Speculative-Growth and the AI "Bubble"

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Abstract

AI technology can generate speculative-growth equilibria. These equilibria are rational but fragile: elevated valuations support rapid capital accumulation, yet persist only as long as beliefs remain coordinated. Because AI capital is labor-like, it expands effective labor and dampens the normal decline in the marginal product of capital as the capital stock grows. The gains from this expansion accrue disproportionately to capitalists, whose saving rate rises with wealth and raises aggregate saving. Building on Caballero et al. (2006), I show that these linked effects generate a funding feedback that can produce multiple equilibria. With intermediate adjustment costs, elevated valuations sustain a transition toward a high-capital equilibrium. Interest rates remain high early in that transition, when resources are reallocated toward the investment boom, and fall sharply only later. At the same time, the mechanism remains fragile: a loss of confidence can precipitate a self-fulfilling crash and reversal.

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1 Introduction

High valuations of AI-exposed firms are often framed in binary terms: either they reflect fundamentals or they reflect a bubble. This paper develops a framework in which that distinction is too rigid. Along one equilibrium path, elevated valuations are consistent with fundamentals and raise Tobin's q, supporting rapid capital accumulation. Yet that same path is fragile, because it depends on coordinated beliefs about future profitability and investment.

The analysis builds on the speculative-growth framework of Caballero et al. (2006, henceforth CFH). In CFH, sustained high valuations require a *funding feedback*: as the economy moves toward the high-capital outcome, the interest rate must eventually decline enough to validate high asset prices. The contribution here is to show why AI technology can generate that feedback in a natural way, and to characterize the transition dynamics that follow. The key primitive is that AI capital is labor-like, and this generates two linked effects:

1. **Production-side effect: a flat marginal product of capital region.** Because AI can substitute for labor across a broad range of tasks (Restrepo, 2025), effective labor expands alongside AI capital, which keeps the capital-to-effective-labor ratio stable and attenuates the usual decline in the marginal product of capital. This flat-MPK region makes multiple steady states more likely.

2. **Distribution-side effect: lower labor share and stronger saving.** The same labor-like feature shifts income away from labor and toward capital as AI is deployed. As wealth concentrates on capitalists, aggregate saving rises with wealth and the interest rate falls (Straub, 2019). This provides the funding feedback that can sustain elevated valuations at high levels of capital.

A third condition is required for the *transition* mechanism to operate, but it is not specific to AI:

3. **Intermediate adjustment costs.** Adjustment costs must lie in an intermediate range: high enough that valuations can depart from replacement cost during the transition, yet low enough to permit rapid accumulation. This condition is not specific to AI, and there is no particular reason to expect AI-related investment frictions to violate it. When it holds, elevated valuations can sustain a transition toward the high-capital equilibrium.

When these forces are present, the economy can feature both a low-capital and a high-capital equilibrium. The distinction is not a permanently higher valuation ratio in the long run—in steady state, valuations return to replacement cost—but a different equilibrium transition. Under coordinated optimism, elevated valuations support rapid accumulation toward the high-capital outcome. Under pessimism, the same economy can reverse course and return to the low-capital equilibrium.

A useful implication of the model concerns the path of interest rates. The mechanism is inherently intertemporal: high valuations raise Tobin's q and support investment during the transition, while the expectation of lower interest rates in the high-capital equilibrium helps validate those valuations ex ante. The model therefore implies a two-stage transition. Early in the boom, resources are reallocated toward investment, so consumption adjusts with a lag; as consumption catches up, its growth remains elevated, which can keep the interest rate high for a time even as valuations rise. Later, as wealth shifts toward high-saving capitalists and aggregate saving increases, the funding feedback strengthens and the interest rate falls sharply. This sequencing is central to the paper's interpretation of speculative growth: high valuations and high investment can coexist with initially tight interest-rate conditions, even though the transition ultimately converges to a low-interest-rate high-capital equilibrium.

Section 2 presents the model. Section 3 characterizes the steady states and conditions for multiplicity. Section 4 studies speculative-growth transitions and fragility. Section 5 concludes. Appendices collect derivations, equilibrium dynamics, and the baseline calibration.

2 Model: AI Technology and the Funding Feedback

This section presents a model built around the labor-like AI primitive described above, and includes adjustment costs, which shape transition dynamics. Derivations and supporting details are in the appendices: Appendix A derives the MPK schedule, Appendix B microfounds the consumption rule, and Appendix C collects the equilibrium dynamics.

2.1 Technology

In a standard neoclassical model, the return to capital falls steadily as capital accumulates. AI technology differs in an important respect. Following the task-based approach synthesized in Restrepo (2025), production involves many discrete tasks. Some tasks are performed by workers, others by machines. Traditional capital can only perform "machine tasks," but AI, like robotization, can also perform worker tasks.

This distinction has implications for diminishing returns. As AI capital accumulates, it does not merely add machines alongside a fixed labor force. Instead, AI operates *as labor*, expanding effective labor (now comprising both humans and AI) that works alongside conventional capital. The result is a region where the MPK is constant. In this "AI deployment" region, each additional unit of AI capital adds effective labor, keeping the effective capital-labor ratio constant. Since the MPK depends on this ratio, diminishing returns are forestalled.

To study the implications of this technology for equilibrium dynamics, I now embed this technology in a continuous-time model. Output is produced with capital and labor:

$$Y = AK_c^{\alpha}N^{1-\alpha},$$

where K_c is conventional capital, N is effective labor, and $\alpha \in (0, 1)$.

Capital can be used in two ways: as conventional capital K_c or as AI capital K_ℓ that substitutes for labor. Total capital is $K = K_c + K_\ell$. AI capital produces "AI labor" at a rate of γ per unit, so effective labor is $N = 1 + \gamma K_\ell$. However, AI deployment faces a capacity constraint $\bar{K}_\ell$—reflecting limits on data, compute, or organizational capacity.

Firms allocate capital optimally between the two uses. As shown in Appendix A, this generates the three-region MPK schedule in Figure 1:

- **Region I** ($K < K_{\text{AI}}$): No AI deployment. Standard diminishing returns $r^K = \alpha AK^{\alpha-1}$.
- **Region II** ($K_{\text{AI}} \leq K < K_{\text{sat}}$): AI deployment phase. The MPK is constant at $r^K = \alpha A \left(\frac{(1-\alpha)\gamma}{\alpha}\right)^{1-\alpha}$.
- **Region III** ($K \geq K_{\text{sat}}$): AI saturated at $K_\ell = \bar{K}_\ell$. Diminishing returns resume $r^K = \alpha A(K - \bar{K}_\ell)^{\alpha-1}(1 + \gamma\bar{K}_\ell)^{1-\alpha}$.

