

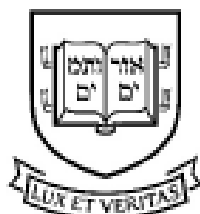
COST BASED NONLINEAR PRICING

By

Dirk Bergemann, Tibor Heumann and Stephen Morris

August 2023

COWLES FOUNDATION DISCUSSION PAPER NO. 2368



COWLES FOUNDATION FOR RESEARCH IN ECONOMICS

YALE UNIVERSITY
Box 208281
New Haven, Connecticut 06520-8281

<http://cowles.yale.edu/>

Cost Based Nonlinear Pricing*

Dirk Bergemann[†] Tibor Heumann[‡] Stephen Morris[§]

August 17, 2023

Abstract

How should a seller offer quantity or quality differentiated products if they have no information about the distribution of demand? We consider a seller who cares about the “profit guarantee” of a pricing rule, that is, the minimum ratio of expected profits to expected social surplus for any distribution of demand.

We show that the profit guarantee is maximized by setting the price markup over cost equal to the elasticity of the cost function. We provide profit guarantees (and associated mechanisms) that the seller can achieve across all possible demand distributions. With a constant elasticity cost function, constant markup pricing provides the optimal revenue guarantee across all possible demand distributions and the lower bound is attained under a Pareto distribution. We characterize how profits and consumer surplus vary with the distribution of values and show that Pareto distributions are extremal. We also provide a revenue guarantee for general cost functions. We establish equivalent results for optimal procurement policies that support maximal surplus guarantees for the buyer given all possible cost distributions of the sellers.

JEL CLASSIFICATION: D44, D47, D83, D84.

KEYWORDS: Nonlinear Pricing, Second Degree Price Discrimination, Profit Guarantees, Competitive Ratio.

*We acknowledge financial support from NSF grants SES-2001208 and SES-2049744, and ANID Fondecyt Iniciación 11200365. An Extended Abstract of this paper appeared in the Conference Proceedings of ACM-EC '23. We thank seminar audiences at Cambridge, Essex, LSE, Oxford, Stanford, UBC, UCL, and UCLA for helpful discussions. We have benefitted from comments by Ben Brooks, Yang Cai, Jason Hartline, Yingkai Li, and Zvika Neeman. We thank Hongcheng Li for excellent research assistance.

[†]Yale University, dirk.bergemann@yale.edu

[‡]Pontificia Universidad Católica de Chile, tibor.heumann@uc.cl

[§]Massachusetts Institute of Technology, semorris@mit.edu

1 Introduction

1.1 Motivation

The arrival of digital commerce has led to an increasing use of personalization and differentiation strategies. With differentiated products along the quality and/or quantity dimension comes the need for nonlinear pricing policies or second degree price discrimination. The optimal pricing policies for quality or quantity differentiated products were first investigated by Mussa and Rosen (1978) and Maskin and Riley (1984), respectively. The optimal pricing strategies were shown to depend heavily on the prior distribution of the willingness-to-pay of the buyers.

Yet, in many circumstances sellers have only very weak and partial information about the demand distribution. The aim of this paper is to devise robust pricing policies that: (i) are independent of the specific demand distribution and (ii) exhibit revenue guarantees across all possible distributions. The main results of this paper construct such robust pricing strategies and show that they can be expressed in terms of very simple and transparent pricing rules. We establish profit guarantees by comparing the profit under the robust pricing policy and any given distribution of private information to the social welfare attainable under the same demand distribution. We then seek to identify the pricing policy that guarantees the highest ratio of these two measures across all feasible distributions with finite expectation. As the social welfare coincides with the profit that the seller could attain under perfect or first degree price discrimination, the ratio has two possible interpretations. In this second perspective we compare the revenue of the seller under incomplete information to the revenue that the seller could attain under complete information (and hence perfect price discrimination).

We consider two broad classes of pricing problems. First, we consider a class of quality differentiated pricing problems as first analyzed by Mussa and Rosen (1978). Here, the type of the buyer is the willingness-to-pay for quality. The cost of production is an increasing and convex function of quality. We focus on iso-elastic cost functions which allow us to express the cost environment of the seller in terms of a single cost elasticity parameter. Second, we consider a class of quantity differentiated pricing problems as first analyzed by Maskin and Riley (1984). Here, the seller has a constant marginal cost of producing additional units of the good and the buyer has concave utility for quantities. In this environment, the elasticity of demand is initially assumed to be constant across all types of the buyers.

For quality differentiated products we exhibit a pricing mechanism that attains a profit guar-

antee as a positive fraction of the social surplus across all possible prior distributions of private information. The profit guarantee is expressed as a ratio that only depends on the cost elasticity η of the quality (Theorem 1). For cost functions with an elasticity η near 1, the ratio is the largest and is given by $1/e$ in the limit as $\eta \rightarrow 1$. As the cost function becomes more convex, the ratio decreases. With a quadratic cost function, i.e., when $\eta = 2$, the ratio is still $1/4$. Eventually as the convexity of the cost becomes more pronounced, the ratio decreases to 0 as $\eta \rightarrow \infty$. We show that these profit guarantees are sharp and are attained by specific instances of distributions, namely Pareto distributions whose shape $\eta/(\eta - 1)$ varies systematically with the elasticity of the cost function (Theorem 2). Interestingly, the optimal mechanism can be implemented by a pricing policy that maintains a constant mark-up: the price of a given quality is η times its production cost (Corollary 1).

The iso-elastic cost functions provide a stylized environment that allows us to describe how pricing should vary with cost to guarantee high profits regardless of demand. What is the natural extension when the cost is not iso-elastic? We propose a mechanism in which the price per unit of quality increases linearly with the elasticity of marginal cost. This pricing scheme coincides exactly with the optimal mechanism we found when the cost was iso-elastic. We refer to this mechanism as marginal-cost elasticity pricing. We show this mechanism allows guaranteeing a high profit guarantee even when the cost elasticity varies with quality (Proposition 1).

We then ask, what is the impact on consumers when the seller uses the optimal profit guarantee mechanism? We compare the consumer surplus generated by the optimal profit guarantee mechanism with the one generated by the Bayesian-optimal mechanism in the iso-elastic cost environment. We show that the consumer surplus generated by the former mechanism is higher than the consumer surplus generated by the latter mechanism for every distribution of values (Theorem 3). Furthermore, consumers get a constant share of the social surplus for every realized value (Corollary 3). We report further results on how social surplus is decomposed in profits, consumer surplus and deadweight loss across distribution of values when the seller uses the Bayesian-optimal mechanism.

We then turn to the model of quantity differentiation. Here the seller has a constant marginal cost to provide additional units of the product. By contrast, the buyer has a concave utility for the product and thus diminishing marginal utility for additional units of the product. The demand elasticity η now summarizes the economics of the environment. The arguments developed in the setting with nonlinear cost can be largely transferred and yield equally sharp results. For every demand elasticity, we obtain a profit guarantee that is polynomial of the demand elasticity alone.

Surprisingly, the robust mechanism is given by a linear pricing mechanism (Theorem 4). Finally, we report results for the procurement environment, where we again make the distinction between quantity and quality differentiation.

1.2 Related Literature

We derive performance guarantees and robust pricing policies that secure these guarantees for large classes of second-degree price discrimination problems as introduced by Mussa and Rosen (1978) and Maskin and Riley (1984). We do this without imposing any restriction on the distribution of the values, such as regularity or monotonicity assumptions, or finite support conditions. We only require that the social surplus of the allocation problem has finite expectation.

The optimal monopoly pricing problem for a single object is a special limiting case of our model when the marginal cost of increasing supply becomes infinitely large. The analysis of a performance guarantee in the single-unit case is also a special case of a result of Neeman (2003). He investigates the performance of English auctions with and without reserve prices. The case of the optimal monopoly pricing is a special case of the auction environment with a single bidder. He establishes a tight bound for the single-bidder case that is given by a “truncated Pareto distribution” (Theorem 5). The bound that he derives is a function of a parameter that is given by the ratio of the bidder’s expected value and the bidder’s maximal value. Without the introduction of this parameter the bound is zero: as this ratio converges to zero, so does the bound. Similarly, Eren and Maglaras (2010) and Hartline and Roughgarden (2014) establish that for distributions with support $[1, h]$, the competitive ratio of the optimal pricing problem is $1 + \ln h$. Thus as h grows, the approximation becomes arbitrarily weak (see Theorem 2.1 of Hartline and Roughgarden (2014)). By contrast, our results obtain a constant approximation for every convex cost function. Thus, the introduction of a convex cost function (or a concave utility function) leads to a noticeable strengthening of the approximation quality.

Carrasco, Luz, Monteiro, and Moreira (2019) consider a different robust version of the nonlinear allocation problem here. The principal faces a privately informed agent and only knows the first moment of the distribution and its finite support (taken to be the unit interval). As in Bergemann and Schlag (2008) they solve the problem by characterizing equilibria of an auxiliary zero-sum game played by the principal and an adversarial nature. Their main result is that in the optimal robust mechanism the ex-post payoff of the principal has to be linear in the agent’s realized type. This property which can be traced to the moment constraint on the mean does not hold for our

solution. Carroll (2017) considers a robust version of multi-item pricing problem. The buyer has an additive value for finitely many items and has private information about the value of each item. There the seller knows the marginal distribution for every item but is uncertain about the joint value distribution. Carroll (2017) considers a minmax problem and shows that separate item-by-item pricing is the robustly optimal pricing policy. Deb and Roesler (2022) consider a related problem under informational robustness. There, the joint distribution of values is given by commonly known distribution but nature or the buyer can choose the optimal information structure. In the corresponding solution of the mechanism design problem, they show that the optimal mechanism is always one of pure bundling.

The focus of this paper is on second-degree price discrimination alone. Bergemann, Brooks, and Morris (2015) consider the limits of third-degree price discrimination. While most of their analysis is focused on the single-unit demand, they present some partial results how market segmentation can affect second-degree price discrimination. Haghpanah and Siegel (2022) present some additional results on the limits of multi-product discrimination for a given prior distribution of values.

We focus here on optimal nonlinear pricing but there are a number of related allocation problems that we could analyze with our methods. We briefly describe the procurement version of our problem in Section 6. Similarly, the model of optimal taxation as proposed by Mirrlees (1971) shares many of the same features. Saez (2001) argues that "unbounded distributions are of much more interest than bounded distribution to address the high optimal income tax and Diamond and Saez (2011) analyzes the optimal tax rate for the setting with a Pareto distribution.

2 Model

2.1 Payoffs

A seller supplies goods of varying quality $q \in \mathbb{R}_+$ to a buyer. The buyer has private information about the willingness-to-pay (or value) $v \in \mathbb{R}_+$ for quality. The utility net of a payment $t \in \mathbb{R}_+$ is:

$$u(v, q, t) = vq - t. \tag{1}$$

The value $v \in \mathbb{R}_+$ is distributed according to $F \in \Delta(\mathbb{R}_+)$.

The seller's cost of providing quality q is given by a cost function $c : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ with the following functional form:

$$c(q) = q^\eta / \eta, \quad \eta \in (1, \infty). \tag{2}$$

Hence, the cost is increasing, convex, and has constant cost elasticity equal to η :

$$\frac{\frac{dc(q)}{dq}}{\frac{c(q)}{q}} = \eta.$$

We relax the assumption that the cost has constant elasticity in Section 4.

2.2 Mechanisms

The seller chooses a menu (or direct mechanism) with qualities $Q(v)$ at prices $T(v)$:

$$M \triangleq \{(Q(v), T(v))\}_{v \in \mathbb{R}_+}. \quad (3)$$

The mechanism has to satisfy incentive compatibility and participation constraints. Thus for all $v, v' \in \mathbb{R}_+$:

$$vQ(v) - T(v) \geq vQ(v') - T(v'); \quad (4)$$

$$vQ(v) - T(v) \geq 0. \quad (5)$$

We denote by \mathcal{M} the set of incentive compatible and individually rational mechanisms. The expected seller's profit and buyer's surplus generated by a distribution F and a mechanism M are respectively:

$$\Pi_{F,M} \triangleq \mathbb{E}[T(v) - c(Q(v))] \quad \text{and} \quad U_{F,M} \triangleq \mathbb{E}[Q(v)v - T(v)].$$

The profit-maximizing mechanism for distribution F is denoted by M_F :

$$M_F \triangleq \arg \max_{M \in \mathcal{M}} \Pi_{F,M}.$$

With some abuse of notation, we denote by

$$\Pi_F = \Pi_{F,M_F} \quad \text{and} \quad U_F = U_{F,M_F}$$

the profit and consumer surplus evaluated at the seller-optimal mechanism M_F given the distribution F .

We shall often use the socially efficient allocation as a benchmark. The socially efficient allocation maximizes the social surplus pointwise for every value v :

$$\widehat{Q}(v) \in \arg \max_{q \in \mathbb{R}_+} \{vq - c(q)\}. \quad (6)$$

The expected social surplus S given a distribution F is given by:

$$S_F \triangleq \mathbb{E}[v\widehat{Q}(v) - c(\widehat{Q}(v))]. \quad (7)$$

We shall consider all value distributions $F \in \Delta(\mathbb{R}_+)$ that generate a finite social surplus.

We denote by $\delta(v)$ the Dirac distribution that places probability one at value v . Whenever total surplus is evaluated at the degenerate distribution $\delta(v)$, we write it as a function of v , that is,

$$S(v) = S_{\delta(v)}.$$

Analogously $\Pi_M(v)$ and $U_M(v)$ are the profits and consumer surplus when the distribution is degenerate.

