Speculative-Growth and the AI "Bubble"

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January 4, 2026

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Abstract

AI technology can generate speculative-growth equilibria. These are rational but fragile: elevated valuations support rapid capital accumulation, yet persist only as long as beliefs remain coordinated. Because AI capital is labor-like, it expands effective labor and dampens the normal decline in the marginal product of capital as the capital stock grows. The gains from this expansion accrue disproportionately to capitalists, whose saving rate rises with wealth, raising aggregate saving. Building on Caballero et al. (2006), I show that these features generate a funding feedback—rising capitalist wealth lowers the required return—that can produce multiple equilibria. With intermediate adjustment costs, elevated valuations are the mechanism that sustains a transition toward a high-capital equilibrium; a loss of confidence can precipitate a self-fulfilling crash and reversal.

JEL Codes: E21, E22, E24, E44, O33, O41

Keywords: AI, speculation, investment boom, crash, multiple equilibria, labor share, non-homothetic preferences, market capitalization

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1 Introduction

Today's high valuations of AI-exposed firms are often described as either fundamentally justified or bubble-like. This paper argues both descriptions can be correct: elevated valuations can be warranted by fundamentals along an optimistic equilibrium path, yet remain fragile.

Drawing on the speculative-growth framework of Caballero et al. (2006, henceforth CFH), I argue that AI makes the CFH mechanism arise naturally. CFH requires a *funding feedback*: some mechanism that lowers the required return as capital expands, allowing elevated valuations to be sustained. Two features of AI technology deliver this:

1. **Labor-like AI and a flat marginal product of capital region.** Because AI can substitute for labor across a broad range of tasks (Restrepo, 2025), effective labor expands alongside AI capital, keeping the capital-to-effective-labor ratio stable and hence the MPK constant. This flat-MPK region makes multiple steady states more likely.

2. **Distribution and the funding feedback.** As AI shifts income toward capitalists and wealth concentrates, aggregate saving rises with wealth and the required return falls (Straub, 2019). This provides the funding feedback that sustains elevated valuations at high levels of capital.

A third condition is required for the *transition* mechanism to operate, but it is not specific to AI:

3. **Intermediate adjustment costs.** Adjustment costs must fall in an intermediate range: high enough that valuations can depart from replacement cost during the transition, yet low enough to permit rapid accumulation. This condition is not specific to AI, and there is no particular reason to expect AI-related investment frictions to violate it. When it holds, elevated valuations can sustain the transition.

When these forces are present, the economy can feature both a low-capital and a high-capital equilibrium. The difference between them is not the long-run valuation level—which returns to replacement cost in steady state—but the *path*: under coordinated optimism, elevated valuations support rapid accumulation toward the high-capital outcome, whereas a loss of confidence can trigger a self-fulfilling crash and reversal.

Section 2 presents the model. Section 3 characterizes the steady states and conditions for multiplicity. Section 4 studies speculative-growth transitions and fragility. Section 5 concludes. Appendices collect derivations, equilibrium dynamics, and the baseline calibration.

2 Model: AI Technology and the Funding Feedback

This section presents a model that delivers the two primitives identified above and includes adjustment costs, which shape transition dynamics. Derivations and supporting details are in the appendices: Appendix A derives the MPK schedule, Appendix B microfounds the consumption rule, and Appendix C collects the equilibrium dynamics.

2.1 Technology

In a standard neoclassical model, the return to capital falls steadily as capital accumulates. AI technology differs in an important respect. Following the task-based approach synthesized in Restrepo (2025), production involves many discrete tasks. Some tasks are performed by workers, others by machines. Traditional capital can only perform "machine tasks," but AI, like robotization, can also perform worker tasks.

This distinction has implications for diminishing returns. As AI capital accumulates, it does not merely add machines alongside a fixed labor force. Instead, AI operates *as labor*, expanding effective labor (now comprising both humans and AI) that works alongside conventional capital.

The result is a region where the MPK is constant. In this "AI deployment" region, each additional unit of AI capital adds effective labor, keeping the effective capital-labor ratio constant. Since the MPK depends on this ratio, diminishing returns are forestalled.

To study the implications of this technology for equilibrium dynamics, I now embed this technology in a continuous-time model. Output is produced with capital and labor:

$$Y = AK_c^{\alpha}N^{1-\alpha},$$

where K_c is conventional capital, N is effective labor, and $\alpha \in (0, 1)$.

Capital can be used in two ways: as conventional capital K_c or as AI capital K_ℓ that substitutes for labor. Total capital is $K = K_c + K_\ell$. AI capital produces "AI labor" at a rate of γ per unit, so effective labor is $N = 1 + \gamma K_\ell$. However, AI deployment faces a capacity constraint $\bar{K}_\ell$—reflecting limits on data, compute, or organizational capacity.

Firms allocate capital optimally between the two uses. As shown in Appendix A, this generates the three-region MPK schedule in Figure 1:

- **Region I** ($K < K_{\text{AI}}$): No AI deployment. Standard diminishing returns $r^K = \alpha AK^{\alpha-1}$.

- **Region II** ($K_{\text{AI}} \leq K < K_{\text{sat}}$): AI deployment phase. The MPK is constant at

$r^K = \alpha A \left(\frac{(1-\alpha)\gamma}{\alpha}\right)^{1-\alpha}$.

- **Region III** ($K \geq K_{\text{sat}}$): AI saturated at $K_\ell = \bar{K}_\ell$. Diminishing returns resume $r^K = \alpha A(K - \bar{K}_\ell)^{\alpha-1}(1 + \gamma \bar{K}_\ell)^{1-\alpha}$.

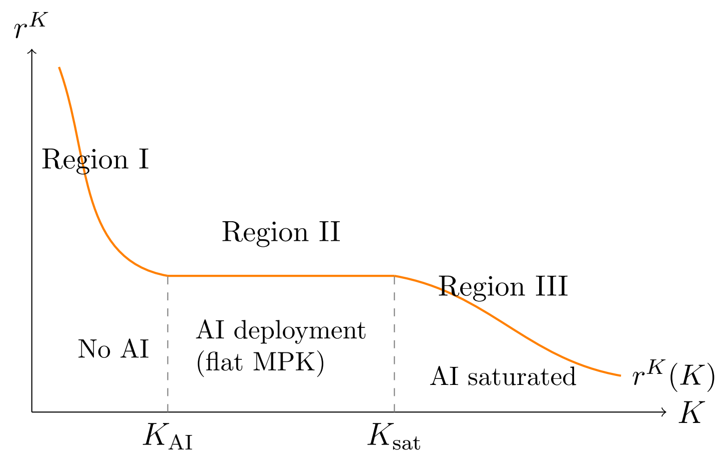

Figure 1: The marginal product of capital with AI technology. In Region II, AI deployment keeps the MPK flat as capital accumulates.