The thresholds K_{AI} and K_{sat} mark, respectively, the onset of AI deployment and the full utilization of AI capacity. Both are derived in Appendix A.

2.2 Households

The introduction described how AI activates the funding feedback. I now formalize this with a two-group household structure: workers and capitalists.

Workers supply labor, earn wages, hold no assets, and consume their entire income: $c_w = w$.

Capitalists own all capital and have non-homothetic preferences. Following Straub (2019), their consumption is given by:

$$c = \kappa W^{\phi}, \qquad \kappa > 0, \quad 0 < \phi < 1, \tag{1}$$

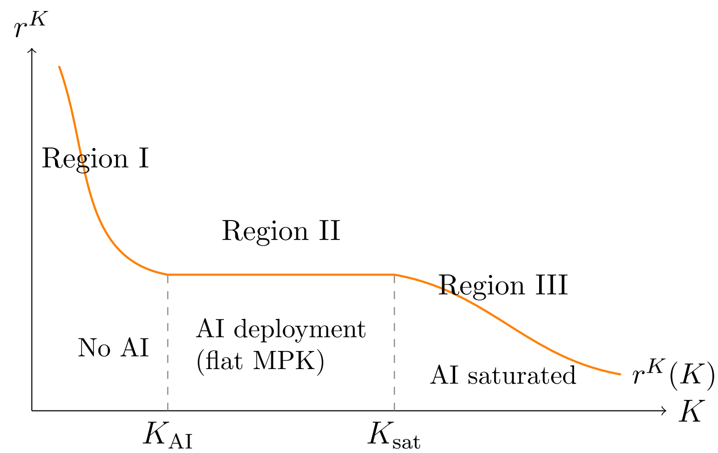

Figure 1: The marginal product of capital with AI technology. In Region II, AI deployment keeps the MPK flat as capital accumulates.

where W is wealth and κ, ϕ are parameters. The key feature is $\phi < 1$: consumption rises less than proportionally with wealth. Equivalently, the saving rate rises with wealth.

This specification is a tractable approximation to optimal behavior under non-homothetic preferences. Appendix B provides microfoundations and shows how to calibrate κ and ϕ to match steady-state behavior exactly.

2.3 Investment and Asset Pricing

Investment faces adjustment costs. Let q denote Tobin's q—the ratio of market value to replacement cost of capital. The investment rate responds to q, so that capital growth is driven by:

$$\frac{\dot{K}}{K} = \psi \ln q - \delta,$$

where $\psi > 0$ governs the responsiveness of investment to valuations and δ is the depreciation rate. When $q > 1$ (market value exceeds replacement cost), gross investment is positive; when $q < 1$, gross investment is negative (scrapping).

Asset pricing requires that the return on holding capital equals the required return:

$$\underbrace{\frac{\dot{q}}{q} - \delta}_{\text{capital gain}} + \underbrace{\frac{r^K(K)}{q}}_{\text{dividend yield}} = R,$$

where R is the required return. Since there is no explicit risk in the model, this required return is the equilibrium interest rate. It depends on capitalist wealth through the saving behavior implied by (1). Across steady states, higher wealth means a lower interest rate—this is the funding feedback.

3 Multiple Steady States

The flat-MPK region and the funding feedback combine to produce multiple steady states. This section characterizes these steady states; the next section asks whether and how the economy can transition between them.

At a steady state, investment exactly covers depreciation ($\dot{K} = 0$), which requires $\psi \ln q = \delta$, hence:

$$\bar{q} = e^{\delta/\psi}.$$

At this valuation, asset market clearing requires that the MPK equals the rental rate implied by the interest rate. Setting $\dot{q} = 0$ gives $r^K(K)/q = R + \delta$, or equivalently $r^K(K) = [R + \delta]q$. At a steady state where $q = \bar{q}$ and $W = \bar{q}K$:

$$r^K(K) = [R(\bar{q}K) + \delta]\bar{q}. \tag{2}$$

The left side is the MPK; the right side is the rental rate implied by the interest rate.

Figure 2 plots both sides against K. The MPK follows the "down-flat-down" pattern from Figure 1. The rental rate schedule is strictly decreasing in K: higher capital means more wealth, which raises saving, lowers the interest rate R, and hence lowers the rental rate.

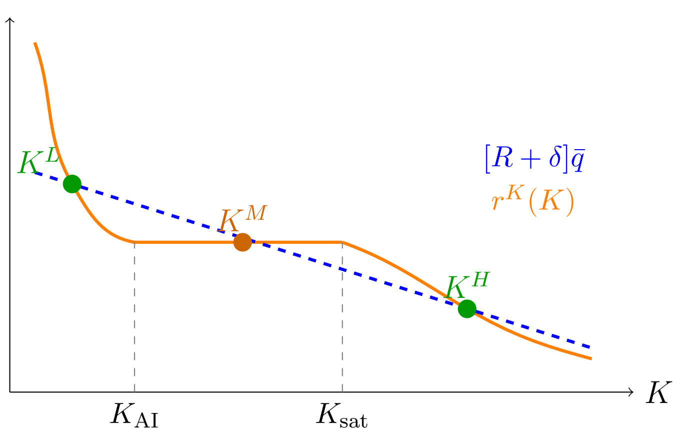

Figure 2: Multiple steady states. The economy can rest at K^L (low-capital, high interest rate) or at K^H (high-capital, low interest rate). The middle intersection K^M is unstable.

The curves cross three times, generating three steady states:

- K^L: Low-capital, no AI, high interest rate.

- K^M: Middle-capital, partial AI, intermediate interest rate. Unstable.
- K^H: High-capital, saturated AI, low interest rate.

The flat region in the MPK schedule enables three crossings. In Region II, the MPK holds steady while the required rental rate continues to fall with K. This allows the curves to cross, separate, and cross again.

Appendix D provides the formal conditions for three steady states and proves that K^L and K^H are saddle-path stable while K^M is unstable.

Although q equals $\bar{q}$ at both stable steady states, total market capitalization $\bar{q}K$ is substantially higher at K^H. The high-capital equilibrium thus features not only a larger capital stock but also greater aggregate wealth.

Having characterized the steady states, I now turn to equilibrium selection and transition dynamics.

4 Speculative Growth and Fragility

Section 3 established that multiple steady states can exist. But can the economy transition from K^L to K^H? Whether such a transition exists depends on the magnitude of adjustment costs, which I discuss next.

