ON THE ALIGNMENT OF CONSUMER SURPLUS AND TOTAL SURPLUS UNDER COMPETITIVE PRICE DISCRIMINATION

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On the Alignment of Consumer Surplus and Total Surplus Under Competitive Price Discrimination*

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Producers of heterogeneous goods with heterogeneous costs compete in prices. When producers know their own production costs and the consumer knows their values, consumer surplus and total surplus are aligned: the information structure and equilibrium that maximize consumer surplus also maximize total surplus. We report when alignment extends to the case where either the consumer is uncertain about their own values or producers are uncertain about their own costs, and we also give examples showing when it does not. Less information for either producers or consumer may intensify competition in a way that benefits the consumer but results in inefficient production. We also characterize the information for consumer and producers that maximizes consumer surplus in a Hotelling duopoly.

KEYWORDS: Information, competition, price discrimination, segmentation, oligopoly.

JEL CLASSIFICATION: C72, D44, D82, D83.

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1 Introduction

1.1 Overview of Results

An elementary observation about discriminatory pricing is that it can be extremely beneficial in terms of the total welfare of society: a monopolist who can perfectly price discriminate will charge a price equal to each consumer's willingness to pay, and a sale will take place whenever the consumer's value is above cost. The resulting outcome, while socially efficient, is absolutely dismal for the consumer, who obtains zero net value from their purchase.[1] For a long time, this was the only known mechanism by which discriminatory pricing could result in socially efficient outcomes. From that state of affairs, one might conclude that there is a fundamental trade-off between consumer surplus and total surplus, and that in order for markets to operate efficiently, the consumer must suffer.

But contrary to this conventional wisdom, Bergemann, Brooks, and Morris (2015b), hereafter BBM, showed that actually there are many ways in which discriminatory pricing might yield a socially efficient outcome. In fact, there are even ways of segmenting a market, so that the resulting outcome is socially efficient, but the monopolist does not benefit from discriminatory pricing at all, and all of the gains in surplus from segmentation go to the consumer. Thus, consumer surplus and total surplus are *aligned*, in that the segmentations that maximize consumer surplus must also maximize total surplus. Moreover, consumer surplus and producer surplus are *opposed,* in that the segmentations that maximize consumer surplus also minimize producer surplus. At a high level, these outcomes are achieved by pooling together high value types of consumer with low value types, in such a way that the monopolist is just barely willing to lower prices. The resulting outcome is efficient, but the high value consumers reap all of the benefits from lower prices.

[1] Throughout our exposition, we refer to a single representative consumer. All of our results can be interpreted as applying to a market consisting of a mass of non-atomic consumers. Our representative consumer's information and value should then be interpreted as the empirical distribution of information and values in the population.

The result of BBM considers the welfare consequences of the producer's information. A closely related result was subsequently obtained by Roesler and Szentes (2017), hereafter RS, concerning the consumer's information: In their main model, they suppose that there is no segmentation of the market, but that the consumer may have imperfect information about their value. RS compute maximum consumer surplus across all models of consumer information, and they show that consumer surplus could be even higher than that obtained through market segmentation. RS also observe that their solution would remain optimal even if we could also optimize over all feasible market segmentations.[2] Moreover, consumer surplus is again maximized in an efficient outcome, thus extending alignment to the case where both consumer and producer information are allowed to vary.

The present paper extends these analyses beyond the monopoly case, to a setting in which there are a number of producers engaged in Bertrand competition. In the special case where the producers' goods are perfect substitutes for one another and when all of the producers have the same cost of production, then in equilibrium, price is competed down to cost, the outcome is socially efficient, and consumers obtain all of the gains from trade. But if the goods are differentiated and costs are heterogeneous, as we suppose, then in general the equilibrium outcome with no segmentation and the consumer knowing their value is neither efficient nor need it be especially good for the consumer: in an extreme case, it could be that the goods are not at all substitutable for one another, and we are effectively back to monopoly. In between, there is a rich plethora of possibilities, in which both market segmentation and consumer information could play an important role in equilibrium and welfare.

Our primary focus is on whether the aforementioned results of BBM and RS extend to oligopoly: are consumer surplus and total surplus aligned, and are consumer surplus and producer surplus opposed? And more broadly, what are the limits of consumer welfare?

[2] Thus, in the case of monopoly, when it is possible to freely choose consumer's information, market segmentation is not needed to maximize consumer surplus. This result does not extend to the case of more than one producer, as we explain below.

Throughout our analysis, we hold fixed the joint distribution of producers' costs and the consumer's values for the different producers' goods. We first assume that the consumer knows their values and producers know their costs. We consider the effect of segmentations of the market, in that each producer observes a "signal" about the consumer's willingness to pay for their product, as well as possibly about the consumer's willingness to pay for other producers' products, other producers' cost of supplying the good, and other producers' signals. This signal represents any characteristics of the consumer or other producers on which the producer is able to condition prices. We refer to a specification of these signals for all producers as an information structure. Given the information structure, the producers play an equilibrium of the game in which producers simultaneously set prices based on their signals, and the consumer buys from whichever producer offers them the most surplus, with ties broken uniformly. For our main result, we restrict attention to "undominated" strategy profiles in which producers set prices above their own costs. Theorem 1 shows that just as in the monopoly case, consumer surplus and total surplus are aligned, and consumer surplus and producer surplus are opposed. Specifically, we construct an information structure and equilibrium that simultaneously maximize consumer surplus, maximize total surplus, and minimize producer surplus.

Maximum consumer surplus is easy to describe. Recall that producers are assumed to price above costs. Thus, a worst-case for each producer is that their competitors price as aggressively as possible, and set their prices equal to their respective costs. A producer can always price optimally against this worst case and guarantee themselves a lower bound on profit. We show that there is an information structure and equilibrium in which each producer's surplus is precisely this lower bound. The outcome is also efficient, and hence also maximizes consumer surplus. Note that if there were no segmentation at all, producers would generally all price above cost, and producer surplus would be higher. Thus, the segmentation of the market serves both to induce producers to price more aggressively and drive down profits, and also to facilitate an efficient outcome without giving extra rents to producers.

The information structure that achieves this outcome has the following structure: First, all producers observe the identity of the producer that can generate the most surplus, who we call the *efficient producer*. In our formal analysis, we use a fixed uniform tie breaking rule, and care is taken to construct mixed strategies for the producers so that ties are broken in favor of the efficient producer (as is also the case in equilibria of asymmetric complete information models of Bertrand price competition). But for the purposes of exposition, we can assume for now that all ties are broken in favor of the efficient producer. Under this assumption, in equilibrium, the inefficient producers all price at cost. Now, without further information, the efficient producer would generally best respond by pricing above cost, and the construction would unravel. However, we now invoke the result of BBM to construct the rest of the segmentation. Specifically, given that the inefficient producers price at cost, there is an induced "residual" willingness to pay for the efficient producer's product. We may then regard the efficient producer as if they are a monopolist facing a fixed demand curve, which is determined by the distribution of the *residual willingness to pay*. The main result of BBM implies that there is a further segmentation of (i.e., signal about) this residual demand curve, and associated optimal pricing by the efficient producer, such that the resulting outcome will be efficient. Thus, the efficient producer always prices below the residual willingness to pay but does not benefit at all from the additional information. And because the efficient producer prices below the residual willingness to pay, none of the inefficient producers can make a sale without dropping price below cost. This completes the description of the information structure and equilibrium.

Our main result relies on the assumptions that the consumer knows their values for all of the goods and that producers know their costs and never price below cost. After proving our main theorem, we consider what happens when these assumptions are dropped.

We first analyze what happens if there is market segmentation but the consumer has only partial information about their values for the goods, thus generalizing the model of RS to oligopoly. An immediate implication of our main result is that consumer surplus and total surplus are *interim aligned*, in the sense that if we hold fixed the consumer's information, then the market segmentation and producer strategies that maximize consumer surplus also maximize total surplus (Proposition 2). However, it may be that the associated outcome is still inefficient ex post, simply because the consumer may not even know which product generates the highest surplus or if that highest surplus is positive. If the goods are perfect substitutes and there is common knowledge of gains from trade (there is at least one producer whose cost is less than the value), then this issue is moot, and the efficient outcome is feasible regardless of the consumer's information. In this case, we show that consumer surplus and total surplus are aligned (Theorem 3). However, if either of these assumptions fails, we give examples showing that consumer surplus could be maximized with consumer information that renders the ex post efficient outcome infeasible, and hence consumer surplus and total surplus are not aligned.

More broadly, we wish to understand the structure of optimal information, even when consumer surplus and total surplus are not aligned. To that end, we completely characterize consumer optimal information in a canonical duopoly setting where the products are differentiated along a Hotelling line. Our main result here, Theorem 4, is a description of the optimal form of segmentation and consumer information and the resulting welfare. In particular, the optimal distribution of consumer's interim values has a two-sided Pareto shape, and the market is divided into two segments, corresponding to which producer's good has the higher interim value. One can view this model as the natural generalization of the characterization of RS to a duopoly environment. The qualitative insight is that when goods are heterogeneous, consumer surplus is maximized when the market learns which firm is interim efficient, but it is generally optimal to muddle the consumer's information about which good is ex post efficient, so as to make the goods more substitutable and intensify price competition.

Finally, we ask what happens if producers do not know their own costs. In this case, pricing below cost need not be a dominated strategy, but we maintain the requirement that producers not set prices that they know are below their cost with probability one. Obviously this makes no difference when there is no uncertainty about costs. However, we show in Theorem 5, that if values are homogeneous, there are two or more producers, and the support for costs is sufficiently rich, then it is possible to attain the same welfare outcomes as if we dropped weak dominance altogether (as in Theorem 2): consumer surplus is arbitrarily close to the efficient total surplus, and producer surplus is arbitrarily close to zero.

1.2 Related Literature

We analyze a model of competitive price discrimination where N producers with heterogeneous products and heterogeneous costs compete for one consumer with unit demand. Relative to the seminal model of oligopoly with product differentiation and uncertain willingness-to-pay of Perloff and Salop (1985), we also allow for uncertainty and private information regarding the production costs and consumer values.

A model of price setting by competing producers is a reverse (or procurement) auction. Reverse auction results have immediate counterparts in standard auction settings. In particular, consider a standard single-unit first-price auction with the twist that the auctioneer has a heterogeneous cost of delivery to the winning bidder (not necessarily known by bidders) and a bid wins if the net bid (bid minus delivery cost) exceeds other bidders' net bids. Now producers' costs are like bidders' values and the auctioneer's delivery costs are like the consumer's heterogeneous values. Our benchmark assumptions that producers' know their costs and the consumer knows their heterogeneous values correspond to assuming that bidders know their values and the auctioneer knows the heterogeneous delivery costs. In the discussion of the literature below, we reinterpret all results that were originally stated for standard first-price auctions, in particular Bergemann, Brooks, and Morris (2015a) and (2017), within the current framework of price competition.

In the special case where products are homogeneous with a commonly known value, our main result (Theorem 1) was proved in Theorem 3 of our working paper Bergemann, Brooks, and Morris (2015a).[3] In the special case where production costs are commonly known and normalized to zero for all producers, our Theorem 1 was proved independently in Theorem 1 of Elliott et al. (2024). Thus, a contribution of this paper is to show that alignment is satisfied whether or not there is common knowledge of homogeneous values or common knowledge of homogeneous costs. Both these papers build on the third degree price discrimination result of BBM.

The case where the consumer does not know their value was studied by RS for the case of one producer. Consistent with our Theorem 3, RS showed alignment when there is common knowledge of gains from trade. The contribution of our Theorem 3 is to extend this result to multiple competing producers (when values are unknown but commonly known to be homogeneous).

We also consider what happens when the assumptions of Theorem 3 fail. Consistent with Theorem 3, RS show (in their appendix) that alignment fails when there is not common knowledge of gains from trade. The Hotelling model is a leading example for the case of heterogeneous values. It corresponds to the special case of our general model, which follows Perloff and Salop (1985), when there are two producers whose costs are commonly known to be zero and whose goods' values to the consumer are perfectly negatively correlated. Armstrong and Zhou (2022) characterize the information structure of the consumer that maximizes consumer surplus, assuming the producers have no information about the consumer's values beyond the prior, restricting attention to pure strategy equilibria. By contrast, we consider the impact of information on both sides of the market. The two-sided nature of the information design is important in our work, and Theorem 1 would not hold if producers had no information about their competitors. We show in Section 5 how additional information for producers leads to more consumer surplus and more efficient allocations than when producers have no information (i.e., the setting of Armstrong and Zhou (2022)). The

[3] Theorem 3 of Bergemann, Brooks, and Morris (2015a) is unpublished and briefly discussed in Section 5.4 of the published version (Bergemann, Brooks, and Morris, 2017).

specific information that the producers receive in the optimal information structure is simply to learn whether or not they are the efficient producer.

Bergemann, Brooks, and Morris (2017) considered the case where producers do not know their costs but values are homogeneous and common knowledge. Their Theorem 2 is closely related to our Theorem 4, as we discuss further below. Kartik and Zhong (2023) consider a one producer setting where the consumer and producer have partial information about cost, which has a one-to-one relationship with value. They establish that consumer surplus and total surplus are aligned, with a single producer and under the assumption of common knowledge of gains from trade.