2.3 Profit Guarantee as Competitive Ratio

Our objective is to derive a pricing policy that can attain a profit guarantee even in the absence of a prior distribution of the willingness-to-pay. Without any restriction on the demand such as statistical moment or support restrictions, it is clear that any profit guarantee cannot be absolute but rather has to be relative or proportional to some benchmark. We define a profit guarantee as a competitive ratio of realized profit against optimal profit at hindsight, thus profit under complete information. We now provide a formal definition of the benchmark.

If the seller were to know the realization of each value v of the buyer then the seller could use first degree price discrimination. The seller would then offer the socially efficient allocation and extract the entire surplus through the transfer. Using the earlier definition (6) of the first best allocation, the first-degree price discrimination policy would be $\overline{M} = (\overline{Q}(v), \overline{T}(v))$ with:

$$\overline{Q}(v) = \widehat{Q}(v), \quad \overline{T}(v) = v\widehat{Q}(v), \quad \forall v \in \mathbb{R}_+. \quad (8)$$

The resulting expected profit across all value realization would then be:

$$\overline{\Pi}_F \triangleq \mathbb{E}[v\overline{Q}(v) - c(\overline{Q}(v))].$$

The competitive ratio of a given mechanism M is now defined as the lowest ratio of realized profit $\Pi_{F,M}$ and the profit under complete information $\overline{\Pi}_F$ across all distribution of values F :

$$\inf_F \frac{\Pi_{F,M}}{\overline{\Pi}_F}. \quad (9)$$

Our first goal is to find a mechanism M^* which attains the highest possible competitive ratio:

$$M^* = \arg \max_{M \in \mathcal{M}} \inf_F \frac{\Pi_{F,M}}{\bar{\Pi}_F}. \quad (10)$$

Thus, we seek to identify a mechanism which achieves the largest competitive ratio against the benchmark given by first degree price discrimination.¹ We refer to this as the optimal profit-guarantee mechanism. As a direct by-product we will find the distribution F of values that minimizes the competitive ratio, or:

$$\inf_F \max_M \frac{\Pi_{F,M}}{\bar{\Pi}_F}. \quad (11)$$

In fact, we will show that the Minimax Theorem applies in our setting, so (10) and (11) will be tightly connected.

In the definition of the competitive ratio, there is often a choice in terms of the benchmark. Here we use the profit under complete information which is the profit of first-degree price discrimination. As we saw in (8), the profit of first-degree price discrimination coincides with the efficient social surplus. For this reason we have

$$\frac{\Pi_{F,M}}{\bar{\Pi}_F} = \frac{\Pi_{F,M}}{S_F},$$

and we often refer to the ratio of expected profit against social surplus as the profit share (or normalized profit). Similarly, we speak about the consumer surplus share as:

$$\frac{U_{F,M}}{\bar{\Pi}_F} = \frac{U_{F,M}}{S_F},$$

and refer to this quantity also as normalized consumer surplus.

3 Optimal Profit-Guarantee Mechanism

In Section 3.1 we establish the optimal profit guarantee as the solution to a minmax problem. In Section 3.2 we show that the optimal profit guarantee can be implemented by an indirect mechanism that charges a constant mark-up relative to the cost of the quality produced. We conclude this section by discussing an alternative benchmark for the competitive ratio.

¹The term competitive ratio originated in the analysis of online vs offline algorithms, see Borodin and El-Yaniv (1998). There, the offline algorithm is allowed to choose the allocation after the realization of all samples whereas the online algorithm is required to make the decision in response to each sample realization.

3.1 Solution to the Minmax Problem

We begin with a given mechanism $M = (Q(v), T(v))$ and ask how well is the mechanism M performing across different demand distributions F . In particular against a most challenging distribution F^* that minimizes the competitive ratio

$$\inf_F \frac{\Pi_{F,M}}{\bar{\Pi}_F}.$$

The competitive ratio is stated in terms of expectations which we now state explicitly

$$\inf_F \frac{\Pi_{F,M}}{\bar{\Pi}_F} = \inf_F \left\{ \frac{\int (T(v) - c(Q(v))) dF(v)}{\int (\bar{T}(v) - c(\bar{Q}(v))) dF(v)} \right\}.$$

Given a menu $M = \{T(v), Q(v)\}$, it is as if nature chooses demand F that lowers the profit guarantee. In particular, nature puts weight on values v where the guarantee is weak:

$$\inf_v \left\{ \frac{T(v) - c(Q(v))}{\bar{T}(v) - c(\bar{Q}(v))} \right\}.$$

To defend against such attacks the seller seeks to find a menu M where pointwise (local) guarantee is as high as possible, uniformly across all v :

$$\frac{T(v) - c(Q(v))}{\bar{T}(v) - c(\bar{Q}(v))} = k, \quad \forall v.$$

How might we identify a mechanism that can maintain a high guarantee everywhere. Now, the first-degree discrimination is supported by the efficient social choice $\bar{Q}(v)$. This suggest that the seller may maintain a profit guarantee by staying with a constant proportion k of $\bar{Q}(v)$:

$$k \cdot \bar{Q}(v), \quad k \in (0, 1).$$

Now, the gross revenue of such a policy would grow at the rate k . The cost would however increase at a rate k^η reflecting the convexity of the cost function. This might suggest that the seller finds the optimal trade-off by identifying a k^* that solves:

$$\max_k \{k - k^\eta\} \Leftrightarrow k^* = \left(\frac{1}{\eta}\right)^{\frac{1}{\eta-1}}.$$

Our first result then pursues this logic and establishes that a mechanism that provides a proportion k^* of the socially efficient allocation:

$$Q^*(v) = k^* \cdot \bar{Q}(v) = k^* \cdot \hat{Q}(v) = \left(\frac{1}{\eta}\right)^{\frac{1}{\eta-1}} \cdot v^{\frac{1}{\eta-1}},$$

can indeed attain a profit guarantee. In the next step we show that the informal intuition can be made exact and that the above allocation attains the highest possible profit guarantee.

A mechanism M is incentive compatible and meets the individual rationality constraint at $v = 0$ if and only if the qualities $\{Q(v)\}_{v \in \mathbb{R}}$ are non-decreasing and the corresponding transfers are given by:

$$T(v) = vQ(v) - \int_0^v Q(s)ds.$$

Since the optimal mechanism is uniquely defined by the qualities, we often refer to a mechanism as a set of qualities $\{Q(v)\}_{v \in \mathbb{R}}$ omitting the prices. The profits generated by a mechanism are:

$$\Pi_{F,Q} \triangleq \int_0^\infty (vQ(v) - c(Q(v)))dF(v) - \int_0^\infty Q(v)(1 - F(v))dv. \quad (12)$$

We then have that the profit can be computed as a function of the qualities $\{Q(v)\}_{v \in \mathbb{R}}$ alone.

Theorem 1 (Profit Guarantee Mechanism)

The mechanism M^ with quantities:*

$$Q^*(v) = \left(\frac{v}{\eta}\right)^{\frac{1}{\eta-1}}, \quad (13)$$

attains for every F a profit share:

$$\frac{\Pi_{F,M^*}}{\bar{\Pi}_F} = \left(\frac{1}{\eta}\right)^{\frac{\eta}{\eta-1}}. \quad (14)$$

Proof. We first prove that, for every distribution F , the profits generated by qualities (13) are:

$$\Pi_{F,M^*} = \frac{1}{\eta^{\frac{\eta}{\eta-1}}} \bar{\Pi}_F.$$

Note that we write Π_{F,M^*} instead of Π_F because these are the profits generated by (13), which in general differs from the profits generated by the Bayesian-optimal mechanism under F . Replacing (13) into (12), we get:

$$\Pi_{F,M^*} = \int \left(z - \frac{z^\eta}{\eta}\right) v^{\frac{\eta}{\eta-1}} dF(v) - \int z v^{\frac{1}{\eta-1}} (1 - F(v)) dv,$$

where $z = 1/\eta^{1/(\eta-1)}$. Integrating by parts the second term we get:

$$\Pi_{F,M^*} = \int \left(z - \frac{z^\eta}{\eta}\right) v^{\frac{\eta}{\eta-1}} dv - \frac{\eta-1}{\eta} \int z v^{\frac{\eta}{\eta-1}} dF(v).$$

Collecting terms, we get:

$$\Pi_{F,M^*} = \int \left(z - \frac{z^\eta}{\eta} - \frac{\eta-1}{\eta} z \right) v^{\frac{\eta}{\eta-1}} dF(v).$$

Replacing back $z = 1/\eta^{1/(\eta-1)}$ we get that:

$$\Pi_{F,M^*} = \frac{1}{\eta^{\frac{\eta}{\eta-1}}} \frac{\eta-1}{\eta} \int v^{\frac{\eta}{\eta-1}} dF(v).$$

We also note that the social surplus S_F under distribution F is:

$$S_F = \frac{\eta-1}{\eta} \int v^{\frac{\eta}{\eta-1}} dF(v).$$

Since the total surplus equals the profits under first-degree price discrimination policy, we get that this strategy guarantees a fraction $1/\eta^{\frac{\eta}{\eta-1}}$ of the profits under first-degree price discrimination policy.

■

The profit guarantee (14) is therefore equal to the proportional share k^* raised to the power of the elasticity η . The profit guarantee is the same across all feasible distributions and in fact is the same across all value realizations v . Since (13) attains a fraction of the social surplus given by (14), the maxmin value (attained by (10)) cannot be smaller than this. We now show that this is indeed the optimal profit-guarantee mechanism by showing that this mechanism is also the Bayesian optimal mechanism for a specific distribution of values.

If the profit-guarantee menu M^* is optimal and there exists a saddle point that solves the minimax problem, then M^* must also be Bayes-optimal against some distribution F . In other words, given F , M^* solves

$$\arg \max_{M \in \mathcal{M}} \frac{\Pi_{F,M}}{\bar{\Pi}_F} \Leftrightarrow \arg \max_{M \in \mathcal{M}} \Pi_{F,M}.$$

The candidate menu M^* offers an optimal quality Q^* that provides a constant share k^* of the socially efficient quality $\bar{Q}(v)$. If the mechanism is Bayes optimal, then the candidate optimal quality Q^* is obtained by a virtual value $\phi(v)$ that must be proportional to the value v . This leads us directly to the Pareto distribution, which is given by:

$$P_\alpha(v) \triangleq 1 - \frac{1}{v^\alpha}.$$

If $\alpha > 1$, then the expectation of the willingness-to-pay under the Pareto distribution is finite, and if $\alpha > \eta/(\eta-1)$, then the social surplus remains finite. When we make reference to a Pareto

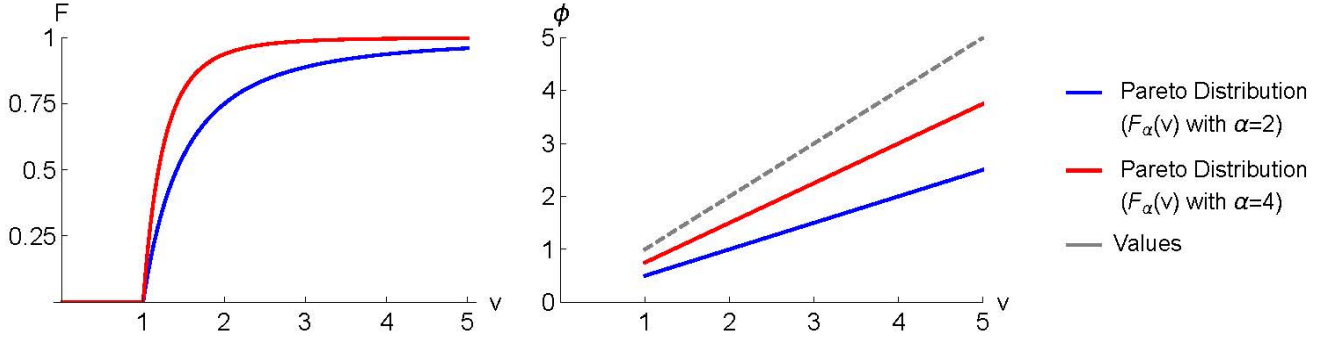


Figure 1: Pareto distribution with shape $\alpha = 2$ and $\alpha = 4$ and associated virtual values

distribution with shape parameter $\alpha = \eta/(\eta - 1)$ we will formally take the limit as the shape parameter approaches this value.

The Pareto distribution uniquely generates a virtual value $\phi(v)$ that is linear in value v . Under the Pareto distribution, the gross virtual values are given by:

$$\phi(v) \triangleq \frac{\alpha - 1}{\alpha} v \quad (15)$$

Hence the qualities $Q(v)$ supplied by the seller under the Bayes optimal mechanism are given by:

$$Q(v) = \left(\frac{\alpha - 1}{\alpha} v\right)^{\frac{1}{\eta-1}}, \quad (16)$$

see Mussa and Rosen (1978) or Myerson (1981). We display in Figure 1 two Pareto distributions with different shape parameters α , and their associated linear virtual values.

We now establish that the profit guarantee given by (14) is indeed the optimal—the highest—profit-guarantee that can be provided. For this, we first give the explicit solution to the Bayes-optimal mechanism (15) when the Pareto distribution has the shape parameter $\alpha = \eta/(\eta - 1)$ and the Bayes optimal mechanism equals the profit-guarantee established in Theorem 1.