4.1 The Role of Adjustment Costs

Multiple steady states are necessary but not sufficient for a speculative-growth transition. As in CFH, the transition requires adjustment costs in an intermediate range. If adjustment costs are very low, q remains close to replacement cost and capital gains are too small to support a valuation-driven boom. If adjustment costs are very high, sustaining rapid accumulation would require valuations so large as to be implausible. I treat intermediate adjustment costs as given here and relegate the formal characterization (in terms of ψ) to Appendix F.

4.2 The Speculative Growth Path

Figure 3 plots the (K, q) phase diagram. The dashed horizontal line is the $\dot{K} = 0$ locus, which lies at the steady-state valuation $\bar{q} = e^{\delta/\psi}$. The non-monotonic orange curve is the $\dot{q} = 0$ locus. The green curve is the stable manifold of the high-capital steady state $(K^H, \bar{q})$; it describes the speculative-growth trajectory.

Capital cannot jump, but asset prices can. Starting from the low-capital steady state $(K^L, \bar{q})$, a speculative-growth episode proceeds as follows:

1. **Expectations shift.** Agents coordinate on optimistic beliefs.

2. **Valuations jump.** q rises discretely from $\bar{q}$ to $q_0 > \bar{q}$.

3. **Investment booms.** The rise in q makes investment profitable and capital starts to accumulate.

4. **AI deploys.** As K crosses K_{AI}, firms deploy AI. Because AI is labor-like, this both sustains the flat-MPK region and raises the capital share, concentrating wealth.

5. **The interest rate falls.** As capitalists become wealthier, their saving rate rises, strengthening the funding feedback and lowering the interest rate.

6. **Convergence.** The economy converges to $(K^H, \bar{q})$: capital reaches its high steady state and valuations eventually return to $\bar{q}$, now consistent with a lower interest rate.

The phase diagram also makes clear why elevated valuations are integral to the transition. At $(K^L, \bar{q})$ the economy is at rest; to induce capital accumulation one must have $q > \bar{q}$. Moreover, reaching K^H requires staying on the stable manifold, which lies above $\bar{q}$ throughout the transition. *High valuations are therefore not a symptom of irrational exuberance; they are the equilibrium mechanism that makes the transition feasible.*

4.3 Time Paths Along the Speculative Growth Trajectory

Figure 4 shows the corresponding time paths along the speculative-growth trajectory. The figure includes a short pre-jump interval (so the initial discrete jump is visible) and then traces the post-jump dynamics on the speculative-growth stable manifold. The key real-side mechanism is the investment response to valuations: the jump in q_t produces an immediate increase in the investment rate. Along the manifold, q_t may continue to rise for a time (even as K_t increases) because the prospect of lower future interest rates feeds back into asset prices. Eventually, once the decline in rates has done most of its work, q_t peaks and mean-reverts toward $\bar{q}$ as the economy approaches the high-capital steady state.

Interest Rate. In this model, there is no explicit risk, so R_t is the equilibrium interest rate relevant for valuation and intertemporal allocation. Aggregate wealth $W_t = q_t K_t$ pins down this rate. Using $c_t = \kappa W_t^{\phi}$ and the Euler equation in Appendix C yields:

$$R(W) = \frac{\rho - \phi\,\kappa W^{\phi-1} - \lambda\,\kappa W^{\phi}}{1-\phi}. \tag{3}$$

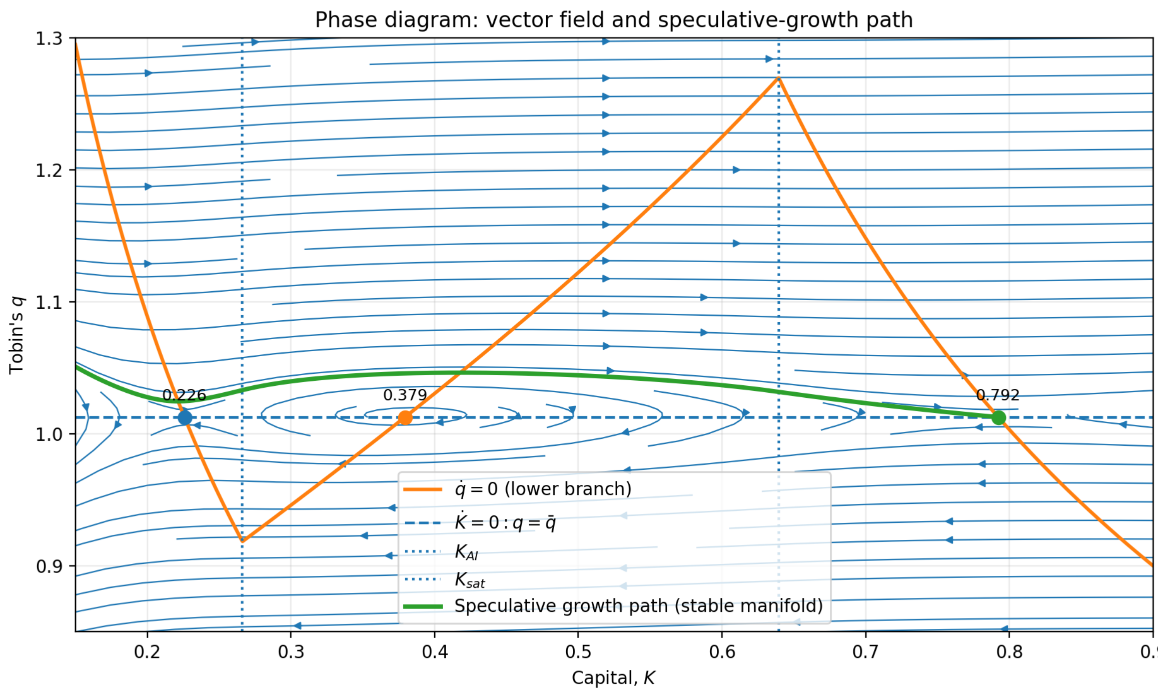

Figure 3: Phase diagram and speculative-growth path. Starting from $(K^L, \bar{q})$, optimistic beliefs trigger an upward jump in q. The economy then converges along the stable manifold (green) to $(K^H, \bar{q})$.

Equation (3) is useful because it makes transparent two forces that operate in opposite directions along the transition. The $W^{\phi-1}$ term captures the consumption-growth component: faster consumption growth raises the interest rate through the Euler equation. The W^{ϕ} term captures the wealth effect on desired saving: as wealth rises, desired saving increases and the interest rate declines. Differentiating yields

$$R'(W) = \phi\kappa W^{\phi-2}\left[1 - \frac{\lambda W}{1-\phi}\right]. \tag{4}$$

Near the low-capital steady state, $R(W)$ is locally flat because these two forces—the consumption-growth component and the wealth effect—nearly offset. This is why the jump in q_t at $t = 0$ (and hence in W_t) generates only a modest immediate change in panel (a) under the baseline calibration.