Our focus in this paper is on maximizing consumer surplus across information structures and equilibria. Some of the papers described above and others in the literature characterize information structures and equilibria maximizing producer surplus. While maximum producer surplus is not a focus of this paper, we summarize these results for context. Bergemann, Brooks, and Morris (2017) characterize maximum producer surplus and minimum consumer surplus when there is common knowledge of homogeneous values but producers may not even know their own costs. Bergemann, Brooks, and Morris (2021) characterize maximum producer surplus in a model where there is common knowledge of homogeneous values and producers know their own cost, which is either high or low. The high cost is above the value of the good and low cost is below he value of the good. [4] In this model, the outcome is always socially efficient, regardless of producers' information. Both no information and complete information maximize consumer surplus, but information structures between these two extremes lead to higher producer surplus. Elliott et al. (2024) consider this setting but with many possible values for the consumer. They provide conditions under which producers can extract the efficient total surplus, under the maintained assumption that costs are homogeneous and commonly known. Armstrong and Vickers (2019) offer a related model of

[4] Bergemann, Brooks, and Morris (2021) offer a consumer search interpretation, in which the "high" cost for a producer's good corresponds to an outcome in which the consumer does not know of the producer's existence.

duopoly and compare outcomes under complete information and no information. The analysis in Bergemann, Brooks, and Morris (2021) shows that asymmetric information between these two extreme information structure impacts the pricing policy and increases the profits substantially. Armstrong and Vickers (2022) generalizes the analysis to many producers but restrict attention to the case where producers have no information.

The rest of this paper proceeds as follows. Section 2 presents the baseline model with known values and known costs. Section 3 contains our main results on the alignment of consumer surplus and total surplus and the opposition of consumer surplus and producer surplus. Section 4 and 5 present extensions involving unknown values and unknown costs, respectively Section 6 concludes the paper. The Appendix contains omitted proofs and an additional example.

2 Model

There are producers $i = 1, \dots, N$ and a single representative consumer.[5] The consumer demands a unit of a good which may be purchased from at most one producer. The consumer's value for producer i's good is v_i. The cost to producer i of supplying the good is c_i. The fundamental uncertainty about values and costs is described by a Borel probability measure $\mu(dv, dc) \in \Delta\left(\mathbb{R}_+^{2N}\right)$. For analytical simplicity, we assume that values are bounded above by $\overline{v} < \infty$. We also assume that the support for costs is finite.

The producers simultaneously choose prices $p_1, \dots, p_i, ..., p_N$. The consumer does not purchase if $v_i < p_i$ for all i. Otherwise, the consumer buys from a producer i that maximizes $v_i - p_i$, breaking ties uniformly. Thus, an implicit assumption of our model is that the consumer knows their values perfectly at the time they make a purchase. This assumption will be relaxed in Section 4. We write $W(p, v)$ for the set of producers that the consumer is willing to purchase from and $q_i(v, p)$ for the likelihood that producer i makes a sale when

[5] All of our results have an equivalent interpretation where there is a mass of non-atomic consumers, and probability distributions are reinterpreted as the population distribution of types.

the prices are $p = (p_1, \dots, p_i, ..., p_N)$ and values are $v = (v_1, \dots, v_i, ..., v_N)$, that is,

$$
\begin{aligned}
W(p, v) &\equiv \{i | v_i - p_i = \max\{0, v_1 - p_1, \dots, v_N - p_N\}\}; \\
q_i(v, p) &\equiv \begin{cases} \frac{1}{|W(p,v)|} & \text{if } i \in W(p, v); \\ 0 & \text{otherwise.} \end{cases}
\end{aligned}
$$

At the time of setting prices, each producer knows their cost and may have additional information about values and others' costs. This is described by an *information structure* (S, ϕ), where $S = \prod_i S_i$ is a product space of signal profiles (and each S_i is a measurable space), and ϕ is a joint probability measure:

$$\phi(ds, dv, dc) \tag{1}$$

whose marginal on (v, c) is μ.

A *strategy* for producer i is a measurable function ρ_i that associates to each $(s_i, c_i) \in S_i \times \mathbb{R}_+$ a probability measure on $\{p_i \in \mathbb{R}_+ | p_i \geq c_i\}$. In other words, we assume that producers price weakly above cost. We identify a strategy profile $\rho = (\rho_1, \dots, \rho_N)$ with the measurable function that maps each (s, c) into the product measure $\rho(dp|s, c) = \prod_i \rho_i(dp_i|s_i, c_i)$.

Given an information structure (S, ϕ) and strategy profile ρ, the resulting ex ante expected surplus for producer i, consumer surplus, and total surplus are respectively

$$
\begin{aligned}
PS_i(S, \phi, \rho) &\equiv \int_{s,v,c,p} (p_i - c_i) q_i(v, p) \rho(dp|s, c) \phi(ds, dv, dc); \\
CS(S, \phi, \rho) &\equiv \sum_{i=1}^{N} \int_{s,v,c,p} (v_i - p_i) q_i(v, p) \rho(dp|s, c) \phi(ds, dv, dc); \\
TS(S, \phi, \rho) &\equiv \sum_{i=1}^{N} \int_{s,v,c,p} (v_i - c_i) q_i(v, p) \rho(dp|s, c) \phi(ds, dv, dc).
\end{aligned}
$$

Ex ante expected producer surplus is $PS(S, \phi, \rho) \equiv \sum_i PS_i(S, \phi, \rho)$. Note that $PS + CS = TS$.

The strategy profile ρ is a *(Bayes Nash) equilibrium* if $PS_i(S,\phi,\rho) \geq PS_i(S,\phi,\rho_i',\rho_{-i})$ for every i and strategy ρ_i'. Note that in any information structure and strategy profile, total surplus is bounded above by the efficient total surplus $\overline{TS}$:

$$TS(S,\phi,\rho) \leq \overline{TS} \equiv \int_{v,c} \max\{0, v_1 - c_1, \ldots, v_N - c_N\}\, \mu(dv, dc). \tag{2}$$

We say that consumer surplus and total surplus are *aligned* if there exists an information structure and equilibrium (S,ϕ,ρ) that simultaneously maximizes both welfare criteria. Consumer surplus and producer surplus are *opposed* if there is an information structure and equilibrium (S,ϕ,ρ) that simultaneously maximizes consumer surplus and minimizes producer surplus. The primary objective of our analysis is to characterize when consumer surplus and total surplus are aligned. A secondary objective is to understand when consumer surplus and producer surplus are opposed.

3 The Alignment of Consumer Surplus and Total Surplus

We now exposit our main results for the model just described. First, we define a lower bound on producer surplus in any information structure and equilibrium. Then we construct an information structure and equilibrium in which this lower bound is attained and the outcome is socially efficient.

3.1 Main Result

To that end, we now state a lower bound on producer surplus given by

$$\underline{PS}_i \equiv \sup_{f:\mathbb{R}_+ \to \mathbb{R}_+} \int_{v,c} (f(c_i) - c_i)\, q_i(v, f(c_i), c_{-i})\, \mu(dv, dc). \tag{3}$$

In other words, this is the highest producer surplus that producer i can obtain if the other producers are pricing at cost, and producer i chooses a best response $f : \mathbb{R}_+ \to \mathbb{R}_+$ that conditions on their own cost. Let $\underline{PS} \equiv \sum_i \underline{PS}_i$. As the following result shows, $\underline{PS}_i$ is a lower bound on producer i's profit in any equilibrium under any information structure:

Proposition 1 (Lower Bound for Producer Surplus)**.**
For any (S, ϕ) and equilibrium ρ, $PS_i(S, \phi, \rho) \geq \underline{PS}_i$.

Proof. Observe that $q_i(p, v)$ is non-decreasing in p_{-i}, and since $p_{-i} \geq c_{-i}$, we have that $q_i(p, v) \geq q_i(p_i, c_{-i}, v)$. Let f be a function that attains a value of $\underline{PS}_i - \varepsilon$ for some $\varepsilon > 0$, and let ρ_i' be a strategy that for every (s_i, c_i) puts probability one on $f(c_i)$. Since ρ is an equilibrium, we have

$$\begin{aligned}
PS_i(S, \phi, \rho) &\geq \int_{s,v,c,p} (f(c_i) - c_i)\, q_i(v, f(c_i), p_{-i})\, \rho(dp|s, c)\, \phi(ds, dv, dc) \\
&\geq \int_{s,v,c,p} (f(c_i) - c_i)\, q_i(v, f(c_i), c_{-i})\, \rho(dp|s, c)\, \phi(ds, dv, dc) \\
&= \int_{v,c,p} (f(c_i) - c_i)\, q_i(v, f(c_i), c_{-i})\, \mu(dv, dc) \\
&\geq \underline{PS}_i - \varepsilon.
\end{aligned}$$

Since ε was arbitrary, the result follows. □

What the proof effectively shows is that each producer always has the option to ignore their signal and just price as a function of their own cost, and best respond as if other producers were pricing at cost. The resulting worst-case payoff is then a lower bound on what a producer can achieve, when a producer has more information available and others' prices are weakly greater than costs.We now present our main result:

Theorem 1 (Alignment)**.**
Consumer surplus and total surplus are aligned. Consumer surplus and producer surplus are opposed. Moreover, there is an information structure and an equilibrium in which each producer's surplus is $\underline{PS}_i$, total surplus is $\overline{TS}$, and consumer surplus is $\overline{TS} - \sum_i \underline{PS}_i$.

The formal proof of Theorem 1 is in the Appendix. We will here motivate and sketch the construction of the information structure and equilibrium that simultaneously maximize consumer surplus, maximize total surplus, and minimize producer surplus.

Competitive pricing under no information and complete information To start, let us consider two natural benchmarks for the producers' information, namely the case of no information and the case of complete information. First, suppose that the producers had no information beyond knowing their own costs. This would correspond to a case where $|S_i| = 1$ for each i; in other words, there is no variation in s_i, so the information contained in (s_i, c_i) is the same as that in c_i alone, and each producer's price only depends on their own costs.[6] Except under extreme distributional assumptions, this would clearly result in an inefficient outcome: producers would have to price strictly above cost in order to earn positive profits, so that they might not make sales even if the consumer's value is above the production cost. Thus, in order to get to an efficient outcome, producers would need additional information about the consumer's values, so that it is feasible to target prices in such a manner that the consumer buys whenever it is efficient to do so.

Another natural benchmark is complete information: In addition to knowing their own costs, the signals reveal *all* of the other producers' costs and the consumer's values. Formally, we can represent this with $S_i = {\mathbb{R}_+}^N \times \mathbb{R}_+^{N-1}$, with typical element $\left(\tilde{v}^i, \tilde{c}^i_{-i}\right)$, and the joint distribution ϕ is such that with probability one $\left(\tilde{v}^i, \tilde{c}^i_{-i}\right) = (v, c_{-i})$ for each i. There are lots of equilibria of the complete information game, but they all share some key attributes. First, a bit of terminology. Given a subset of producers $\tilde{\mathcal{N}}$, the (ex post) *efficient surplus among producers in* $\tilde{\mathcal{N}}$ is

$$TS\left(v_{\tilde{\mathcal{N}}}, c_{\tilde{\mathcal{N}}}\right) \equiv \max_j \left\{v_j - c_j, 0 | j \in \tilde{\mathcal{N}}\right\}.$$

A producer $j \in \tilde{\mathcal{N}}$ is *efficient among* $\tilde{\mathcal{N}}$ if $v_j - c_j = TS\left(v_{\tilde{\mathcal{N}}}, c_{\tilde{\mathcal{N}}}\right)$. Dropping the qualifier "in $\tilde{\mathcal{N}}$" means that $\tilde{\mathcal{N}} = \mathcal{N}$. Now, under complete information, there is common knowledge of (v, c).

[6] Generally, producers might have to play mixed strategies in equilibrium, but again the mixing behavior would only depend on the producer's own cost.

In equilibrium, if no producer is efficient (meaning that $v_i < c_i$ for each i) then producers can set any prices above cost and the consumer does not purchase. If $TS(v, c) > 0$, then one of the producers that is efficient, say producer i, must set a price equal to the consumer's *residual willingness to pay* r_i for producer i's good, given that $p_{-i} = c_{-i}$:

$$r_i \equiv v_i - TS(v_{-i}, c_{-i}). \tag{4}$$

There is considerable multiplicity as to the remaining producers' equilibrium behavior. But what is always true is that a subset of the *runner-up* producers that are efficient among $-i$ must price aggressively enough to induce the efficient producer to price at r_i. If we could break ties in favor of producer i, then this would be straightforward: The remaining producers all price at cost, producer i prices at r_i, and the consumer buys from producer i. But with the standard tie breaking rule—one that assigns the object with uniform and symmetric probability—unless there is more than one efficient producer (so that $r_i = c_i$), these strategies will not be an equilibrium. The reason is that the efficient producer would sometimes lose the tie break by pricing exactly at r_i, and if this price is strictly above c_i, then producer i would not have a best response. So in order to break ties in favor of producer i, the runner-up producers must randomize over prices just above their costs, in order to prevent producer i from profitably deviating to higher prices.

There are lots of mixed strategies for runner-up producers that would induce the efficient producer to price at r_i. As an example, we select a particular runner-up producer j to mix over p_j according to the cumulative distribution H_ε:

$$H_\varepsilon(p_j) = \begin{cases} 0 & \text{if } p_j < c_j, \\ 1 - \frac{p_i - c_i}{p_i - c_i + p_j - c_j} & \text{if } c_j \leq p_j \leq c_j + \varepsilon, \\ 1 & \text{otherwise;} \end{cases} \tag{5}$$

and the other producers set any prices that offer less than ε surplus to the consumer. As a result, there is zero probability of a tie where $v_i - r_i = v_j - p_j$ (and in fact producer i is made indifferent between all prices in $[p_i, p_i + \varepsilon]$). In spite of the complex mixing needed to break ties the right way, the outcome is morally the same as what would obtain if producers $-i$ priced at cost and we broke ties in favor of producer i.