Theorem 2 (Minimax Distribution)

The profit-guarantee mechanism M^* is the Bayes optimal mechanism for the Pareto distribution with shape parameter:

$$\alpha = \frac{\eta}{\eta - 1},$$

and attains the infimum:

$$\inf_F \frac{\Pi_F}{S_F} = \frac{\Pi_{P_\alpha}}{S_{P_\alpha}} \Big|_{\alpha=\frac{\eta}{\eta-1}} = \frac{1}{\eta^{\frac{\eta}{\eta-1}}}. \quad (17)$$

Proof. We can now compute the profit and social surplus in closed-form to conclude that, when $\alpha > \eta/\eta - 1$:

$$\frac{\Pi_{P_\alpha}}{S_{P_\alpha}} = \left(\frac{\alpha - 1}{\alpha} \right)^{\frac{\eta}{\eta-1}}.$$

This equation is valid only when $\alpha > \eta/(\eta - 1)$ because these are the Pareto distributions for which social surplus remains finite. However, by taking limit $\alpha \rightarrow \eta/(\eta - 1)$, we get (17).

Theorem 2 then follows from Theorem 1. If the seller can guarantee a fraction $1/\eta^{\frac{\eta}{\eta-1}}$ of the social surplus as profit, and this is in fact the best the seller can do for some distribution of values, then this distribution of values minimizes the fraction of the social surplus that the seller can generate as profit. ■

Thus the Pareto distribution with shape $\alpha = \eta/(\eta - 1)$ allows the seller to generate the least amount of normalized profit when the seller uses the Bayesian optimal mechanism. Theorem 1 and Theorem 2 provide a saddle point to the minmax problem (10)-(11).

In the nonlinear pricing environment we thus provide a positive profit guarantee even in the absence of any support restriction. This stands in stark contrast to the analysis of a single-unit demand as in Neeman (2003), Eren and Maglaras (2010) and Hartline and Roughgarden (2014) where there is no positive profit guarantee in the absence of a support restriction.

The improvement in the profit guarantee with a differentiated demand relative to the single unit demand is perhaps best explained through the double role that the menu of offers plays in the model of differentiated demand. With quality (or quantity) differentiated demand, a menu of choices is offered already in the socially efficient solution that matches willingness to pay and quality provided. A menu guarantees incentive compatibility and gives rise to both information rent to consumers and profit to the seller. In contrast with a single unit-demand, the socially optimal (as well as the revenue optimal) solution against a given distribution is a constant solution at a given price. Interestingly the optimal policy to attain a profit guarantee against unknown demand is using a menu in the form of a lottery of random prices. But with the single demand, the menu does not improve the social surplus but rather secures a competitive ratio. With differentiated demand, a menu is already necessary to establish a socially efficient or approximate efficient solution and then can be further used to maintain a high profit guarantee. Thus a menu serves both roles and in conjunction attains a positive profit guarantee even in the absence of support restrictions.

3.2 Constant Mark-Up Mechanism

We now give an alternative representation of the optimal profit-guarantee mechanism as an indirect mechanism in the form of a tariff. An incentive compatible mechanism $\{Q(v), T(v)\}$ can be implemented by offering a tariff that charges price $P(q)$ for quality q .

Corollary 1 (Constant Mark-Up Mechanism)

The profit-guarantee mechanism M^ can be implemented by offering quality q at a price that increases linearly with cost at rate η , that is, the mechanism consists of a constant mark-up price:*

$$\frac{P(q) - c(q)}{c(q)} = \eta - 1. \quad (18)$$

To prove the corollary, we rewrite the indirect mechanism in terms of its marginal price for quality, the price-per-quality increment:

$$p(q) \triangleq P'(q). \quad (19)$$

Any given incentive compatible mechanism that implements qualities $Q(v)$ can be implemented by a price-per-quality equal to:

$$p(q) = Q^{-1}(q). \quad (20)$$

The corollary then follows from integrating $Q^{*-1}(v)$.

Thus, the price in the profit guarantee mechanism refers only to cost information and charges a constant mark-up relative to the cost. We can alternatively express the constant mark-up pricing policy in terms of the Lerner's index which is frequently used as a measure of market power:

$$\frac{P(q) - c(q)}{P(q)} = \frac{\eta - 1}{\eta}. \quad (21)$$

It is informative to contrast the profit guarantee policy with the Bayesian optimal policy for a given prior distribution F . Following Corollary 1, the optimal profit-guarantee mechanism maintains a constant mark-up for each *additional or incremental unit* of quality:

$$\frac{p(q) - c'(q)}{c'(q)} = \eta - 1.$$

In the Bayes-optimal mechanism the qualities are solved by the first-order condition with respect to the virtual utility. Supposing for the moment that F is regular, Mussa and Rosen (1978) solve:

$$Q(v) \in \arg \max_q \left\{ \left(v - \frac{1 - F(v)}{f(v)} \right) q - c(q) \right\}.$$

The first-order condition is given by:

$$\frac{v - c'(Q(v))}{v} = \frac{1 - F(v)}{f(v)v},$$

where we have simply taken the first-order condition pointwise and rearranged terms. As before, the mechanism can be implemented using a price-per-quality (20) and the demand for quality q at price-per-quality p is given by:

$$D(p) \triangleq 1 - F(p).$$

We can thus write the first-order condition as follows:

$$\frac{p(q) - c'(q)}{p(q)} = -\frac{D(p(q))}{D'(p(q))p(q)},$$

The right-hand-side is the negative of the reciprocal of the demand elasticity: this is the classic formula for the Lerner's index.

The Bayes-optimal mechanism is determined entirely by the demand elasticity which can be expressed in terms of the product of the value v and the hazard rate $f(v)/(1 - F(v))$. By contrast, the profit-guarantee mechanism is determined only by the elasticity of the cost function and does not refer to either the willingness-to-pay v or the distribution of the willingness-to-pay. As the profit-guarantee is attained across all possible demand distribution, it does not refer to any specific distribution, but rather uses the cost information to offer a uniform menu for all possible demand distributions.

For the analysis of the profit guarantee mechanism we did not impose any restriction on the distribution of values, such as monotonicity or regularity restriction. We also did not impose any support restrictions on F . Indeed, we showed that the critical demand function is a Pareto distribution with unbounded support. An implication of the unbounded support is that the “no distortion at the top” result fails to hold, rather we have a constant mark-up everywhere as derived in Corollary 1. This absence of a “no distortion at the top” result is related to insights in the literature on optimal taxation in the presence of agents with unbounded income that we briefly discussed in the Introduction.

3.3 Alternative Benchmark

So far we have studied the mechanism that maximizes the profit share, defined as the ratio of profits to the profits generated by the first-degree price discrimination policy. An alternative variation of

the benchmark would be to choose as the denominator of the ratio the profits generated by the Bayesian-optimal mechanism when the distribution is F , thus

$$\max_M \inf_F \frac{\Pi_{F,M}}{\Pi_F}. \quad (22)$$

Now, for a given distribution F , the Bayes optimal profit Π_F is typically smaller than the first-degree price discrimination profit $\bar{\Pi}_F$, or

$$\Pi_F \leq \bar{\Pi}_F,$$

with equality if and only if F is degenerate. We can check that the competitive ratio is however the same under this variation as well.

The profit guarantee is the same across all feasible distributions and in fact is the same across all value realizations v . It is then possible to show that the mechanism also attains the same profit guarantee if we consider the alternative benchmark in which we use the profits under the Bayesian optimal mechanism in the denominator. For every degenerate distribution the seller can extract the full surplus, so $\bar{\Pi}_{\delta(v)} = \Pi_{\delta(v)}$. So, we have that:

$$\max_{M \in \mathcal{M}} \inf_F \frac{\Pi_{F,M}}{\bar{\Pi}_F} = \max_{M \in \mathcal{M}} \inf_{v \in [0,1]} \frac{\Pi_{\delta(v),M}}{\bar{\Pi}_{\delta(v)}} = \max_{M \in \mathcal{M}} \inf_{v \in [0,1]} \frac{\Pi_{\delta(v),M}}{\Pi_{\delta(v)}} \geq \max_{M \in \mathcal{M}} \inf_F \frac{\Pi_{F,M}}{\Pi_F}.$$

However, since $\bar{\Pi}_F \geq \Pi_F$ for every F , we get:

$$\max_{M \in \mathcal{M}} \inf_F \frac{\Pi_{F,M}}{\bar{\Pi}_F} = \max_{M \in \mathcal{M}} \inf_F \frac{\Pi_{F,M}}{\Pi_F}.$$

Hence, this mechanism guarantees the same profits if we consider the alternative benchmark (22). However, under this alternative benchmark a saddle point will not exist. In fact, we have that:

$$\inf_F \max_{M \in \mathcal{M}} \frac{\Pi_{F,M}}{\Pi_F} = \inf_F \frac{\max_{M \in \mathcal{M}} \Pi_{F,M}}{\Pi_F} = 1,$$

which follows from the definition of Π_F .

4 Beyond Constant Cost Elasticity

In Theorem 1 we characterized the optimal profit-guarantee mechanism for constant elasticity cost functions. We now turn to the analysis of the optimal menu in the presence of a general convex cost function without requiring constant elasticity. Throughout this section we assume that $c(q)$ is increasing, convex, and $c(0) = 0$ (but we do not assume that it has constant elasticity).

We first introduce a mechanism that uses the pricing rules that are informed by the constant elasticity pricing derived in Corollary 1. We will use the elasticity of the marginal cost function as a pointwise approximation. Hence, as we describe the problem with a general cost function, it is the elasticity of the marginal cost that is critical rather than the cost elasticity. Of course, for the case of the constant cost elasticity, the two notions are just apart by one unit. We refer to the mechanism we introduce as marginal-cost elasticity pricing. We then provide bounds on the normalized profits that are attained by the marginal-cost elasticity pricing. We conclude this section by discussing the optimal profit-guarantee mechanism under general cost functions and when the seller knows bounds on the support of the distribution of values.

4.1 Marginal-Cost Elasticity Pricing

We consider the class of increasing and convex cost functions $c : \mathbb{R}_+ \rightarrow \mathbb{R}_+$, with $c(0) = 0$. For a given cost function $c(q)$ we denote the pointwise cost elasticity by:

$$\eta(q) = \frac{dc(q)}{dq} \frac{q}{c(q)}.$$

As we consider cost functions with varying elasticity, it is often helpful to work with the elasticity of the marginal cost rather than the cost itself. Thus we define

$$\gamma(q) = \frac{dc'(q)}{dq} \frac{q}{c'(q)}.$$

For cost functions with constant elasticity, we have a simple relationship between these cost elasticities, $\eta(q)$ and $\gamma(q)$, respectively:

$$\gamma(q) = \eta(q) - 1.$$

For general convex cost functions there is no immediate relationship between these two notions of elasticity. The marginal cost elasticity is closely related to the supply elasticity.²

We now consider a mechanism that uses a tariff proportional to the elasticity of the marginal cost function. We introduced earlier the tariff $P(q)$ and the price for quality increment, $p(q) = P'(q)$,

²The supply function $Q(p)$ of a firm is defined by the price p and its cost function. The optimal supply for the firm is the solution to the profit maximization problem $\max_q \{qp - c(q)\}$ and thus to supply quality q until the marginal cost is equal to marginal revenue $p : p = c'(q)$. The supply function is thus $Q(p) = (c')^{-1}(p)$ and the supply elasticity is defined by

$$(dQ(p)/dp) / (Q(p)/p).$$

see (19). We now define a tariff in terms of the quality increment that is proportional everywhere to the elasticity of the marginal cost function:

$$\tilde{p}(q) \triangleq (\gamma(q) + 1)c'(q). \quad (23)$$

Hence, the tariff can be expressed in terms of the markup as:

$$\frac{\tilde{p}(q) - c'(q)}{c'(q)} = \gamma(q).$$

When the cost function has constant elasticity, i.e. $c(q) \propto q^{1+\gamma}$ for some $\gamma \in \mathbb{R}_+$, then the markup is constant. In particular, the above pricing rule is exactly equal to the constant-mark up policy that we derived in Corollary 1 when the cost has constant elasticity.

4.2 Profit Guarantee with Varying Cost Elasticity

Given the indirect mechanism (23), we can identify a direct and incentive compatible mechanism. We denote by \tilde{Q} the incentive compatible allocation rule implemented by the tariff (23), which corresponds to inverting the tariff. At every marginal price $\tilde{p}(q)$, the marginal buyer is given by the willingness to pay equal to their value, or

$$v = \tilde{p}(q),$$

and hence

$$\tilde{Q}(v) \triangleq \tilde{p}^{-1}(v). \quad (24)$$

Hence, this is the allocation function, which can be implemented using transfers constructed using the Envelope condition. We first show that pricing according to the marginal-cost elasticity (as in (23)) generates a profit that corresponds to the first-best total surplus generated by a downward shifted value.

Lemma 1 (Profit as Downward Shifted Social Surplus)

The tariff (23) generates a profit equal to:

$$\Pi(v) = S(w(v)). \quad (25)$$

where $w(v)$ is the following downward shift in values:

$$w(v) \triangleq \frac{v}{\gamma(\tilde{Q}(v)) + 1}. \quad (26)$$

Proof. The profit that the seller realizes from selling to value v , with $\tilde{Q}(v) = \tilde{p}^{-1}(v)$ is given by:

$$\begin{aligned}\Pi(v) &= \int_0^{\tilde{Q}(v)} c'(q)(\gamma(q) + 1) - c'(q) dq = \int_0^{\tilde{Q}(v)} c'(q)\gamma(q) dq \\ &= \int_0^{\tilde{Q}(v)} c''(q)q dq = c'(\tilde{Q}(v))\tilde{Q}(v) - c(\tilde{Q}(v)) = S(c'(\tilde{Q}(v))) = S\left(\frac{v}{\gamma(\tilde{Q}(v)) + 1}\right).\end{aligned}$$

■

Hence, the profit generated from a value v is equal to the socially efficient surplus when the value is $v/(\gamma(\tilde{Q}(v)) + 1)$. We thus have that the tariff (23) generates a profit as if the seller can extract the full surplus of a buyer with a fraction of the value. Since the profit and the social surplus is monotone in v , it is not surprising that a monotone relationship between profit and social surplus exists for every v and $w(v) < v$. It is a feature of the incentive compatible mechanism that the relationship between v and $w(v)$ can be expressed in terms of a constant fraction of the marginal cost elasticity at v . The marginal cost elasticity also allows us to bound the variation in the social surplus.