The same decomposition also clarifies the broader transition dynamics. Early in the speculative-growth episode, the economy is financing an investment boom: elevated valuations raise investment, and resources are reallocated toward capital accumulation. Consumption therefore adjusts only gradually; as it catches up, consumption growth remains elevated, which keeps the $\dot{c}/c$ term in the Euler equation high and can sustain an elevated

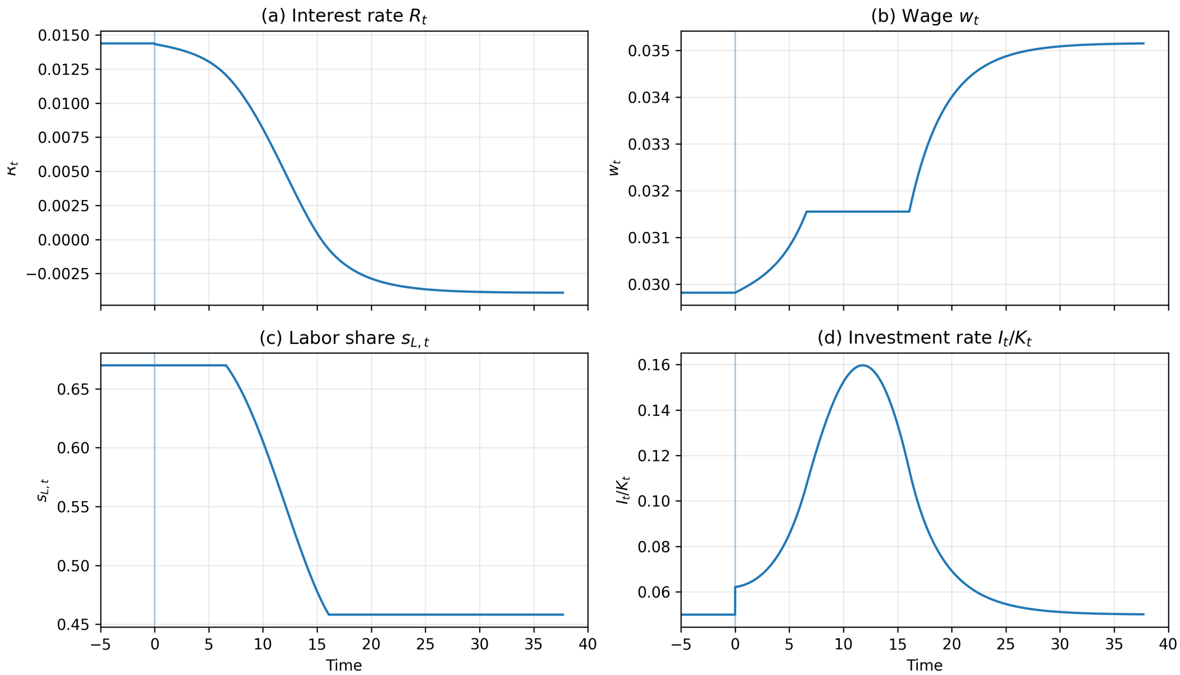

Figure 4: Time paths along the speculative-growth trajectory. The economy starts at the low-capital steady state and jumps onto the speculative-growth manifold. Panel (a): interest rate R_t; Panel (b): wage w_t; Panel (c): labor share $s_{L,t}$; Panel (d): investment rate I_t/K_t.

interest rate even as valuations rise. The wealth effect is already present, but it does not yet dominate the financing needs of the transition.

The key change occurs in Region II. There, the flat MPK allows q_t to remain elevated while K_t expands rapidly, which accelerates the rise in capitalist wealth and strengthens the saving feedback. As wealth rises further, the wealth effect dominates the consumption-growth component and the interest rate falls sharply. As the economy approaches K^H and $q_t \to \bar{q}$, wealth stabilizes and R_t converges to its new, lower steady-state level.

Wages and the labor share. Wages w_t and the labor share $s_{L,t}$ depend on the economy's effective labor N_t. As the trajectory enters the AI-deployment region, N_t rises because AI expands effective labor. Output increases while wages initially stagnate, compressing the labor share:

$$s_{L,t} = \frac{w_t L}{Y_t} = \frac{1-\alpha}{N_t}.$$

Once AI saturates (Region III), N_t stabilizes and the labor share converges to a permanently lower level, while wages resume rising with capital deepening.

Investment dynamics. The path of I_t/K_t follows the evolution of q_t along the speculative-growth manifold. The initial jump in q_t produces an immediate increase in investment. In

Region I, investment continues to rise as valuations build in the expected transition toward the high-capital regime. In Region II, the flat MPK region helps keep valuations elevated despite ongoing capital deepening, so investment remains high. Eventually, as q_t peaks and begins to mean-revert toward $\bar{q}$, the investment rate declines gradually and converges to its steady-state level.

4.4 Fragility: The Crash

The speculative-growth path is fragile. The same mechanism that enables the boom also enables its reversal.

Consider an economy partway through the transition to the high-capital equilibrium: capital has accumulated to some $K > K^L$ and valuations remain elevated at $q > \bar{q}$. Suppose confidence weakens—due to negative news about AI capabilities, a financial shock, or a shift in sentiment.

If valuations decline—even absent any change in fundamentals—the economy can depart from the speculative-growth path. A sufficiently large decline places the economy on the only alternative equilibrium path—one converging to K^L rather than K^H.

Figure 5 illustrates this scenario. The red segment represents a crash: q drops discretely to the stable manifold associated with the low-capital steady state. The red path shows the subsequent dynamics: investment collapses, capital decumulates, and the economy returns to K^L.

The crash is self-fulfilling: a downward revision in beliefs lowers valuations today, which reduces investment and reverses capital accumulation. The weaker capital path then validates the pessimistic beliefs.

This analysis clarifies the sense in which AI valuations can be simultaneously "not a bubble" and fragile. They are not a bubble in the traditional sense because the growth and wealth they generate can ultimately validate those valuations. They are fragile because that validation requires sustained confidence throughout a potentially lengthy transition.

5 Conclusion

This paper develops a model of speculative growth in which elevated asset valuations and rapid capital accumulation can reinforce one another. In the AI context, this mechanism can produce multiple equilibria, including a high-capital equilibrium sustained by optimistic valuations and a fragile transition path vulnerable to self-fulfilling reversals.

