Speculative Growth and the AI "Bubble"

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Abstract

I develop a model in which temporary optimism about AI returns can induce investment that permanently raises productive capacity, even after a correction. AI capital supplies labor services, allowing firms to expand conventional capital and effective labor together while keeping the marginal product of capital constant during deployment. Income from AI services accrues to investors, whose propensity to save rises with wealth. Sufficient AI deployment capacity generates low- and high-capital steady states separated by a threshold. There is no rational equilibrium path from the low-capital steady state to the high-capital steady state. However, strong and persistent optimism can induce enough investment for the economy to converge to the high-capital steady state after a full correction. Later corrections can be larger without reversing accumulation. Investment demand puts upward pressure on the interest rate during the expansion. At the high-capital steady state, investors' greater wealth lowers the interest rate, wages are higher, and the labor share is lower. Workers' consumption rises even when the expansion is temporary. Under sufficient conditions, investors who finance a permanent expansion have lower ex post welfare despite higher wealth.

JEL Codes: E21, E22, E24, E44, O33, O41.

Keywords: AI, speculative growth, optimism, investment transition, multiple steady states, interest rates, labor share, wealth-in-utility, saving glut.

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1 Introduction

> "While some part of the investment which was going on in the world at large was doubtless ill-judged and unfruitful, there can, I think, be no doubt that the world was enormously enriched by the constructions of the quinquennium from 1925 to 1929; its wealth increased in these five years by as much as in any other ten or twenty years of its history." —Keynes (1931)

Temporary optimism about AI returns can induce an investment expansion that permanently raises productive capacity, even when asset prices subsequently fall. High valuations of AI-related firms are commonly assessed by asking whether they reflect fundamentals or a bubble. Their economic consequences also depend on the investment they induce: the capital installed while valuations are high changes production and the income available to investors after a correction. I study an economy in which investment changes both productive capacity and investors' saving, so the expansion itself affects how much capital can be sustained after a correction.

AI capital performs tasks otherwise performed by workers. Firms allocate capital between conventional production and the provision of effective labor services. At low capital stocks, they allocate all capital to conventional production, where its marginal product is higher; AI deployment begins when capital accumulation makes the marginal products in the two uses equal. During deployment, firms expand conventional and AI capital together, keeping conventional capital per unit of effective labor constant. The marginal product of capital therefore remains constant over this range of accumulation. Before deployment begins, effective labor consists of the fixed supply of workers. After deployment capacity is exhausted, the AI contribution to effective labor is fixed as well. Further accumulation therefore raises conventional capital per unit of effective labor and lowers its marginal product. AI deployment capacity determines how much capital can accumulate without diminishing returns.

Income generated by AI services accrues to investors, whose propensity to save rises with wealth. During AI deployment, the marginal product of capital remains constant, while greater wealth lowers the steady-state interest rate. With sufficient AI deployment capacity, these production and saving responses generate low- and high-capital steady states separated by a middle steady state. At the low-capital steady state, AI deployment has not begun. At the high-capital steady state, deployment is complete, investors' wealth is greater, and the interest rate is lower.

There is no rational equilibrium path from the low-capital steady state to the high-capital steady state. The rational path from each capital stock is unique. The middle steady state is the capital threshold: below it, the path converges to the low-capital steady state; above it, the path converges to the high-capital steady state. At the low-capital steady state, the rational price implies replacement investment and the economy remains there.

I represent optimism by a belief wedge: investors overestimate the return on capital by a fixed amount, which they regard as permanent until an unanticipated correction. Optimistic investors value installed capital more highly. Firms respond to the higher valuation by

investing more, with convex adjustment costs limiting the speed of accumulation. The belief wedge must be large enough for investment to exceed replacement throughout the accumulation needed to cross the capital threshold. It must also persist long enough for that capital to be installed. The price adjusts immediately to a change in beliefs, whereas capital accumulates over time. Strong but short-lived optimism can end before the threshold is crossed; with persistent but insufficient optimism, capital converges to a steady state with no AI deployment.