Lemma 2 (Rate of Increase of Total Surplus)

For every given $v < v'$ we have that:

$$\left(\frac{v}{v'}\right)^{\frac{g+1}{g}} \leq \frac{S(v)}{S(v')}, \quad (27)$$

where g is given by:

$$g = \min\{\gamma(q) : q \in [0, \bar{Q}(v')]\}.$$

Proof. We can define the marginal surplus elasticity as

$$\sigma(v) = \frac{\frac{S'(v)}{dv}}{\frac{S'(v)}{v}} = \frac{S''(v)v}{S'(v)}.$$

We first show that that marginal surplus elasticity is given for every v by:

$$\frac{d \log S'(v)}{d \log v} = \frac{1}{\gamma(\bar{Q}(v))}. \quad (28)$$

To prove this, we first note that:

$$S(v) = \bar{Q}(v)v - c(\bar{Q}(v)).$$

We thus have that

$$\frac{d^2 S(v)}{dv^2} = (v - c'(\bar{Q}(v))) \frac{d^2 \bar{Q}(v)}{dv^2} + 2 \frac{d\bar{Q}(v)}{dv} - \frac{d^2 c(\bar{Q}(v))}{dq^2} \left(\frac{d\bar{Q}(v)}{dv} \right)^2.$$

The first term is 0 because of the first-order condition and using the Envelope theorem we have that $S'(v) = \bar{Q}(v)$, so we get that:

$$\frac{v}{S'(v)} \frac{d^2 S(v)}{dv} = \frac{v}{\bar{Q}(v)} \left(2 \frac{d\bar{Q}(v)}{dv} - \frac{d^2 c(\bar{Q}(v))}{dq^2} \left(\frac{d\bar{Q}(v)}{dv} \right)^2 \right).$$

We also have that:

$$\frac{v}{\bar{Q}(v)} \frac{d\bar{Q}(v)}{dv} = \frac{1}{\gamma(\bar{Q}(v))}; \quad \frac{v}{\bar{Q}(v)} \frac{d^2 c(\bar{Q}(v))}{dq^2} \left(\frac{d\bar{Q}(v)}{dv} \right)^2 = \frac{1}{\gamma(\bar{Q}(v))}.$$

The second term follows from the fact that $v = c'(\bar{Q}(v))$ and so:

$$\frac{v}{\bar{Q}(v)} \left(\frac{d\bar{Q}(v)}{dv} \right) = \frac{1}{\gamma(\bar{Q}(v))} \quad \text{and} \quad \frac{d^2 c(\bar{Q}(v))}{dq^2} \frac{\bar{Q}(v)}{v} = \gamma(\bar{Q}(v)).$$

We thus prove (28).

We now note that (28) implies that for any $v < v'$,

$$\left(\frac{v}{v'} \right)^{\frac{1}{g}} \leq \frac{S'(v)}{S'(v')}. \quad (29)$$

We thus get that:

$$\begin{aligned} S(v) &= \int_0^v S'(t) dt = S'(v) \int_0^v \frac{S'(t)}{S'(v)} dt \\ &\geq S'(v) \int_0^v \left(\frac{t}{v} \right)^{\frac{1}{g}} dt = \frac{S'(v)}{v^{\frac{1}{g}}} v^{\frac{g+1}{g}} \frac{g}{g+1} \end{aligned}$$

Therefore:

$$\frac{S'(v)}{S(v)v} \leq \frac{g+1}{g}.$$

So the surplus elasticity is less than $(g+1)/g$, which gives the bound in (27) is satisfied. ■

We now use Lemma 1 and 2 to find a bound on normalized profits.

Proposition 1 (Lower Bound)

The normalized profits at any v in mechanism (23) are bounded below by:

$$\frac{\Pi(v)}{S(v)} \geq \left(\frac{1}{g'+1} \right)^{\frac{g+1}{g}}, \quad (30)$$

where $g' = \gamma(\bar{Q}(w))$ and $g = \min\{\gamma(q) : q \in [0, \bar{Q}(v)]\}$.

This bound is a straightforward application of Lemma 1 and 2. The right-hand-side of (30) is increasing in g and decreasing in g' . The intuition is simple. A larger marginal-cost elasticity implies that the value-distortion in (23) is larger: this is reflected in the fact that the lower bound is decreasing in g' . On the other hand, a larger marginal cost elasticity implies that the surplus decreases more slowly with the value-distortion (23) has less impact on normalized profits: this is reflected in the fact that the lower bound is increasing in g . We can then bound the normalized profits using the upper and lower bound on the marginal cost elasticity.

Corollary 2 (Bounded Elasticity)

Suppose the elasticity is bounded $\gamma(q) \in [\underline{\gamma}, \bar{\gamma}]$ for every q , then

$$\frac{\Pi(v)}{S(v)} \geq \left(\frac{1}{\bar{\gamma} + 1} \right)^{\frac{\gamma+1}{\underline{\gamma}}} . \tag{31}$$

Thus, relative to the analysis with constant elasticity, the lower bound given by (31) is weaker as base and exponent of the term are formed by the upper and lower bound of the cost elasticity. The bound coincides with the bound we provided for the case of constant elasticity in Theorem 2 as we then have $\underline{\gamma} = \bar{\gamma} = \gamma = \eta - 1$.

Beyond Marginal-Cost Elasticity Pricing So far we have verified that the marginal-cost elasticity pricing guarantees a significant profit share. We could further ask what we can say about the mechanism that guarantees the maximum profit share. We briefly report the main insights here and relegate the formal results to the appendix. We focus on the case where there are finite bounds on the support of the distribution $0 < \underline{v} < \bar{v} < \infty$. We proceed with this assumption to guarantee the existence of a solution via Sion’s Minmax theorem. It also presents a natural way to introduce partial information about the demand that is relevant for the seller.³

The optimal mechanism now described by a differential equation which keeps the competitive ratio between realized profit and social surplus constant at every point in the support of the demand (as in the case of constant cost elasticity and unbounded support). However, as there is a finite and

³Alternative ways to introduce information about demand is through data samples or restriction on the demand distribution F , say in terms of the moments. In a recent paper, Anunrojwong, Balseiro, and Besbes (2023) show how information about the support can inform the design of robust pricing policies with a single buyer or auction policies with more than one buyer (bidder). In related work, Allouah, Bahamou, and Besbes (2021), (2022) show how in the case of optimal pricing of a single good, additional information in terms of samples from the value distribution can improve the competitive ratio.

common upper bound \bar{v} for all possible distributions, the classic "no-distortion at the top" result will reemerge and thus there will be zero mark-up at the upper bound. More broadly, the markup will no longer be determined purely by the cost function, but also by the information about demand (via the bounds on the support). By taking the limits as the \underline{v} converge to 0 and \bar{v} diverges to infinity, we get a solution that depends only on the cost function.

5 Consumer Surplus and Surplus Sharing

The profit guarantee mechanism M^* can secure a profit guarantee for the seller as established by Theorem 1 and 2. Surprisingly, the guarantee is not only a lower bound for some distribution of values, but the mechanism enables the seller to attain this guarantee uniformly across all distributions. But as the profit-guarantee mechanism is chosen to attain the highest possible profit level, we might be concerned that the profit guarantee mechanism is succeeding in obtaining a high profit share by depressing or even minimizing consumer surplus among all incentive compatible mechanisms. We now examine the implications of the optimal pricing for consumer surplus.

We show that the consumer surplus generated by the optimal profit guarantee mechanism is in fact (weakly) higher than under any Bayesian optimal mechanism across all distributions. As by-product, we characterize the upper frontier of surplus sharing between seller and consumers across all distributions when the seller uses the Bayesian optimal mechanism. We conclude this section by discussing results that further characterize what are the possible normalized profits and consumer surplus that can arise across all distributions when the seller uses the Bayesian-optimal mechanism.

5.1 Surplus Sharing Under the Profit Guarantee Mechanism

The minmax solution of the profit guarantee mechanism M^* and Pareto distribution generate particular pairs of surplus sharing among seller and consumers. We begin by describing the consumer share generated by the profit guarantee optimal mechanism.

Corollary 3 (Consumer Surplus in the Profit Guarantee Mechanism)

The profit-guarantee mechanism M^ generates a constant consumer share surplus*

$$\frac{U_{F,M^*}}{\bar{\Pi}_F} = \left(\frac{1}{\eta}\right)^{\frac{1}{\eta-1}} \tag{32}$$

across all distributions F .

Hence, the optimal profit-guarantee mechanism provides a consumer surplus that is a constant fraction of the socially efficient allocation. Since (32) also applies to degenerate distributions, the consumer surplus of every realized value is a constant fraction of the social surplus that this value could generate. We might have expected the uniformity of the profit guarantee across all distributions from the minmax property of the mechanism. Indeed, in Hartline and Roughgarden (2014), the optimal single unit monopoly pricing policy also has the property that it generates a uniform profit guarantee across all distributions. Yet, in the single unit monopoly pricing, the consumer surplus share is not uniform across all distributions. For a given value v , the net utility of a buyer is $(v \ln v) / (1 + \ln v)$ and thus the expected consumer surplus share depends on the distribution of v . More precisely, in the profit-guarantee mechanism, each consumer receives the same share of the efficient surplus. By contrast, in the single unit monopoly pricing model, the share of the consumer surplus is increasing in the willingness to pay.

Thus we find that the profit share decreases with the cost elasticity η and the consumer surplus share increases with the cost elasticity. The limit of $\eta \rightarrow 1$ corresponds to the case of selling a good with constant marginal cost. In this limit

$$\lim_{\eta \rightarrow 1} \frac{\Pi_F}{S_F} = \lim_{\eta \rightarrow 1} \frac{U_F}{S_F} = \frac{1}{e} \approx 0.37.$$

The limit of $\eta \rightarrow \infty$ corresponds to selling an indivisible good: in this case, in the limit, the cost of selling less than 1 unit is 0 and the cost of selling more than 1 unit is infinite. In this limit, we have:

$$\lim_{\eta \rightarrow \infty} \frac{\Pi_F}{S_F} = 0, \quad \lim_{\eta \rightarrow \infty} \frac{U_F}{S_F} = 1.$$

In Figure 2 we display how profit and consumer surplus share are affected by the cost elasticity η . The social surplus that is generated by the profit guarantee mechanism has a lower bound of $2/e \approx 0.74$ which arises in the limit of $\eta \rightarrow 1$. The realized social surplus is increasing in the cost elasticity η .

The property of a uniform consumer surplus share is interesting in its own right. But we might ask how the consumer surplus guarantee compares to levels of consumer surplus that can be attained across all Bayes optimal mechanisms. This is what we pursue next.

5.2 Surplus Sharing Under the Bayesian-Optimal Mechanisms

We now show that the optimal profit-guarantee mechanism generates a consumer surplus that is at least as large as the consumer surplus that would be obtained if the seller used the Bayesian-optimal

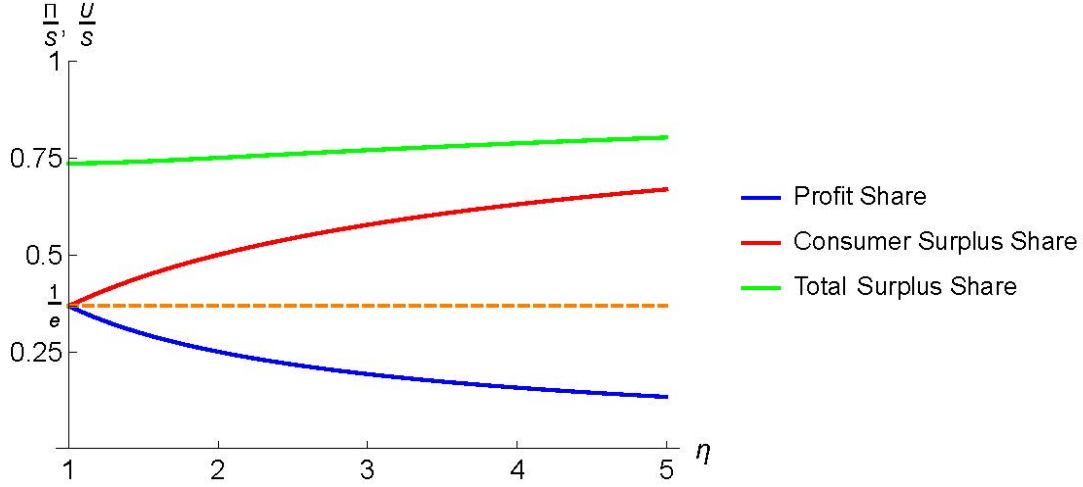


Figure 2: Profit and consumer surplus share in the profit guarantee mechanism M^* and comparative statics in cost elasticity η .

mechanism (regardless of the distribution of values).

