Speculative Growth and the AI "Bubble"

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Abstract

Are today's high AI valuations a bubble? I argue the answer may be "both yes and no"—in a precise economic sense. Drawing on the speculative growth framework developed in Caballero et al. (2006), I claim that AI technology plausibly satisfies the conditions for multiple equilibria. The core mechanism in that framework is a funding feedback: as wealth accumulates, interest rates eventually fall, validating the high valuations that set the process in motion. AI technology reinforces this mechanism through a flat marginal product of capital region—arising from AI's ability to substitute for labor across a broad range of tasks—which allows substantial capital accumulation without rapidly eroding returns. The concentration of AI gains among high-saving capital owners provides the funding feedback, while intermediate adjustment costs in building AI capacity allow asset prices to generate the capital gains that sustain an investment boom, while at the same time permit a rapid expansion in AI capital. When these conditions hold, the economy can sustain either a low-capital equilibrium with high interest rates, or a high-capital equilibrium with low interest rates, high market capitalization and, ultimately, high wages. Crucially, the transition to the high-capital equilibrium requires elevated valuations throughout: high asset prices finance the investment boom that ultimately validates the optimism. Yet this transition is fragile—a loss of confidence can trigger a self-fulfilling crash. The favorable outcome and high valuations are inseparable along the journey.

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1 Introduction

The rise of artificial intelligence has been accompanied by extraordinary valuations of technology companies. Some observers view these valuations as a bubble, while others contend they reflect fundamentals. I argue that both views may be right.

To make this point, I use the "speculative growth" framework developed in Caballero et al. (2006, henceforth CFH). The core mechanism in that paper is a *funding feedback*: as wealth accumulates, the saving rate rises, pushing down interest rates and validating the high valuations that set the process in motion. Under certain conditions, this feedback can generate multiple equilibria—a low-capital outcome and a high-capital outcome—with the selection between them driven by self-fulfilling expectations. The transition to the high-capital equilibrium requires elevated asset prices that finance the investment boom. However, the same mechanism that enables the boom also makes it fragile: a loss of confidence can trigger a self-fulfilling reversal.

This note argues that AI technology plausibly satisfies the conditions for this mechanism to operate:

1. **The funding feedback.** As AI displaces labor income, wealth concentrates among capital owners. Because their saving rate typically rises with wealth (Straub, 2019), aggregate saving increases and interest rates fall as the economy grows richer. This is essential: at the high-capital steady state, the marginal product of capital is lower, so interest rates *must* fall to support asset valuations.

2. **A flat marginal product of capital region.** Unlike traditional capital, AI can substitute for labor across a broad range of tasks (Restrepo, 2025). As AI capital accumulates, it replaces labor rather than facing immediate diminishing returns. This creates a region where the economy can absorb substantial investment before returns fall—reinforcing the funding feedback by reducing how much interest rates must decline to sustain multiple equilibria.

3. **Intermediate adjustment costs.** Adjustment costs in building AI capacity must be high enough that asset prices can exceed replacement cost and generate capital gains, but low enough to permit a rapid expansion in AI capital. This allows expectations of future growth to be capitalized into current valuations.

When these conditions hold, the economy can sustain either a low-capital equilibrium with high interest rates, or a high-capital equilibrium with low interest rates, high market capitalization and, ultimately, high wages. The transition to the high-capital equilibrium requires elevated valuations throughout, and is inherently fragile.

The remainder of this note proceeds as follows. Section 2 develops the first two conditions, which determine whether multiple steady states exist. Section 3 presents the model. Section 4 characterizes the multiple steady states. Section 5 introduces the third condition—adjustment costs—and analyzes the speculative-growth path and its fragility. Section 6 concludes. The appendix contains all technical details needed to reproduce the results.

2 Conditions for Multiple Steady States

This section develops the two conditions that determine whether multiple steady states exist. The core mechanism, inherited from CFH, is the *funding feedback*: as wealth accumulates, the saving rate rises, pushing down interest rates and validating high valuations. For multiple equilibria to arise, the MPK schedule and the required-return schedule must cross more than once (as illustrated in Figure 2 below).

AI technology facilitates this through a flat-MPK region: because AI can substitute for labor, capital can accumulate without rapidly diminishing returns. This reduces the burden on the funding feedback—interest rates need not fall as much to generate multiple crossings, since the required return continues to fall while the MPK holds steady.

The third condition mentioned in the introduction—intermediate adjustment costs—does not affect the existence of multiple steady states. It determines whether a *transition* between them is feasible, and is discussed in Section 5.

2.1 The Flat MPK Region

In a standard neoclassical model, the return to capital falls steadily as capital accumulates. AI technology differs in an important respect. Following the task-based approach synthesized in Restrepo (2025), consider production as involving many discrete tasks. Some tasks are performed by workers, others by machines. Traditional capital can only perform "machine tasks," but AI—like robotization—can perform *both*: it can substitute for labor across a wide range of "worker tasks."

This distinction has implications for diminishing returns. As AI capital accumulates, it does not merely add machines alongside a fixed labor force. Instead, AI *replaces* some labor, effectively expanding the pool of "workers" (now comprising both humans and AI) alongside which conventional capital operates.

The result is a region where the marginal product of capital (MPK) is approximately flat. In this "AI deployment" region, each additional unit of AI capital displaces some labor, keeping the effective capital-labor ratio roughly constant. Diminishing returns are postponed.

Figure 1 illustrates. For low capital (Region I), the economy has no AI and standard diminishing returns apply. For intermediate capital (Region II), AI deployment keeps the MPK flat. For high capital (Region III), AI capacity is saturated and diminishing returns resume.