The key primitive is that AI capital is labor-like, and this feature generates two linked

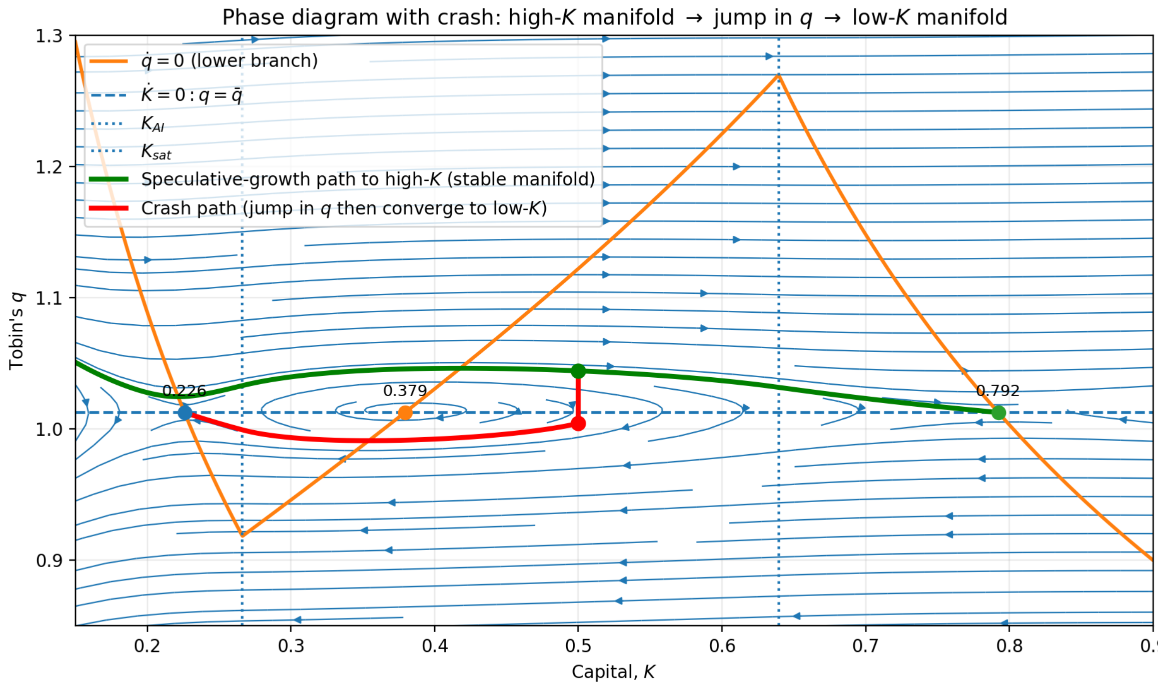

Figure 5: A crash along the speculative-growth path. A drop in valuations places the economy on a trajectory (red) leading back to K^L.

effects. On the production side, it attenuates the usual decline in the marginal product of capital as the capital stock expands, supporting high valuations. On the distribution side, it reduces the labor share and shifts income and wealth toward capitalists, whose saving rises with wealth. The resulting increase in aggregate saving lowers the interest rate in the high-capital equilibrium, further supporting valuations.

The model also implies a distinctive transition path for interest rates. Elevated valuations raise Tobin's q and stimulate investment during the transition, even though the lower interest rates that help support those valuations materialize only later. Early in the boom, resources are reallocated toward investment, so consumption adjusts with a lag; as consumption catches up, its growth remains elevated, which can keep the interest rate high for a time even as valuations rise. Interest rates fall sharply only later, as wealth accumulation and higher saving dominate near the high-capital equilibrium.

These results suggest that AI-driven valuation booms may be both more persistent and more fragile than standard narratives imply. They can be sustained by forward-looking beliefs about future productivity and future low interest rates, yet they remain vulnerable to shifts in confidence that can trigger self-fulfilling crashes and reversals.

To be clear, this is a possibility argument. My goal is to isolate a coherent mechanism that could rationalize the joint behavior of valuations and investment, not to provide conclusive evidence. Nor do I mean to imply that the valuations and investment rates we are currently observing are fully consistent with a rational expectations model. Rather, the point is that behavioral narratives that align with a rational-expectations equilibrium will tend to persist. This happens not because agents understand the underlying mechanism, but because the equilibrium itself sustains beliefs that happen to point in the right direction.

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A Technology Details

A.1 Setup

Total capital $K = K_c + K_\ell$ is divided between conventional capital $K_c \geq 0$ and AI capital $K_\ell \geq 0$. Effective labor is

$$N = 1 + \gamma \min\{K_\ell, \bar{K}_\ell\},$$

where $\gamma > 0$ is the labor equivalence of AI and $\bar{K}_\ell$ is the AI capacity constraint.

Production is Cobb-Douglas: $Y = AK_c^\alpha N^{1-\alpha}$.

A.2 Optimal Allocation

Given K, firms choose K_c and K_ℓ to maximize output. At an interior solution:

$$\frac{\partial Y}{\partial K_c} = \frac{\partial Y}{\partial K_\ell} \quad \Rightarrow \quad \frac{\alpha}{K_c} = \frac{(1-\alpha)\gamma}{N}.$$

Define $b \equiv (1-\alpha)\gamma/\alpha$. The optimality condition becomes $N = bK_c$, yielding:

$$K_c = \frac{\gamma K + 1}{\gamma + b}, \qquad K_\ell = \frac{bK - 1}{\gamma + b}.$$

This interior solution is valid when $K_\ell \in [0, \bar{K}_\ell]$, i.e., when $K \in [K_{\text{AI}}, K_{\text{sat}}]$ where:

$$K_{\text{AI}} = \frac{1}{b} = \frac{\alpha}{(1-\alpha)\gamma}, \qquad K_{\text{sat}} = \frac{1 + (\gamma + b)\bar{K}_\ell}{b}.$$

A.3 The MPK Schedule

The marginal product of capital is:

$$r^K(K) = \begin{cases} \alpha A K^{\alpha-1}, & K < K_{\text{AI}}, \\ \alpha A b^{1-\alpha} \equiv r^K_{\text{flat}}, & K_{\text{AI}} \leq K < K_{\text{sat}}, \\ \alpha A (K - \bar{K}_\ell)^{\alpha-1}(1 + \gamma \bar{K}_\ell)^{1-\alpha}, & K \geq K_{\text{sat}}. \end{cases}$$

In Region II, the MPK is constant because as K rises, K_ℓ rises proportionally, keeping the capital-to-effective-labor ratio $K_c/N = 1/b$ constant.

B Microfoundations for the Consumption Rule

B.1 Preferences

Capitalists maximize:

$$\int_0^\infty e^{-\rho t} \left[\ln c_t + \lambda W_t \right] dt,$$

subject to $\dot{W}_t = R_t W_t - c_t$. The term λW_t captures direct utility from wealth (wealth as a "luxury good").