Theorem 3 (Maximum Consumer Surplus)

The consumer surplus generated by a Bayesian-optimal mechanism is bounded above by the consumer surplus generated by the optimal profit-guarantee mechanism for every F :

$$U_F \leq U_{F,M^*}.$$

We thus obtain that the consumer surplus generated by the optimal profit guarantee mechanism is larger than that generated by the Bayesian-optimal mechanism for any distribution of values. Surprisingly, the mechanism M^* that guarantees the seller the highest profit guarantee across all distributional environments is also the mechanism that offers the buyer the highest expected consumer surplus across all Bayesian-optimal mechanism for all distributional environments. Thus, the mechanism that guarantees the seller the highest revenue does so by conceding the most consumer surplus and offering a nearly efficient mechanism.

The proof of Theorem 3 is a direct corollary of a more general characterization of the upper frontier of the feasible consumer surplus and profit share across all distributions. We now characterize:

$$\sup_F \left\{ \frac{U_F}{S_F} : \frac{\Pi_F}{S_F} = \beta \right\}. \quad (33)$$

We refer to the upper frontier as the surplus frontier. In other words, we seek to identify the maximum consumer surplus given that the profit is greater than or equal to some fraction $\beta \in [0, 1]$

of the social surplus. Of course, problem (33) is well defined when β can be attained. In particular, when β is the minimum profit share that can be attained (i.e., β equals to (14)) we will find the maximum consumer surplus across all distribution of values. We now provide a complete description of the surplus frontier.

Proposition 2 (Surplus Frontier)

The surplus frontier is given by:

$$\sup_F \left\{ \frac{U_F}{S_F} : \frac{\Pi_F}{S_F} = \beta \right\} = \frac{\eta}{\eta - 1} \left(\beta^{\frac{1}{\eta}} - \beta \right). \tag{34}$$

The constraint is feasible if and only if $\beta \in [1/\eta^{\frac{\eta}{\eta-1}}, 1]$.

Proof. The gross virtual values are given by:

$$\phi(v) \triangleq v - \frac{1 - F(v)}{f(v)}.$$

We denote the ironed virtual values by $\bar{\phi}$. We can find a collection of intervals $\{(\underline{v}_i, \bar{v}_i)\}_{i \in I}$ such that $\phi(v) = \bar{\phi}(v)$ in $[\underline{v}_i, \bar{v}_i]$ and outside these intervals (i.e., in each interval of the form $(\bar{v}_i, \underline{v}_{i+1})$) we have that $\phi(v) < \bar{\phi}(v)$ and $\bar{\phi}(v)$ remains constant. The ironed-virtual values outside these intervals are given by:

$$\bar{\phi}(v) = \frac{\int_{\bar{v}_i}^{\underline{v}_{i+1}} (v - \frac{1-F(v)}{f(v)}) f(v) dv}{F(\underline{v}_{i+1}) - F(\bar{v}_i)}.$$

The optimal quality offered by the optimal mechanism is given by (see Toikka (2011)):

$$Q(v) = \max\left\{ \frac{\eta - 1}{\eta} \bar{\phi}(v)^{\frac{\eta}{\eta-1}}, 0 \right\}.$$

As it is standard in the literature, the quality is constant in $(\bar{v}_i, \underline{v}_{i+1})$, so to avoid confusion we write:

$$q_i \triangleq Q(\bar{v}_i) = Q(\underline{v}_{i+1}).$$

And types with a negative ironed virtual value will be excluded. We then have that:

$$\begin{aligned} U_F &= \int \bar{\phi}(v)^{\frac{1}{\eta-1}} (1 - F(v)) dv; \\ \Pi_F &= \frac{\eta - 1}{\eta} \int \bar{\phi}(v)^{\frac{\eta}{\eta-1}} f(v) dv. \end{aligned} \tag{35}$$

Finally, we note that the first-best surplus is given by:

$$S_F = \int \frac{\eta - 1}{\eta} v^{\frac{\eta}{\eta-1}} f(v) dv.$$

This corresponds to solving (7) explicitly.

We now note that the normalized profit can be written as follows:

$$\Pi_F = \frac{\eta - 1}{\eta} \int \bar{\phi}(v)^{\frac{1}{\eta-1}} \left(v - \frac{1 - F(v)}{f(v)} \right) f(v) dv. \quad (36)$$

To verify this, we note that in any regular interval $[\underline{v}_i, \bar{v}_i]$ we have that

$$\phi(v) = \bar{\phi}(v) = \left(v - \frac{1 - F(v)}{f(v)} \right).$$

Hence, we have that:

$$\bar{\phi}(v)^{\frac{1}{\eta-1}} \left(v - \frac{1 - F(v)}{f(v)} \right) = \bar{\phi}(v)^{\frac{\eta}{\eta-1}}.$$

In any non-regular interval $[\bar{v}_i, \underline{v}_{i+1}]$ we have that

$$\bar{\phi}(v) = \frac{\int_{\bar{v}_i}^{\underline{v}_{i+1}} \left(v - \frac{1 - F(v)}{f(v)} \right) f(v) dv}{F(\underline{v}_{i+1}) - F(\bar{v}_i)}.$$

Hence, we have that:

$$\int_{\bar{v}_i}^{\underline{v}_{i+1}} \bar{\phi}(v)^{\frac{1}{\eta-1}} \left(v - \frac{1 - F(v)}{f(v)} \right) f(v) dv = \int_{\bar{v}_i}^{\underline{v}_{i+1}} \bar{\phi}(v)^{\frac{\eta}{\eta-1}} f(v) dv.$$

We thus prove that (36) is satisfied and we can write the normalized consumer surplus as follows:

$$U_F = \int \bar{\phi}(v)^{\frac{1}{\eta-1}} v f(v) dv - \frac{\eta}{\eta - 1} \Pi_F \quad (37)$$

Using Hölder's inequality we get that:

$$\int \bar{\phi}(v)^{\frac{1}{\eta-1}} v f(v) dv \leq \left(\int \bar{\phi}(v)^{\frac{\eta}{\eta-1}} f(v) dv \right)^{\frac{1}{\eta}} \left(\int v^{\frac{\eta}{\eta-1}} f(v) dv \right)^{\frac{\eta-1}{\eta}}.$$

We thus have that:

$$U_F \leq \frac{\eta}{\eta - 1} \left((\Pi_F)^{\frac{1}{\eta}} (S_F)^{\frac{\eta-1}{\eta}} - \Pi_F \right).$$

Dividing by S_F we obtain an upper bound on normalized consumer surplus:

$$\frac{U_F}{S_F} \leq \frac{\eta}{\eta - 1} \left(\left(\frac{\Pi_F}{S_F} \right)^{\frac{1}{\eta}} - \frac{\Pi_F}{S_F} \right). \quad (38)$$

We note that this function $h(x) = x^{1/\eta} - x$ is decreasing in x for all $x \in [1/\eta^{\frac{\eta}{\eta-1}}, 1]$. Proposition 2 states that $\Pi_F/S_F \geq 1/\eta^{\frac{\eta}{\eta-1}}$, so we obtain that the right-hand-side of (34) is an upper bound. We now need to show the inequality (38) is tight.

To prove the inequality is tight, consider a Pareto distribution $F(v) = 1 - v^{-\alpha}$ with $\alpha = \frac{1}{1-\beta^{\frac{\eta-1}{\eta}}}$. We get that:

$$\frac{\Pi_F}{S_F} = \beta \quad \text{and} \quad \frac{U_F}{S_F} = \frac{\eta}{\eta-1}(\beta^{\frac{1}{\eta}} - \beta). \quad (39)$$

This proves that (38) is tight. Note that replacing $\Pi_F/S_F = 1/\eta^{\frac{\eta}{\eta-1}}$ in (34) we obtain:

$$\frac{U_F}{S_F} = \frac{1}{\eta^{\frac{1}{\eta-1}}}.$$

■

We illustrate the frontier for different values of η in Figure 3. All pairs of equilibrium surplus that are on the frontier are generated by Pareto distributions with different shape parameters $\alpha \geq \eta/(\eta-1)$. The point with the largest normalized consumer surplus is attained with the profit guarantee mechanism M^* of Theorem 1 and under the Bayesian-optimal mechanism when the distribution of values is Pareto with shape $\alpha = \eta/(\eta-1)$ (Theorem 2). That is,

$$\sup_F \frac{U_F}{\Pi_F} = \left(\frac{1}{\eta}\right)^{\frac{1}{\eta-1}},$$

where the upper bound coincides with (32). Hence, as a direct corollary of Proposition 2, we obtain Theorem 3. It is then also clear that we obtain that in general the buyer can capture a greater share of the social surplus when the cost is more elastic.

5.3 Beyond the Efficient Frontier

In Proposition 2 we characterized the efficient frontier of producer surplus and consumer surplus that can be attained by optimal mechanisms across all possible distributions and all possible cost elasticities. We may ask what are the pairs of normalized profits and consumer surplus that can be attained outside the efficient frontier. In the Appendix we also provide in Proposition 5 a lower bound on the social surplus generated by a Bayesian-optimal mechanism across all distribution of values. That is, we find the minimum social surplus:

$$\inf_F \frac{U_F + \Pi_F}{S_F} = \frac{1}{\eta}. \quad (40)$$

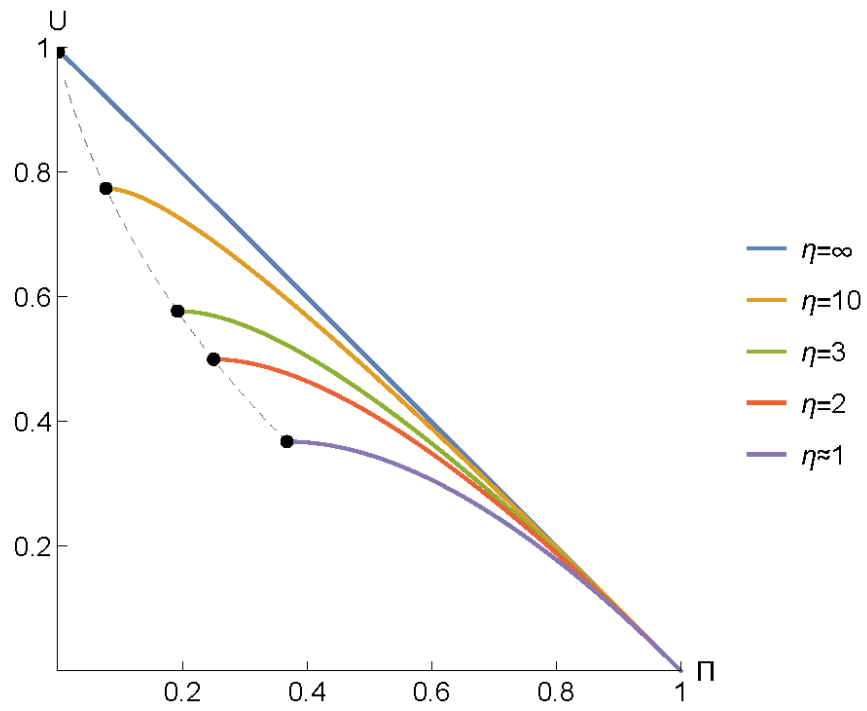


Figure 3: Efficient equilibrium frontier for different iso-elastic cost functions. The highest consumer surplus represented by the dot on each curve is the equilibrium payoff under the saddle point (M^*, F^*) .

The bound is valid only if the marginal cost is convex (that is, if $\eta \geq 2$). As the cost becomes more inelastic (η grows), the lower bound becomes smaller. When the cost is quadratic, then the optimal mechanism always generates at least 1/2 of the social surplus.

Finally, we might be interested in a complete characterization of the set of feasible surplus pairs. For the case of the quadratic cost function we provide such a description in the Appendix (Proposition 6). The reason that $\eta = 2$ is particularly easy to analyze is that in this case we can compute all qualities in closed-form. We provide an illustration of the result below in Figure 4. We established that the upper boundary of the surplus sharing between seller and buyers is attained by the family of Pareto distributions with $\alpha \geq 2$. To describe the rest of the boundary, we introduce the following truncated version of the Pareto distribution is defined by:

$$P_{\alpha,k}(v) \triangleq \begin{cases} 1 - \frac{1}{v^\alpha} & \text{if } v < k; \\ 1 & \text{if } v \geq k. \end{cases} \quad (41)$$

That is, $P_{\alpha,k}(v)$ is the same as a Pareto distribution for values $v < k$ and it has a mass point at k . The lower bound along the segment that provides positive consumer surplus is attained by taking the limit $k \rightarrow \infty$ of truncated Pareto distributions with shape parameter $1 \leq \alpha \leq 2$. Finally, the lower segment with zero surplus is attained by truncated Pareto distribution with shape parameter $\alpha = 1$ and different truncation values $k \in [1, \infty]$. Here, the seller offers the product only to the buyers with a value in the mass point of the truncated distribution. As a consequence, the seller can extract all the surplus and provides the efficient allocation for those buyers at the upper mass point. The buyers who get served receive zero net surplus.

This class of truncated Pareto distribution with shape parameter $\alpha = 1$ arises in Roesler and Szentes (2017) as the consumer-optimal distribution of (expected) values when the seller has an indivisible good for sale. In the present context of nonlinear pricing, this same family of distributions generate zero consumer surplus. This difference arise because when the seller has zero cost, then the product can be allocated even when the virtual value is zero (in fact, the seller is indifferent between allocating or not allocating). However, when the cost is not zero, the seller will supply zero quality when the virtual value is equal to zero.