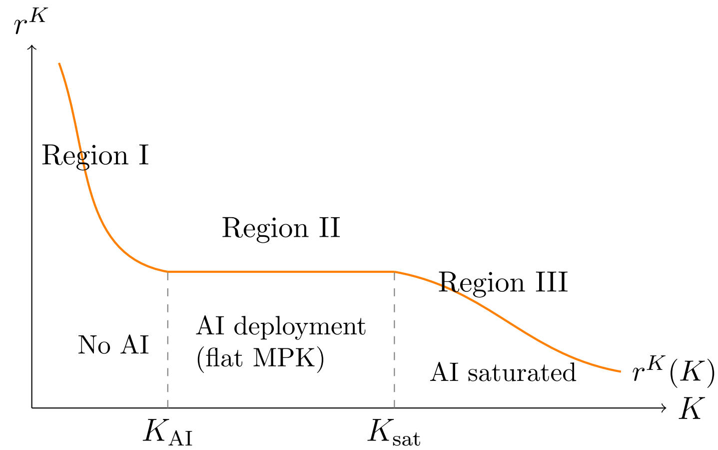

Figure 1: The marginal product of capital with AI technology. In Region II, AI deployment keeps the MPK flat as capital accumulates.

2.2 The Funding Feedback

The funding feedback is the core mechanism from CFH. This occurs naturally in the context of AI. As AI displaces labor, the capital share rises and wealth concentrates among capital owners. Because high-saving capital owners' saving rates rise with wealth (Straub, 2019), aggregate saving increases and the interest rate required to clear asset markets falls.

This mechanism is essential for multiple equilibria. At high levels of capital, the marginal product of capital is lower (diminishing returns). For a high-capital outcome to be sustainable, the required return must also be sufficiently low—otherwise investors would not hold the capital stock. The funding feedback provides exactly this: higher capital raises wealth, which increases saving and lowers interest rates.

The flat-MPK region reinforces this mechanism. Without it, the MPK would fall steeply as capital accumulates, requiring a correspondingly steep decline in interest rates to generate multiple crossings. The flat region reduces this burden: over the range $[K_{\text{AI}}, K_{\text{sat}}]$, the MPK holds steady while the required return continues to fall, making multiple equilibria easier to sustain.

3 A Simple Model

This section presents a minimal model. Technical details are in the Appendix: Appendix A derives the MPK schedule, Appendix B provides microfoundations for the consumption rule, and Appendix C collects the equilibrium dynamics.

3.1 Technology

Time is continuous. Output is produced with capital and labor:

$$Y = AK_c^{\alpha}N^{1-\alpha},$$

where K_c is conventional capital, N is effective labor, and $\alpha \in (0, 1)$.

Capital can be used in two ways: as conventional capital K_c or as AI capital K_ℓ that substitutes for labor. Total capital is $K = K_c + K_\ell$. AI capital produces "AI labor" at rate γ per unit (i.e., one unit of AI capital is equivalent to γ units of labor), so effective labor is $N = 1 + \gamma K_\ell$. However, AI deployment faces a capacity constraint $\bar{K}_\ell$—think of limits in data, compute, or organizational capacity.

Firms allocate capital optimally between the two uses. As shown in Appendix A, this generates the three-region MPK schedule in Figure 1:

- **Region I** ($K < K_{\text{AI}}$): No AI deployment. Standard diminishing returns: $r^K = \alpha AK^{\alpha-1}$.
- **Region II** ($K_{\text{AI}} \leq K < K_{\text{sat}}$): AI deployment phase. The MPK is constant: $r^K = \alpha A \left(\frac{(1-\alpha)\gamma}{\alpha}\right)^{1-\alpha}$.
- **Region III** ($K \geq K_{\text{sat}}$): AI saturated at $K_\ell = \bar{K}_\ell$. Diminishing returns resume: $r^K = \alpha A(K - \bar{K}_\ell)^{\alpha-1}(1 + \gamma\bar{K}_\ell)^{1-\alpha}$.

The threshold K_{AI} is the capital level at which AI deployment becomes profitable; K_{sat} is the level at which AI capacity $\bar{K}_\ell$ is fully utilized. Both are derived in Appendix A.

3.2 Households

There are two types of households: workers and capitalists.

Workers supply labor, earn wages, and consume their entire income: $c_w = w$. They hold no assets.

Capitalists own all capital and have non-homothetic preferences. Following Straub (2019), I assume their consumption follows:

$$c = \kappa W^{\phi}, \qquad \kappa > 0, \quad 0 < \phi < 1, \tag{1}$$

where W is wealth and κ, ϕ are parameters. The key feature is $\phi < 1$: consumption rises less than proportionally with wealth. Equivalently, the saving rate rises with wealth.

This specification is a tractable approximation to optimal behavior under non-homothetic preferences. Appendix B provides microfoundations and shows how to calibrate κ and ϕ to match steady-state behavior exactly.

Given (1), richer capitalists save a larger fraction of their income. As the capital share rises during AI deployment, aggregate saving rises, pushing down the interest rate required to clear asset markets at high valuations.

3.3 Investment and Asset Pricing

Investment faces adjustment costs. Let q denote Tobin's q—the market value of installed capital. The investment rate responds to q:

$$\frac{\dot{K}}{K} = \psi \ln q - \delta,$$

where $\psi > 0$ governs the responsiveness of investment to valuations and δ is depreciation. When $q > 1$ (market value exceeds replacement cost), gross investment is positive; when $q < 1$, gross investment is negative.

Asset pricing requires that the return on holding capital equals the required return:

$$\frac{\dot{q}}{q} + \frac{r^K(K)}{q} - \delta = R,$$

where R is the interest rate. The interest rate depends on capitalist wealth through the saving behavior embedded in (1). Across steady states, higher wealth means lower interest rates—this is the funding feedback.