B.2 Euler Equation

The Hamiltonian is $H = \ln c + \lambda W + \mu(RW - c)$. First-order conditions:

$$\frac{\partial H}{\partial c} = 0 \quad \Rightarrow \quad \frac{1}{c} = \mu,$$

$$\dot{\mu} = \rho\mu - \frac{\partial H}{\partial W} = \rho\mu - \lambda - \mu R.$$

Combining: $\dot{\mu}/\mu = \rho - \lambda c - R$. Since $\mu = 1/c$, we have $\dot{\mu}/\mu = -\dot{c}/c$, yielding:

$$R = \rho + \frac{\dot{c}}{c} - \lambda c.$$

B.3 Steady-State Consumption

At a steady state, $\dot{c} = \dot{W} = 0$, so $R^{ss} = \rho - \lambda c^{ss}$ and $c^{ss} = R^{ss}W$. Solving:

$$c^{ss}(W) = \frac{\rho W}{1 + \lambda W}, \qquad R^{ss}(W) = \frac{\rho}{1 + \lambda W}.$$

B.4 The Isoelastic Approximation

The consumption-wealth elasticity at a steady state is:

$$\phi(W) \equiv \frac{d \ln c^{ss}}{d \ln W} = \frac{1}{1 + \lambda W} \in (0, 1).$$

I approximate the optimal policy with $c = \kappa W^{\phi}$, calibrating ϕ and κ at a reference wealth W^*:

$$\phi = \frac{1}{1 + \lambda W^*}, \qquad \kappa = \rho\phi(W^*)^{1-\phi}.$$

Equivalently, calibrating to match the exact steady-state policy at W^*,

$$\kappa = \frac{c^{ss}(W^*)}{(W^*)^{\phi}} = \frac{\rho W^*/(1 + \lambda W^*)}{(W^*)^{\phi}} = \rho\,\phi\,(W^*)^{1-\phi},$$

where the last equality uses $1 + \lambda W^* = 1/\phi$.

This approximation is exact at W^* and first-order accurate nearby. Taking $W^* = \bar{q}K^L$ (the low steady state) yields an upper bound on ϕ for the transition, since $\phi(W)$ is decreasing.

C Equilibrium Dynamics

C.1 Laws of Motion

Capital accumulates according to:

$$\dot{K} = (\psi \ln q - \delta)K.$$

C.2 Required Return

Using the consumption rule $c = \kappa W^{\phi}$ with $W = qK$:

$$\dot{c} = \phi\kappa W^{\phi-1}\dot{W} = \phi\kappa W^{\phi-1}(RW - c) = \phi\kappa W^{\phi}(R - \kappa W^{\phi-1}).$$

Substituting into the Euler equation $R = \rho + \dot{c}/c - \lambda c$:

$$R = \rho + \phi(R - \kappa W^{\phi-1}) - \lambda\kappa W^{\phi}.$$

Solving for R:

$$R(W) = \frac{\rho - \phi\kappa W^{\phi-1} - \lambda\kappa W^{\phi}}{1-\phi}.$$

At a steady state where $\dot{c} = 0$, this simplifies to $R^{ss}(W) = \rho/(1 + \lambda W)$.

C.3 Asset Pricing

The return on holding capital equals the required return:

$$\frac{\dot{q}}{q} + \frac{r^K(K)}{q} - \delta = R(qK).$$

Rearranging:

$$\dot{q} = [R(qK) + \delta]q - r^K(K).$$

C.4 Phase Diagram Loci

The $\dot{K} = 0$ locus is $q = \bar{q} = e^{\delta/\psi}$ (horizontal).

The $\dot{q} = 0$ locus is $[R(qK) + \delta]q = r^K(K)$, which varies with the three-region MPK structure.

C.5 Output, Wages, and Labor Share

Output in each region:

$$Y = \begin{cases} AK^{\alpha}, & K < K_{\text{AI}}, \\ AK_c^{\alpha}N^{1-\alpha} \text{ with } K_c = \frac{\gamma K+1}{\gamma+b}, N = bK_c, & K_{\text{AI}} \le K < K_{\text{sat}}, \\ A(K - \bar{K}_{\ell})^{\alpha}(1 + \gamma\bar{K}_{\ell})^{1-\alpha}, & K \ge K_{\text{sat}}. \end{cases}$$

The wage equals the marginal product of human labor:

$$w = (1-\alpha)\frac{Y}{N}.$$

The labor share (human labor's share of output) is:

$$s_L = \frac{wL}{Y} = \frac{1-\alpha}{N},$$

where $L = 1$ is human labor supply. In Region II, N rises with K, so s_L falls.

D Proof of Three Steady States

Define $\Delta(K) \equiv r^K(K) - [R^{ss}(K) + \delta]\bar{q}$. A steady state exists where $\Delta(K) = 0$.

Here $R^{ss}(K)$ is shorthand for the required return $R^{ss}(W)$ evaluated at steady-state wealth $W = \bar{q}K$, i.e., $R^{ss}(K) \equiv R^{ss}(\bar{q}K)$.

Region I: $\Delta(K) \to +\infty$ as $K \to 0$ (since $r^K \to \infty$). If the multiplicity condition holds, $\Delta(K_{\text{AI}}) < 0$. By continuity, $\exists K^L \in (0, K_{\text{AI}})$ with $\Delta(K^L) = 0$.

Region II: r^K is constant while $R^{ss}(K)$ is decreasing, so Δ is increasing. The multiplicity condition implies $\Delta(K_{\text{AI}}) < 0$ and $\Delta(K_{\text{sat}}) > 0$. By continuity, $\exists K^M \in (K_{\text{AI}}, K_{\text{sat}})$ with $\Delta(K^M) = 0$.

Region III: At K_{sat}, $\Delta > 0$. As $K \to \infty$, $r^K \to 0$ while $[R^{ss} + \delta]\bar{q} \to \delta\bar{q} > 0$, so $\Delta < 0$. By continuity, $\exists K^H \in (K_{\text{sat}}, \infty)$ with $\Delta(K^H) = 0$.

