. Now let $\delta \to 0$. Let $\rho \in \Delta(\Pi(S^*))$ be a limit point of ρ^δ as $\delta \to 0$. Then by payoff continuity, we have

$$0 \leq \sum_{\gamma \in \Pi(S^*)} \rho(\gamma) f_i(S(i, \gamma))$$

for all $i \in S^*$. That is, $\rho \in \Delta(\Pi(S^*))$ satisfies sequential obedience.

Similarly, for each player $i \in I \setminus S^*$, if he observes a signal $x_i = \alpha - \delta$ $(< \xi_i^{*\kappa})$, we have

$$\begin{aligned}
0 &\geq \mathbb{E}[d_i(\{j \in I \setminus \{i\} \mid x_j \geq \xi_j^{*\kappa}\}, x_i)|x_i = \alpha - \delta] \\
&= \mathbb{E}[d_i(\{j \in (I \setminus S^*) \setminus \{i\} \mid \kappa\varepsilon_j - \xi_j^{*\kappa} \geq \kappa\varepsilon_i - \xi_i^{*\kappa}\} \cup S^*, x_i)|x_i = \alpha - \delta] \\
&= \mathbb{E}[d_i(\{j \in (I \setminus S^*) \setminus \{i\} \mid \kappa\varepsilon_j - \xi_j^{*\kappa} \geq \kappa\varepsilon_i - \xi_i^{*\kappa}\} \cup S^*, \alpha - \delta)] \\
&= \sum_{\gamma \in \Pi(I\setminus S^*)} \rho^{0,\delta}(\gamma) d_i(S^0(i, \gamma), \alpha - \delta) \\
&> \sum_{\gamma \in \Pi(I\setminus S^*)} \rho^{0,\delta}(\gamma) f_i(S^0(i, \gamma)),
\end{aligned}$$

where the second equality holds due to the assumption of the uniform prior, and the last inequality follows from state monotonicity. Therefore, $\rho^{0,\delta} \in \Delta(\Pi(I \setminus S^*))$ satisfies strict reverse sequential obedience (for any $\delta \leq \frac{\alpha}{3}$). □

6.2. **Proof of the "If" Part of Theorem 1.** The "if" part of Theorem 1 immediately follows from the following:

Lemma 2. *For any BAS game* $\mathbf{f}$ *and for any* $S^* \subset I$ *and any* $\rho \in \Delta(\Pi(S^*))$ *and* $\rho^0 \in \Delta(\Pi(I \setminus S^*))$*, there exists a noise distribution* ψ *such that*

(1) *if ρ satisfies sequential obedience and ρ^0 satisfies strict reverse sequential obedience in* $\mathbf{f}$*, then* $S^* = \overline{S}(\mathbf{f}, \psi)$*, and*

(2) *if ρ satisfies strict sequential obedience and ρ^0 satisfies reverse sequential obedience in* $\mathbf{f}$*, then* $S^* = \underline{S}(\mathbf{f}, \psi)$.

The proof below is an adaptation of the argument in Oyama and Takahashi (2020, Supplemental Material).

Proof. Let any BAS game $\mathbf{f}$ be given, and fix any $S^* \subset I$. Let $\rho \in \Delta(\Pi(S^*))$ and $\rho^0 \in \Delta(\Pi(I \setminus S^*))$ be given. We transform these discrete distributions to obtain a continuous density function ψ on $[-\frac{1}{2}, \frac{1}{2}]^I$. First, permutations $\gamma \in \Pi(S^*)$ and $\gamma^0 \in \Pi(I \setminus S^*)$ are drawn according to ρ and ρ^0, respectively. The discrete noise profile $(\ell_i)_{i \in I}$ is then given by

$$\ell_i = \begin{cases} |S^*| - \ell(i, \gamma) + 1 & \text{if } i \in S^*, \\ \ell(i, \gamma^0) & \text{if } i \in I \setminus S^*, \end{cases}$$

where for $\gamma = (i_1, \ldots, i_k)$ and for i that appears in γ, $\ell(i, \gamma) = \ell$ if $i = i_\ell$. To make the noises continuous, add to $(\ell_i)_{i \in I}$ i.i.d. continuous random variables $(\zeta_i)_{i \in I}$ independent of $(\ell_i)_{i \in I}$ where each ζ_i has a density function $3 - 9|z|$ with support $[-\frac{1}{3}, \frac{1}{3}]$. Then map $(\ell_i + \zeta_i)_{i \in I}$ into $[-\frac{1}{2}, \frac{1}{2}]^I$ by letting $\varepsilon_i = \frac{1}{2}(\ell_i + \zeta_i)/(|I| + \frac{1}{3})$ for each $i \in I$. Finally, let ψ be the density function of $(\varepsilon_i)_{i \in I}$, which is continuous and whose support is contained in $[-\frac{1}{2}, \frac{1}{2}]^I$. Note that, by construction, for $i, j \in S^*$, $i \neq j$, we have $x_j \geq x_i$ if and only if $\ell_j < \ell_i$, and for $i, j \in I \setminus S^*$, $i \neq j$, we have $x_j \geq x_i$ if and only if $\ell_j > \ell_i$.