The thresholds K_{AI} and K_{sat} mark, respectively, the onset of AI deployment and the full utilization of AI capacity. Both are derived in Appendix A.

2.2 Households

The introduction described how AI activates the funding feedback. I now formalize this with a two-group household structure: workers and capitalists.

Workers supply labor, earn wages, hold no assets, and consume their entire income: $c_w = w$.

Capitalists own all capital and have non-homothetic preferences. Following Straub (2019), their consumption is given by:

$$c = \kappa W^\phi, \qquad \kappa > 0, \quad 0 < \phi < 1, \tag{1}$$

where W is wealth and κ, ϕ are parameters. The key feature is $\phi < 1$: consumption rises less than proportionally with wealth. Equivalently, the saving rate rises with wealth.

This specification is a tractable approximation to optimal behavior under non-homothetic preferences. Appendix B provides microfoundations and shows how to calibrate κ and ϕ to match steady-state behavior exactly.

2.3 Investment and Asset Pricing

Investment faces adjustment costs. Let q denote Tobin's q—the ratio of market value to replacement cost of capital. The investment rate responds to q:

$$\frac{\dot{K}}{K} = \psi \ln q - \delta,$$

where $\psi > 0$ governs the responsiveness of investment to valuations and δ is the depreciation rate. When $q > 1$ (market value exceeds replacement cost), gross investment is positive; when $q < 1$, gross investment is negative (scrapping).

Asset pricing requires that the return on holding capital equals the required return:

$$\underbrace{\frac{\dot{q}}{q} - \delta}_{\text{capital gain}} + \underbrace{\frac{r^K(K)}{q}}_{\text{dividend yield}} = R,$$

where R is the required return. The latter depends on capitalist wealth through the saving behavior implied by (1). Across steady states, higher wealth means a lower required return—this is the funding feedback.

3 Multiple Steady States

The flat-MPK region and the funding feedback combine to produce multiple steady states. This section characterizes these steady states; the next section asks whether and how the economy can transition between them.

At a steady state, investment exactly covers depreciation ($\dot{K} = 0$), which requires $\psi \ln q = \delta$, hence:

$$\bar{q} = e^{\delta/\psi}.$$

At this valuation, asset market clearing requires that the MPK equals the required rental rate. Setting $\dot{q} = 0$ gives $r^K(K)/q = R + \delta$, or equivalently $r^K(K) = [R + \delta]q$. At a steady state where $q = \bar{q}$ and $W = \bar{q}K$:

$$r^K(K) = [R(\bar{q}K) + \delta]\bar{q}. \tag{2}$$

The left side is the MPK; the right side is the required rental rate.

Figure 2 plots both sides against K. The MPK follows the "down-flat-down" pattern from Figure 1. The required rental rate is strictly decreasing in K: higher capital means more wealth, which raises saving, lowers R, and hence lowers the required rental rate.

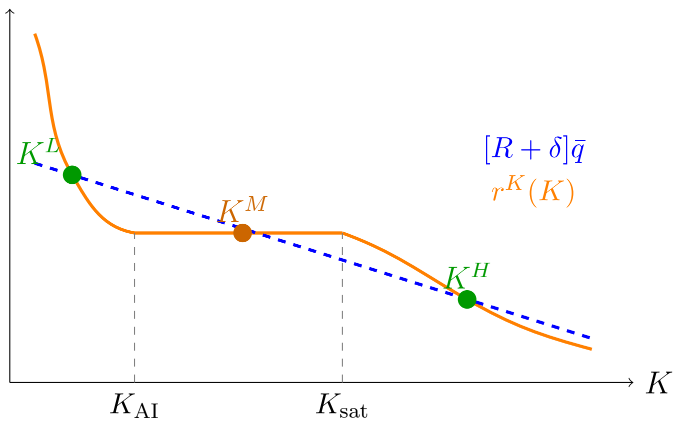

Figure 2: Multiple steady states. The economy can rest at K^L (low-capital, high required return) or at K^H (high-capital, low required return). The middle intersection K^M is unstable.

The curves cross three times, generating three steady states:

- K^L: Low-capital, no AI, high required return.
- K^M: Middle-capital, partial AI, intermediate required return. Unstable.
- K^H: High-capital, saturated AI, low required return.

The flat region in the MPK schedule enables three crossings. In Region II, the MPK holds steady while the required rental rate continues to fall with K. This allows the curves to cross, separate, and cross again.

Appendix D provides the formal conditions for three steady states and proves that K^L and K^H are saddle-path stable while K^M is unstable.

Although q equals $\bar{q}$ at both stable steady states, total market capitalization $\bar{q}K$ is substantially higher at K^H. The high-capital equilibrium thus features not only a larger capital stock but also greater aggregate wealth.

Having characterized the steady states, I now turn to equilibrium selection and transition dynamics.

4 Speculative Growth and Fragility

Section 3 established that multiple steady states can exist. But can the economy transition from K^L to K^H? Whether such a transition exists depends on the magnitude of adjustment costs, which I discuss next.

4.1 The Role of Adjustment Costs

Multiple steady states are necessary but not sufficient for a speculative-growth transition. As in CFH, the transition requires adjustment costs in an intermediate range. If adjustment costs are very low, q remains close to replacement cost and capital gains are too small to support a valuation-driven boom. If adjustment costs are very high, sustaining rapid accumulation would require valuations so large as to be implausible. I treat intermediate adjustment costs as given here and relegate the formal characterization (in terms of ψ) to Appendix F.

4.2 The Speculative Growth Path

Figure 3 plots the (K, q) phase diagram. The dashed horizontal line is the $\dot{K} = 0$ locus, which lies at the steady-state valuation $\bar{q} = e^{\delta/\psi}$. The non-monotonic orange curve is the $\dot{q} = 0$ locus. The green curve is the stable manifold of the high-capital steady state $(K^H, \bar{q})$; it describes the speculative-growth trajectory.