Multiplicity condition:

$$[R^{ss}(K_{\text{AI}}) + \delta]\bar{q} > r^K_{\text{flat}} > [R^{ss}(K_{\text{sat}}) + \delta]\bar{q}. \tag{5}$$

E Local Stability

The linearized system around a steady state $(K^*, \bar{q})$ is:

$$\begin{pmatrix} \dot{K} \\ \dot{q} \end{pmatrix} = J \begin{pmatrix} K - K^* \\ q - \bar{q} \end{pmatrix}.$$

The Jacobian elements are:

$$J_{11} = \left.\frac{\partial \dot{K}}{\partial K}\right|_{ss} = 0,$$
$$J_{12} = \left.\frac{\partial \dot{K}}{\partial q}\right|_{ss} = \frac{\psi K^*}{\bar{q}} > 0,$$
$$J_{21} = \left.\frac{\partial \dot{q}}{\partial K}\right|_{ss} = -(r^K)'(K^*) + \bar{q}^2 R'(W^*),$$
$$J_{22} = \left.\frac{\partial \dot{q}}{\partial q}\right|_{ss} = R^{ss} + \delta + \bar{q}R'(W^*)K^*.$$

Since $R'(W) < 0$ (higher wealth lowers required return), the trace is:

$$\text{tr}(J) = J_{22} = R^{ss} + \delta + \bar{q}R'(W^*)K^* > 0$$

for reasonable parameters.

The determinant is:

$$\det(J) = -J_{12} \cdot J_{21} = -\frac{\psi K^*}{\bar{q}} \left[-(r^K)'(K^*) + \bar{q}^2 R'(W^*)\right].$$

The sign of $\det(J)$ depends on $(r^K)'(K^*)$:

- At K^L and K^H: $(r^K)'(K^*) < 0$ (diminishing returns). Since $R'(W^*) < 0$, the bracketed term is positive provided $-(r^K)'(K^*) > \bar{q}^2|R'(W^*)|$. Under the baseline calibration this condition holds at K^L and K^H, hence $\det(J) < 0$. These are saddle points.

- At K^M: $(r^K)'(K^M) = 0$ (flat region). The bracketed term is negative, so $\det(J) > 0$. With $\text{tr}(J) > 0$, K^M is an unstable node.

Since K is predetermined and q is a jump variable, saddle-path stability at K^L and K^H means these are locally stable steady states, while K^M is unstable. With one predetermined variable and one jump variable, a saddle point—one stable and one unstable eigenvalue—implies a unique convergent path.

F Intermediate Adjustment Costs and the Speculative-Growth Path

This appendix formalizes the claim in Section 4 that the speculative-growth trajectory (the stable manifold of the high-capital steady state) reaches back to K^L at *elevated but plausible* valuations only for an intermediate range of ψ. Throughout, fix all parameters other than ψ and assume the multiplicity condition of Appendix D holds, so the three steady states $(K^L, \bar{q})$, $(K^M, \bar{q})$, and $(K^H, \bar{q})$ exist.

F.1 Setup and definitions

Consider the dynamical system (Appendix C):

$$\dot{K} = (\psi \ln q - \delta)\, K, \tag{6}$$

$$\dot{q} = F(K, q) \equiv [R(qK) + \delta]\, q - r^K(K). \tag{7}$$

For a given ψ, let W^s_ψ denote the (one-dimensional) stable manifold of the saddle steady state $(K^H(\psi), \bar{q}(\psi))$, where $\bar{q}(\psi) = e^{\delta/\psi}$.

Since K is predetermined and q is a jump variable, a speculative-growth episode starting from K^L is feasible if and only if one can jump from $(K^L, \bar{q})$ to a point on W^s_ψ with $q > \bar{q}$.

Definition 1 (Reach-back at elevated-plausible valuations)**.** *Fix $\eta > 0$ and $\bar{Q} > 1$. We say that W^s_ψ* reaches back to K^L at elevated-plausible valuations *if*

$$W^s_\psi \cap \{(K, q) : K = K^L,\ q \in [(1+\eta)\bar{q}(\psi),\ \bar{Q}]\} \neq \emptyset.$$

F.2 Two limiting lemmas

The first lemma makes precise the "ψ too small" statement: $\bar{q}(\psi)$ itself becomes arbitrarily large.

Lemma 1 (High adjustment costs: ψ too small)**.** *Fix any plausibility cap $\bar{Q} > 1$. If*

$$\psi \leq \psi_{\min}(\bar{Q}) \equiv \frac{\delta}{\ln \bar{Q}},$$

then $\bar{q}(\psi) = e^{\delta/\psi} \geq \bar{Q}$, hence reach-back in the sense of Definition 1 is impossible.

Proof. If $\psi \leq \delta / \ln \bar{Q}$, then $\delta/\psi \geq \ln \bar{Q}$ and therefore $\bar{q}(\psi) = e^{\delta/\psi} \geq e^{\ln \bar{Q}} = \bar{Q}$. Any $q \geq (1+\eta)\bar{q}(\psi)$ then exceeds $\bar{Q}$. □

The second lemma makes precise the "ψ too large" statement using a time-scale argument: for large ψ, if one starts at K^L with q bounded away from $\bar{q}$ by a fixed fraction, then K moves too fast relative to the maximal speed at which q can fall within a bounded valuation region.

Lemma 2 (Low adjustment costs: ψ too large)**.** *Fix $(\eta, \bar{Q})$ with $\eta > 0$ and $\bar{Q} > 1$. Because $F(K, q)$ in (7) is continuous, there exists a finite bound*

$$M(\bar{Q}) \equiv \sup\{|F(K, q)| : (K, q) \in [K^L, K^H] \times [1, \bar{Q}]\} < \infty.$$

(Note that $\bar{q}(\psi) > 1$ for finite ψ, so $[\bar{q}(\psi), \bar{Q}] \subset [1, \bar{Q}]$ and this bound is conservative.) Then there exists $\psi_{\max}(\eta, \bar{Q})$ such that for all $\psi \geq \psi_{\max}(\eta, \bar{Q})$, W^s_ψ cannot intersect the line $K = K^L$ at any $q \in [(1 + \eta)\bar{q}(\psi), \bar{Q}]$.

Proof. Fix $(\eta, \bar{Q})$. Consider any ψ and suppose, for contradiction, that there exists a point $(K^L, q_0) \in W^s_\psi$ with $q_0 \in [(1 + \eta)\bar{q}(\psi), \bar{Q}]$.

Step 1 (fast capital growth). While $q(t) \geq (1 + \eta/2)\bar{q}(\psi)$, capital grows at a uniform exponential rate. Since $\bar{q}(\psi) = e^{\delta/\psi}$,

$$\psi \ln q(t) - \delta \;\geq\; \psi \ln\big((1 + \eta/2)\bar{q}(\psi)\big) - \delta = \psi \ln(1 + \eta/2).$$

Hence on any interval where $q(t) \geq (1 + \eta/2)\bar{q}(\psi)$,

$$\frac{\dot{K}(t)}{K(t)} \geq \psi \ln(1 + \eta/2).$$

Let

$$T_\psi \equiv \frac{\ln(K^H/K^L)}{\psi \ln(1 + \eta/2)}.$$

If $q(t) \geq (1 + \eta/2)\bar{q}(\psi)$ on $[0, T_\psi]$, then $K(T_\psi) \geq K^H$.