4 Multiple Steady States

The flat-MPK region and the funding feedback interact to generate multiple steady states. This section characterizes these steady states; the next section asks whether and how the economy can transition between them.

At a steady state, investment exactly covers depreciation ($\dot{K} = 0$), which requires $\psi \ln q = \delta$, hence:

$$\bar{q} = e^{\delta/\psi}.$$

At this valuation, asset market clearing requires that the MPK equal the required rental rate. Setting $\dot{q} = 0$ in the asset pricing equation and rearranging:

$$r^K(K) = [R(\bar{q}K) + \delta]\bar{q}. \tag{2}$$

The left side is the MPK; the right side is the required rental rate given the equilibrium interest rate.

Figure 2 plots both sides against K. The MPK follows the "down-flat-down" pattern from Figure 1. The required rental rate is strictly decreasing in K: higher capital means more wealth, which raises saving, lowers R, and hence lowers the required rental.

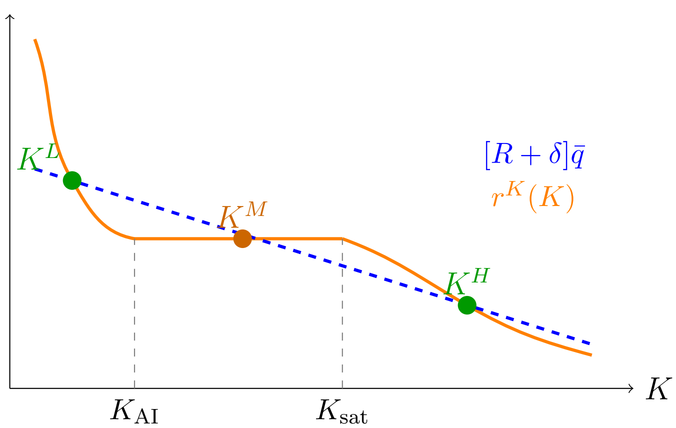

Figure 2: Multiple steady states. The economy can rest at K^L (low capital, high interest rates) or K^H (high capital, low interest rates). The middle intersection K^M is unstable.

The curves cross three times, generating three steady states:

- K^L: Low capital, no AI, high interest rates.
- K^M: Middle capital, partial AI, intermediate interest rates. Unstable.
- K^H: High capital, saturated AI, low interest rates.

The existence of three crossings is facilitated by the flat region in the MPK schedule. In Region II, the MPK holds steady while the required rental rate continues to fall with K. This allows the curves to cross, separate, and cross again.

Appendix D provides the formal conditions for three steady states and proves that K^L and K^H are saddle-path stable while K^M is unstable.

Although Tobin's q equals $\bar{q}$ at both stable steady states, total market capitalization $\bar{q}K$ differs substantially: it is higher at K^H than at K^L in proportion to the capital stocks. The high-capital equilibrium thus features not only a larger capital stock but also greater aggregate wealth.

Having characterized the steady states, I now turn to equilibrium selection and transition dynamics.

5 Speculative Growth and Fragility

Section 4 established that multiple steady states can exist. But can the economy transition from K^L to K^H? That is, can a speculative-growth episode arise? This requires the third condition mentioned in the introduction: intermediate adjustment costs.

5.1 The Role of Adjustment Costs

The existence of multiple steady states does not guarantee a feasible path between them. For a speculative-growth path to exist, investment must face adjustment costs—but not too large and not too small.

The key insight, emphasized in CFH, is that adjustment costs must be large enough to *decouple the return on investment from the marginal product of capital in the short run.* With adjustment costs, asset prices can exceed replacement cost. Returns then derive partly from capital gains rather than solely from the MPK, allowing expectations of future growth to be capitalized into current valuations.

The parameter ψ governs how responsive investment is to Tobin's q. For a speculative-growth path to exist, ψ must be intermediate:

If ψ is too large (low adjustment costs), the decoupling disappears. The stable manifold of K^H—the set of (K, q) points from which the economy converges to K^H— does not reach back to K^L at elevated valuations. Thus there is no capital gain to offset a declining MPK.

If ψ is too small (high adjustment costs), the investment boom is too costly to sustain.

With intermediate ψ, the stable manifold of K^H reaches back to K^L at valuations that are elevated but plausible.

5.2 The Speculative Growth Path

Figure 3 plots the (K, q) phase diagram. The dotted horizontal line is the $\dot{K} = 0$ locus, which occurs at the steady-state valuation $\bar{q} = e^{\delta/\psi}$. The non-monotonic orange curve is the $\dot{q} = 0$ locus. The green curve is the stable manifold of the high-capital steady state $(K^H, \bar{q})$; it describes the speculative-growth trajectory.

Capital cannot jump, but asset prices can. Starting from the low-capital steady state $(K^L, \bar{q})$, a speculative-growth episode unfolds as follows:

1. **Expectations shift.** Agents coordinate on optimistic beliefs.

2. **Valuations jump.** Tobin's q rises discretely from $\bar{q}$ to $q_0 > \bar{q}$.

3. **Investment booms.** The rise in q makes investment profitable and capital starts to accumulate.

4. **AI deploys.** As K crosses K_{AI}, firms deploy AI, raising the capital share and concentrating wealth.

5. **Interest rates fall.** As capitalists become wealthier, their saving rate rises, lowering the required return.

6. **Convergence.** The economy converges to $(K^H, \bar{q})$: capital reaches its high steady state and valuations eventually return to $\bar{q}$, now consistent with a lower interest rate.