Capital cannot jump, but asset prices can. Starting from the low-capital steady state $(K^L, \bar{q})$, a speculative-growth episode proceeds as follows:

1. **Expectations shift.** Agents coordinate on optimistic beliefs.
2. **Valuations jump.** q rises discretely from $\bar{q}$ to $q_0 > \bar{q}$.
3. **Investment booms.** The rise in q makes investment profitable and capital starts to accumulate.
4. **AI deploys.** As K crosses K_{AI}, firms deploy AI, raising the capital share and concentrating wealth.
5. **The required return falls.** As capitalists become wealthier, their saving rate rises, lowering the required return.
6. **Convergence.** The economy converges to $(K^H, \bar{q})$: capital reaches its high steady state and valuations eventually return to $\bar{q}$, now consistent with a lower required return.

The phase diagram also makes clear why elevated valuations are integral to the transition. At $(K^L, \bar{q})$ the economy is at rest; to induce capital accumulation one must have $q > \bar{q}$. Moreover, reaching K^H requires staying on the stable manifold, which lies above $\bar{q}$ throughout the transition. *High valuations are therefore not a symptom of irrational exuberance; they are the equilibrium mechanism that makes the transition feasible.*

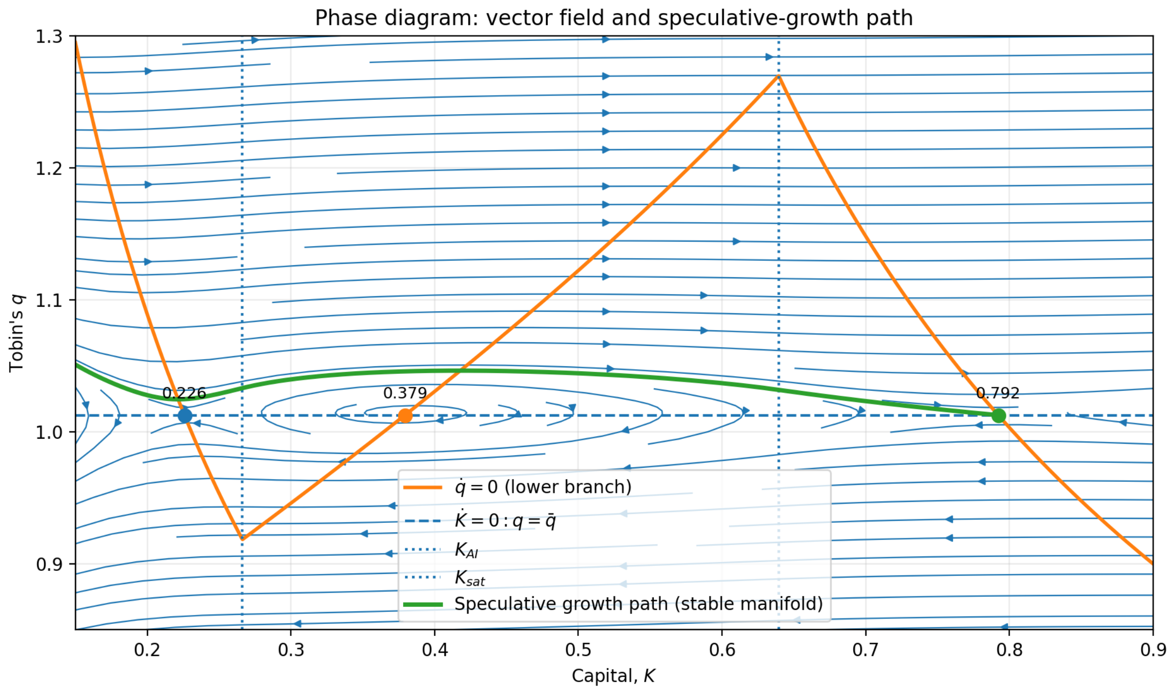

Figure 3: Phase diagram and speculative-growth path. Starting from $(K^L, \bar{q})$, optimistic beliefs trigger an upward jump in q. The economy then converges along the stable manifold (green) to $(K^H, \bar{q})$.

4.3 Time Paths Along the Speculative Growth Trajectory

Figure 4 shows the corresponding time paths along the speculative-growth trajectory. The economy is shown for a short pre-jump interval (so the initial discrete jump is visible) and then follows the post-jump path on the (speculative-growth) stable manifold. The key real-side driver is the investment response to valuations. Since

$$\frac{I_t}{K_t} = \psi \ln q_t,$$

the jump in q_t produces an immediate increase in the investment rate. What happens thereafter depends on the evolution of q_t along the manifold: q_t can continue rising for a time (even as K_t increases) before eventually peaking and mean-reverting toward $\bar{q}$ as the economy approaches the high-capital steady state.

The Required Return. Aggregate wealth $W_t = q_t K_t$ pins down the required return. Using $c_t = \kappa W_t^{\phi}$ and the Euler equation in Appendix C:

$$R(W) = \frac{\rho - \phi\,\kappa W^{\phi-1} - \lambda\,\kappa W^{\phi}}{1-\phi}. \tag{3}$$