In certain ways, complete information and the associated equilibrium seem to be an improvement on no information: The outcome is socially efficient, and all producers except for the efficient producer are pricing (nearly) at cost, which is in a sense as aggressive as possible. An important caveat, though, is that the efficient producer may still be earning significant rents. In fact, under complete information, each producer receives their entire marginal contribution to total surplus, since

$$p_i - c_i = r_i - c_i = v_i - TS\left(v_{-i}, c_{-i}\right) - c_i = TS\left(v, c\right) - TS\left(v_{-i}, c_{-i}\right). \tag{6}$$

In that sense, producers still retain quite a bit of monopoly power.

Producer Pricing under Partial Information We can do even better for the consumer by applying the ideas from third degree price discrimination, as analyzed by BBM. Consider a monopoly setting, where there is a single producer with a given cost of production, and a consumer whose value for the good is uncertain. A segmentation of the market, in the sense of third degree price discrimination, is simply a signal about the consumer's value upon which the producer can condition prices. Clearly, the monopolist always has the option to ignore their information and set the optimal price under no information, which we denote by p^*. The associated outcome is generally inefficient, but it yields a lower bound on the monopolist's surplus. There is also an associated upper bound on consumer surplus, which is the efficient surplus less the lower bound on producer surplus. Theorem 1 of BBM says that there exists a signal and associated optimal pricing strategy with the property that producer surplus is the same as if the monopolist has no information (and indeed, for every signal realization, the monopolist is indifferent to pricing at p^*), but at the same time, the induced outcome is socially efficient. Hence, consumer surplus attains the upper bound, and must therefore be maximized.

It is not necessary to understand the proof of this result for us to apply it in the oligopoly context. But for the sake of completeness, we will give some intuition in the special case where there are discrete values and then present a fully worked out example in Section 3.2. Consider the set of distributions over the consumer's value for which it is optimal for the monopolist to set the price p^*. This is a convex set, and its extreme points turn out to have useful structure: A distribution is an extreme point if and only if (i) a price p is optimal if and only if it is in the support of the distribution, and (ii) the value p^* is itself in the support.[7] Now, the ex ante value distribution can always be written as a weighted average of such extreme points. These weighted averages can be naturally interpreted as a signal about the value, where the weights are the (ex ante) likelihood of each signal realization, and the extremal market is the posterior distribution of the value conditional on the signal. Under this information structure, by properties (i) and (ii), setting a price of p^* is optimal, no matter the realized signal, so the monopolist does not benefit at all from the information. At the same time, because of property (i), it is also optimal to set a price equal to the lowest value in the support of the posterior value distribution. Under that optimal strategy, the outcome is socially efficient, and therefore consumer surplus is maximized. Even though this sketch uses discreteness, BBM take limits to establish an analogous result for general distributions. Effectively, what is happening is that we pool a relatively large proportion of low-value types with some higher-value types of the consumer in such a way that the monopolist is just barely willing to drop the price, and the higher value consumer types reap all the gains in total surplus.

[7]There are various proofs of this fact, but one is via counting constraints: For every value v other than p^*, the likelihood must be non-negative, and also the profit from price v must be weakly less than profit from price p^*. Clearly, at most one of these constraints can hold as an equality for each $v \neq p^*$, and the only way to have enough equations to fully determine the distribution is if exactly one of the non-negativity and optimality constraints holds as an equality for each $v \neq p^*$.

From Monopoly to Oligopoly We now return to the oligopoly problem, where there are many producers and heterogeneous costs and values. Fix the identity i and cost c_i of the efficient producer. As we have already observed, if all of the producers $-i$ price at cost, then there is an induced *residual (willingness to pay)*, denoted r_i, which largely plays the same role as does the value v_i if producer i were a monopolist. There is an associated lower bound on profit, which is achieved by setting a price $p_i^* (c_i)$, which is the best response when other producers price at cost, producer i has no additional information beyond their own cost, and all ties are broken in favor of producer i. This last assumption is problematic, but continuing with it for the moment, we may then invoke the result of BBM to conclude that there is a signal for producer i about r_i such that they would still be willing to price at $p_i^* (c_i)$ (and therefore do not benefit from the information). Moreover, producer i is also willing to set a price equal to the lowest value of the residual willingness to pay r_i that is in the support of the posterior distribution. The resulting outcome is socially efficient, and hence the bounds on surplus in Theorem 1 are achieved. Moreover, since producer i sets a price $p_i \leq r_i$ with probability one, we have that the consumer's willingness to pay for the good of producer $j \neq i$ is at most

$$v_j - (v_i - p_i) \leq v_j - (v_i - r_i) = v_j - TS\,(v_{-i}, c_{-i}) \leq v_j - (v_j - c_j) = c_j,$$

so that producer j can only make a sale by pricing weakly below their cost. Hence, the inefficient producers have no profitable deviation either, and we are done.

The only problem with this argument is the presumption that ties are broken in favor of the efficient producer, whereas in fact they are broken uniformly. But we can finesse this issue using the same kind of mixing as in the complete information case. This is precisely what is done in the formal proof of Theorem 1.

To summarize, the information structure that we construct does the following: (i) it publicly reveals the identity of the efficient producer; (ii) it generates a signal for the efficient producer i about r_i, using the construction of BBM, so that under the premise that $p_{-i} = c_{-i}$, producer i would get the payoff $\underline{PS}_i$, but they also make a sale whenever it is efficient to do so; and (iii) there is an additional signal for any producer j that might tie with the efficient producer i that tells them an interval over which to randomize their price just above their cost, to break ties in favor of the efficient producer. The associated strategies are such that the efficient producer i sets a price equal to the lowest possible value of r_i conditional on their information, and the inefficient producers either price at cost or randomize as per case (iii). The resulting outcome is efficient, and producers are held down to their lower bound surplus, and hence consumer surplus is maximized.

We can also relate the results visually in the matrix below. In an environment with know values and costs, the pricing equilibrium is given by the complete information Bertrand equilibrium. If the cost is private information to each seller, then Bergemann, Brooks, and Morris (2015a) provide the characterization of the consumer surplus maximizing information structure and equilibrium. Elliott et al. (2024) consider the case when there is private information about the value but the cost is commonly known and equal to a common constant. Theorem 1 establishes the consumer surplus maximizing information and equilibrium when both value and cost are private information, thus encompassing the earlier results on competition and private information as summarized in the matrix below:

		value $(v_1, ..., v_N)$	
		Known	Unknown
cost $(c_1, ..., c_N)$	Known	Complete Information Bertrand	Elliott et al. (2024)
	Unknown	Bergemann, Brooks, and Morris (2015a)	Theorem 1

3.2 Competitive Price Discrimination: An Example

We now illustrate how this construction works with a simple example:

Example 1. There are two producers that offer differentiated products with uncertain cost $c_i \in \{0, 1\}$ and value $v_i \in \{1, 4\}$. Each profile of costs and values (v_1, v_2, c_1, c_2) is equally likely. The entries in the following table show the surplus generated from purchasing from each producer, as a function of the value/ cost profile:

surplus vector $(v_i - c_i, v_j - c_j)$

$(v_i, c_i) \setminus (v_j, c_j)$	$(1,1)$	$(1,0)$	$(4,1)$	$(4,0)$
$(1,1)$	$(0,0)$	$(0,1)$	$(0,3)$	$(0,4)$
$(1,0)$	$(1,0)$	$(1,1)$	$(1,3)$	$(1,4)$
$(4,1)$	$(3,0)$	$(3,1)$	$(3,3)$	$(3,4)$
$(4,0)$	$(4,0)$	$(4,1)$	$(4,3)$	$(4,4)$

We now apply the construction underlying Theorem 1 to obtain the consumer surplus maximizing information structure. Note that both producers are efficient on the diagonal, and producer i is efficient only in the region including and below the diagonal. The corresponding residual willingness to pay r_i, as defined earlier in (4), for producer i are:

residual willingness to pay r_i

$(v_i, c_i) \setminus (v_j, c_j)$	$(1,1)$	$(1,0)$	$(4,1)$	$(4,0)$
$(1,1)$	1	0	-2	-3
$(1,0)$	1	0	-2	-3
$(4,1)$	4	3	1	0
$(4,0)$	4	3	1	0

Thus, producer 1's cost is less than the residual willingness to pay, or residual for brevity, on and below the diagonal, which is precisely when they are the efficient producer.

We now construct the information structure and pricing policy as outlined in Theorem 1. If the producers have the same profile (v_i, c_i), then a signal that informs them of the competitive nature of the market yields prices equal to cost, and the consumer receives all the surplus. It thus suffices to consider the entries off the diagonal in the preceding tables. If each producer were only to observe their own private cost c_i and were to receive a signal when they are the efficient producer, then the row producer would receive the signal in the profile realizations in the portion of the matrix below the diagonal:

residual willingness to pay r_i

$(v_i, c_i) \setminus (v_j, c_j)$	$(1,1)$	$(1,0)$	$(4,1)$	$(4,0)$
$(1,1)$				
$(1,0)$	1			
$(4,1)$	4	3		
$(4,0)$	4	3	1	

The signal of being the efficient producer conditional on the cost c_i would then inform the producer about two possible segments. These segments are represented as rows in the following table, where we report the total likelihood of the segment and the conditional likelihood of each residual:

		r_i		
c_i	Prob	1	3	4
1	$\frac{1}{3}$	0	$\frac{1}{2}$	$\frac{1}{2}$
0	$\frac{2}{3}$	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{4}$

Now, suppose for the moment that the signal of being the efficient producer and the cost c_i would be the only information that producer i would get. Then the optimal pricing policy would depend on the cost realization. If the cost is low, $c_i = 0$, then the optimal price is $p_i = 3$, as it would generate a revenue of $3/2$, which is higher than either alternative price $p_i = 1$ or $p_i = 4$ which would generate revenues equal to $1/2$ and 1, respectively. Thus, uniform pricing would lead to an inefficient allocation. If the cost were high, $c_i = 1$, then the optimal price would be $p_i = 3$ and would lead to an efficient allocation.

We can ask what a consumer surplus maximizing segmentation of the residual willingness to pay would look like through the lens of BBM. For the case of $c_i = 0$, the following segmentation (and associated prices p_i) increase consumer surplus and form an equilibrium:

		r_i		
Segment	Prob	1	3	4
$p_i = 1$	$\frac{3}{4}$	$\frac{2}{3}$	$\frac{1}{6}$	$\frac{1}{6}$
$p_i = 3$	$\frac{1}{4}$	0	$\frac{1}{2}$	$\frac{1}{2}$
Total	1	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{4}$

As a consequence of the segmentation, the expected price charged by the winning producer with low cost $c_i = 0$ would decrease from 3 to $(3/4)\,1 + (1/4)\,3 = 3/2$. The profit of the low cost producer however would stay constant at 3/4 due to construction of the indifference segments, lower prices are compensated through a higher probability of sale. The corresponding consumer surplus increases from $(1/4)\,(4-3) = 1/4$ to $(1/4)\,(3/4)\,((4-1)+(3-1)) = 15/16$. Thus, segmentation increases consumer surplus as well as total surplus.

It only remains to describe the pricing strategy of the competing but inefficient producer. Here we can follow the construction of the proof to identify a competitive strategy that preserves the outcome and incentives for the efficient producer, while breaking ties efficiently.

3.3 Relaxing Weak Dominance

The lower bound $\underline{PS}_i$ on producer i's surplus relies on the hypothesis that producers do not price below cost. If we allow producers to price below cost, then some rather extreme welfare outcomes can be supported in equilibrium.

Theorem 2 (Weakly dominated equilibria)**.**
Without weak dominance, consumer surplus and total surplus are aligned, and consumer surplus and producer surplus are opposed. Moreover, for every $\varepsilon > 0$, there exists an information structure (S, ϕ) and equilibrium strategies ρ so that $PS \leq \varepsilon$ and $CS \geq \overline{TS} - \varepsilon$.

The formal proof is in the Appendix, but the idea is quite simple. These extreme outcomes can be sustained when the producers have complete information about (v, c). The efficient producer prices at the minimum of $c_i + \epsilon$ and whatever price would tie with the runner-up producer. The runner-up producer either prices at cost (when there is a tie) or randomizes over prices (below their own cost) so that the residual willingness to pay is distributed on $[c_i + \epsilon, c_i + 2\epsilon]$. Moreover, we can pick the shape of this distribution so that pricing at $c_i + \epsilon$ is a best response for the efficient producer.

Thus, without weak dominance, it is possible to sustain hypercompetitive outcomes in equilibrium, where producers know that they are pricing well below cost, but they are willing to do so because they expect to not make a sale. Imposing weak dominance is a straightforward and intuitive way to rule out such implausible scenarios.

4 Market Segmentation and Unknown Values

We now consider what happens if the consumer may have only partial information about their value for the products. We first observe that the logic of Theorem 1 goes through, holding fixed the consumer's information. This immediately delivers a result, Proposition 2 on interim alignment of consumer surplus and total surplus, and interim opposition of consumer surplus and producer surplus. Then Theorem 3 establishes that our main result, Theorem 1 holds under the ex ante notion of efficiency, under the hypotheses that there is common knowledge of gains from trade and the goods are homogeneous. We then show how misalignment can occur wheren the goods are not homogeneous anymore. We provide a complete analysis of the optimal information structure in the Hotelling model of competition in Theorem 4. In this canonical model of horizontal differentiation the values of the consumer are heterogeneous as they depend on the location of the consumer.