The phase diagram also makes clear why elevated valuations are integral to the transition. At $(K^L, \bar{q})$ the economy is at rest; to induce capital accumulation one must have $q > \bar{q}$. Moreover, reaching K^H requires staying on the stable manifold, which lies above $\bar{q}$ throughout. **High valuations are therefore not a symptom of irrational exuberance; they are the equilibrium mechanism that makes the transition feasible.**

5.3 Time Paths Along the Speculative Growth Trajectory

Figure 4 reports the corresponding time paths along the speculative-growth trajectory, starting from the post-jump point on the stable manifold. The key real-side driver is the investment response to valuations. Since

$$\frac{I_t}{K_t} = \psi \ln q_t,$$

the jump in q_t produces an immediate increase in the investment rate. What happens thereafter depends on the evolution of q_t along the manifold: q_t can continue rising for a

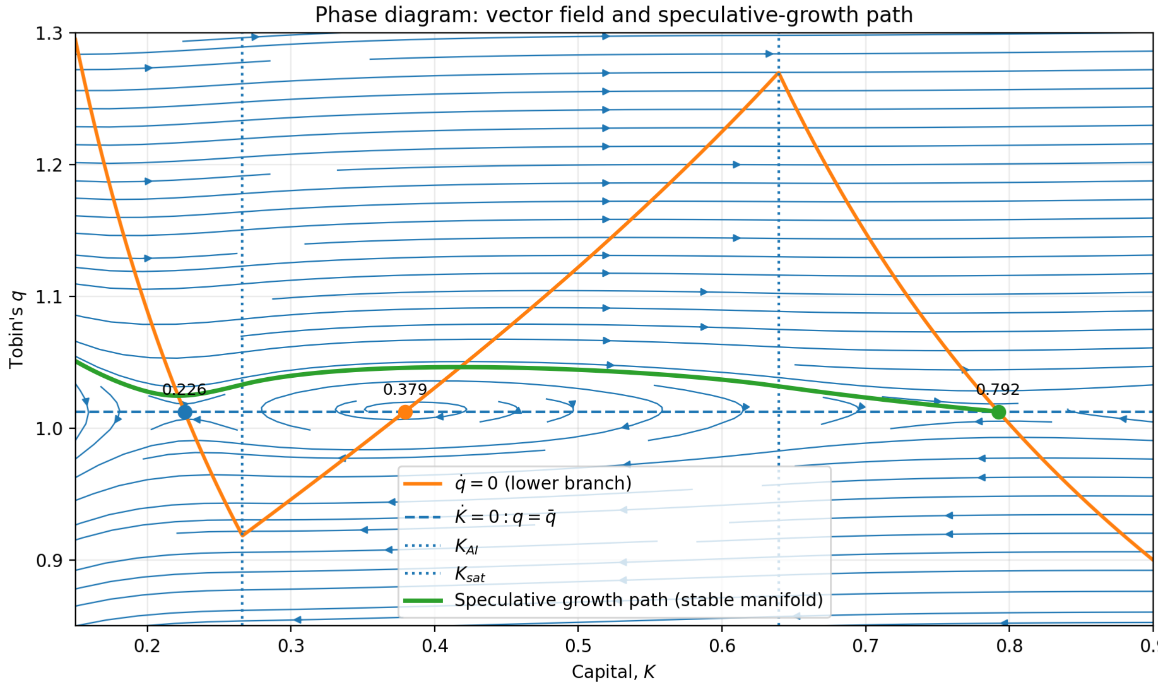

Figure 3: Phase diagram and speculative-growth path. Starting from $(K^L, \bar{q})$, optimistic expectations trigger an upward jump in q. The economy then converges along the stable manifold (green) to $(K^H, \bar{q})$.

time (even as K_t increases) before eventually peaking and mean-reverting toward $\bar{q}$ as the economy approaches the high-capital steady state.

Interest rates. The interest rate R_t is governed by aggregate wealth $W_t = q_t K_t$. Early in the transition, as capital deepens through Region I, the marginal product of capital falls and wealth rises only gradually, so the interest rate declines slowly. The key change occurs once the economy enters Region II: because the marginal product of capital stabilizes there, valuations q_t can continue to rise even as K_t increases. Higher q_t strengthens Tobin's q incentives and accelerates capital accumulation; simultaneously, the rising valuation of the expanding capital stock speeds up the growth of aggregate wealth W_t. In equilibrium, this faster wealth accumulation strengthens the funding feedback—pushing the required return down and thereby sustaining the elevated valuations that keep investment high. As the economy approaches K^H and q_t converges back to $\bar{q}$, the interest rate converges to its new, lower steady-state level.

Wages and the labor share. Wages w_t and the labor share $s_{L,t}$ move with the economy's effective labor input N_t. As the trajectory enters the AI-deployment region, N_t rises as AI substitutes for labor. Output increases while wages initially stagnate, compressing the