6 Related Screening Problems

So far, we considered a model of quality discrimination in the spirit of Mussa and Rosen (1978). We now investigate corresponding results for quantity discrimination in the spirit of Maskin and

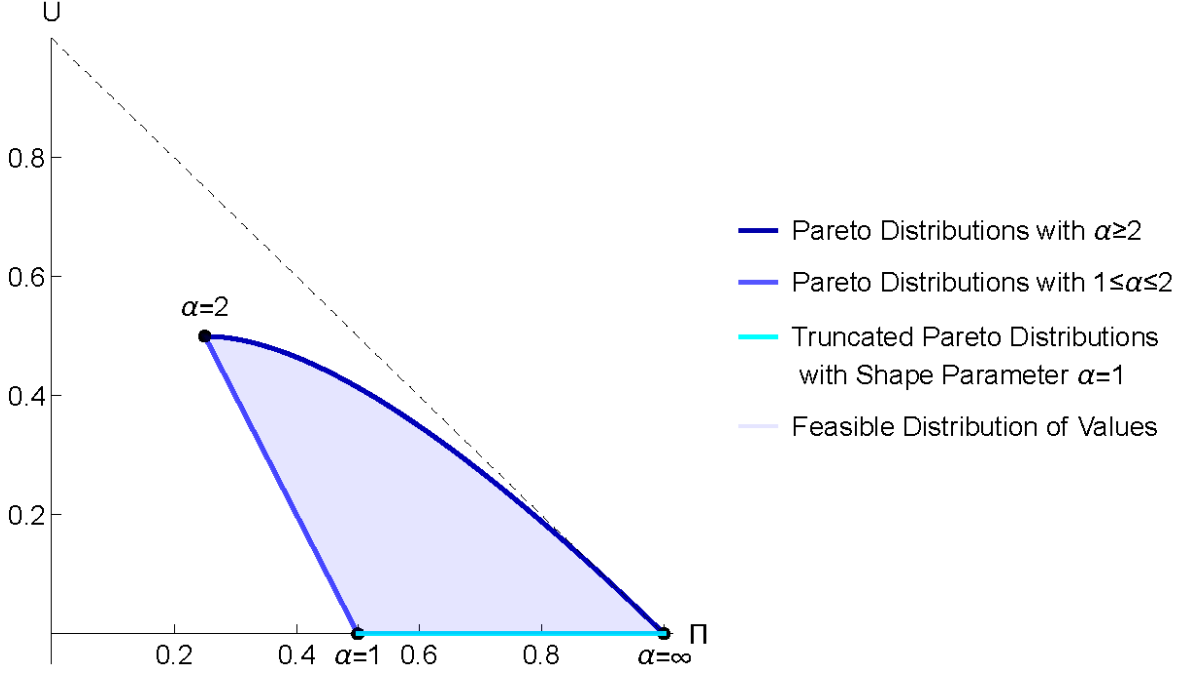


Figure 4: Set of all equilibrium profit and consumer surplus share for with quadratic cost: $\eta = 2$.

Riley (1984). We then analyze procurement environments.

6.1 Quantity Discrimination

We now assume that the utility function is given by

$$u(v, q) = v \frac{\eta}{\eta + 1} q^{\frac{\eta+1}{\eta}},$$

for some $\eta \in (-\infty, -1)$. Thus the utility function for a higher quantity is increasing and concave. The cost of production is linear $c(q) = cq$. In this section we focus on the case of multiplicatively separable utility functions (as specified above) and extend it to nonlinear environments without separability conditions in the Appendix.

The individual demand is defined by the inverse of the marginal utility:

$$D(v, p) \triangleq u_q^{-1}(v, p),$$

where the subscript q denotes the partial derivative with respect to q . With the above parametrization we find that the demand elasticity is

$$\frac{\partial D(v, p)}{\partial p} \frac{p}{D(v, p)} \triangleq \eta.$$

We note that as we shift from cost elasticity to demand elasticity, we maintain η as the parameter of the elasticity. However, now η is a negative number $\eta \in (-\infty, -1)$.

As earlier in the case of quality discrimination, the Pareto distribution with the shape parameter $\alpha \in (1, \infty)$ is playing a critical role for the minmax problem.

Theorem 4 (Profit Guarantee with Quantity Discrimination)

*The profit guarantee mechanism is a uniform-price mechanism $t = p^*q$ with*

$$p^* = \frac{\eta}{\eta + 1}. \tag{42}$$

It generates profits:

$$\Pi^* = \left(\frac{\eta}{\eta + 1} \right)^\eta S, \tag{43}$$

for every F . Furthermore, the profit-guarantee mechanism is the Bayes optimal mechanism against the Pareto distribution with shape parameter $\alpha = |\eta|$.

Proof. In the baseline model of Section 2, we have a buyer with utility function:

$$u(v, q, t) = vq - t,$$

and a seller with a cost function:

$$c(q) = q^\eta / \eta.$$

Now consider the following change of variables

$$\hat{q} \triangleq \frac{q^\eta}{\eta}, \quad \hat{\eta} \triangleq -\frac{\eta}{\eta - 1}, \quad \hat{v} \triangleq v \left(\frac{\hat{\eta}}{\hat{\eta} + 1} \right)^{\frac{1}{\hat{\eta}}}.$$

We then have that the utility functions and cost functions are given by:

$$u(v, q, t) = \hat{v} \left(\frac{\hat{\eta}}{\hat{\eta} + 1} \right) \hat{q}^{\frac{\hat{\eta} + 1}{\hat{\eta}}} - t,$$

and $c(q) = \hat{q}$. With this change of variable we then obtain the model of this section. The profit guarantee result of Theorem 1 and 2 then follow immediately. Following Corollary 1 the optimal mechanism is given by:

$$P(q) = \eta c(q) = \frac{\hat{\eta}}{\hat{\eta} + 1} \hat{q}.$$

Hence, the optimal mechanism is a uniform-price mechanism with price-per-unit equal to:

$$p^* = \frac{\hat{\eta}}{\hat{\eta} + 1}.$$

Finally, the uniform price mechanism is Bayes optimal against the Pareto distribution with parameter

$$\alpha = \frac{\eta}{\eta - 1} = -\hat{\eta}.$$

This proves the result. ■

Thus, in the case of concave utility functions and linear cost functions we obtain a profit guarantee mechanism. The mechanism maintains the constant mark-up property that we saw earlier, but in the presence of linear costs, we now have that a linear pricing mechanism, a uniform per unit price, generates the profit guarantee. Thus, the profit guarantee can be established with an even simpler mechanism. With the change in variable suggested in the proof of Theorem 4, it also follows immediately that the profit guarantee mechanism generates a uniform profit and consumer surplus share for all distributions F . Thus, the profit guarantee mechanism maintains a uniform sharing of surplus between buyers and seller across all distributions.

In the Appendix we consider a class of nonlinear utility functions in which willingness-to-pay and quantity can interact in a nonlinear manner and without the former multiplicative separability condition. Thus, we assume that the utility net of the payment $t \in \mathbb{R}_+$ is:

$$u(v, q, t) = h(v, q) - t,$$

where h is concave in q given v . Proposition 7 gives a profit guarantee for an environment where the demand elasticity may vary within a limited range $[\bar{\eta} - 1, \bar{\eta}]$ across willingness-to-pay and price.

6.2 Procurement

We focused throughout on the classic problem of nonlinear pricing where the seller is uncertain about the demand of the buyers who have private information regarding their willingness-to-pay for varying quality or quantity. Alternatively, we might be interested in the robust procurement policies where a single large buyer wishes to procure from sellers who have private information about their cost condition. We can then ask what are the robust procurement policies as measured by the competitive ratio. As the selling and procurement problem are closely connected, we indeed find that a similar characterization as in Theorem 1 and 4 exists. As before, a distinction between quantity differentiation and quality differentiation proves to be useful.

Quality Differentiation A single buyer procures an object of variable quality from a seller with unknown cost. The buyer receives value q from quality $q \in \mathbb{R}_+$ and the seller has a cost $\theta \cdot c(q)$

to provide a good of quality q . The parameter $\theta \in \mathbb{R}_+$ is private information for the seller and described by a distribution F . The cost function is given by a constant elasticity function:

$$c(q) = q^\eta/\eta.$$

The efficient social surplus is generated by finding the optimal quality given the prevailing cost function:

$$S(\theta) = \max_q \{q - \theta c(q)\}.$$

The efficient quality is inversely related to the cost parameter θ :

$$q^* = \left(\frac{1}{\theta}\right)^{\frac{1}{\eta-1}},$$

and generates a social surplus of:

$$S(\theta) = \frac{\eta - 1}{\eta} \left(\frac{1}{\theta}\right)^{\frac{1}{\eta-1}}.$$

If the buyer offers a constant price p for every marginal unit of quality, then the seller will optimally offer a quality:

$$\theta c'(q) = p \iff q = \left(\frac{p}{\theta}\right)^{\frac{1}{\eta-1}}.$$

Corollary 4 (Surplus Guarantee Mechanism)

The optimal surplus guarantee mechanism offers a constant unit price $p^ = 1/\eta$ for incremental quality and the buyer is guaranteed a constant share of the efficient social surplus:*

$$\left(\frac{1}{\eta}\right)^{\frac{1}{\eta-1}}.$$

Thus, the robust optimal pricing policy is a uniform unit price for quality at which the seller can then deliver the optimal quality. The surplus guarantee is increasing in the elasticity of the cost function of the seller.

Quantity Differentiation We can alternatively consider the case where the buyer has a declining marginal utility for quantity and the seller has a constant marginal cost of producing additional units of the product. The buyer thus has a utility function $u(q)$, where q is the quantity of the good and

$$u(q) = \eta q^{(\eta+1)/\eta} / (\eta + 1) \tag{44}$$

with a demand elasticity:

$$\eta \in (-\infty, -1). \quad (45)$$

The utility function is increasing and concave in the above range with

$$u'(q) = q^{\frac{1}{\eta}} > 0, u''(q) = \frac{q^{\frac{1}{\eta}-1}}{\eta} < 0.$$

(For $\eta > -1$, the above parametrization gives a negative gross utility.) The seller has cost

$$c(q) = \theta \cdot q,$$

where the marginal cost θ is private information for the seller and given by a common prior distribution. The first-best surplus is:

$$S(\theta) = \max_q \{u(q) - \theta q\}.$$

The efficient quantity is:

$$q^* = \theta^\eta$$

and the social surplus is

$$S(\theta) = -\frac{\theta^{\eta+1}}{\eta+1}.$$

Corollary 5 (Surplus Guarantee Mechanism)

The optimal surplus guarantee mechanism offers a constant mark-up

$$p^*(q) = \frac{\eta+1}{\eta} q^{1/\eta}$$

for quantity and the buyer is guaranteed a constant share of the optimal social surplus:

$$\left(\frac{\eta}{\eta+1}\right)^{\eta+1}.$$

If the buyer sets a constant mark-up $p(q) = zq^{1/\eta}$, the seller will sell

$$\theta = p(q).$$

So, $q = (\frac{\theta}{z})^\eta$. So the buyer surplus will be:

$$u(q) - \int_0^q p(y)dy = \frac{\eta}{\eta+1} \left(\frac{\theta}{z}\right)^{\eta+1} - z \frac{\eta}{\eta+1} \left(\frac{\theta}{z}\right)^{\eta+1}$$

We get that the optimal mark-up is:

$$z = \frac{\eta + 1}{\eta}.$$

Thus the buyer offers a constant mark-up pricing policy that the price per unit of quantity delivered is proportional to the marginal value.

Thus, for a large buyer, the optimal policy is not a constant unit price but a constant mark-up price that is decreasing in quantity. (This is easy to implement and offers a trade-off between commitment and insurance.)

7 Conclusion

We established robust pricing and menu policies for environments with second degree price discrimination. We showed that simple pricing policies, namely constant mark-up policies for the case of quality differentiation and linear pricing rules for the case of quantity differentiation attain the highest profit guarantee for the seller.

We established the optimality of these pricing policy for constant elasticity of cost or demand functions. But we showed that the features of the policies enable us to establish bounds even beyond the case of constant elasticity when we merely assume convexity for cost functions or concavity for demand functions.

In the analysis we focused on the classic optimal selling problem where the seller is designing a menu of choices to screen the buyers with private information regarding their willingness to pay. We further showed that the same arguments and associated mechanisms allow us to establish utility guarantees in procurement settings. Here, the buyer seeks to derive optimal purchasing policies against vendors with private information regarding their cost or marginal cost of producing quantities or qualities. Thus, we derived robust profit or utility guarantees across a wide spectrum of nonlinear pricing problems.

We formulated the profit guarantee in terms of a competitive ratio relative to the socially efficient surplus. As part of the minmax problem, the Pareto distribution emerged as critical distribution that minimizes the revenue of the seller. As critical value distribution, the Pareto distribution generates a linear virtual utility. This suggests that a version of the Pareto distribution may also play an important role in related problems. For example, Bergemann, Brooks, and Morris (2015) consider the limits of third-degree price discrimination where the consumers all have unit-demand. It is an open problem how market segmentation can impact the surplus distribution in

the context of nonlinear pricing problems, thus when we allow jointly for second and third-degree price discrimination.

8 Appendix

The appendix collects the proofs omitted in the main text and some additional material.

8.1 Finite Support and Additional Information

We have established an optimal menu in the absence of any demand information. One might ask how the optimal menu may change if the seller were to have some information about the nature of demand. The additional information could come in the form of information about the possible support of the demand, in particular lower and/or upper bounds on the demand realization. It could also come in the form of data samples or restriction on the demand distribution F , say in terms of the moments. Here, we shall start this investigation with lower and upper bounds on the support of the value distribution, thus $0 < \underline{v} < \bar{v} < \infty$.