4.1 Interim Alignment

We model partial information of the consumer by generalizing our definition of an information structure. We say that a distribution $\mu'(dv', dc)$ is a *value garbling* of μ if there is a probability transition kernel $\eta : \mathbb{R}_+{}^2 \to \Delta(\mathbb{R}_+)$ such that

$$\mu(dv, dc) = \int_{v'} \mu'(dv', dc)\, \eta(dv|v', c)$$

and

$$\int_{v} v\, \eta(dv|v', c) = v'.$$

In other words, the distribution $\mu(dv, dc)$ is obtained from $\mu'(dv', dc)$ by adding noise to v' that has mean zero conditional on (v', c). This noise represents the consumer's residual uncertainty about the value. An *unknown values information structure* is an information structure as defined in Section 2, except that we only require that the marginal of the joint distribution (of the information structure) ϕ on (v, c) is a value garbling of μ. (We previously required that this marginal of ϕ is exactly μ.)

This definition of an unknown values information structure builds in a non-trivial restriction. Namely the producers only have information about the consumer's interim expected value, and not directly about the ex post value of the consumer. Without this assumption, it could be that producers know more about the true value than does the consumer. And if producers can price based on such information, then the consumer might end up with a non-trivial inference problem about their true value, given the prices they observe. Our assumption that the consumer knows everything the producers know about v shuts down this signaling channel.[8]

We say that consumer surplus and total surplus are *interim aligned* if holding fixed the marginal on (v, c), there is an information structure and equilibrium that simultaneously

[8]For a discussion of what might happen with such signaling through prices in the monopoly context, see Kartik and Zhong (2023).

maximizes both consumer surplus and total surplus. Similarly, we say that consumer surplus and producer surplus are *interim opposed* if holding fixed the marginal on (v, c), there is an information structure and equilibrium that simultaneously maximizes consumer surplus and minimizes producer surplus. In particular, let us define interim analogues of the bounds from Theorem 1:

$$\overline{TS}\left(\mu'\right) \equiv \int_{v,c} \max\left\{0, v_1 - c_1, \ldots, v_N - c_N\right\} \mu'\left(dv, dc\right),$$

$$\underline{PS}_i\left(\mu'\right) \equiv \sup_{f:\mathbb{R}_+\to\mathbb{R}_+} \int_{v,c} \left(f\left(c_i\right) - c_i\right) q_i\left(f\left(c_i\right), c_{-i}, v\right) \mu'\left(dv, dc\right);$$

Our first result on the unknown values model is the following:

Proposition 2 (Interim Alignment)**.**

Consumer surplus and total surplus are interim aligned, and consumer surplus and producer surplus are interim opposed. In particular, if there is an optimal information structure such that the marginal on (v, c) is μ', then there is a consumer surplus maximizing information structure and equilibrium in which each producer's surplus is $\underline{PS}_i(\mu')$, total surplus is $\overline{TS}(\mu')$, and consumer surplus is $\overline{TS}\left(\mu'\right) - \sum_i \underline{PS}_i\left(\mu'\right)$.

Proof. Applying Theorem 1 to the case where the prior is μ', we conclude that holding fixed μ', there is an information structure and equilibrium that simultaneously maximizes consumer surplus, maximizes total surplus, and minimizes producer surplus, and attains the welfare outcome in the statement of the proposition. The result then follows immediately. □

4.2 Homogenous Values

We now give conditions under which consumer surplus and total surplus are aligned, even when there are unknown values. We say that values are *homogeneous* if $v_1 = \cdots = v_N$ μ-almost surely. We say that there is *common knowledge of gains from trade* if $\max_i v_i - c_i \geq 0$ μ-almost surely.

Theorem 3 (Alignment with Unknown Values)**.**

Suppose that values are unknown and homogeneous and there is common knowledge of gains from trade. Then consumer surplus and total surplus are aligned, and consumer surplus and producer surplus are opposed. In particular, if consumer surplus is maximized when the marginal on (v, c) is μ', then there is a consumer surplus maximizing information structure and equilibrium in which each producer's surplus is $\underline{PS}_i(\mu')$, total surplus is $\overline{TS}(\mu') = \overline{TS}$, and consumer surplus is $\overline{TS} - \sum_i \underline{PS}_i(\mu')$.

Proof. Because of homogeneous values, we have that for all (v', c) in the support of μ',

$$v'_i = \int_v v_i \eta\left(dv|v', c\right) = \int_v v_j \eta\left(dv|v', c\right) = v'_j,$$

so that μ' satisfies homogeneous values as well. Moreover, under common knowledge of gains from trade, for all (v', c) in the support of μ', we have

$$\begin{aligned}
\max_i \left(v'_i - c_i\right) &= v'_1 - \min_i c_i \\
&= \int_v v_1 \eta\left(dv|v', c'\right) - \min_i c_i \\
&= \int_v \left(v_1 - \min_i c_i\right) \eta\left(dv|v', c'\right) \\
&= \int_v \max_i \left(v_i - c_i\right) \eta\left(dv|v', c'\right) \geq 0.
\end{aligned}$$

Hence, μ' also satisfies common knowledge of gains from trade. Thus,

$$\begin{aligned}
\overline{TS}\left(\mu'\right) &= \int_{v',c} \max \{0\} \cup \{v'_1 - c_1, \ldots, v'_N - c_N\} \, \mu'\left(dv', dc\right) \\
&= \int_{v',c} \left(v'_1 - \min_i c_i\right) \mu'\left(dv', dc\right) \\
&= \int_{v,c} \left(v_1 - \min_i c_i\right) \mu\left(dv, dc\right) = \overline{TS}.
\end{aligned}$$

It then follows immediately from Proposition 2 that consumer surplus and total surplus are aligned.

Now suppose that there is another information structure and equilibrium in which $PS < \sum_i \underline{PS}_i(\mu')$. Let μ'' be the marginal on (v,c) associated with this information structure. By the argument in the preceding paragraph, $\overline{TS}(\mu'') = \overline{TS}$. By Proposition 2, $PS \geq \sum_i \underline{PS}_i(\mu'')$, and also there is an information structure and equilibrium in which the outcome is efficient and producer surplus is precisely $\underline{PS}_i(\mu'')$. In this outcome, consumer surplus is therefore $\overline{TS} - \sum_i \underline{PS}_i(\mu'') \geq \overline{TS} - PS > \overline{TS} - \sum_i \underline{PS}_i(\mu')$, which contradicts the hypothesis that μ' corresponds to a consumer surplus maximizing information structure. Thus, it must be that $\sum_i \underline{PS}_i(\mu')$ is also minimum producer surplus, and consumer surplus and producer surplus are opposed. □

Theorem 3 shows that when values are unknown and goods are homogeneous, consumer surplus and total surplus are aligned. However, the theorem does not provide a detailed characterization of the optimal interim value distribution for the consumer. For the special case of one producer, RS characterize the consumer surplus maximizing information: The consumer's interim expected value has a truncated Pareto distribution, so that the producer is willing to price at the bottom of the support, and the parameters of that distribution minimize the price subject to the constraint that the interim value distribution is a mean-preserving contraction of the prior.

Beyond the monopoly case, we are not aware of a general characterization of the consumer-surplus maximizing information. In the working paper version, Bergemann et al. (2023) we fully solve the following example with two producers:

Example 2 (Duopoly with unknown and homogenous values)**.** A consumer's ex post value is in the interval $[0,1]$ and has distribution F. Producer has cost $c_1 = 0$, and producer 2 has cost $c_2 \in [0,1]$.

Note that this example satisfies the hypotheses of Theorem 3, so that consumer surplus and total surplus are aligned, and consumer surplus will be maximized at an outcome that is ex post efficient. It is straightforward to see that producer 2 will price at cost, so that the consumer's willingness to pay for producer 1's good is the minimum of their interim value v and producer 2's cost, c_2. Even in this simple case, the optimal information of the consumer departs significantly from the solution of RS. The reason is that what matters for producer 1 is the interim residual willingness to pay $\min\{v, c_2\}$, and the mean-preserving constraint on v imposes only weak restrictions on the distribution $\min\{v, c_2\}$.

As suggested by Theorem 3, even when values are homogenous, alignment may fail when there is not common knowledge of gains from trade. The simplest example of this is when there is a single producer, μ puts probability one on a particular cost c_1, and there is positive probability that $v_1 < c_1$. In fact, this model is one that has been studied by RS. While their baseline model assumes common knowledge of gains from trade, their Appendix contains an extension to the case where the consumer's value is less than the producer's cost with positive probability, and they find that the information that maximizes consumer surplus can result in inefficient trade, as the following example shows:

Example 3 (Monopoly without common knowledge of gains from trade)**.** The monopolist's cost is $c_1 = 1$ and $v_1 \in \{0, 3\}$, with both values equally likely. In order for trade to be efficient, the consumer must learn their value exactly. But in that case, the optimal price is $p_1 = 3$, so that consumer surplus is zero. Now consider the following value garbling: With probability α, the consumer learns their value, and otherwise they don't learn anything. Then the interim value distribution is $v_1 = 0$ with probability $\alpha/2$, $v_1 = 3$ with probability $\alpha/2$, and $v_1 = 3/2$ with probability $1-\alpha$. The producer's payoff from $p_1 = 3/2$ is $(1 - \alpha/2)$, and the payoff from $p_1 = 3$ is $3\alpha/2$. Hence, as long as $3(\alpha/2) \leq (3/2)(1 - \alpha/2) \Leftrightarrow \alpha \leq 2/3$, the producer will set a price of $p_1 = 3/2$, and consumer surplus is $3\alpha/4 > 0$. However, with probability $(1 - \alpha)/2$, the consumer buys even though their value is 0, which is inefficient.

4.3 Heterogeneous Values and the Hotelling Model

We now consider what happens with heterogeneous values, with our primary focus being on the Hotelling duopoly model. Producers $i = 1, 2$ have zero marginal cost of production, $c_i = 0$, and $v_i \in [0, \overline{v}]$. The consumer's values are symmetrically distributed and perfectly negatively correlated, with $v_1 + v_2 = \overline{v}$. We recall that we earlier defined r_i as the residual willingness to pay for the product of firm i, see (4). We can now write

$$r_i = v_i - v_j \in [-\overline{v}, \overline{v}],$$

for the difference between values and denote by F the distribution of the residual willingness to pay $r_i \in [-\overline{v}, \overline{v}]$. By the assumed symmetry of the producers, the residual willingness has the same distribution for r_1 and r_2, and thus we drop the subscript i on r for the remainder of this Section. Thus r is the residual willingness to pay for the representative producer.

This model can be viewed as a generalization of RS to more than one producer. It was also recently studied by Armstrong and Zhou (2022), who only considered the role of consumer information. In contrast, and like RS, we study both the role of consumer information and market segmentation in shaping welfare. Our main result is a complete characterization of the information and equilibrium that maximize consumer surplus. In RS, market segmentation plays no role, and maximum consumer surplus can be achieved without any market segmentation. As we will see, with more than one firm, non-trivial market segmentation plays a key role in pinning down maximum consumer surplus.

Now, to see why consumer surplus and total surplus may not be aligned, we can consider the following simple binary example:

Example 4 (Hotelling with binary values)**.** The value profiles $(v_1, v_2) \in \{(0, 1), (1, 0)\}$ are both equally likely, so that r is equally likely to be ± 1. For the outcome to be efficient, the consumer would have to learn which producer gives them the higher value. In that case, each producer knows that the consumer's residual for their good is equally likely to be 0 and 1, so the optimal price is $p_i = 1$, and therefore consumer surplus is zero. On the other hand, if the consumer has no information about the value, then their expected value is $1/2$ for both producers. The producers will compete the price down to cost, and $p_1 = p_2 = 0$. Consumer surplus is equal to $1/2$, which is also the total surplus, so consumer surplus is positive but the outcome is inefficient. The first takeaway then is that by creating uncertainty about the consumer's ex post value, it is possible to generate more competitive pricing which raises consumer surplus but at the cost of lowering efficiency.[9]

In Appendix B.1, we also consider the Hotelling model in which r is uniformly distributed on $[-1, 1]$.

We now turn to a general characterization of the information structure that maximizes consumer surplus in the Hotelling model. We first establish that a generalization of the censored Pareto distribution suffices to maximize consumer surplus. We then describe some of the welfare implications. Proposition 2 established that the welfare is entirely pinned down by the interim distribution of values μ'. Moreover, Proposition 2 implies that the low value producer i will price at 0. Thus, the high value producer faces a problem akin to a monopoly problem on the residual demand given by r. This will allow us to generalize the monopoly analysis of Roesler and Szentes (2017) to the symmetric segments of the efficient producers.

We start with the symmetric distribution of the residual willingness to pay r denoted by F with $F \in \Delta[-\overline{v}, \overline{v}]$. We first identify a class of extremal distribution functions that suffice to maximize consumer surplus. Let us write G for the distribution of interim expectations of r. The distribution $G \in \Delta[-\overline{v}, \overline{v}]$ has to form a mean-preserving contraction of the underlying distribution F, thus:

$$\int_{x=r}^{\infty} (G(x) - F(x))dx \geq 0, \ \forall r. \tag{7}$$

[9]In Bergemann et al. (2023) we also provide a complete solution to the Hotelling model in which r is uniformly distributed on $[-1, 1]$.

By the symmetry of the problem, it is without loss to consider symmetric interim residual value distributions that satisfy $G(-r) = 1 - G(r)$ for $r \geq 0$.