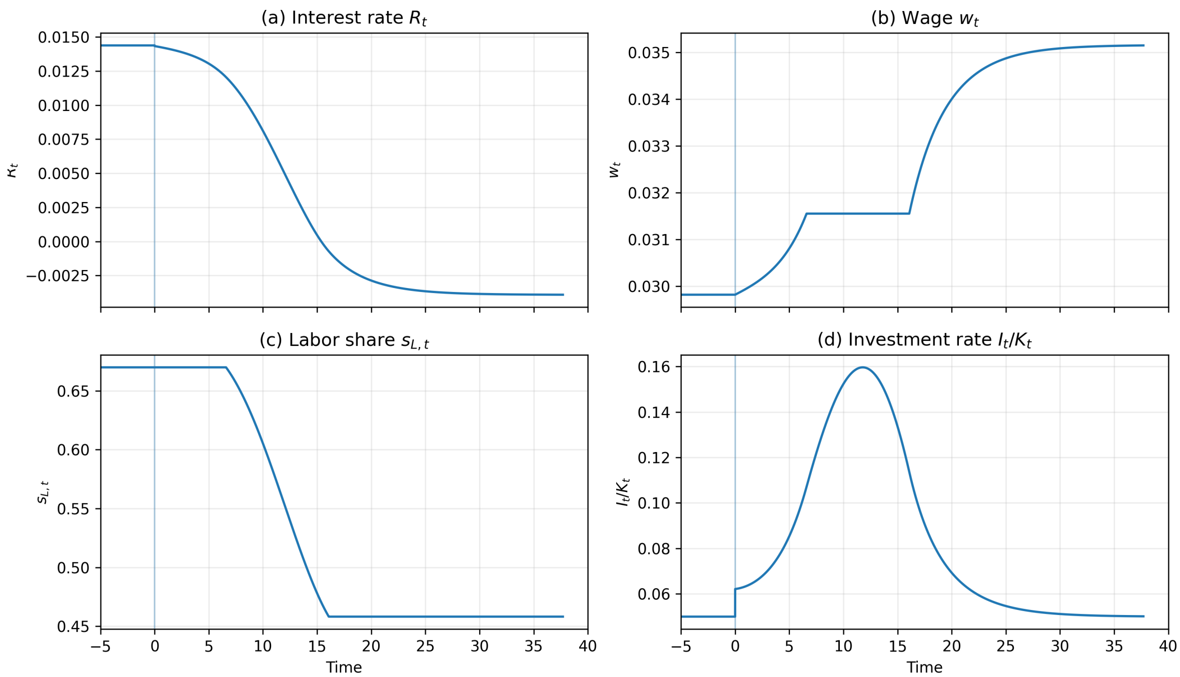

Figure 4: Time paths along the speculative-growth trajectory. The economy starts at the low-capital steady state and jumps onto the speculative-growth manifold. Panel (a): interest rate R_t; Panel (b): wage w_t; Panel (c): labor share $s_{L,t}$; Panel (d): investment rate I_t/K_t.

labor share:

$$s_{L,t} = \frac{w_t L}{Y_t} = \frac{1-\alpha}{N_t}.$$

Once AI saturates (Region III), N_t stabilizes and the labor share converges to a permanently lower level, while wages resume rising with capital deepening.

Investment dynamics. The evolution of I_t/K_t reflects the forward-looking behavior of q_t along the speculative-growth path. The economy first experiences an upward jump in q_t when it shifts onto the stable manifold. Crucially, q_t continues to rise already in Region I: investors anticipate the stronger valuation support that becomes available once the economy reaches Region II, where the marginal product of capital stabilizes. The resulting increase in q_t strengthens Tobin's q incentives and drives a further rise in the investment rate. In Region II, the stabilization of the marginal product of capital allows q_t to remain elevated—and often to keep rising for some time despite ongoing capital deepening—thereby sustaining high investment. Eventually, as the economy approaches the high-capital steady state, q_t peaks and mean-reverts toward $\bar{q}$, bringing I_t/K_t down gradually until it converges to its steady-state level.

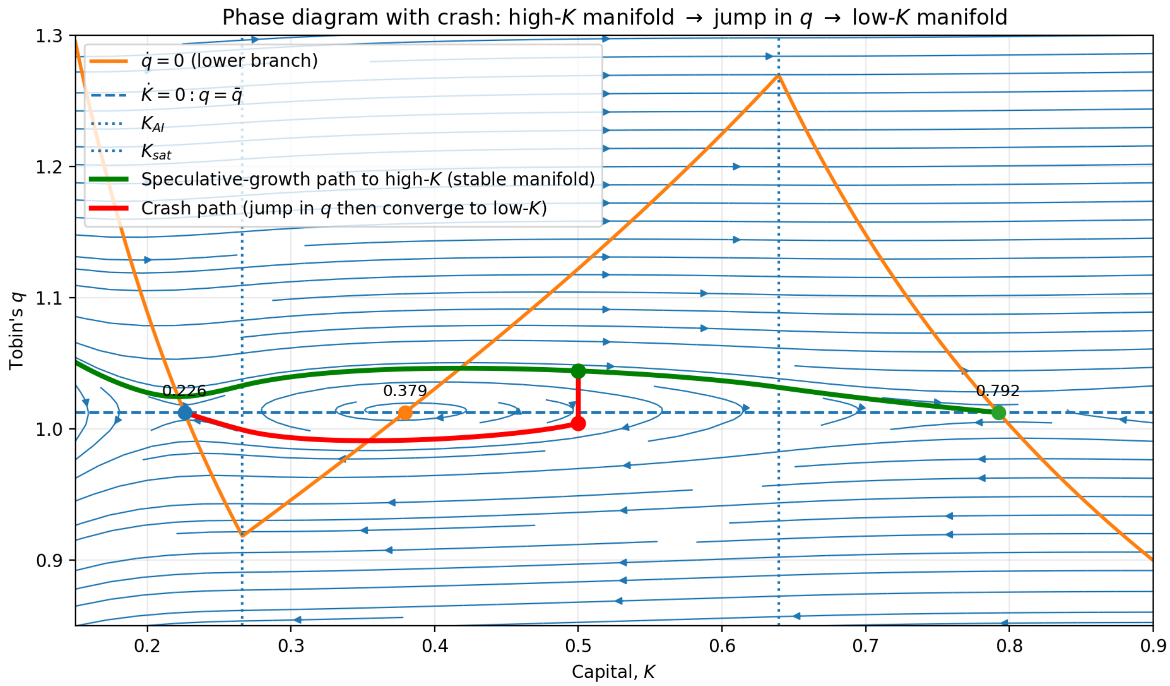

Figure 5: A crash along the speculative-growth path. A drop in valuations places the economy on a trajectory (red) leading back to K^L.