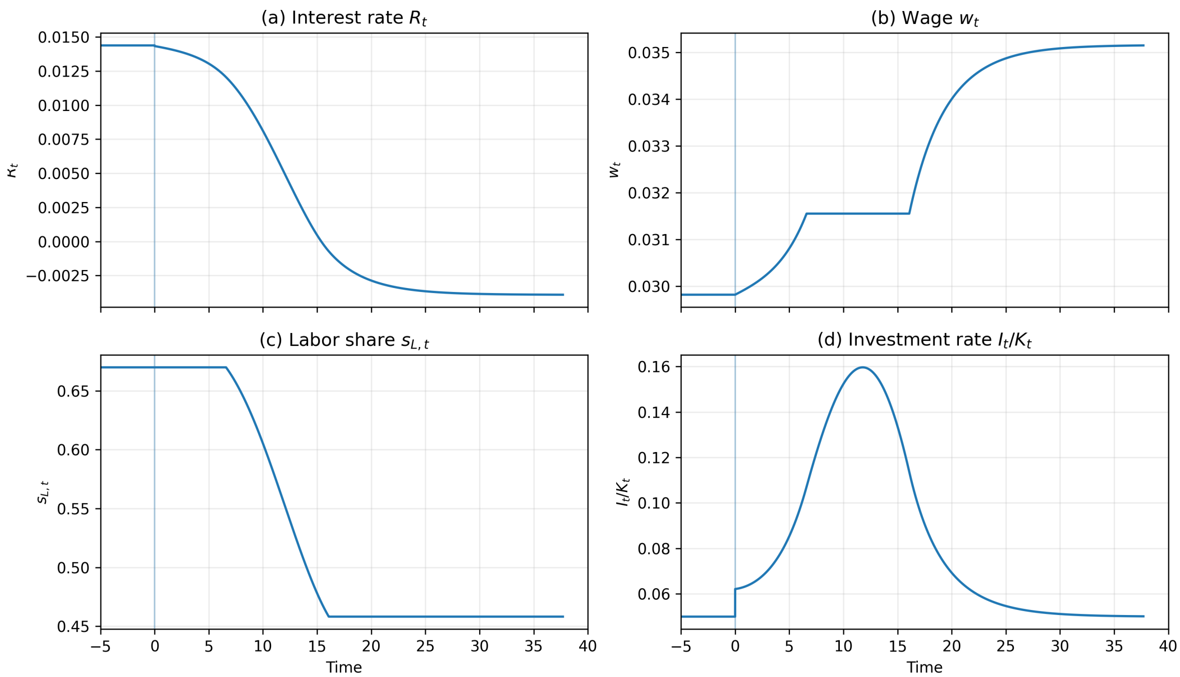

Figure 4: Time paths along the speculative-growth trajectory. The economy starts at the low-capital steady state and jumps onto the speculative-growth manifold. Panel (a): required return R_t; Panel (b): wage w_t; Panel (c): labor share $s_{L,t}$; Panel (d): investment rate I_t/K_t.

Differentiating yields

$$R'(W) = \phi\kappa W^{\phi-2}\left[1 - \frac{\lambda W}{1-\phi}\right]. \tag{4}$$

Near the low-capital steady state, $R(W)$ is locally flat. The $W^{\phi-1}$ term reflects consumption growth—faster consumption growth requires a higher return—while the W^{ϕ} term captures the wealth effect on desired saving. Around W^L these forces nearly offset.

Early in Region I, wealth therefore rises with only a gradual decline in R_t. The jump in q_t at $t = 0$ does raise W_t discretely, but the implied change in R_t is small under the baseline calibration—hence not visually salient in panel (a).

The key change occurs in Region II. There, the flat MPK allows q_t to stay elevated while K_t expands rapidly, accelerating wealth growth. Since $R(W)$ steepens as W rises, this produces a sharper decline in the required return. As the economy approaches K^H and $q_t \to \bar{q}$, wealth stabilizes and R_t converges to its new, lower steady-state level.

Wages and the labor share. Wages w_t and the labor share $s_{L,t}$ depend on the economy's effective labor N_t. As the trajectory enters the AI-deployment region, N_t rises because AI expands effective labor. Output increases while wages initially stagnate, compressing the labor share:

$$s_{L,t} = \frac{w_t L}{Y_t} = \frac{1-\alpha}{N_t}.$$

Once AI saturates (Region III), N_t stabilizes and the labor share converges to a permanently lower level, while wages resume rising with capital deepening.

Investment dynamics. The evolution of I_t/K_t reflects the forward-looking behavior of q_t along the speculative-growth path. The economy first experiences an upward jump in q_t onto the stable manifold. Crucially, q_t begins rising already in Region I: investors anticipate Region II, where the flat MPK will allow valuations to remain elevated without being eroded by falling returns. The resulting increase in q_t strengthens Tobin's q incentives and drives a further rise in the investment rate.

In Region II, the stabilization of the marginal product of capital allows q_t to remain elevated—and often to keep rising for some time despite ongoing capital deepening—thereby sustaining high investment. Eventually, as the economy approaches the high-capital steady state, q_t peaks and mean-reverts toward $\bar{q}$, bringing I_t/K_t down gradually until it converges to its steady-state level.

4.4 Fragility: The Crash

The speculative-growth path is fragile. The same mechanism that enables the boom also enables its reversal.

Consider an economy partway through the transition to the high-capital equilibrium: capital has accumulated to some $K > K^L$ and valuations remain elevated at $q > \bar{q}$. Suppose confidence weakens—due to negative news about AI capabilities, a financial shock, or a shift in sentiment.

If valuations decline—even absent any change in fundamentals—the economy can depart from the speculative-growth path. A sufficiently large decline places the economy on the only alternative equilibrium path—one converging to K^L rather than K^H.

Figure 5 illustrates this scenario. The red segment represents a crash: holding K fixed, q drops discretely to the stable manifold associated with the low-capital steady state. The red path shows the subsequent dynamics: investment collapses, capital decumulates, and the economy returns to K^L.

The crash is self-fulfilling: a downward revision in beliefs lowers valuations today, which reduces investment and reverses capital accumulation. The weaker capital path then validates the pessimistic beliefs.

This analysis clarifies the sense in which AI valuations can be simultaneously "not a bubble" and fragile. They are not a bubble in the traditional sense because the growth and