Step 2 (bounded speed of q). As long as $q(t) \in [1, \bar{Q}]$ we have $|\dot{q}(t)| \leq M(\bar{Q})$, so

$$q(t) \geq q_0 - M(\bar{Q})\, t \quad \text{for } t \geq 0.$$

Choose ψ large enough so that

$$M(\bar{Q})\, T_\psi \leq \frac{\eta}{2}\bar{q}(\psi).$$

A sufficient condition is

$$\psi \geq \psi_{\max}(\eta, \bar{Q}) \equiv \frac{2M(\bar{Q}) \ln(K^H/K^L)}{\eta \ln(1 + \eta/2)}.$$

Then for all $t \in [0, T_\psi]$,

$$q(t) \geq q_0 - M(\bar{Q})T_\psi \geq (1 + \eta)\bar{q}(\psi) - \frac{\eta}{2}\bar{q}(\psi) = (1 + \eta/2)\bar{q}(\psi).$$

Combining with Step 1 yields $K(T_\psi) \geq K^H$ and $q(T_\psi) > (1+\eta/2)\bar{q}(\psi) > \bar{q}(\psi)$.

Step 3 (overshoot contradiction). For ψ large enough, Steps 1–2 imply that at time T_ψ the trajectory satisfies $K(T_\psi) \geq K^H$ and $q(T_\psi) > \bar{q}(\psi)$. Suppose, toward a contradiction, that this trajectory lies on the stable manifold W^s_ψ.

By standard local dynamics around a saddle, the stable manifold passes through $(K^H, \bar{q})$ with negative slope. In our system,

$$J_{12} = \frac{\partial \dot{K}}{\partial q}\Big|_{(K^H,\bar{q})} = \frac{\psi K^H}{\bar{q}} > 0, \quad \lambda_s < 0,$$

so the slope of the stable eigenvector in (K,q)–space is

$$m_s \equiv \frac{dK}{dq}\Big|_{\text{stable}} = \frac{\lambda_s}{J_{12}} < 0.$$

Thus, in a neighborhood of $(K^H, \bar{q})$, points on W^s_ψ with $K > K^H$ must satisfy $q < \bar{q}$, while points with $K < K^H$ must satisfy $q > \bar{q}$.

However, for large ψ the point $(K(T_\psi), q(T_\psi))$ lies in the region $K \geq K^H$ and $q > \bar{q}$, which is inconsistent with this local characterization of W^s_ψ. Hence $(K(T_\psi), q(T_\psi))$ cannot belong to the stable manifold, contradicting the assumption that $(K^L, q_0) \in W^s_\psi$. □

F.3 Intermediate ψ

Proposition 1 (Intermediate ψ is necessary (and locally sufficient))**.** *Fix $(\eta, \bar{Q})$. Then:*

1. *If $\psi \leq \psi_{\min}(\bar{Q})$, reach-back in the sense of Definition 1 is impossible.*

2. *If $\psi \geq \psi_{\max}(\eta, \bar{Q})$, reach-back in the sense of Definition 1 is impossible.*

Moreover, if for some $\psi^\star \in (\psi_{\min}(\bar{Q}), \psi_{\max}(\eta, \bar{Q}))$ there exists an elevated-plausible intersection of $W^s_{\psi^\star}$ with the line $K = K^L$ that is transverse*, then the intersection (hence reach-back) persists for all ψ in an open neighborhood of $\psi^\star$.*

Proof. The necessity statements are Lemmas 1–2.

For persistence, $(K^H(\psi), \bar{q}(\psi))$ is a hyperbolic saddle over the multiplicity range, so by the Stable Manifold Theorem the local stable manifold depends smoothly on parameters. A transverse intersection with the vertical line $K = K^L$ implies the corresponding defining equation has a nonzero derivative with respect to the local manifold parameter, so the Implicit Function Theorem yields a locally unique intersection point that varies continuously with ψ. □

G Baseline Calibration

G.1 Parameter Values

The figures use:

$$A = 0.0729, \quad \alpha = 0.33, \quad \gamma = 1.85, \quad \bar{K}_\ell = 0.25, \quad \rho = 0.08, \quad \lambda = 20, \quad \delta = 0.05, \quad \psi = 3.0.$$

G.2 Derived Quantities

From the technology block:

$$b = \frac{(1-\alpha)\gamma}{\alpha} = \frac{0.67 \times 1.85}{0.33} \approx 3.76.$$

The boundaries of the flat-MPK region:

$$K_{\text{AI}} = \frac{1}{b} = 0.266, \qquad K_{\text{sat}} = \frac{1 + (\gamma + b)\bar{K}_\ell}{b} = 0.641.$$

The flat-region MPK:

$$r^K_{\text{flat}} = \alpha A b^{1-\alpha} = 0.33 \times 0.0729 \times 3.76^{0.67} \approx 0.058.$$

The steady-state valuation:

$$\bar{q} = e^{\delta/\psi} = e^{0.05/3.0} \approx 1.0168.$$

G.3 Computing Steady States

At a steady state with wealth $W = \bar{q}K$, the required return $R^{ss}(W)$ evaluated at this wealth level is:

$$R^{ss}(\bar{q}K) = \frac{\rho}{1 + \lambda \bar{q} K}.$$

Steady states solve $r^K(K) = [R^{ss}(\bar{q}K) + \delta]\bar{q}$. For the baseline calibration, the three solutions are:

$$K^L = 0.224, \qquad K^M = 0.384, \qquad K^H = 0.790.$$

G.4 Verifying the Multiplicity Condition

The multiplicity condition (5) requires:

$$[R^{ss}(K_{\text{AI}}) + \delta]\bar{q} > r^K_{\text{flat}} > [R^{ss}(K_{\text{sat}}) + \delta]\bar{q}.$$

At the boundaries:

$$[R^{ss}(K_{\text{AI}}) + \delta]\bar{q} = \left[\frac{0.08}{1 + 20 \times 1.0168 \times 0.266} + 0.05\right] \times 1.0168 \approx 0.062,$$
$$[R^{ss}(K_{\text{sat}}) + \delta]\bar{q} = \left[\frac{0.08}{1 + 20 \times 1.0168 \times 0.641} + 0.05\right] \times 1.0168 \approx 0.056.$$

Since $0.062 > 0.058 > 0.056$, the multiplicity condition is satisfied.