5.4 Fragility: The Crash

The speculative-growth path is inherently fragile. The same mechanism that enables the boom also enables its reversal.

Consider an economy partway through the transition to the high capital equilibrium: capital has accumulated to some $K > K^L$ and valuations remain elevated at $q > \bar{q}$. Suppose confidence weakens—due to negative news about AI capabilities, a financial shock, or a shift in sentiment.

If valuations decline—even absent any change in fundamentals—the economy can depart from the speculative-growth path. A sufficiently large decline places the economy on a trajectory converging back to K^L rather than forward to K^H.

Figure 5 illustrates this scenario. The red segment represents a crash: at fixed capital, q drops discretely. The red path shows the subsequent dynamics: capital decumulates and the economy returns to the low-capital steady state.

The crash is self-fulfilling: agents revise downward their expectations of future capital, which lowers expected returns and hence lowers valuations today. Lower valuations reduce investment, causing capital to decumulate. Capital decumulation in turn validates the pessimistic expectations.

This analysis clarifies the sense in which AI valuations can be simultaneously “not a bubble” and “fragile.” They are not a bubble in the traditional sense because they can be validated by the investment and growth they enable. They are fragile because that validation requires sustained confidence throughout a potentially lengthy transition.

6 Conclusion

I have applied the speculative growth framework of CFH to AI. The core mechanism in that framework is a funding feedback: as wealth accumulates, saving rises and interest rates fall, validating high valuations. AI technology plausibly satisfies the conditions for this mechanism to generate multiple equilibria and a feasible transition path between them: (i) concentration of AI gains among high-saving capital owners provides the funding feedback; (ii) AI’s ability to substitute for labor creates a flat-MPK region that reinforces the mechanism by allowing capital accumulation without rapidly diminishing returns; and (iii) intermediate adjustment costs in building AI capacity enable expectations of future growth to be capitalized into current valuations.

Several implications emerge:

- **Elevated valuations may be integral to the transition.** The investment boom that validates optimistic expectations requires those expectations to manifest in high asset prices. Characterizing AI valuations as simply a bubble overlooks the self-fulfilling nature of the transition.

- **The transition is fragile.** The same multiplicity that enables the boom also enables its reversal. A loss of confidence—for any reason—can derail the transition and return the economy to the low-capital steady state.

- **The high-capital outcome and high valuations are inseparable.** One cannot reach the high-capital equilibrium without traversing a path of elevated asset prices. This interdependence is what makes the current situation both an opportunity and a source of macroeconomic risk.

It goes without saying that this is a possibility argument. My goal is to isolate a coherent mechanism that could rationalize the joint behavior of valuations and investment, not to claim conclusive evidence. I also do not mean to imply that the valuations and investment rates we are currently observing are fully consistent with a rational expectations model. Rather, the point is that if one can write down a rational expectations benchmark that rationalizes these patterns, then behaviorally motivated narratives that move in the same direction are more likely to be self-sustaining—and therefore to survive and persist—because they are aligned with an internally coherent equilibrium logic.

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A Technology Details

A.1 Setup

Total capital $K = K_c + K_\ell$ is divided between conventional capital $K_c \geq 0$ and AI capital $K_\ell \geq 0$. Effective labor is

$$N = 1 + \gamma \min\{K_\ell, \bar{K}_\ell\},$$

where $\gamma > 0$ is the labor-equivalent productivity of AI and $\bar{K}_\ell$ is the AI capacity constraint.

Production is Cobb-Douglas: $Y = AK_c^\alpha N^{1-\alpha}$.

A.2 Optimal Allocation

Given K, firms choose K_c and K_ℓ to maximize output. At an interior solution:

$$\frac{\partial Y}{\partial K_c} = \frac{\partial Y}{\partial K_\ell} \quad \Rightarrow \quad \frac{\alpha}{K_c} = \frac{(1-\alpha)\gamma}{N}.$$

Define $b \equiv (1-\alpha)\gamma/\alpha$. The optimality condition becomes $N = bK_c$, yielding:

$$K_c = \frac{\gamma K + 1}{\gamma + b}, \qquad K_\ell = \frac{bK - 1}{\gamma + b}.$$

This interior solution is valid when $K_\ell \in [0, \bar{K}_\ell]$, i.e., when $K \in [K_{\text{AI}}, K_{\text{sat}}]$ where:

$$K_{\text{AI}} = \frac{1}{b} = \frac{\alpha}{(1-\alpha)\gamma}, \qquad K_{\text{sat}} = \frac{1 + (\gamma + b)\bar{K}_\ell}{b}.$$

A.3 The MPK Schedule

The marginal product of capital is:

$$r^K(K) = \begin{cases} \alpha A K^{\alpha-1}, & K < K_{\text{AI}}, \\ \alpha A b^{1-\alpha} \equiv r^K_{\text{flat}}, & K_{\text{AI}} \le K < K_{\text{sat}}, \\ \alpha A(K - \bar{K}_\ell)^{\alpha-1}(1 + \gamma \bar{K}_\ell)^{1-\alpha}, & K \ge K_{\text{sat}}. \end{cases}$$

In Region II, the MPK is constant because as K rises, K_ℓ rises proportionally, keeping the capital-to-effective-labor ratio $K_c/N = 1/b$ constant.

B Microfoundations for the Consumption Rule

B.1 Preferences

Capitalists maximize:

$$\int_0^\infty e^{-\rho t} \left[\ln c_t + \lambda W_t\right] dt,$$

subject to $\dot{W}_t = R_t W_t - c_t$. The term λW_t captures direct utility from wealth (wealth as a "luxury good").