The next set of result will show that the main insights of Theorem 1 and 2 remain in the presence of additional support information. Namely, there exists a minmax solution. In addition, the competitive ratio between realized profit and social surplus will be constant at every point in the support of the demand. Some of the results will change with a finite support. In particular, the optimal menu will not display a constant elasticity anymore. As there is a finite and common upper bound \bar{v} for all possible distributions, the classic "no-distortion at the top" result will reemerge and thus there will be zero mark-up at the upper bound. Correspondingly, the ratio of consumer surplus to social surplus will now typically vary and not be constant anymore as in the case of unrestricted support.

The general robust-pricing mechanism, consists of finding a mechanism that maximizes:

$$M^* = \arg \max_{M \in \mathcal{M}} \inf_{F \in \Delta[\underline{v}, \bar{v}]} \frac{\Pi_{F,M}}{\bar{\Pi}_F}. \quad (46)$$

The only difference with (10) is that F has bounded support. As before, the mechanism is completely determined by the allocation rule so we refer to a mechanism by the allocation it induces $Q(v)$. We denote a typical solution by Q^* . We now describe how to find Q^* for a given finite interval and how we obtain a constant competitive ratio.

We define a family of allocations, parameterized by $\beta \in [0, 1]$, denoted by $Q_\beta : [\underline{v}, \bar{v}] \rightarrow \mathbb{R}$ and defined implicitly as follows:

$$\frac{\Pi_{Q_\beta}(v)}{\bar{\Pi}(v)} = \beta. \quad (47)$$

The upper bound of the domain v_β is the upper bound on which this equation can be satisfied. That is:

$$v_\beta \triangleq \max\{v \in [\underline{v}, \bar{v}] : \text{there exists } Q_\beta \text{ such that } \frac{\Pi_{Q_\beta}(v)}{\bar{\Pi}(v)} = \beta, \text{ for all } v \in [\underline{v}, v_\beta)\}.$$

Thus, this the largest domain over which it is possible to maintain a constant normalized profits.

We can write (47) as a differential equation as follows:

$$\frac{dQ_\beta(v)}{dv} = \frac{\beta c'^{-1}(v)}{v - c'(Q_\beta(v))} \quad (48)$$

with the initial condition:

$$\frac{\underline{v}Q_\beta(\underline{v}) - c(Q_\beta(\underline{v}))}{\bar{\Pi}(\underline{v})} = \beta. \quad (49)$$

Hence, we analyze this differential equation. In every rectangle $[v_1, v_2] \times [Q_1, Q_2] \subset \mathbb{R}^2$ such that:

$$v - c'(q) > 0, \text{ for all } (v, q) \in [v_1, v_2] \times [Q_1, Q_2]$$

the right-hand-side of (48) is Lipschitz continuous. Following Picard-Lindelöf theorem, in any such interval, the differential equation with initial condition $(v_0, Q_0) \in [v_1, v_2] \times [Q_1, Q_2]$ has a solution. And, we have that at v_β :

$$v_\beta - c'(Q_\beta(v_\beta)) = 0.$$

This is the point where the right-hand-side of (48) is discontinuous. We can now discuss how Q_β and v_β change with β .

Lemma 3 (Constant Profit-Surplus Ratio)

The allocation rule Q_β is increasing in β and v_β is decreasing in β .

Proof. Consider $\beta_1 < \beta_2$. The difference between the profits generated by both mechanisms can be written as follows:

$$\Pi_{Q_{\beta_1}}(\hat{v}) - \Pi_{Q_{\beta_2}}(\hat{v}) = (Q_{\beta_1}(\hat{v})\hat{v} - c(Q_{\beta_1}(\hat{v}))) - (Q_{\beta_2}(\hat{v})\hat{v} - c(Q_{\beta_2}(\hat{v}))) - \int_0^{\hat{v}} (Q_{\beta_1}(v) - Q_{\beta_2}(v))dv.$$

Following (49), we must have that:

$$Q_{\beta_1}(\underline{v}) < Q_{\beta_2}(\underline{v}).$$

Suppose there exists \hat{v} such that:

$$Q_{\beta_1}(\hat{v}) \geq Q_{\beta_2}(\hat{v}).$$

Without loss of generality, assume that for all $v \in [\underline{v}, \hat{v})$:

$$Q_{\beta_1}(v) < Q_{\beta_2}(v).$$

We have that:

$$0 > \int_0^{\hat{v}} (Q_{\beta_1}(v) - Q_{\beta_2}(v)) dv; \quad (50)$$

$$0 \leq (Q_{\beta_1}(\hat{v})\hat{v} - c(Q_{\beta_1}(\hat{v}))) - (Q_{\beta_2}(\hat{v})\hat{v} - c(Q_{\beta_2}(\hat{v}))). \quad (51)$$

Thus,

$$\Pi_{Q_{\beta_1}}(\hat{v}) - \Pi_{Q_{\beta_2}}(\hat{v}) > 0,$$

so we reach a contradiction. It thus also follow that $v_{\beta_1} > v_{\beta_2}$ ■

We can now show that the optimal profit-guarantee mechanism is $Q^* = Q_\beta$ for some β .

Proposition 3 (Constant Profit-Surplus Ratio)

The profit-guarantee mechanism Q^* , is given by $Q^*(v) = Q_\beta(v)$, with β such that $v_\beta = \bar{v}$

Proof. We assume that $Q^*(v) \neq Q_\beta(v)$ and reach a contradiction by showing that Q^* cannot guarantee a competitive ratio equal to β . Let \hat{v} be the first value such that Q^* and Q_β differ:

$$\hat{v} \triangleq \sup\{v \in [0, 1] : \text{such that } Q^*(v') = Q_\beta(v') \text{ for all } v' \in [\underline{v}, v]\}.$$

Suppose that $Q^*(v) < Q_\beta(v)$ for all v in some neighborhood $[\hat{v}, \hat{v} + \varepsilon]$. We would then have that in this neighborhood:

$$\Pi_{Q^*}(v) < \Pi_{Q_\beta}(v) = \beta.$$

We thus reach a contradiction. On the contrary, suppose that $Q^*(v) > Q_\beta(v)$ for all v in some neighborhood $[\hat{v}, \hat{v} + \varepsilon]$. We then define:

$$\hat{v}' \triangleq \sup\{v \in (\hat{v}, 1] : \text{such that } Q^*(v') \geq Q_\beta(v') \text{ for all } v' \in [0, v]\}.$$

We thus have that $Q^*(\hat{v}') \leq Q_\beta(\hat{v}')$ and $\hat{v}' > \hat{v}$. We again have that:

$$\Pi_{Q^*}(\hat{v}') < \Pi_{Q_\beta}(\hat{v}') = \beta.$$

The inequality is obtained analogously to (50) and (51) (replacing $Q_{\beta_2} = Q_\beta$ and $Q_{\beta_1} = Q^*$). We thus reach a contradiction. ■

With a finite support we cannot provide a compact and explicit solution for the competitive ratio even when the cost function has constant elasticity. Yet we can provide an upper bound by means of a Bayes optimal mechanism when the cost has constant elasticity η . The mechanism has the property that it converges to the exact solution as the upper bound of the support diverges to ∞ . Moreover, numerically it appears to be close to the optimal solution whose existence we proved in Proposition 3.

Proposition 4 (Bounded Support)

There exists a distribution F with support in $[\underline{v}, \bar{v}]$ such that the Bayesian optimal mechanism generates normalized profits:

$$\frac{\Pi}{S} = \frac{1}{\eta^{\frac{\eta}{\eta-1}}} + \left(1 - \frac{1}{\eta^{\frac{\eta}{\eta-1}}}\right) \frac{1}{1 + \frac{\eta}{\eta-1} \log(\bar{v})}.$$

Hence, this constitutes an upper bound on the optimal profit guarantee.

Proof. Throughout the proof we normalize $\underline{v} = 1$ (of course, the only thing that matters is \bar{v}/\underline{v}). Consider distribution:

$$F(v) = \begin{cases} 0 & \text{if } v \in [0, 1]; \\ 1 - v^{-\alpha} & \text{if } v \in [1, \bar{v}]; \\ 1 & \text{if } v \in [\bar{v}, \infty), \end{cases}$$

with $\alpha = \eta/(\eta - 1)$. The first-best total surplus is given by:

$$S_F = \int_1^{\bar{v}} \frac{\eta - 1}{\eta} v^{\frac{\eta}{\eta-1}} \alpha v^{-\alpha-1} dF(v) + (\bar{v}^\alpha) \bar{v}^{\frac{\eta}{\eta-1}} = \frac{\eta - 1}{\eta} + \log(\bar{v})$$

$$\Pi_F = \int_1^{\bar{v}} \frac{\eta - 1}{\eta} \left(\frac{\alpha - 1}{\alpha} v\right)^{\frac{\eta}{\eta-1}} \alpha v^{-\alpha-1} dF(v) + (\bar{v}^\alpha) \bar{v}^{\frac{\eta}{\eta-1}} = \frac{\eta - 1}{\eta} + \frac{1}{\eta^{\frac{\eta}{\eta-1}}} \log(\bar{v})$$

Thus, across all optimal profit-guarantee mechanisms:

$$\frac{\Pi_F}{S_F} \leq \frac{1}{\eta^{\frac{\eta}{\eta-1}}} + \left(1 - \frac{1}{\eta^{\frac{\eta}{\eta-1}}}\right) \frac{1}{1 + \frac{\eta}{\eta-1} \log(\bar{v})}.$$

In the limit $\bar{v} \rightarrow \infty$ we get the same as we have obtained with unbounded support. ■

We illustrate the behavior of the profit share in Figure 5.

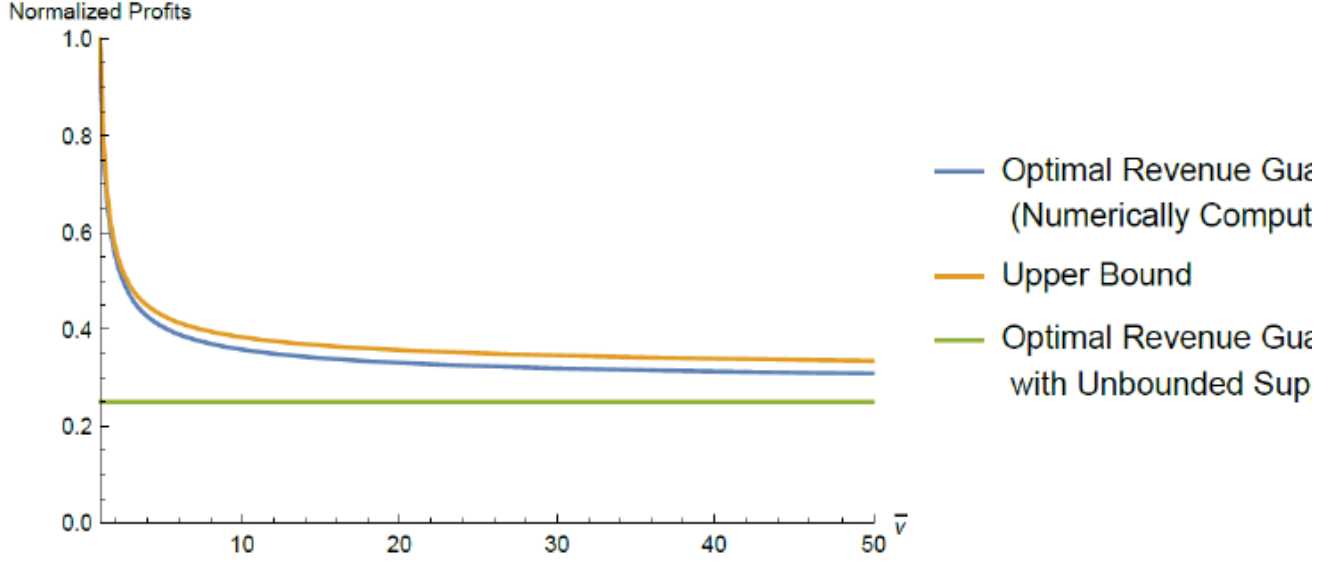


Figure 5: Profit share as a function of the upper bound of the support, \bar{v} for $\eta = 2$ and $\underline{v} = 1$.

8.2 Additional Results on Welfare Bounds

We now provide a lower bound on the distortions generated by any mechanism. Before we provide the result, we note that:

$$\left. \frac{U_{P_\alpha}}{S_{P_\alpha}} \right|_{\alpha=1} = 0 \quad \text{and} \quad \left. \frac{\Pi_{P_\alpha}}{S_{P_\alpha}} \right|_{\alpha=1} = \frac{1}{\eta}.$$

That is, when the distribution of values is the Pareto distribution with shape parameter $\alpha = 1$ the consumer's surplus is 0 (in fact, it is 0 for every truncated Pareto distribution $P_{1,k}$, not only in the limit), and the normalized profit is $1/\eta$. We thus have that:

$$\left. \frac{U_{P_\alpha} + \Pi_{P_\alpha}}{S_{P_\alpha}} \right|_{\alpha=1} = \frac{1}{\eta}. \tag{52}$$

In other words, the generated social surplus is a fraction $1/\eta$ of the efficient social surplus.