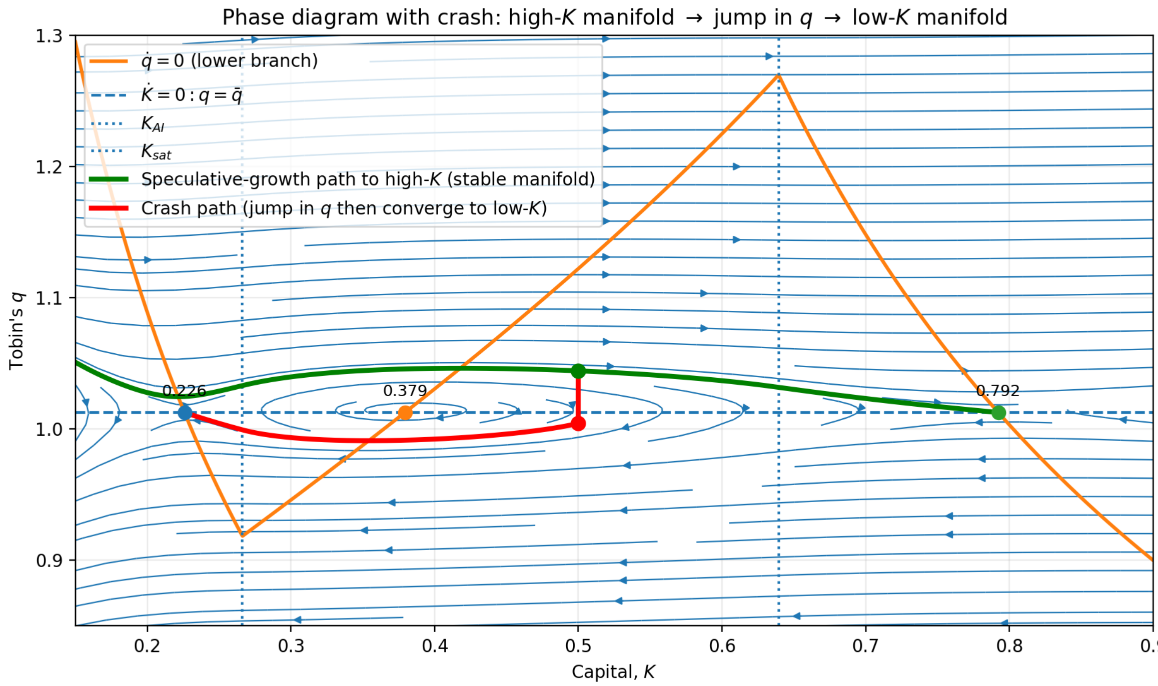

Figure 5: A crash along the speculative-growth path. A drop in valuations places the economy on a trajectory (red) leading back to K^L.

wealth they generate can ultimately validate those valuations. They are fragile because that validation requires sustained confidence throughout a potentially lengthy transition.

5 Conclusion

I have applied the speculative-growth framework of CFH to an AI-driven macroeconomic environment. AI activates CFH's funding feedback through two channels: labor-like capital that dampens diminishing returns, and a distributional shift toward capitalists that raises aggregate saving and lowers the required return.

Several implications emerge:

- **Elevated valuations may be integral to the transition.** The investment boom that validates optimistic expectations requires those expectations to manifest in high asset prices. Labeling AI valuations as merely a bubble overlooks the self-fulfilling nature of the transition.

- **The transition is fragile.** The same multiplicity that enables the boom also enables its reversal. A loss of confidence—whatever its source—can derail the transition and return the economy to the low-capital steady state.

- **The high-capital outcome and elevated valuations are inseparable.** One cannot reach the high-capital equilibrium without traversing a path of elevated asset prices. This interdependence is what makes the current situation both an opportunity and a source of macroeconomic risk.

To be clear, this is a possibility argument. My goal is to isolate a coherent mechanism that could rationalize the joint behavior of valuations and investment, not to provide conclusive evidence. Nor do I mean to imply that the valuations and investment rates we are currently observing are fully consistent with a rational expectations model. Rather, the point is that behavioral narratives aligned with a rational-expectations equilibrium will tend to persist. This happens not because agents understand the underlying mechanism, but because the equilibrium itself sustains beliefs that happen to point in the right direction.

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A Technology Details

A.1 Setup

Total capital $K = K_c + K_\ell$ is divided between conventional capital $K_c \geq 0$ and AI capital $K_\ell \geq 0$. Effective labor is

$$N = 1 + \gamma \min\{K_\ell, \bar{K}_\ell\},$$

where $\gamma > 0$ is the labor equivalence of AI and $\bar{K}_\ell$ is the AI capacity constraint.

Production is Cobb-Douglas: $Y = AK_c^\alpha N^{1-\alpha}$.

A.2 Optimal Allocation

Given K, firms choose K_c and K_ℓ to maximize output. At an interior solution:

$$\frac{\partial Y}{\partial K_c} = \frac{\partial Y}{\partial K_\ell} \quad \Rightarrow \quad \frac{\alpha}{K_c} = \frac{(1-\alpha)\gamma}{N}.$$

Define $b \equiv (1-\alpha)\gamma/\alpha$. The optimality condition becomes $N = bK_c$, yielding:

$$K_c = \frac{\gamma K + 1}{\gamma + b}, \qquad K_\ell = \frac{bK - 1}{\gamma + b}.$$

This interior solution is valid when $K_\ell \in [0, \bar{K}_\ell]$, i.e., when $K \in [K_{\text{AI}}, K_{\text{sat}}]$ where:

$$K_{\text{AI}} = \frac{1}{b} = \frac{\alpha}{(1-\alpha)\gamma}, \qquad K_{\text{sat}} = \frac{1 + (\gamma + b)\bar{K}_\ell}{b}.$$

A.3 The MPK Schedule

The marginal product of capital is:

$$r^K(K) = \begin{cases} \alpha A K^{\alpha-1}, & K < K_{\text{AI}}, \\ \alpha A b^{1-\alpha} \equiv r^K_{\text{flat}}, & K_{\text{AI}} \leq K < K_{\text{sat}}, \\ \alpha A (K - \bar{K}_\ell)^{\alpha-1}(1 + \gamma \bar{K}_\ell)^{1-\alpha}, & K \geq K_{\text{sat}}. \end{cases}$$

In Region II, the MPK is constant because as K rises, K_ℓ rises proportionally, keeping the capital-to-effective-labor ratio $K_c/N = 1/b$ constant.

B Microfoundations for the Consumption Rule

B.1 Preferences

Capitalists maximize:

$$\int_0^\infty e^{-\rho t} \left[\ln c_t + \lambda W_t\right] dt,$$

subject to $\dot{W}_t = R_t W_t - c_t$. The term λW_t captures direct utility from wealth (wealth as a "luxury good").