B.2 Euler Equation

The Hamiltonian is $H = \ln c + \lambda W + \mu(RW - c)$. First-order conditions:

$$\frac{\partial H}{\partial c} = 0 \quad \Rightarrow \quad \frac{1}{c} = \mu,$$

$$\dot{\mu} = \rho\mu - \frac{\partial H}{\partial W} = \rho\mu - \lambda - \mu R.$$

Combining: $\dot{\mu}/\mu = \rho - \lambda c - R$. Since $\mu = 1/c$, we have $\dot{\mu}/\mu = -\dot{c}/c$, yielding:

$$R = \rho + \frac{\dot{c}}{c} - \lambda c.$$

B.3 Steady-State Consumption

At a steady state, $\dot{c} = \dot{W} = 0$, so $R^{ss} = \rho - \lambda c^{ss}$ and $c^{ss} = R^{ss} W$. Solving:

$$c^{ss}(W) = \frac{\rho W}{1 + \lambda W}, \qquad R^{ss}(W) = \frac{\rho}{1 + \lambda W}.$$

B.4 The Isoelastic Approximation

The consumption-wealth elasticity at a steady state is:

$$\phi(W) \equiv \frac{d \ln c^{ss}}{d \ln W} = \frac{1}{1 + \lambda W} \in (0, 1).$$

I approximate the optimal policy with $c = \kappa W^{\phi}$, calibrating ϕ and κ at a reference wealth W^*:

$$\phi = \frac{1}{1 + \lambda W^*}, \qquad \kappa = \rho\phi(W^*)^{1-\phi}.$$

Equivalently, calibrating to match the exact steady-state policy at W^*,

$$\kappa = \frac{c^{ss}(W^*)}{(W^*)^{\phi}} = \frac{\rho W^*/(1 + \lambda W^*)}{(W^*)^{\phi}} = \rho\,\phi\,(W^*)^{1-\phi},$$

where the last equality uses $1 + \lambda W^* = 1/\phi$.

This approximation is exact at W^* and accurate to first order nearby. Taking $W^* = \bar{q} K^L$ (the low steady state) yields an upper bound on ϕ for the transition, since $\phi(W)$ is decreasing.

C Equilibrium Dynamics

C.1 Laws of Motion

Capital accumulates according to:

$$\dot{K} = (\psi \ln q - \delta) K.$$

C.2 Required Return

Using the consumption rule $c = \kappa W^{\phi}$ with $W = qK$:

$$\dot{c} = \phi\kappa W^{\phi-1}\dot{W} = \phi\kappa W^{\phi-1}(RW - c) = \phi\kappa W^{\phi}(R - \kappa W^{\phi-1}).$$

Substituting into the Euler equation $R = \rho + \dot{c}/c - \lambda c$:

$$R = \rho + \phi(R - \kappa W^{\phi-1}) - \lambda\kappa W^{\phi}.$$

Solving for R:

$$R(W) = \frac{\rho - \phi\kappa W^{\phi-1} - \lambda\kappa W^{\phi}}{1 - \phi}.$$

At a steady state where $\dot{c} = 0$, this simplifies to $R^{ss}(W) = \rho/(1 + \lambda W)$.

C.3 Asset Pricing

The return on holding capital equals the required return:

$$\frac{\dot{q}}{q} + \frac{r^K(K)}{q} - \delta = R(qK).$$

Rearranging:

$$\dot{q} = [R(qK) + \delta]q - r^K(K).$$

C.4 Phase Diagram Loci

The $\dot{K} = 0$ locus is $q = \bar{q} = e^{\delta/\psi}$ (horizontal).

The $\dot{q} = 0$ locus is $[R(qK) + \delta]q = r^K(K)$, which varies with the three-region MPK structure.

C.5 Output, Wages, and Labor Share

Output in each region:

$$Y = \begin{cases} AK^\alpha, & K < K_{\text{AI}}, \\ AK_c^\alpha N^{1-\alpha} \text{ with } K_c = \frac{\gamma K+1}{\gamma+b}, N = bK_c, & K_{\text{AI}} \le K < K_{\text{sat}}, \\ A(K - \bar{K}_\ell)^\alpha (1 + \gamma \bar{K}_\ell)^{1-\alpha}, & K \ge K_{\text{sat}}. \end{cases}$$

The wage equals the marginal product of human labor:

$$w = (1 - \alpha)\frac{Y}{N}.$$

The labor share (human labor's share of output) is:

$$s_L = \frac{wL}{Y} = \frac{1 - \alpha}{N},$$

where $L = 1$ is human labor supply. In Region II, N rises with K, so s_L falls.

D Proof of Three Steady States

Define $\Delta(K) \equiv r^K(K) - [R^{ss}(K) + \delta]\bar{q}$. A steady state exists where $\Delta(K) = 0$.

Here $R^{ss}(K)$ is shorthand for the required return $R^{ss}(W)$ evaluated at steady-state wealth $W = \bar{q}K$, i.e., $R^{ss}(K) \equiv R^{ss}(\bar{q}K)$.

Region I: $\Delta(K) \to +\infty$ as $K \to 0$ (since $r^K \to \infty$). If the multiplicity condition holds, $\Delta(K_{\text{AI}}) < 0$. By continuity, $\exists K^L \in (0, K_{\text{AI}})$ with $\Delta(K^L) = 0$.

Region II: r^K is constant while $R^{ss}(K)$ is decreasing, so Δ is increasing. The multiplicity condition implies $\Delta(K_{\text{AI}}) < 0$ and $\Delta(K_{\text{sat}}) > 0$. By continuity, $\exists K^M \in (K_{\text{AI}}, K_{\text{sat}})$ with $\Delta(K^M) = 0$.