Proposition 5 (Lower Bound on Social Surplus)

When $\eta \geq 2$, social surplus is bounded below by:

$$\inf_F \frac{U_F + \Pi_F}{S_F} = \left. \frac{U_{P_\alpha} + \Pi_{P_\alpha}}{S_{P_\alpha}} \right|_{\alpha=1} = \frac{1}{\eta}.$$

Proof. Following expression (37) we obtain:

$$\frac{\eta - 1}{2\eta - 1}U_F + \frac{\eta}{2\eta - 1}\Pi_F = \frac{\eta - 1}{2\eta - 1} \int \bar{\phi}(v)v^{\frac{1}{\eta-1}}vf(v)dv.$$

We first note that, $\bar{\phi}(v) \leq v$, so when $\eta \geq 2$, we have that:

$$\bar{\phi}(v)v^{\frac{1}{\eta-1}}v \geq \bar{\phi}(v)v^{\frac{1}{\eta-1}}.$$

We now note that:

$$\int \bar{\phi}(v)v^{\frac{1}{\eta-1}}f(v)dv \geq \int \phi(v)v^{\frac{1}{\eta-1}}f(v)dv = \frac{1}{\eta} \int v^{\frac{\eta-1}{\eta}}f(v)dv. \quad (53)$$

The inequality follows from the fact that, by construction of the ironed virtual values, for any increasing function $h(v)$ we have that $\int \bar{\phi}(v)h(v)f(v)dv \geq \int \phi(v)h(v)f(v)dv$ (see Kleiner, Moldovanu, and Strack (2021)). The equality follows from integrating by parts. Then the right-hand-side of (53) is exactly equal to S_F , so we have that:

$$\frac{\eta - 1}{2\eta - 1}U_F + \frac{\eta}{2\eta - 1}\Pi_F \geq \frac{1}{2\eta - 1}S_F. \quad (54)$$

We can now show the inequality is tight. Using (52) we get that:

$$\frac{\eta - 1}{2\eta - 1}U_F + \frac{\eta}{2\eta - 1}\Pi_F = \frac{1}{2\eta - 1}S_F.$$

This corresponds to the lower bound (54). We thus have that:

$$\inf_F \frac{\frac{\eta-1}{2\eta-1}U_F + \frac{\eta}{2\eta-1}\Pi_F}{S_F} = \frac{\eta}{2\eta-1} \frac{1}{\eta}.$$

Since $\frac{\eta}{2\eta-1} > 1/2$ and the infimum is attained at an information structure that generates 0 consumer surplus, we must also have that:

$$\inf_F \frac{\frac{1}{2}U_F + \frac{1}{2}\Pi_F}{S_F} = \frac{1}{2} \frac{1}{\eta}.$$

Multiplying by 2, we obtain the result. ■

The feasible normalized profit and consumer surplus are:

$$\mathcal{F} = \{(x, y) \in \mathbb{R} \mid \text{there exists } F \text{ such that } x = \frac{U_F}{S_F} \text{ and } y = \frac{\Pi_F}{S_F}\}.$$

We recall that the truncated Pareto distribution is defined in (41). When $\eta = 2$, for a fixed k , we have that:

$$\frac{U_{P_{\alpha,k}}}{S_{P_{\alpha,k}}} = \frac{2(\alpha - 1)(k^{\alpha-2} - 1)}{\alpha(\alpha k^{\alpha-2} - 2)} \quad \text{and} \quad \frac{\Pi_{P_{\alpha,k}}}{S_{P_{\alpha,k}}} = \frac{(\alpha - 1)^2 k^{\alpha-2} - 1}{\alpha(\alpha k^{\alpha-2} - 2)}.$$

Taking the limit $k \rightarrow \infty$, we get that:

$$\left(\frac{U_{P_\alpha}}{S_{P_\alpha}}, \frac{\Pi_{P_\alpha}}{S_{P_\alpha}}\right) = \begin{cases} \left(\frac{2(\alpha-1)}{\alpha^2}, \frac{(\alpha-1)^2}{\alpha^2}\right), & \text{for } \alpha \in [2, \infty); \\ \left(\frac{\alpha-1}{\alpha}, \frac{1}{2\alpha}\right), & \text{for } \alpha \in [1, 2]. \end{cases} \quad (55)$$

The curve for $\alpha \geq 2$ is the one characterized in Proposition 2; the curve for $\alpha \in [1, 2]$ does not have a direct counter part in the results we have provided thus far (except for $\alpha = 1$).

Proposition 6 (Feasible Normalized Profits and Utilities)

The closure of \mathcal{F} is given by the area enclosed by the curves in (55).

Proof. The upper boundary was characterized in Proposition 2. We now characterize the lower boundary of \mathcal{F} . Writing (54) for $\eta = 2$, we get:

$$\frac{1}{3}U_F + \frac{2}{3}\Pi_F \geq \frac{1}{3}S_F.$$

However, the limit of Pareto distributions with parameter $\alpha \in [1, 2]$ (see (39)) gives exactly that:

$$\frac{\frac{1}{3}U_{P_\alpha} + \frac{2}{3}\Pi_{P_\alpha}}{S_{P_\alpha}} = \frac{1}{3}.$$

Hence, these distributions give the lower frontier of the set of feasible consumer surplus and profit.

■
The proposition gives a full characterization of the set of normalized profit and consumer surplus generated by any distribution of values.

8.3 Quantity Differentiation with Nonlinear Utility

We now consider a class of nonlinear utility functions in which willingness-to-pay and quantity can interact in a nonlinear manner and without the former multiplicative separability condition. Thus, we assume that the utility net of the payment $t \in \mathbb{R}_+$ is:

$$u(v, q, t) = h(v, q) - t,$$

where h is concave in q given v . The willingness-to-pay parameter v remains distributed according to F and the cost of production remains linear $c(q) = cq$ and we normalize $c = 1$ without loss of generality. The demand function is then defined by the inverse of the marginal utility:

$$D(v, p) \triangleq h_q^{-1}(v, p),$$

where the subscript q denotes the partial derivative with respect to q . The demand elasticity is then given by:

$$\eta(v, p) \triangleq \frac{\partial D(v, p)}{\partial p} \frac{p}{D(v, p)},$$

where the demand elasticity is assumed to be negative $\eta(v, p) < 0$ for all v, p . We assume that, for all $p \in [1, \infty)$,

$$\eta(v, p) \text{ is non-increasing in } p \text{ and } \eta(v, p) \in [\bar{\eta} - 1, \bar{\eta}], \quad (56)$$

for some $\bar{\eta} \in (-\infty, -1)$. We next present a robust profit guarantee that holds as long as the demand elasticity $\eta(v, p)$ is within the range $[\bar{\eta} - 1, \bar{\eta}]$ for some upper bound $\bar{\eta} < -1$.

For a given demand function $D(v, p)$, the optimal uniform price is given by:

$$\hat{p} = \arg \max_p D(v, p)(p - c).$$

The first-order condition can be written as follows:

$$\hat{p} = c \frac{\eta(v, \hat{p})}{\eta(v, \hat{p}) + 1}.$$

We then have that:

$$\hat{p} \leq c \frac{\bar{\eta}}{\bar{\eta} + 1}.$$

Since the upper bound will be relevant for our analysis, it is useful to denote by Π^* the profit generated by the uniform price mechanism with price p^* :

$$\Pi^* \triangleq D(v, p^*)(p^* - c).$$

Proposition 7 (Robust Profit-Guarantee Mechanism)

*The uniform-price mechanism $t = p^*q$, where*

$$p^* = \frac{\bar{\eta}}{\bar{\eta} + 1}c$$

guarantees a profit share of the social surplus:

$$\Pi^* \geq \left(\frac{\bar{\eta}}{\bar{\eta} + 1} \right)^{\bar{\eta}} S. \quad (57)$$

Proof. The profit generated by a uniform price mechanism is given by:

$$\Pi^* = D(v, p^*)(p^* - c).$$

The social surplus is given by:

$$S = \int_c^\infty D(v, p) dp.$$

The demand satisfies:

$$\log\left(\frac{D(v, p)}{D(v, p^*)}\right) = \int_{p^*}^p \frac{\eta(v, s)}{s} ds.$$

Since the price elasticity is non-increasing, we have that

$$\log\left(\frac{D(v, p)}{D(v, p^*)}\right) = \int_{p^*}^p \frac{\eta(v, s)}{s} ds \leq \eta(v, p^*) \int_{p^*}^p \frac{1}{s} ds = \eta(v, p^*) \log\left(\frac{p}{p^*}\right).$$

We thus have that:

$$D(v, p) \leq D(v, p^*) \left(\frac{p}{p^*}\right)^{\eta(v, p^*)}.$$

We then have that:

$$S \leq D(v, p^*) \left(\frac{1}{p^*}\right)^{\eta(v, p^*)} \int_c^\infty p^{\eta(v, p^*)} dp = D(v, p^*) \left(\frac{1}{p^*}\right)^{\eta(v, p^*)} \frac{-1}{\eta(v, p^*) + 1} c^{\eta(v, p^*) + 1}.$$

We then have that

$$\frac{\Pi^*}{S} \geq \frac{-(\eta(v, p^*) + 1)(p^*)^{\eta(v, p^*)}(p^* - c)}{c^{\eta(v, p^*) + 1}}.$$

We now note that the function $g(\eta) \triangleq -(\eta + 1)p^\eta$ is quasi-concave in η . We also have that we assumed that $\eta(v, p^*) \in [\underline{\eta}, \bar{\eta}]$ and:

$$-(\bar{\eta} + 1)(p^*)^{\bar{\eta}} = -(\underline{\eta} + 1)(p^*)^{\underline{\eta}}.$$

where $\underline{\eta} = \bar{\eta} - 1$. We thus have that:

$$\frac{\Pi^*}{S} \geq \frac{-(\bar{\eta} + 1)(p^*)^{\bar{\eta}}(p^* - c)}{c^{\eta(v, p^*) + 1}} = \left(\frac{\bar{\eta}}{\bar{\eta} + 1}\right)^{\bar{\eta}}.$$

Now, since the bound was established pointwise for every v , it holds in aggregate across all v . ■

Proposition 7 gives a profit guarantee for an environment where the demand elasticity may vary within a limited range across willingness-to-pay and price. Thus in contrast to the earlier results, it does not require a constant demand elasticity. The robustness of the profit guarantee is perhaps of more relevance when we consider demand rather than cost elasticity. After all, when the seller lacks information about the willingness-to-pay of the buyer, the seller may also lack information about the demand elasticity. Correspondingly, the bound that we obtain is somewhat weaker as it refers only to the upper bound in the demand elasticity. Similarly, we do not establish that the uniform price mechanism is Bayes optimal for arbitrary nonlinear demand functions that satisfy the above elasticity condition (56).

References

- ALLOUAH, A., A. BAHAMOU, AND O. BESBES (2021): “Pricing with Samples,” .
- (2022): “Optimal Pricing with a Single Point,” Discussion paper.
- ANUNROJWONG, J., S. BALSEIRO, AND O. BESBES (2023): “Roubst Auction Design with Support Information,” Discussion paper.
- BERGEMANN, D., B. BROOKS, AND S. MORRIS (2015): “The Limits of Price Discrimination,” *American Economic Review*, 105, 921–957.
- BERGEMANN, D., AND K. SCHLAG (2008): “Pricing Without Priors,” *Journal of the European Economic Association Papers and Proceedings*, 6, 560–569.
- BORODIN, A., AND R. EL-YANIV (1998): *Online Computation and Competitive Analysis*. Cambridge University Press, Cambridge.
- CARRASCO, V., V. F. LUZ, P. MONTEIRO, AND H. MOREIRA (2019): “Roubst Mechanisms: The Curvature Case,” *Economic Theory*, 68, 203–222.
- CARROLL, G. (2017): “Robustness and Separation in Multidimensional Screening,” *Econometrica*, 85, 453–488.
- DEB, R., AND A. ROESLER (2022): “Multi-dimensional Screening: Buyer-Optimal Learning and Informational Robustness,” Discussion paper, University of Toronto.
- DIAMOND, P., AND E. SAEZ (2011): “From Basic Research to Policy Recommendations: The Case for a Progressive Tax,” *Journal of Economic Perspectives*, 25, 165–190.
- EREN, S., AND C. MAGLARAS (2010): “Monopoly Pricing with Limited Demand Information,” *Journal of Revenue and Pricing Management*, 9, 23–48.
- HAGHPANAH, N., AND R. SIEGEL (2022): “The Limits of Multi-Product Discrimination,” *American Economic Review: Insights*, 4, 443–458.
- HARTLINE, J., AND T. ROUGHGARDEN (2014): “Optimal Platform Design,” Discussion paper, Northwestern University.

- KLEINER, A., B. MOLDOVANU, AND P. STRACK (2021): “Extreme Points and Majorization: Economic Applications,” *Econometrica*, 89, 1557–1593.
- MASKIN, E., AND J. RILEY (1984): “Monopoly with Incomplete Information,” *RAND Journal of Economics*, 15, 171–196.
- MIRRLEES, J. (1971): “An Exploration in the Theory of Optimum Income Taxation,” *Review of Economic Studies*, 38, 175–208.
- MUSSA, M., AND S. ROSEN (1978): “Monopoly and Product Quality,” *Journal of Economic Theory*, 18, 301–317.
- MYERSON, R. (1981): “Optimal Auction Design,” *Mathematics of Operations Research*, 6, 58–73.
- NEEMAN, Z. (2003): “The Effectiveness of English Auctions,” *Games and Economic Behavior*, 43, 214–238.
- ROESLER, A., AND B. SZENTES (2017): “Buyer-Optimal Learning and Monopoly Pricing,” *American Economic Review*, 107, 2072–2080.
- SAEZ, E. (2001): “Using Elasticities to Derive Optimal Income Tax Rates,” *Review of Economic Studies*, 68, 205–229.
- TOIKKA, J. (2011): “Ironing without Control,” *Journal of Economic Theory*, 146(6), 2510–2526.