B.2 Euler Equation

The Hamiltonian is $H = \ln c + \lambda W + \mu(RW - c)$. First-order conditions:

$$\frac{\partial H}{\partial c} = 0 \quad \Rightarrow \quad \frac{1}{c} = \mu,$$

$$\dot{\mu} = \rho\mu - \frac{\partial H}{\partial W} = \rho\mu - \lambda - \mu R.$$

Combining: $\dot{\mu}/\mu = \rho - \lambda c - R$. Since $\mu = 1/c$, we have $\dot{\mu}/\mu = -\dot{c}/c$, yielding:

$$R = \rho + \frac{\dot{c}}{c} - \lambda c.$$

B.3 Steady-State Consumption

At a steady state, $\dot{c} = \dot{W} = 0$, so $R^{ss} = \rho - \lambda c^{ss}$ and $c^{ss} = R^{ss} W$. Solving:

$$c^{ss}(W) = \frac{\rho W}{1 + \lambda W}, \qquad R^{ss}(W) = \frac{\rho}{1 + \lambda W}.$$

B.4 The Isoelastic Approximation

The consumption-wealth elasticity at a steady state is:

$$\phi(W) \equiv \frac{d \ln c^{ss}}{d \ln W} = \frac{1}{1 + \lambda W} \in (0, 1).$$

I approximate the optimal policy with $c = \kappa W^{\phi}$, calibrating ϕ and κ at a reference wealth W^*:

$$\phi = \frac{1}{1 + \lambda W^*}, \qquad \kappa = \rho\phi(W^*)^{1-\phi}.$$

Equivalently, calibrating to match the exact steady-state policy at W^*,

$$\kappa = \frac{c^{ss}(W^*)}{(W^*)^{\phi}} = \frac{\rho W^*/(1 + \lambda W^*)}{(W^*)^{\phi}} = \rho\,\phi\,(W^*)^{1-\phi},$$

where the last equality uses $1 + \lambda W^* = 1/\phi$.

This approximation is exact at W^* and first-order accurate nearby. Taking $W^* = \bar{q}K^L$ (the low steady state) yields an upper bound on ϕ for the transition, since $\phi(W)$ is decreasing.

C Equilibrium Dynamics

C.1 Laws of Motion

Capital accumulates according to:

$$\dot{K} = (\psi \ln q - \delta)K.$$

C.2 Required Return

Using the consumption rule $c = \kappa W^{\phi}$ with $W = qK$:

$$\dot{c} = \phi\kappa W^{\phi-1}\dot{W} = \phi\kappa W^{\phi-1}(RW - c) = \phi\kappa W^{\phi}(R - \kappa W^{\phi-1}).$$

Substituting into the Euler equation $R = \rho + \dot{c}/c - \lambda c$:

$$R = \rho + \phi(R - \kappa W^{\phi-1}) - \lambda\kappa W^{\phi}.$$

Solving for R:

$$R(W) = \frac{\rho - \phi\kappa W^{\phi-1} - \lambda\kappa W^{\phi}}{1-\phi}.$$

At a steady state where $\dot{c} = 0$, this simplifies to $R^{ss}(W) = \rho/(1 + \lambda W)$.

C.3 Asset Pricing

The return on holding capital equals the required return:

$$\frac{\dot{q}}{q} + \frac{r^K(K)}{q} - \delta = R(qK).$$

Rearranging:

$$\dot{q} = [R(qK) + \delta]q - r^K(K).$$

C.4 Phase Diagram Loci

The $\dot{K} = 0$ locus is $q = \bar{q} = e^{\delta/\psi}$ (horizontal).

The $\dot{q} = 0$ locus is $[R(qK) + \delta]q = r^K(K)$, which varies with the three-region MPK structure.

C.5 Output, Wages, and Labor Share

Output in each region:

$$Y = \begin{cases} AK^{\alpha}, & K < K_{\text{AI}}, \\ AK_c^{\alpha}N^{1-\alpha} \text{ with } K_c = \frac{\gamma K+1}{\gamma+b}, N = bK_c, & K_{\text{AI}} \leq K < K_{\text{sat}}, \\ A(K-\bar{K}_\ell)^{\alpha}(1+\gamma\bar{K}_\ell)^{1-\alpha}, & K \geq K_{\text{sat}}. \end{cases}$$

The wage equals the marginal product of human labor:

$$w = (1-\alpha)\frac{Y}{N}.$$

The labor share (human labor's share of output) is:

$$s_L = \frac{wL}{Y} = \frac{1-\alpha}{N},$$

where $L = 1$ is human labor supply. In Region II, N rises with K, so s_L falls.

D Proof of Three Steady States

Define $\Delta(K) \equiv r^K(K) - [R^{ss}(K) + \delta]\bar{q}$. A steady state exists where $\Delta(K) = 0$.

Here $R^{ss}(K)$ is shorthand for the required return $R^{ss}(W)$ evaluated at steady-state wealth $W = \bar{q}K$, i.e., $R^{ss}(K) \equiv R^{ss}(\bar{q}K)$.

Region I: $\Delta(K) \to +\infty$ as $K \to 0$ (since $r^K \to \infty$). If the multiplicity condition holds, $\Delta(K_{\text{AI}}) < 0$. By continuity, $\exists K^L \in (0, K_{\text{AI}})$ with $\Delta(K^L) = 0$.

Region II: r^K is constant while $R^{ss}(K)$ is decreasing, so Δ is increasing. The multiplicity condition implies $\Delta(K_{\text{AI}}) < 0$ and $\Delta(K_{\text{sat}}) > 0$. By continuity, $\exists K^M \in (K_{\text{AI}}, K_{\text{sat}})$ with $\Delta(K^M) = 0$.

Region III: At K_{sat}, $\Delta > 0$. As $K \to \infty$, $r^K \to 0$ while $[R^{ss} + \delta]\bar{q} \to \delta\bar{q} > 0$, so $\Delta < 0$. By continuity, $\exists K^H \in (K_{\text{sat}}, \infty)$ with $\Delta(K^H) = 0$.

Multiplicity condition:

$$[R^{ss}(K_{\text{AI}}) + \delta]\bar{q} > r^K_{\text{flat}} > [R^{ss}(K_{\text{sat}}) + \delta]\bar{q}. \tag{5}$$

E Local Stability

The linearized system around a steady state $(K^*, \bar{q})$ is:

$$\begin{pmatrix} \dot{K} \\ \dot{q} \end{pmatrix} = J \begin{pmatrix} K - K^* \\ q - \bar{q} \end{pmatrix}.$$

The Jacobian elements are:

$$\begin{aligned} J_{11} &= \left.\frac{\partial \dot{K}}{\partial K}\right|_{ss} = 0, \\ J_{12} &= \left.\frac{\partial \dot{K}}{\partial q}\right|_{ss} = \frac{\psi K^*}{\bar{q}} > 0, \\ J_{21} &= \left.\frac{\partial \dot{q}}{\partial K}\right|_{ss} = -(r^K)'(K^*) + \bar{q}^2 R'(W^*), \\ J_{22} &= \left.\frac{\partial \dot{q}}{\partial q}\right|_{ss} = R^{ss} + \delta + \bar{q} R'(W^*) K^*. \end{aligned}$$

Since $R'(W) < 0$ (higher wealth lowers required return), the trace is:

$$\mathrm{tr}(J) = J_{22} = R^{ss} + \delta + \bar{q} R'(W^*) K^* > 0$$

for reasonable parameters.