Now, the total surplus can be written as the sum over the expectation of the unconditional value and the residual value of the representative producer, thus

$$\overline{TS}\left(G\right) \equiv \int_{v=0}^{\overline{v}} v\mu(dv) + \int_{r=0}^{\overline{v}} rG(dr). \tag{8}$$

In particular, the first integral is a constant and independent of the choice of the optimal information structure with interim residual value distribution G. The sum of the producers' surplus is the sum of the revenue across the efficient producers:

$$\underline{PS}\left(G\right) = \max_{p\geq 0}\left\{pG^{-}\left(-p\right)\right\} + \max_{p\geq 0}\left\{p\left(1 - G^{-}\left(p\right)\right)\right\},$$

where G^{-} denotes the limit from the left, and optimal consumer surplus is $\overline{TS}\left(G\right) - \underline{PS}\left(G\right)$. Now, if the sum of the producers' surplus is p (and thus jointly the producers sell with probability one), then conditional on being the efficient producer, a producer's surplus must be at most p. This is equivalent to the interim distribution G satisfying:

$$\begin{aligned} r\frac{1 - G^{-}(r)}{1/2} &\leq p, \ \forall r \geq 0, \\ -r\frac{G^{-}(r)}{1/2} &\leq p, \ \forall r \leq 0; \end{aligned}$$

in which case the above constraints are equivalent to

$$\begin{aligned} G^{-}(r) &\geq 1 - \frac{p}{2r}, \ \forall r \geq 0, \\ G^{-}(r) &\leq -\frac{p}{2r}, \ \forall r \leq 0. \end{aligned}$$

So, we can focus on choosing $G(r)$, subject to the aforementioned pricing constraints and mean-preserving contraction constraints. We look for a solution $G_p^B(r)$ of the form:

$$G_p^B(r) \equiv \begin{cases} 0 & \text{if } r \leq -B; \\ -\frac{p}{2r} & \text{if } -B < r \leq -p; \\ 1/2 & \text{if } -p < r \leq p; \\ 1-\frac{p}{2r} & \text{if } p < r \leq B; \\ 1 & \text{if } p > B; \end{cases} \tag{9}$$

The distribution $G_p^B(r)$ defines a symmetric distribution that on each side of 0 – $[-\overline{v}, 0]$, and $[0, \overline{v}]$, respectively – is formed by a truncated Pareto distribution with bounds $\pm p$ and $\pm B$. Each segment has a mass point at $|B|$ and the distribution G_p^B is constant between $[-p, p]$.

Theorem 4 (Consumer Surplus Maximizing Information Structure in Hotelling Model)**.** *In the Hotelling model, there exists a p and B such that the interim value distribution G_p^B maximizes consumer surplus.*

Proof. The proof closely follows that of Lemma 1 of Roesler and Szentes (2017) separately for each efficient producer and then joins the solution across the segments.

First, suppose that there is an interim value distribution G for which producer surplus is p. We claim that there is a B such that G_p^B is a symmetric mean-preserving contraction of G. To prove the claim, first note that conditional on $r \geq 0$, the distribution G first-order stochastically dominates $G_p^{\overline{v}}$, and G is first-order stochastically dominated by G_p^p. Hence, conditional on $r \geq 0$, the expectation under G is between the expectations under $G_p^{\overline{v}}$ and G_p^p. Because the expectation under G_p^B is continuous in B, by the intermediate value theorem, there is a $B \in [p, \overline{v}]$ such that the expectation of r conditional on $r \geq 0$ is the same under G and G_p^B, and in particular,

$$\int_{x=0}^{\overline{v}} (G_p^B(x) - G(x))dx = 0.$$

Since $G(r) \geq G_p^B(r)$ for $r < B$ and $G(r) \leq G_p^B(r)$ for all $r \geq B$, we conclude that for all $r \geq 0$,

$$\int_{x=r}^{\infty} (G_p^B(x) - G(x))dx \geq 0.$$

By symmetry, we conclude that G_p^B is a mean-preserving contraction of G, and hence is also a mean-preserving contraction of F.

Note that $\underline{PS}\left(G_p^B\right) = p$, so the lower bound on producer surplus has not changed. Moreover, because G_p^B is separately a mean-preserving contraction of G on either side of zero, we have not changed the expectation of $|r|$, and hence total surplus has not changed as well . Thus, it is without loss to optimize consumer surplus over distributions of the form G_p^B that are mean-preserving contractions of F. □

It should be noted that the consumer surplus maximizing parameters (p, B) are generally distinct from those that minimize producer surplus, and hence consumer surplus and producer surplus are not opposed in the Hotelling model.

Returning to binary values of Example 4, we now derive the distribution of values that maximizes consumer surplus using Theorem 4. Recall that the residual willingness to pay under complete information is equally likely to be $r \in \{-1, +1\}$. Thus, the mean-preserving contraction constraints are automatically satisfied by G_p^B as long as $B \leq 1$. Note that following the decomposition of the total surplus given earlier by (8), we have

$$\overline{TS}\left(G_p^B\right) = \frac{1}{2} + \int_{r=p}^{1} r G_p^B(dr) = \frac{1}{2}\left(1 + p + p\left(\ln B - \ln p\right)\right).$$

Hence, consumer surplus is

$$\overline{TS}\left(G_p^B\right) - p = \frac{1}{2}\left(1 - p + p\left(\ln B - \ln p\right)\right). \tag{10}$$

The optimal information structure sets $B^* = 1$ and $p^* = 1/e^2 \approx 0.07$, and the maximized consumer surplus is $1/2(1 + e^{-2}) \approx 0.57$. Note that the total surplus is $1/2(1 + 3e^{-2}) \approx 0.70$, whereas the efficient surplus is 1. Thus, with probability $1/2(1+3e^{-2}) \approx 0.70$, the ex-post efficient producer makes the sale, but with the complementary probability of $1/2(1-3e^{-2}) \approx 0.30$, the ex-post inefficient producer makes the sale in the consumer surplus maximizing equilibrium.

Our analysis of the Hotelling model is related to that of Armstrong and Zhou (2022), who also characterize the information structure that maximizes consumer surplus. Their analysis of competition under optimal consumer information leads them to impose the additional constraints that (i) producers receive no information and (ii) the producers use pure strategies.[10] Thus, as they impose additional restrictions on the optimal information structure relative to the current setting, we can expect their results to lead to (weakly) lower consumer surplus in equilibrium.

Indeed, they also find that the distribution of interim residual values has a censored Pareto shape. Yet, the shape of the distribution differ in noticable ways across the two regimes. In Armstrong and Zhou (2022), the producers do not have any information about who might be the efficient producer. Thus, the producers offer a *single price*. In order to avoid the natural separation into two local monopolies, the distribution of interim residual value must pile up around zero. For the binary example here, the analysis of Armstrong and Zhou (2022) shows that consumer surplus is maximized at $p^* \approx 0.05$. The resulting total surplus is ≈ 0.57 and the resulting consumer surplus is ≈ 0.52. Thus, total surplus, producer surplus, and consumer surplus are all lower when the producers have no information than in our case where the producers are informed about the identity of the efficient producer.

In Figure 1, we display the resulting density of the residual values in the optimal information structure under these two distinct informational regimes. In our setting, the producers are informed who is the efficient producer and the realized residual values are realized away from 0. When the producers do not receive any information, as required by Armstrong and

[10] Armstrong and Zhou (2022) also consider maximum producer surplus, whereas our focus is on consumer surplus.

Zhou (2022), then the realized residual values are centered around 0 in order to sustain competition.

The logic underlying Theorem 4 readily generalizes to a considerably larger class of models. First, it is not essential that values are perfectly negatively correlated. Suppose that the values are distributed according to $\mu(v_1, v_2)$, with both v_1 and v_2 being non-negative. By Proposition 2, it is still the case that in the consumer surplus maximizing information structure, the producers learn which of them is efficient, and the residual willingness to pay for the efficient producer i's good is $r_i = v_i - v_j$. Thus, only information about the residual is strategically relevant to the producers, and the variation in levels of values is only important insofar as it contributes to the total surplus. Indeed, the efficient surplus can be more generally written as

$$\overline{TS}(\mu') = \int_{(v_1,v_2)} \left[\frac{v_1 + v_2}{2} + \frac{|v_1 - v_2|}{2} \right] \mu'(dv_1, dv_2).$$

In addition, while we assumed that the distribution of residuals was symmetric, this was not essential to our argument. The construction of the mean-preserving contraction in the proof of Theorem 4 was done separately conditional on the identity of the efficient producer. In fact, the argument could even be applied with more than two producers: All that matters is the consumer's interim expectation of their residual r_i for the efficient producer i's good, assuming the other producers price at cost, and it is without loss to consider distributions of r_i that have the censored Pareto shape.

5 Extension: Unknown Costs

We now explore the case in which the consumer knows their values but producers may not know their own costs. Operationally, what this means is that each producer i's strategy can only depend on their signal s_i, and cannot depend directly on their cost, i.e., a strategy ρ_i associates to each s_i a distribution over prices. We will continue to require that producers

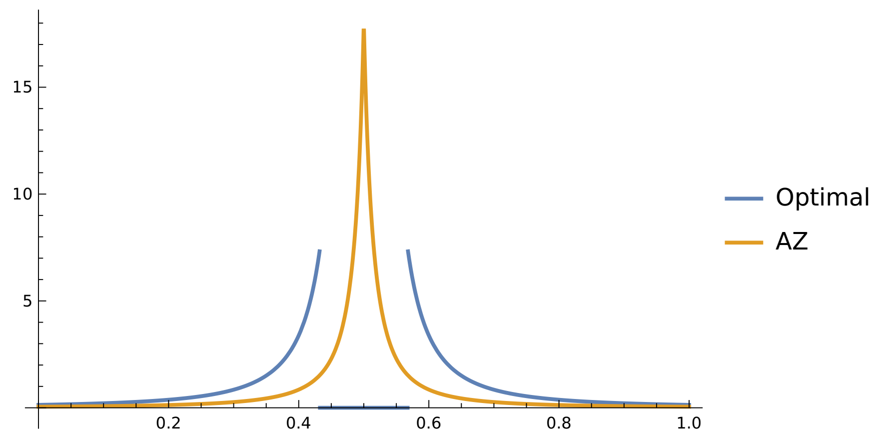

Figure 1: Probability density of consumer surplus maximizing interim values

not play weakly dominated strategies, although now we must provide a more general definition, that does not rely on the assumption that costs are known. In particular, under the information structure (S, π), we say that a strategy ρ_i is *undominated* if for any function $f : S_i \to \mathbb{R}$ such that $\pi\left(\{(s, v, c) \,|c_i \geq f(s_i)\}\right) = 1$, we have that

$$\int_{(v,c)} \rho_i\left([f(s_i), \infty)|s_i\right) \pi\left(ds, dv, dc\right) = 1.$$

In other words, an undominated strategy is one for which there is probability zero that producers price strictly below a lower bound on their cost, where the lower bound depends only on their own signal.

Obviously, in the special case where producers' costs are certain, this notion of dominance reduces to the requirement that producers price above cost, and our existing results would go through without modification. However, we will argue that with even a small amount of uncertainty, weak dominance loses much of its bite. In fact, Theorem 5 shows that there are cases in which it is possible to approximate in undominated strategies the same hypercompetitive outcomes as those obtained in Theorem 2, where we dropped the weak dominance restriction altogether. The critical issue is that producers may be frequently pricing below cost, but that behavior is undominated because producers cannot distinguish it from when they would also be setting similarly low prices as the efficient producer.

We say that the prior μ is *weakly competitive* if whenever there is positive probability that producer i is *uniquely efficient*—meaning that they are the only efficient producer—and has cost $c_i = x$, then there exists a producer $j \neq i$ such that there is positive probability that producer j is uniquely efficient and has cost $c_j = x$. The substantive implication of weak competitiveness is that a producer cannot infer the identity of the efficient producer just from knowing the efficient producer's cost: For any given efficient cost, there are always at least two producers who could be uniquely efficient with that cost. This condition would be trivially satisfied if the prior distribution of (v, c) is exchangeable.

Theorem 5 (Alignment with Unknown Costs)**.** *Suppose that $N \geq 2$, costs are unknown, values are homogeneous, and the prior is weakly competitive. Then consumer surplus and total surplus are aligned, and consumer surplus and producer surplus are opposed. In particular, for any $\varepsilon > 0$, there exists an information structure and equilibrium in which $TS = \overline{TS}$, $PS < \varepsilon$ and $CS \geq \overline{TS} - \varepsilon$.*

To prove the theorem, we construct an information structure and equilibrium of the following form: Each producer's signal is a "recommended" price, and in equilibrium, producers set prices equal to their signals. Because values are homogeneous, the efficient producer is simply the producer with the lowest cost. The low cost producer i is recommended a random price $p_i \in [c_i, c_i + \varepsilon]$, where $c_i + \varepsilon < \min_{j \neq i} c_j$. By weak competitiveness, there is a producer $j \neq i$ who also is sometimes uniquely efficient with the cost c_i. That producer is recommended a random price in $[p_i, c_i + \varepsilon]$, according to a distribution that makes producer i prefer p_i to prices in $[p_i, c_i + \varepsilon]$. This incentivizes producer i to price close to c_i, and moreover, the strategy of following the recommendation is not weakly dominated, since producer j cannot tell whether they are recommended such a price because they are efficient, or because they are inefficient and being used to pressure the efficient producer to price close to cost.[11]

[11] Theorem 5 generalizes Theorem 2 of Bergemann, Brooks, and Morris (2017), presented in the setting of a private-value first-price auction. The result in that paper corresponds to the special case in which values

Note that the outcome described in Theorem 5 simultaneously maximizes consumer surplus and total surplus, which shows that consumer surplus and total surplus are aligned. However, the theorem also shows that unknown costs are consistent with some rather extreme and hypercompetitive outcomes in which producer surplus is driven down to zero.