Now let $\mathbf{d}$ be any base global game that embeds $\mathbf{f}$, and consider the simplified global games $G^{*\kappa}(\mathbf{d}, \psi)$. Let $(\xi_i^{*\kappa})_{i \in I}$ be the equilibrium cutoff profile of $G^{*\kappa}(\mathbf{d}, \psi)$, and let $\xi_i^* = \lim_{\kappa \to 0} \xi_i^{*\kappa}$. Note that at the equilibrium of $G^{*\kappa}(\mathbf{d}, \psi)$, each player i must be indifferent between the two actions if he observes a signal $x_i = \xi_i^{*\kappa}$ by the continuity of the expected payoffs in the signal x_i, which follows from the continuity of the prior density and the noise density ψ. We only prove part (1); the proof of part (2) is symmetric. Suppose that ρ satisfies sequential obedience and ρ^0 satisfies strict reverse sequential obedience in $\mathbf{f}$. We want to show that $S^* = \overline{S}(\mathbf{f}, \psi)$ $(= \{i \in I \mid \xi_i^* \leq 0\})$.

First, we show that $\max_{i \in S^*} \xi_i^* \leq 0$. Assume to the contrary that $\max_{i \in S^*} \xi_i^* > 0$. Let $\kappa > 0$ be sufficiently small that $\max_{i \in S^*} \xi_i^{*\kappa} > 0$. Let $i \in S^*$ be such that $\xi_i^{*\kappa} \geq \xi_j^{*\kappa}$ for all $j \in S^*$, where $\xi_i^{*\kappa} > 0$. For such i, we have

$$0 = \mathbb{E}[d_i(\{j \in I \setminus \{i\} \mid x_j \geq \xi_j^{*\kappa}\}, x_i) | x_i = \xi_i^{*\kappa}]$$

$$
\begin{aligned}
&\geq \mathbb{E}[d_i(\{j \in S^* \setminus \{i\} \mid x_j \geq x_i\}, x_i)|x_i = \xi_i^{*\kappa}] \\
&= \mathbb{E}[d_i(\{j \in S^* \setminus \{i\} \mid \ell_j < \ell_i\}, x_i)|x_i = \xi_i^{*\kappa}] \\
&= \mathbb{E}[d_i(\{j \in S^* \setminus \{i\} \mid \ell_j < \ell_i\}, \xi_i^{*\kappa})] \\
&> \mathbb{E}[f_i(\{j \in S^* \setminus \{i\} \mid \ell_j < \ell_i\})] \\
&= \sum_{\gamma \in \Pi(S^*)} \rho(\gamma) f_i(S(i,\gamma)),
\end{aligned}
$$

which contradicts the sequential obedience of ρ, where the weak inequality follows from strategic complementarities, the second last equality holds due to the uniform prior, and the strict inequality follows from state monotonicity.

Second, we show that $\min_{i \in I \setminus S^*} \xi_i^* > 0$. Assume to the contrary that $\min_{i \in I \setminus S^*} \xi_i^* \leq 0$. Let $\delta > 0$, and let $\kappa > 0$ be sufficiently small that $\min_{i \in I \setminus S^*} \xi_i^{*\kappa} \leq \delta$. Let $i \in I \setminus S^*$ be such that $\xi_i^{*\kappa} \leq \xi_j^{*\kappa}$ for all $j \in I \setminus S^*$, where $\xi_i^{*\kappa} \leq \delta$. For such i, we have

$$
\begin{aligned}
0 &= \mathbb{E}[d_i(\{j \neq i \mid x_j \geq \xi_j^{*\kappa}\}, x_i)|x_i = \xi_i^{*\kappa}] \\
&\leq \mathbb{E}[d_i(\{j \in (I \setminus S^*) \setminus \{i\} \mid x_j \geq x_i\} \cup S^*, x_i)|x_i = \xi_i^{*\kappa}] \\
&= \mathbb{E}[d_i(\{j \in (I \setminus S^*) \setminus \{i\} \mid \ell_j > \ell_i\} \cup S^*, x_i)|x_i = \xi_i^{*\kappa}] \\
&= \mathbb{E}[d_i(\{j \in (I \setminus S^*) \setminus \{i\} \mid \ell_j > \ell_i\} \cup S^*, \xi_i^{*\kappa})] \\
&\leq \mathbb{E}[d_i(\{j \in (I \setminus S^*) \setminus \{i\} \mid \ell_j > \ell_i\} \cup S^*, \delta] \\
&= \sum_{\gamma \in \Pi(I \setminus S^*)} \rho^0(\gamma) d_i(S^0(i,\gamma), \delta),
\end{aligned}
$$

where the first inequality follows from strategic complementarity, the second last equality holds due to the uniform prior, and the strict inequality follows from state monotonicity. Then let $\delta \to 0$. By payoff continuity, we then have

$$
0 \leq \sum_{\gamma \in \Pi(I \setminus S^*)} \rho^0(\gamma) f_i(S^0(i,\gamma)),
$$

which contradicts the strict reverse sequential obedience of ρ^0.

Thus, we have shown that $S^* = \{i \in I \mid \xi_i^* \leq 0\}$. □

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