Region III: At K_{sat}, $\Delta > 0$. As $K \to \infty$, $r^K \to 0$ while $[R^{ss} + \delta]\bar{q} \to \delta\bar{q} > 0$, so $\Delta < 0$. By continuity, $\exists K^H \in (K_{\text{sat}}, \infty)$ with $\Delta(K^H) = 0$.

Multiplicity condition:

$$[R^{ss}(K_{\text{AI}}) + \delta]\bar{q} > r^K_{\text{flat}} > [R^{ss}(K_{\text{sat}}) + \delta]\bar{q}. \tag{3}$$

E Local Stability

The linearized system around a steady state $(K^*, \bar{q})$ is:

$$\begin{pmatrix} \dot{K} \\ \dot{q} \end{pmatrix} = J \begin{pmatrix} K - K^* \\ q - \bar{q} \end{pmatrix}.$$

The Jacobian elements are:

$$\begin{aligned}
J_{11} &= \left.\frac{\partial \dot{K}}{\partial K}\right|_{ss} = 0, \\
J_{12} &= \left.\frac{\partial \dot{K}}{\partial q}\right|_{ss} = \frac{\psi K^*}{\bar{q}} > 0, \\
J_{21} &= \left.\frac{\partial \dot{q}}{\partial K}\right|_{ss} = -(r^K)'(K^*) + \bar{q}^2 R'(W^*), \\
J_{22} &= \left.\frac{\partial \dot{q}}{\partial q}\right|_{ss} = R^{ss} + \delta + \bar{q}R'(W^*)K^*.
\end{aligned}$$

Since $R'(W) < 0$ (higher wealth lowers required return), the trace is:

$$\text{tr}(J) = J_{22} = R^{ss} + \delta + \bar{q}R'(W^*)K^* > 0$$

for reasonable parameters.

The determinant is:

$$\det(J) = -J_{12} \cdot J_{21} = -\frac{\psi K^*}{\bar{q}} \left[-(r^K)'(K^*) + \bar{q}^2 R'(W^*) \right].$$

The sign of $\det(J)$ depends on $(r^K)'(K^*)$:

- At K^L and K^H: $(r^K)'(K^*) < 0$ (diminishing returns). Since $R'(W^*) < 0$, the bracketed term is positive provided $-(r^K)'(K^*) > \bar{q}^2 |R'(W^*)|$ (equivalently $|(r^K)'(K^*)| > \bar{q}^2 |R'(W^*)|$). Under the baseline calibration this condition holds at K^L and K^H, hence $\det(J) < 0$. These are saddle points.
- At K^M: $(r^K)'(K^M) = 0$ (flat region). The bracketed term is negative, so $\det(J) > 0$. With $\mathrm{tr}(J) > 0$, K^M is an unstable node.

Since K is predetermined and q is a jump variable, saddle-path stability at K^L and K^H means these are locally stable steady states, while K^M is unstable. With one predetermined variable and one jump variable, a saddle point (one stable and one unstable eigenvalue) implies a unique convergent path.

F Baseline Calibration

F.1 Parameter Values

The figures use:

$$A = 0.0729, \quad \alpha = 0.33, \quad \gamma = 1.85, \quad \bar{K}_\ell = 0.25, \quad \rho = 0.08, \quad \lambda = 20, \quad \delta = 0.05, \quad \psi = 3.0.$$

F.2 Derived Quantities

From the technology block:

$$b = \frac{(1-\alpha)\gamma}{\alpha} = \frac{0.67 \times 1.85}{0.33} \approx 3.76.$$

The boundaries of the flat-MPK region:

$$K_{\mathrm{AI}} = \frac{1}{b} = 0.266, \qquad K_{\mathrm{sat}} = \frac{1 + (\gamma + b)\bar{K}_\ell}{b} = 0.641.$$

The flat-region MPK:

$$r^K_{\mathrm{flat}} = \alpha A b^{1-\alpha} = 0.33 \times 0.0729 \times 3.76^{0.67} \approx 0.058.$$

The steady-state valuation:

$$\bar{q} = e^{\delta/\psi} = e^{0.05/3.0} \approx 1.0168.$$

F.3 Computing Steady States

At a steady state with wealth $W = \bar{q}K$, the required return $R^{ss}(W)$ evaluated at this wealth level is:

$$R^{ss}(\bar{q}K) = \frac{\rho}{1 + \lambda\bar{q}K}.$$

Steady states solve $r^K(K) = [R^{ss}(\bar{q}K) + \delta]\bar{q}$. For the baseline calibration, the three solutions are:

$$K^L = 0.224, \qquad K^M = 0.384, \qquad K^H = 0.790.$$

F.4 Verifying the Multiplicity Condition

The multiplicity condition (3) requires:

$$[R^{ss}(K_{\text{AI}}) + \delta]\bar{q} > r^K_{\text{flat}} > [R^{ss}(K_{\text{sat}}) + \delta]\bar{q}.$$

At the boundaries:

$$[R^{ss}(K_{\text{AI}}) + \delta]\bar{q} = \left[\frac{0.08}{1 + 20 \times 1.0168 \times 0.266} + 0.05\right] \times 1.0168 \approx 0.062,$$
$$[R^{ss}(K_{\text{sat}}) + \delta]\bar{q} = \left[\frac{0.08}{1 + 20 \times 1.0168 \times 0.641} + 0.05\right] \times 1.0168 \approx 0.056.$$

Since $0.062 > 0.058 > 0.056$, the multiplicity condition is satisfied.