A critical assumption of Theorem 5 is that there are at least two producers. The case of a single producer has been studied by Kartik and Zhong (2023) and looks quite different. They showed that as long as there is common knowledge of gains from trade, there is an information structure and and optimal strategy for the producer which results in an efficient outcome, but where the producer does not benefit from the information at all. Hence, with monopoly producer, an analogue of the main result of BBM obtains, and consumer surplus and total surplus are aligned. But when there is a single producer and there is not common knowledge from gains from trade, then consumer surplus and total surplus may not be aligned, as the following example shows:

Example 5 (Monopoly without alignment)**.** The value cost profile (v, c) is either $(3, 3+\varepsilon)$ or $(2, 0)$, both equally likely, and where ε is close to zero. In an efficient outcome, it would have to be that the producer always sets a price above 3 when the value is 3, and sets a price below 2 when the value is 2. Clearly, this would require the producer to learn the consumer's value exactly, in which case the producer will set a price equal to 2 when $v = 2$, so that consumer surplus is zero. On the other hand, under no information, the producer will optimally price at 2 and earn a producer surplus of $2 - (3+\varepsilon)/2 > 0$, and the resulting consumer surplus is $5/2$. In effect, by pooling efficient and inefficient outcomes, the producer is forced to sometimes sell at a loss, in a manner that benefits the consumer.

The issue of whether or not there is common knowledge of gains from trade becomes moot when there are at least two producers and if the prior is weakly competitive, because the producers drive one another's prices down to cost. By focusing on the case of homoge-

are homogenous and certain, i.e., there is a commonly known v which is the value for every producer's cost. Moreover, Bergemann, Brooks, and Morris (2017) assumed that the prior distribution is exchangeable, which implies weak competitiveness. The structure of the proofs is largely the same.

neous values in Theorem 5, we have opted for simplicity of exposition rather than providing the most general conditions under which this kind of hypercompetitive outcome can be supported. A necessary condition to be able to drive producer surplus to zero is that whenever the efficient producer has cost c_i, there is another producer who can be induced to price in a way that the residual willingness to pay for producer i's product is arbitrarily close to c_i. This would entail an inefficient producer setting prices $p_j \approx v_j - v_i + c_i$, would are necessarily below producer j's cost c_j. In principle, we could still construct the information and equilibrium so that producer j prices at this level without knowing for sure that they are pricing below cost, as long as there is also positive probability that producer j is efficient and has a cost $c_j' = v_j - v_i + c_i$. However, it is easy to exhibit distributions for which this assumption is not satisfied, such as whenever costs are certain and there is non-trivial heterogeneity in values.

Moreover, if we drop homogeneous values and weak competitiveness, it may be that consumer surplus and total surplus are not aligned, even though there is common knowledge of gains from trade. This is demonstrated by the following example:

Example 6 (Duopoly without alignment)**.** There are two producers, $(v_2, c_2) = (1, 1-2\varepsilon)$ with probability one, and (v_1, c_1) is equally likely to be $(2, 0)$ and $(3, 3-\varepsilon)$. Thus, it is always efficient to trade, but trade should be with producer 1 when $(v_1, c_1) = (2, 0)$ and trade should be with producer 2 when $(v_1, c_1) = (3, 3-\varepsilon)$. Note that producer 2 will never set a price less than $1-2\varepsilon$, and hence will never offer more than 2ε in surplus to the consumer. Thus, the residual willingness to pay r_1 is at least $3-2\varepsilon$ when $v_1 = 3$, and r_1 is at most 2 when $v_1 = 2$. As a result, for trade to be efficient, producer 1 must be setting a price less than 2 when $v_1 = 2$ and must be setting a price greater than $3-2\varepsilon$ when $v_1 = 3$. Hence, producer 1 must learn exactly the consumer's value, and therefore consumer surplus is at most 2ε. However, under no information, there is an equilibrium in which producer 1 offers a price of $2-2\varepsilon$, producer 2 randomizes on an interval, say, $[1-2\varepsilon, \varepsilon]$, and the consumer always buys from producer 1. In this equilibrium, consumer surplus is $1/2+\varepsilon$. This example is quite similar to the one that we presented above with a single producer, except that now it is the option of trading with producer 2 that determines whether or not it is efficient to trade with producer 1, rather than the possibility of not purchasing at all.

The takeaway from this analysis is that a lot of things can happen when costs are unknown. When costs are certain, we are back in the world of our baseline model and Theorem 1, whereas when goods are homogeneous and the prior is weakly competitive, weak dominance loses all bite, and the welfare outcome is the same as in Theorem 2. In both of these extreme cases, consumer surplus and total surplus are aligned. And yet, examples show that there is a rich plethora of cases in between, with intermediate welfare outcomes, and where consumer surplus and total surplus may not be aligned. The task of providing a more complete characterization of possible welfare outcomes under unknown costs is an interesting direction for future work.

6 Conclusion

The purpose of this paper has been to investigate the role of information and competition in determining welfare in models of price competition with differentiated products. In the monopoly setting, BBM showed that consumer surplus and total surplus are aligned, and consumer surplus and producer surplus are opposed. Our main result dramatically extends this finding to the oligopoly setting: It is possible for information to simultaneously maximize consumer surplus and total surplus, while the producers are no better off than if they had no information *and if their competitors priced as aggressively as possible.* A takeaway is that there is no inherent conflict between consumer surplus and total surplus. We have considered whether this finding extends when the consumer may have partial information about their values and when producers have partial information about their costs. In both cases, consumer surplus and total surplus may or may not be aligned, depending on what additional assumptions we make about the distribution of values and costs. For settings with unknown values and/or unknown costs, we have stopped short of a complete and general characterization of the information that maximizes consumer surplus. More broadly, even with known values and known costs, we have focused on characterizing maximum consumer surplus and total surplus. It remains an open question what is the whole set of welfare outcomes that are achievable with information and competition, even when values and costs are known.

References

Armstrong, M. and J. Vickers (2019): "Discriminating Against Captive Consumers," *American Economic Review: Insights*, 1, 257–272.

——— (2022): "Patterns of Price Competition," *Econometrica*, 90, 153–191.

Armstrong, M. and J. Zhou (2022): "Consumer Information and the Limits to Competition," *American Economic Review*, 112, 534–577.

Bergemann, D., B. Brooks, and S. Morris (2015a): "First Price Auctions with General Information Structures: Implications for Bidding and Revenue," Tech. Rep. 2018R, Cowles Foundation for Research in Economics, Yale University.

——— (2015b): "The Limits of Price Discrimination," *American Economic Review*, 105, 921–957.

——— (2017): "First Price Auctions with General Information Structures: Implications for Bidding and Revenue," *Econometrica*, 85, 107–143.

——— (2021): "Search, Information and Prices," *Journal of Political Economy*, 129, 2275–2319.

——— (2023): "On the Alignment of Consumer Surplus and Total Surplus under Competitive Price Discrimination," Tech. Rep. CFDP 2373, Cowles Foundation for Research in Economics.

Elliott, M., A. Galeotti, A. Koh, and W. Li (2024): "Market Segmentation Through Information," Tech. rep.

Kartik, N. and W. Zhong (2023): "Lemonade from Lemons: Information Design and Adverse Selection," Tech. rep.

Perloff, J. and S. Salop (1985): "Equilibrium with Product Differentiation," *Review of Economic Studies*, 52, 107–120.

Roesler, A. and B. Szentes (2017): "Buyer-Optimal Learning and Monopoly Pricing," *American Economic Review*, 107, 2072–2080.

A Appendix

A.1 Proof of Theorem 1

The information structure we construct has the form

$$S_i = \{0\} \cup (\{1, \ldots, N\} \times \mathbb{R} \times \Delta(\mathbb{R}) \times \{0, 1\}).$$

Thus, each producer either gets a signal 0 or a signal that is a tuple $s_i = (k_i, \tilde{c}_i, x_i, l_i)$. Moreover, the first three components of the signal are public, meaning that with probability one $k_1 = \cdots = k_N$, $\tilde{c}_1 = \cdots = \tilde{c}_N$, and $x_1 = \cdots = x_N$, and hence we will drop the subscript and just write $(k, \tilde{c}, x)$.

First, the producers' signals are all 0 with likelihood $\left(1 - \sum_{k>0} q_k(v, c)\right) \mu(dv, dc)$. (Recall that $1 - \sum_k q_k(v, c)$ is either zero or one, and it is one if and only if production is inefficient.)

Now we describe the signals when production is efficient. We first construct the joint distribution of (k, v, c) to be $q_k(v, c) \mu(dv, dc)$ for $k > 0$. In other words, k is the identity of the producer that the consumer would choose to purchase from if all producers priced at cost, with ties broken uniformly. We define, for all i,

$$r_i(v, c_{-i}) \equiv \min_{j \neq i} v_i - v_j + c_j.$$

This is the "residual" willingness to pay of the consumer for producer i's good when other producers price at cost. We can then define a measure $\zeta^i(dr_i, dv, dc)$, according to

$$\zeta^i(X) \equiv \int_{\{(v,c) | (r_i(v,c_{-i}), v, c) \in X\}} q_i(v, c) \mu(dv, dc).$$

This measure can then be disintegrated as $\zeta^i(dr_i, dv, dc) = \eta^i(dc_i) \nu^i(dr_i | c_i) \gamma^i(dv, dc_{-i} | r_i, c_i)$.

Claim: For every i and c_i, there is a solution to

$$\max_{p_i} (p_i - c_i) \int_{r_i} \mathbb{I}_{r_i \geq p_i} \nu^i (dr_i | c_i),$$

which we denote by $p_i^* (c_i)$. This follows from the fact that the integral is simply the upper cumulative distribution of the random variable r_i, which is upper semi-continuous, and the domain of p_i can without loss be restricted to $[c_i, \overline{v}]$ (since $q_i (v, p_i, p_{-i}) = 0$ when $p_i > \overline{v}$, ν^i almost surely).

Claim: For every i,

$$\underline{PS}_i = \int_{c_i} \eta^i (dc_i) (p_i^* (c_i) - c_i) \int_{r_i \geq p_i^*(c_i)} \nu^i (dr_i | c_i).$$

To prove the claim, observe that in (3), it is without loss to restrict attention to f such that $f (c_i) \geq c_i$ for all i, since otherwise the contribution to the right-hand side is necessarily non-positive. Among such functions, let f be one for which the right-hand side of (3) is at least $\underline{PS}_i - \varepsilon$. Note that if $q_i (v, c) = 0$ (meaning that producer i is *not* an efficient producer) then $q_i (v, f (c_i), c_{-i}) = 0$ as well. Thus, the contribution to the right-hand side of the event where producer i is not efficient is zero. Moreover, if $q_i (c, v) \in (0, 1)$, meaning that there is more than one efficient producer, then the contribution must be zero as well. The reason is that if $f (c_i) = c_i$, then the contribution is zero because producer i is pricing at cost, and if $f (c_i) > c_i$, then $q_i (f (c_i), c_{-i}, v) = 0$, because the consumer would not want to buy from producer i at a price strictly higher than c_i. Thus, the contribution to the right-hand side is strictly positive only if producer i is the *unique* efficient producer, and hence

$$\begin{aligned}
&\int_{v,c} \left(f\left(c_i\right)-c_i\right) q_i\left(v, f\left(c_i\right), c_{-i}\right) \mu(dv, dc) \\
=&\int_{v,c} \left(f\left(c_i\right)-c_i\right) q_i\left(v, f\left(c_i\right), c_{-i}\right) q_k(c, v) \mu(dv, dc) \\
\leq&\int_{v,c} \left(f\left(c_i\right)-c_i\right) \mathbb{I}_{r_i\left(v, c_{-i}\right) \geq f\left(c_i\right)} q_k(v, c) \mu(dv, dc) \\
\leq&\int_{v,c} \left(p_i^*\left(c_i\right)-c_i\right) \mathbb{I}_{r_i\left(v, c_{-i}\right) \geq p_i^*\left(c_i\right)} q_k(v, c) \mu(dv, dc) .
\end{aligned}$$

In the first inequality, we used the fact that if $q_i\left(f_i\left(c_i\right), c_{-i}, v\right)>0$, then $r_i\left(v, c_{-i}\right) \geq f\left(c_i\right)$ (otherwise the consumer would not be willing to purchase from producer i with positive probability). To complete the proof of the claim, it only remains to show that there exist f's for which the gap is arbitrarily small. Let $f\left(c_i\right)=p^*\left(c_i\right)-\varepsilon$. Then $r_i\left(v, c_{-i}\right) \geq p_i^*\left(c_i\right)$ implies that $q_i\left(v, f\left(c_i\right), c_{-i}\right)=1$, so

$$\begin{aligned}
&\int_{v,c} \left(f\left(c_i\right)-c_i\right) q_i\left(v, f\left(c_i\right), c_{-i}\right) q_k(c, v) \mu(dv, dc) \\
\geq&\int_{v,c} \left(f\left(c_i\right)-c_i\right) \mathbb{I}_{r_i\left(v, c_{-i}\right) \geq p_i^*\left(c_i\right)} q_k(v, c) \mu(dv, dc) \\
\geq&\int_{v,c} \left(p_i^*\left(c_i\right)-c_i\right) \mathbb{I}_{r_i\left(v, c_{-i}\right) \geq p_i^*\left(c_i\right)} q_k(v, c) \mu(dv, dc)-\varepsilon \\
=&\int_{c_i} \left(p_i^*\left(c_i\right)-c_i\right) \eta^i\left(dc_i\right) \int_{r_i \geq p_i^*\left(c_i\right)} \nu^i\left(dr_i \mid c_i\right)-\varepsilon,
\end{aligned}$$

as desired.