The determinant is:

$$\det(J) = -J_{12} \cdot J_{21} = -\frac{\psi K^*}{\bar{q}} \left[-(r^K)'(K^*) + \bar{q}^2 R'(W^*) \right].$$

The sign of $\det(J)$ depends on $(r^K)'(K^*)$:

- At K^L and K^H: $(r^K)'(K^*) < 0$ (diminishing returns). Since $R'(W^*) < 0$, the bracketed term is positive provided $-(r^K)'(K^*) > \bar{q}^2 |R'(W^*)|$. Under the baseline calibration this condition holds at K^L and K^H, hence $\det(J) < 0$. These are saddle points.

- At K^M: $(r^K)'(K^M) = 0$ (flat region). The bracketed term is negative, so $\det(J) > 0$. With $\mathrm{tr}(J) > 0$, K^M is an unstable node.

Since K is predetermined and q is a jump variable, saddle-path stability at K^L and K^H means these are locally stable steady states, while K^M is unstable. With one predetermined variable and one jump variable, a saddle point—one stable and one unstable eigenvalue—implies a unique convergent path.

F Intermediate Adjustment Costs and the Speculative-Growth Path

This appendix formalizes the claim in Section 4 that the speculative-growth trajectory (the stable manifold of the high-capital steady state) reaches back to K^L at *elevated but plausible* valuations only for an intermediate range of ψ. Throughout, fix all parameters other than ψ and assume the multiplicity condition of Appendix D holds, so the three steady states $(K^L, \bar{q})$, $(K^M, \bar{q})$, and $(K^H, \bar{q})$ exist.

F.1 Setup and definitions

Consider the dynamical system (Appendix C):

$$\dot{K} = (\psi \ln q - \delta)\, K, \tag{6}$$

$$\dot{q} = F(K, q) \equiv [R(qK) + \delta]\, q - r^K(K). \tag{7}$$

For a given ψ, let W^s_ψ denote the (one-dimensional) stable manifold of the saddle steady state $(K^H(\psi), \bar{q}(\psi))$, where $\bar{q}(\psi) = e^{\delta/\psi}$.

Since K is predetermined and q is a jump variable, a speculative-growth episode starting from K^L is feasible if and only if one can jump from $(K^L, \bar{q})$ to a point on W^s_ψ with $q > \bar{q}$.

Definition 1 (Reach-back at elevated-plausible valuations)**.** *Fix $\eta > 0$ and $\bar{Q} > 1$. We say that W^s_ψ* reaches back to K^L at elevated-plausible valuations *if*

$$W^s_\psi \cap \{(K, q) : K = K^L,\ q \in [(1+\eta)\bar{q}(\psi),\ \bar{Q}]\} \neq \emptyset.$$

F.2 Two limiting lemmas

The first lemma makes precise the "ψ too small" statement: $\bar{q}(\psi)$ itself becomes arbitrarily large.

Lemma 1 (High adjustment costs: ψ too small)**.** *Fix any plausibility cap $\bar{Q} > 1$. If*

$$\psi \leq \psi_{\min}(\bar{Q}) \equiv \frac{\delta}{\ln \bar{Q}},$$

then $\bar{q}(\psi) = e^{\delta/\psi} \geq \bar{Q}$, hence reach-back in the sense of Definition 1 is impossible.

Proof. If $\psi \leq \delta / \ln \bar{Q}$, then $\delta/\psi \geq \ln \bar{Q}$ and therefore $\bar{q}(\psi) = e^{\delta/\psi} \geq e^{\ln \bar{Q}} = \bar{Q}$. Any $q \geq (1+\eta)\bar{q}(\psi)$ then exceeds $\bar{Q}$. □

The second lemma makes precise the "ψ too large" statement using a time-scale argument: for large ψ, if one starts at K^L with q bounded away from $\bar{q}$ by a fixed fraction, then K moves too fast relative to the maximal speed at which q can fall within a bounded valuation region.

Lemma 2 (Low adjustment costs: ψ too large)**.** *Fix $(\eta, \bar{Q})$ with $\eta > 0$ and $\bar{Q} > 1$. Because $F(K, q)$ in (7) is continuous, there exists a finite bound*

$$M(\bar{Q}) \equiv \sup\{|F(K,q)| : (K,q) \in [K^L, K^H] \times [1, \bar{Q}]\} < \infty.$$

(Note that $\bar{q}(\psi) > 1$ for finite ψ, so $[\bar{q}(\psi), \bar{Q}] \subset [1, \bar{Q}]$ and this bound is conservative.) Then there exists $\psi_{\max}(\eta, \bar{Q})$ such that for all $\psi \geq \psi_{\max}(\eta, \bar{Q})$, W^s_ψ cannot intersect the line $K = K^L$ at any $q \in [(1+\eta)\bar{q}(\psi), \bar{Q}]$.

Proof. Fix $(\eta, \bar{Q})$. Consider any ψ and suppose, for contradiction, that there exists a point $(K^L, q_0) \in W^s_\psi$ with $q_0 \in [(1+\eta)\bar{q}(\psi), \bar{Q}]$.

Step 1 (fast capital growth). While $q(t) \geq (1+\eta/2)\bar{q}(\psi)$, capital grows at a uniform exponential rate. Since $\bar{q}(\psi) = e^{\delta/\psi}$,

$$\psi \ln q(t) - \delta \;\geq\; \psi \ln\big((1+\eta/2)\bar{q}(\psi)\big) - \delta = \psi \ln(1+\eta/2).$$

Hence on any interval where $q(t) \geq (1+\eta/2)\bar{q}(\psi)$,

$$\frac{\dot{K}(t)}{K(t)} \geq \psi \ln(1+\eta/2).$$

Let

$$T_\psi \equiv \frac{\ln(K^H/K^L)}{\psi \ln(1+\eta/2)}.$$

If $q(t) \geq (1+\eta/2)\bar{q}(\psi)$ on $[0, T_\psi]$, then $K(T_\psi) \geq K^H$.