Now, we invoke Theorem 1B of BBM, which says that for every c_i, there exists a *uniform profit preserving segmentation*, which we write as $\sigma_i\left(\cdot \mid c_i\right) \in \Delta \Delta(\mathbb{R})$ and $\sigma_i\left(dx \mid c_i\right)$, where x is itself a probability measure on the reals, with the properties that for every x in the support of $\sigma_i\left(\cdot \mid c_i\right)$ and p_i in the support of x,

$$\left(p_i-c_i\right) x\left(\left[p_i, \bar{v}\right]\right)=\min \operatorname{supp} x,$$

$p_i^*(c_i) \in \text{supp} x$, and

$$\int_x x(dr_i)\,\sigma_i(dx|c_i) = \nu^i(dr_i|c_i).$$

Now, we define a measure over (k, c_k, x, v, c) according to

$$\phi(k, c_k, dx, dv, dx) = \eta^k(c_k)\,\sigma_k(dx|c_k)\int_{r_k} x(dr_k)\,\gamma^k(dv, dc_{-k}|c_k, r_k).$$

Finally, we describe the private component of the signal, l_i. The purpose of this component is to "alert" producers if they need to randomize, in order to break ties in favor of the efficient producer. If the realized segment x does not have a mass point at $\underline{r} = \min \text{supp} x$, or if there is a mass point at $\underline{r}$ but $\underline{r} = c_k$, then we simply set $l_j = 0$ for each j. On the other hand, if there is a mass point at $\underline{r}$, then we set $l_j = 1$ for any producer j with $\underline{r} = v_i - v_j + c_j$, and $l_j = 0$ otherwise. This completes the construction of the information structure.

We now describe the strategies. First, at the signal (k, x, l_i), let $\underline{r} = \min \text{supp} x$. If $i = k$, then $\rho_i(\underline{r}|k, x, l_i) = 1$. In other words, the efficient producer sets a price equal to the lowest residual willingness to pay in the segment x. If $k \neq i$ and $l_i = 0$, then $\rho_i(c_i|k, x, 0) = 1$. Finally, if $l_i = 1$, then producer i randomizes on an interval just above c_i according to a distribution that we now define. Since $l_i = 1$, there is a mass point at $\underline{r}$. Since the efficient producer is indifferent between different prices in the support, it must be that there is a gap in the support. (If not, then the efficient producer would not be willing to set a price just above $\underline{r}$, which would entail a discrete drop in demand from the consumer with residual willingness to pay $\underline{r}$.) Let $\hat{r}$ be $\min\{r \in \text{supp} x | r > \underline{r}\}$ be the second lowest residual willingness to pay. Then a producer with $l_i = 1$ randomizes according to the distribution

$$\rho([c_i, c_i + \varepsilon]\,|k, x, 1) = \begin{cases} 0 & \text{if } \varepsilon < 0; \\ 1 - \frac{\underline{r} - c_k}{\underline{r} - c_k + \varepsilon} & \text{if } 0 \leq \varepsilon < (\hat{r} - \underline{r})/2; \\ 1 & \text{if } \varepsilon > (\hat{r} - v_1 = 3\underline{r})/2. \end{cases}$$

Note that if $l_i = 1$, then $\underline{r} > c_k$, so that the distribution is non-degenerate.

Now let us verify that these strategies are an equilibrium. We first verify this for the efficient producer. Suppose that producer i is efficient and the realized segment is x. producer i is setting a price $\underline{r} = \min \operatorname{supp} x$, which induces a profit of $\underline{r} - c_i$. If $\underline{r} = c_i$, then it must be that there is a tie for efficient producer, because $c_i = \underline{r} = v_i - v_j + c_j$ for some $j \neq i$. Moreover, that producer j is pricing at cost (because $l_j = 0$ for all j in this case) and the only way for the efficient producer to make a sale is with a price $p_i \leq c_i$, that would induce non-positive profit. Thus, there is no profitable deviation. We now consider what happens if $\underline{r} > c_i$. If there is no mass point on $\underline{r}$, then ties occur with zero probability at $\underline{r}$, and if there is a mass point on $\underline{r}$, then any producer j with $\underline{r} = v_i - v_j + c_j$ received a signal $l_j = 1$, and hence they are randomizing on the interval $[c_j, c_j + (\hat{r} - \underline{r})/2]$, where $\hat{r}$ is the second lowest element of the support of x. This induces a residual demand curve, where the probability of making a sale from a price $p_i \in [\underline{r}, (\underline{r} + \hat{r})/2]$ is $((\underline{r} - c_i)/(p_i - c_i))^L$, where $L = \sum_i l_i \geq 1$. Setting any other price that is not in $\operatorname{supp} x \cup [\underline{r}, (\hat{r} + \underline{r})/2]$ is clearly dominated. From the properties of a uniform profit preserving segmentation, if ties were broken in favor of the efficient producer, then setting any price in the support of x must induce the same profit. Since we break ties uniformly, such prices induce a weakly lower profit than a price of $\underline{r}$. Finally, setting a price $p_i \in [\underline{r}, (\underline{r} + \hat{r})/2]$ induces an interim expected producer surplus of

$$(p_i - c_i)\left(\frac{\underline{r} - c_i}{p_i - c_i}\right)^L \leq (p_i - c_i)\frac{\underline{r} - c_i}{p_i - c_i} = \underline{r} - c_i,$$

as desired.

Next, for any inefficient producer j,

$$p_i \leq r_i = \min_{k \neq i} v_i - v_k + c_k \leq v_i - v_j + c_j.$$

So, for producer j to make a sale, they would have to set a price weakly below cost, and hence they cannot make positive profit. Thus, the proposed strategies are a best response.

Finally, we verify that the welfare outcome is the one described in the theorem. By the properties of a uniform profit preserving segmentation, the efficient producer i is indifferent to pricing at $p_i^*(c_i)$ for any signal realization x. Thus, they are indifferent to *always* pricing at $p_i^*(c_i)$, so that their resulting payoff is $\underline{PS}_i$. But an efficient producer always makes a sale, so that total surplus is $\overline{TS}$. This completes the proof.

A.2 Proof of Theorem 2

We take $S_i = \mathbb{R}^{2N}$, and $\phi(ds, dv, dc)$ puts probability one on $s_i = (v, c)$ for all i, that is, the information structure publicly reveals all of the values and costs. If a sale is inefficient, or if there is more than one efficient producer, then all producers simply price at c. If there is only one efficient producer, who we take to be producer i, then producer i sets a price

$$p_i = \min\{c_i + \varepsilon, (c_i + r_i)/2\},$$

where

$$r_i = v_i - \max_{j \neq i} v_j + c_j$$

The inefficient producers then randomize on the interval $[p_i, (p_i + r_i)/2]$, according to the distribution

$$\rho_j([p_i, x] | s_j) = \begin{cases} 0 & \text{if } x < p_i; \\ 1 - \frac{p_i - c_i}{x - c_i} & p_i \leq x < (p_i + r_i)/2; \\ 1 & \text{if } x \geq (p_i + r_i)/2. \end{cases}$$

By construction, $p_i < r_i \leq v_i - v_j + c_j$ for all $j \neq i$, so the only way for a producer $j \neq i$ to make a sale is by setting a price below cost, which would give non-negative profit. Hence, inefficient producers have no profitable deviations. On the other hand, if the efficient producer prices at $x > (p_i + r_i)/2$, they make zero profit, at any price $x \leq p_i$ they make a sale with probability one and hence profit is weakly lower than at $x = p_i$, and for $x \in [p_i, (p_i + r_i)/2]$, expected profit is

$$(x - c_i) \prod_{j \neq i} \rho_j \left([x, (p_i + r_i)/2] \,| s_j \right) = (x - c_i) \left(\frac{p_i - c_i}{x - c_i} \right)^{N-1}$$
$$\leq (x - c_i) \left(\frac{p_i - c_i}{x - c_i} \right)$$
$$= p_i - c_i.$$

Hence, the efficient producer does not have a profitable deviation either. Since the efficient producer always makes a sale, $TS = \overline{TS}$. But the efficient producer's price is always less than $c_i + \varepsilon$, so $PS \leq \varepsilon$, and therefore $CS \geq \overline{TS} - \varepsilon$, as desired.

A.3 Proof of Theorem 5

Fix $\varepsilon > 0$. Because the support of costs is finite, we may assume that ε is small enough so for any c and c' that are in the support of μ, if $c_i \neq c'_j$, then $\left| c_i - c'_j \right| > \varepsilon$.

Consider the information structure where each producer is recommended a price. If trade is inefficient, or if trade is efficient but there is more than one efficient producer, then all producers are recommended to price at cost. Otherwise, there is a unique efficient producer, and since values are homogeneous, the efficient producer is the one who has the lowest cost. We recommend a price p_i to the efficient producer that is drawn from any full support, non-atomic distribution (say uniform) on $[c_i, c_i + \varepsilon]$. As a result, the price set by the efficient producer is necessarily low enough that other producers would have to price weakly below cost in order to make a sale. By the richness assumption, there is a producer $j \neq i$ who with positive probability is efficient with the same cost. We draw a price p_j for that producer on the interval $[p_i, (p_i + c_i + \varepsilon)/2]$, according to the distribution

$$Prob\,(p_j \leq x) = \begin{cases} 0 & \text{if } x < p_i; \\ 1 - \frac{p_i - c_i}{x - c_i} & p_i \leq x < \frac{p_i + c_i + \varepsilon}{2}; \\ 1 & \text{if } x \geq \frac{p_i + c_i + \varepsilon}{2}. \end{cases}$$

All other producers $k \neq i, j$ are recommended prices $p_k = c_k$.

We claim that under this information structure, it is an equilibrium for each producer to set a price equal to their signal, i.e., to obey the recommendation. To see why, suppose that producer i is recommended to price at p_i. We will consider three events: (i) $p_i = c_i$, (ii) producer i is inefficient and $p_i < c_i$, or (iii) producer i is efficient and $p_i \geq c_i$. In fact, we will argue that a producer would not have a profitable deviation, even if they knew which case (i)–(iii) had obtained. In case (i), then either trade is inefficient, there is more than one efficient producer and all producers are pricing at cost, or there is another producer that is efficient and is setting a price below $p^* (v, c)$. In any of these cases, the only way for producer i to make a sale with positive probability would be to lower their price, which would be to a value less than their cost. Hence, a producer cannot make positive profit on this event by deviating. Case (ii) is similar: By setting the recommended price, producer i will not make a sale. The only way to make a sale is by lowering their price, which is already below cost, so the producer would make negative profit. Finally, in case (iii), producer i is making a sale with probability one by obeying the recommendation. Deviating to a lower price will only result in lower profit, and deviating to a higher price x will result in a sale with probability zero if $x > (p_i + c_i + \varepsilon)/2$, a profit of

$$\frac{p_i - c_i + \varepsilon}{2} \frac{1}{2} \frac{p_i - c_i}{(p_i - c_i + \varepsilon)/2} = \frac{p_i - c_i}{2}$$

if $x = (p_i + c_i + \varepsilon)/2$ (because of the mass point on $(p_i + c_i + \varepsilon)/2$), and otherwise results in profit

$$(x - c_i)(1 - Prob(p_j \leq x)) = p_i - c_i,$$

the same as that obtained by following the recommendation. Thus, there is also no profitable deviation in case (iii).

Finally, we verify that the proposed strategies are undominated. Signals take the form of recommended prices. This will be achieved by demonstrating that any lower bound $f : \mathbb{R} \to \mathbb{R}$ such that $\pi(\{(s, v, c) | c_i \geq f(s_i)\}) = 1$ must satisfy $f(p_i) \leq p_i$ with probability one. Suppose not. Because there are finitely many costs, then there must be some cost x so that the prices for which $f(p_i) > p_i$ occurs with positive probability when the efficient cost is x, meaning that the prices are in the interval $[x, x + \varepsilon]$. Let us compute the conditional distribution of producer i's cost, given a recommendation p_i in this interval. Let γ be the probability that they are recommended such a price when $c_i > x$ (case (ii)), and let γ' be the likelihood of being recommended the price when $c_i = x$ (case (iii)). The conditional probability of the cost being x is therefore

$$\frac{\gamma'/\varepsilon}{\gamma'/\varepsilon + \gamma \int_{y=x}^{p_i} \frac{y-x}{(p_i-x)^2} dy/\varepsilon} = \frac{\gamma'}{\gamma' + \gamma/2} > 0.$$

(It is also possible that in the event that $c_i > x$, the efficient producer was told to set a price y so that $p_i = (y + x + \varepsilon)/2$, in which case there is a conditional mass point on the recommendation of p_i of size $(y - c_i)/(p_i - c_i)$, but since this occurs with probability zero conditional on $c_i > x$, omitting it does not affect the interim belief conditional on the recommendation p_i.) Thus, conditional on a recommendation of $p_i \in [x, x + \varepsilon]$, a producer assigns positive probability to the event that $c_i = x$, and hence $f(p_i) \leq x \leq p_i$, as desired.

B Additional Examples

B.1 Hotelling example with the uniform distribution

In this Appendix, we compute the consumer-surplus maximizing information structure for the Hotelling model with uniformly distributed values:

Example 7 (Hotelling with uniform values)**.** There are two producers, both of whom have a cost of zero. We suppose that v_1 is uniformly distributed on $[0,1]$, and that $v_2 = 1 - v_1$, so that r is uniformly distributed on $[-1,1]$.

In light of the analysis of Section 4.3, it remains only to compute the optimal parameters (p, B) for the case where r is uniformly distributed on $[-1,1]$. Note that

$$\int_{-1}^{r} G_p^B(x)dx = \begin{cases} 0 & \text{if } r \leq -B; \\ \frac{p}{2}\ln\left(-\frac{B}{r}\right) & \text{if } -B < r \leq -p; \\ \frac{p}{2}\ln\left(\frac{B}{p}\right) + \frac{1}{2}(r+p) & \text{if } -p < r \leq p; \\ \frac{p}{2}\ln\left(\frac{B}{r}\right) + r & \text{if } p < r \leq B; \\ r & \text{if } r > B; \end{cases}$$

Moreover,

$$\int_{-1}^{r} F(x)dx = \frac{1}{4}(r+1)^2.$$

We therefore wish to maximize the consumer surplus subject to the above constraints (7). We computed the optimum numerically and found it to be approximately $p^* = 0.1162$ and $B^* = 0.9374$. Total surplus is approximately 0.6794 and the resulting consumer surplus is 0.5632. Note that the efficient surplus is 0.75, so that once again the consumer surplus maximizing outcome is inefficient.

It is natural to compare maximum consumer surplus to what could be attained with solutions that are ex post efficient. To compute the latter, we simply restrict attention to

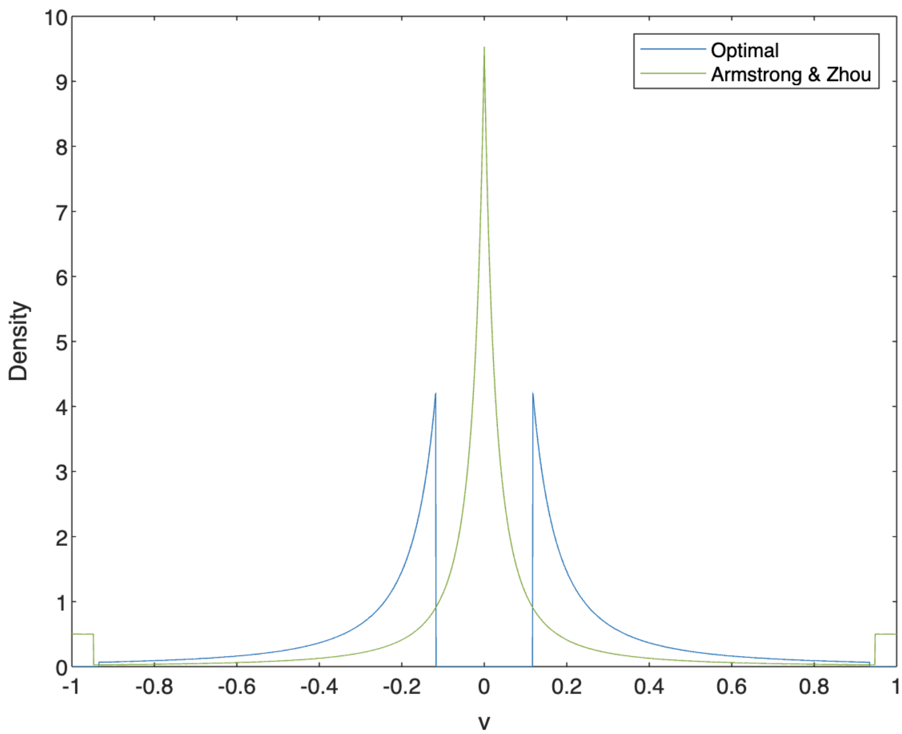

Figure 2: Distribution of posterior values with two-sided vs one-sided optimal information structure

(p, B) such that G_p^B is a mean-preserving contraction of the prior conditional on $r \geq 0$. The optimal values turn out to be $p^* = 0.2045$ and $B^* = 0.8687$, and the resulting consumer surplus is 0.5457, which is actually quite close to maximum consumer surplus.

B.2 Uncertain and known costs

We now fully solve for the consumer surplus maximizing information in Example 2. By Theorem 3, we know that the outcome will be efficient, and so producer 1 will be the only producer to make a sale, and maximizing consumer surplus is equivalent to minimizing producer 1's surplus.

Let $\mathcal{G}_F$ be the set of mean preserving contractions of F, i.e.,

$$\mathcal{G}_F = \left\{ G \,\middle|\, \int_{v=0}^{x} F(v)\, dv \geq \int_{s=0}^{x} G(s)\, ds \text{ for all } x \in [0,1]\,, \text{ with equality for } x = 1 \right\}$$

It is useful to divide this condition into two parts, (i) the inequalities for $x \in [0,1]$; and (ii) the equality for $x = 1$. We will refer to (i) as the SOSD inequalities and (ii) as the mean constraint (since it is equivalent to the requirement that the mean of G is equal to the mean of F).

Our problem is to find $G \in \mathcal{G}_F$ to minimize the surplus of the low cost producer. The novelty, relative to the problem studied in Roesler and Szentes (2017), is that while G is the distribution of the interim value of the consumer, it is *not* the distribution of the consumer's willingness to pay for producer 1's good. The reason is that the consumer has the option to buy the good from producer 2 at a cost of c. Hence, the distribution of the willingness to pay is equal to the distribution of the interim value, censored above c. Effectively, this means that the detailed shape of the distribution on $[c,1]$ does not matter, since all of those values will be collapsed down to c anyway, and we can ignore the SOSD inequalities above c. Note that if the support of F is in $[c,1]$, then the consumer's willingness to pay for producer 1's good is c with probability one, no matter what is the distribution of their interim expected value, and producer 1's profit is c. We henceforth focus on the non-trivial case where there is positive probability that the value is strictly below c.

We now proceed more formally. Let us define

$$G_\pi^c(s) = \begin{cases} 0 & \text{if } s \in [0,\pi]\,; \\ 1 - \frac{\pi}{s} & \text{if } s \in [\pi, c]\,; \\ 1 - \frac{\pi}{c} & \text{if } s \in [c,1); \\ 1 & \text{if } s = 1. \end{cases}$$

Let π^* be the smallest value of π such that G_π^c satisfies the SOSD inequalities, i.e.,

$$\pi^* = \min\left\{ \pi \,\middle|\, \int_{v=0}^{x} F(v)\,dv \geq \int_{s=0}^{x} G_\pi^c(s)\,ds \text{ for all } x \geq \pi \right\}. \tag{11}$$

The value π^* is a lower bound on the producer surplus of producer 1. The reason is that if producer i's surplus is π, then the distribution of the interim value must be above $G_\pi^c(s)$ for all $s \in [0,1]$ (recalling that all values above c are censored at c). Note that G_π^c will not in general satisfy the mean constraint. By our assumption that $F(v) > 0$ for some $v < c$, we must have $\pi^* \in (0,c)$. Also, let x^* be the lowest value of x at which the SOSD inequality of $G_{\pi^*}^c$ is an equality, i.e.,

$$x^* = \min \left\{ x \in [\pi^*, 1] \left| \int_{v=0}^{x} F(v)\, dv = \int_{s=0}^{x} G_{\pi^*}(s)\, ds \right. \right\}.$$

In fact, π^* is precisely the minimum payoff of producer 1. We will prove this by exhibiting a distribution $G \in \mathcal{G}_F$ for which

1. $G(s) = G_{\pi^*}^c(s)$ for all $s \in [0, \min(x^*, c)]$ (and thus $s(1 - G(s)) = \pi^*$ for all $s \in [\pi^*, \min(x^*, c)]$)

2. $G(s) \geq G_{\pi^*}^c(s)$ for all $s \in [0,c]$ (and thus $s(1-G(s)) \leq \pi^*$ for all $s \in [0,c]$).

To that end, for $\pi \in [0,c]$ and $B \in [\pi, 1]$, write $G_{\pi,B}^c$ for the distribution that is equal to G_π^c on the interval $[0,B]$, but jumps to 1 at B. Thus

$$G_{\pi,B}^c(s) \equiv \begin{cases} 0 & \text{if } s \in [0,\pi]\,; \\ 1 - \frac{\pi}{s} & \text{if } s \in [\pi, \min(c,B)]\,; \\ 1 - \frac{\pi}{c} & \text{if } s \in [\min(c,B), B); \\ 1 & \text{if } s = [B,1]\,. \end{cases}$$

Note that we may have $B \in [\pi, c)$ or $B \in [c,1]$; $G_{\pi,B}^c$ is well-defined in either case. Obviously, we cannot have $s \in [c,B)$ if $B < c$. Note that G_{π^*,B^*}^c satisfies conditions (i) and (ii) by construction. It remains to show that there is a B so that $G_{\pi^*,B^*}^c \in \mathcal{G}_F$.

As a first step, we show that there exists a unique $B^* \in [x^*, 1]$ such that the mean constraint is satisfied. To verify this, observe that

$$\int_{s=0}^{1} G^c_{\pi^*,B}\left(s\right) ds$$

is continuous and decreasing in B. In addition,

$$\begin{aligned}\int_{s=0}^{1} G^c_{\pi^*,x^*}\left(s\right) ds &= \int_{s=0}^{x^*} G^c_{\pi^*}\left(s\right) ds + \left(1 - x^*\right) \\ &= \int_{v=0}^{x^*} F\left(v\right) dv + \left(1 - x^*\right), \text{ by definition of } x^* \\ &\geq \int_{v=0}^{1} F\left(v\right) dv\end{aligned}$$

and

$$\begin{aligned}\int_{s=0}^{1} G^c_{\pi^*,1}\left(s\right) ds &= \int_{s=0}^{1} G^c_{\pi^*}\left(s\right) ds \\ &\leq \int_{v=0}^{1} F\left(v\right) dv, \text{ by definition of } x^*\end{aligned}$$

Thus, by the intermediate value theorem, there is a unique B^* with

$$\int_{s=0}^{1} G^c_{\pi^*,B^*}\left(s\right) ds = \int_{v=0}^{1} F\left(v\right) dv$$

Now we verify that $G^c_{\pi^*,B^*}$ satisfies all SOSD inequality constraints, i.e.,

$$\lambda\left(x\right) = \int_{v=0}^{x} F\left(v\right) dv - \int_{s=0}^{x} G^c_{\pi^*,B^*}\left(s\right) ds \geq 0$$

for all x. Observe that

$$\lambda(x) = \int\limits_{v=0}^{x} F(v)\,dv - \int\limits_{s=0}^{x} G^{c}_{\pi^*}(s)\,ds$$

for $x \in [0, B^*]$ and thus $\lambda(x) \geq 0$ by construction of π^*. Moreover, $\lambda(x)$ is decreasing on the interval $[B^*, 1]$, because $G^{c}_{\pi^*,B^*}(s) = 1 \geq F(s)$ for all $s \in (B^*, 1]$. And $\lambda(1) = 0$ by construction. Hence, $\lambda(x) \geq 0$ for all x.

As an example, suppose that F is uniform, so that

$$\int\limits_{v=0}^{x} F(v)\,dv = \int\limits_{v=0}^{x} v\,dv = \frac{1}{2}v^2\Big]_0^x = \frac{1}{2}x^2$$

Now if $x \leq \pi$,

$$\int\limits_{s=0}^{x} G^{c}_{\pi}(v)\,dv = 0$$

If $\pi \leq x \leq c$, then

$$\begin{aligned}
\int\limits_{s=0}^{x} G^{c}_{\pi}(v)\,dv &= \int\limits_{s=\pi}^{x} \left(1 - \frac{\pi}{s}\right) ds \\
&= x - \pi \ln x - \pi + \pi \ln \pi \\
&= x - \pi - \pi \ln \frac{x}{\pi}
\end{aligned}$$

If $c \leq x \leq 1$, then

$$\begin{aligned}
\int\limits_{s=0}^{x} G^{c}_{\pi}(v)\,dv &= \int\limits_{s=\pi}^{c} \left(1 - \frac{\pi}{s}\right) ds + (x - c)\left(1 - \frac{\pi}{c}\right) \\
&= c - \pi - \pi \ln \frac{c}{\pi} + (x - c)\left(1 - \frac{\pi}{c}\right) \\
&= x - \pi - \pi \ln \frac{c}{\pi} - \frac{x\pi}{c} + \pi \\
&= x - \pi \ln \frac{c}{\pi} - \frac{x\pi}{c}
\end{aligned}$$

So

$$\int\limits_{s=0}^{x} G_{\pi}^{c}(v)\,dv = \begin{cases} 0 & \text{if } x \in [0, \pi]\,; \\ x - \pi - \pi \log \frac{x}{\pi} & \text{if } x \in [\pi, c]\,; \\ x - \pi \log \frac{c}{\pi} - \frac{x\pi}{c} & \text{if } x \in [c, 1]\,. \end{cases}$$

Hence,

$$\pi^*(c) = \min \left\{ \pi \;\middle|\; \begin{array}{c} \frac{1}{2}x^2 \geq x - \pi - \pi \ln \frac{x}{\pi} \text{ for all } x \in [\pi, c] \\ \frac{1}{2}x^2 \geq x - \pi \ln \frac{c}{\pi} - \frac{x\pi}{c} \text{ for all } x \in [c, 1] \end{array} \right\}.$$