Step 2 (bounded speed of q). As long as $q(t) \in [1, \bar{Q}]$ we have $|\dot{q}(t)| \leq M(\bar{Q})$, so

$$q(t) \geq q_0 - M(\bar{Q})\, t \quad \text{for } t \geq 0.$$

Choose ψ large enough so that

$$M(\bar{Q})\, T_\psi \leq \frac{\eta}{2}\bar{q}(\psi).$$

A sufficient condition is

$$\psi \geq \psi_{\max}(\eta, \bar{Q}) \equiv \frac{2M(\bar{Q}) \ln(K^H/K^L)}{\eta \ln(1+\eta/2)}.$$

Then for all $t \in [0, T_\psi]$,

$$q(t) \geq q_0 - M(\bar{Q})T_\psi \geq (1+\eta)\bar{q}(\psi) - \frac{\eta}{2}\bar{q}(\psi) = (1+\eta/2)\bar{q}(\psi).$$

Combining with Step 1 yields $K(T_\psi) \geq K^H$ and $q(T_\psi) > (1+\eta/2)\bar{q}(\psi) > \bar{q}(\psi)$.

Step 3 (overshoot contradiction). For ψ large enough, Steps 1–2 imply that at time T_ψ the trajectory satisfies $K(T_\psi) \geq K^H$ and $q(T_\psi) > \bar{q}(\psi)$. Suppose, toward a contradiction, that this trajectory lies on the stable manifold W^s_ψ.

By standard local dynamics around a saddle, the stable manifold passes through $(K^H, \bar{q})$ with negative slope. In our system,

$$J_{12} = \frac{\partial \dot{K}}{\partial q}\Big|_{(K^H,\bar{q})} = \frac{\psi K^H}{\bar{q}} > 0, \quad \lambda_s < 0,$$

so the slope of the stable eigenvector in (K, q)–space is

$$m_s \equiv \frac{dK}{dq}\Big|_{\text{stable}} = \frac{\lambda_s}{J_{12}} < 0.$$

Thus, in a neighborhood of $(K^H, \bar{q})$, points on W^s_ψ with $K > K^H$ must satisfy $q < \bar{q}$, while points with $K < K^H$ must satisfy $q > \bar{q}$.

However, for large ψ the point $(K(T_\psi), q(T_\psi))$ lies in the region $K \geq K^H$ and $q > \bar{q}$, which is inconsistent with this local characterization of W^s_ψ. Hence $(K(T_\psi), q(T_\psi))$ cannot belong to the stable manifold, contradicting the assumption that $(K^L, q_0) \in W^s_\psi$. □

F.3 Intermediate ψ

Proposition 1 (Intermediate ψ is necessary (and locally sufficient))**.** *Fix $(\eta, \bar{Q})$. Then:*

1. *If $\psi \leq \psi_{\min}(\bar{Q})$, reach-back in the sense of Definition 1 is impossible.*

2. *If $\psi \geq \psi_{\max}(\eta, \bar{Q})$, reach-back in the sense of Definition 1 is impossible.*

Moreover, if for some $\psi^\star \in (\psi_{\min}(\bar{Q}), \psi_{\max}(\eta, \bar{Q}))$ there exists an elevated-plausible intersection of $W^s_{\psi^\star}$ with the line $K = K^L$ that is transverse*, then the intersection (hence reach-back) persists for all ψ in an open neighborhood of $\psi^\star$.*

Proof. The necessity statements are Lemmas 1–2.

For persistence, $(K^H(\psi), \bar{q}(\psi))$ is a hyperbolic saddle over the multiplicity range, so by the Stable Manifold Theorem the local stable manifold depends smoothly on parameters. A transverse intersection with the vertical line $K = K^L$ implies the corresponding defining equation has a nonzero derivative with respect to the local manifold parameter, so the Implicit Function Theorem yields a locally unique intersection point that varies continuously with ψ. □

G Baseline Calibration

G.1 Parameter Values

The figures use:

$$A = 0.0729, \quad \alpha = 0.33, \quad \gamma = 1.85, \quad \bar{K}_\ell = 0.25, \quad \rho = 0.08, \quad \lambda = 20, \quad \delta = 0.05, \quad \psi = 3.0.$$

G.2 Derived Quantities

From the technology block:

$$b = \frac{(1-\alpha)\gamma}{\alpha} = \frac{0.67 \times 1.85}{0.33} \approx 3.76.$$

The boundaries of the flat-MPK region:

$$K_{\text{AI}} = \frac{1}{b} = 0.266, \qquad K_{\text{sat}} = \frac{1 + (\gamma + b)\bar{K}_\ell}{b} = 0.641.$$

The flat-region MPK:

$$r^K_{\text{flat}} = \alpha A b^{1-\alpha} = 0.33 \times 0.0729 \times 3.76^{0.67} \approx 0.058.$$

The steady-state valuation:

$$\bar{q} = e^{\delta/\psi} = e^{0.05/3.0} \approx 1.0168.$$

G.3 Computing Steady States

At a steady state with wealth $W = \bar{q}K$, the required return $R^{ss}(W)$ evaluated at this wealth level is:

$$R^{ss}(\bar{q}K) = \frac{\rho}{1 + \lambda \bar{q} K}.$$

Steady states solve $r^K(K) = [R^{ss}(\bar{q}K) + \delta]\bar{q}$. For the baseline calibration, the three solutions are:

$$K^L = 0.224, \qquad K^M = 0.384, \qquad K^H = 0.790.$$

G.4 Verifying the Multiplicity Condition

The multiplicity condition (5) requires:

$$[R^{ss}(K_{\mathrm{AI}}) + \delta]\bar{q} > r^K_{\mathrm{flat}} > [R^{ss}(K_{\mathrm{sat}}) + \delta]\bar{q}.$$

At the boundaries:

$$[R^{ss}(K_{\mathrm{AI}}) + \delta]\bar{q} = \left[\frac{0.08}{1 + 20 \times 1.0168 \times 0.266} + 0.05\right] \times 1.0168 \approx 0.062,$$
$$[R^{ss}(K_{\mathrm{sat}}) + \delta]\bar{q} = \left[\frac{0.08}{1 + 20 \times 1.0168 \times 0.641} + 0.05\right] \times 1.0168 \approx 0.056.$$

Since $0.062 > 0.058 > 0.056$, the multiplicity condition is satisfied.