A full correction eliminates the belief wedge and lowers the price. The subsequent equilibrium path depends on whether capital has crossed the threshold. If capital remains below the threshold, the new price implies investment below replacement and the economy returns to the low-capital steady state. If the threshold has been crossed, the rational equilibrium path converges to the high-capital steady state. A fall in asset prices is consistent with either a temporary or a permanent increase in capital.

A partial correction reduces the belief wedge and can slow accumulation without reversing it. During deployment, the price rises along the optimistic path as capital accumulates, while the price at which investment just replaces depreciation remains constant. Later corrections can therefore be larger without reversing accumulation. The slower pace of accumulation can leave capital below the threshold at the full correction, in which case the economy returns to the low-capital steady state. Higher financing costs can also lower capital prices and investment while investors remain optimistic.

The investment expansion and the accumulation of investors' wealth affect the interest rate in opposite directions. Investment demand puts upward pressure on the rate during the expansion. At the high-capital steady state, investors are willing to hold their greater wealth at a lower interest rate, and the rate is below its initial level. The long-run effect of AI on interest rates therefore depends on who receives its income as well as on the investment it induces.

Wages are higher and the labor share is lower at the high-capital steady state. Conventional capital deepening raises wages before deployment and after capacity is exhausted; during deployment, wages remain constant because conventional capital and effective labor expand together. Income from AI-provided labor services accrues to investors, so the labor share falls during deployment even though the wage level rises over the full transition. Investors are wealthier at the high-capital steady state but bear the capital loss at the correction.

Workers consume more even when an early correction eventually returns the economy to the low-capital steady state. Under sufficient conditions, investors' ex post welfare is lower than if the economy had remained at the low-capital steady state, despite higher wealth when the expansion is permanent. This comparison includes the utility investors derive from market wealth: those benefits do not offset the cost of the consumption forgone during accumulation.

Related literature. The closest antecedent is Caballero et al. (2006), in which asset values and funding conditions affect investment, and capital accumulation affects asset values. They obtain rational speculative growth paths through equilibrium multiplicity for an intermediate range of adjustment costs. Here, AI deployment and investors' saving generate multiple steady states, but the rational equilibrium path from each capital stock is

unique. Starting from the low-capital steady state, convergence to the high-capital steady state after a full correction requires optimism large and persistent enough for capital to cross the threshold.

Capital performs tasks otherwise performed by workers, as in Acemoglu and Restrepo (2018). Here, capital used to provide labor services expands together with conventional capital, producing a range over which the marginal product of capital is constant. Moll, Rachel, and Restrepo (2022) study how automation affects returns to wealth and the distribution of income and wealth. Their baseline has an upward-sloping long-run capital supply, so automation raises returns; here, the comparison is between steady states of a fixed technology, with a lower interest rate when investors hold more wealth. Related work studies transitions to artificial general intelligence (Restrepo, 2025; Korinek and Suh, 2024) and the broader macroeconomics of transformative AI (Trammell and Korinek, 2023; Acemoglu, 2025; Aghion and Bunel, 2024; Jones and Tonetti, 2026; Brynjolfsson et al., 2025).

The saving mechanism draws on work connecting inequality, saving, and interest rates. Wealth-in-utility preferences represent the non-homothetic saving behavior in Straub (2019) and relate to Mian, Straub, and Sufi (2020) on the saving glut of the rich. AI deployment provides an endogenous source of income concentration: income from tasks performed by capital accrues to investors. Their greater wealth lowers the interest rate at which they are willing to hold it.[1]

Pástor and Veronesi (2009) show how learning about a new technology can generate high prices that subsequently fall even when the technology is productive. Investment induced by optimistic valuations changes productive capacity and investors' saving and can therefore alter the steady state reached after a full correction. The relation between expectations and investment is supported by the evidence in Gennaioli, Ma, and Shleifer (2016). De Long and Shleifer (1991) use closed-end fund premia to study overvaluation in the late-1920s stock market.

Martin and Ventura (2016) show how credit bubbles can expand investment by increasing the credit available to entrepreneurs. Caballero and Krishnamurthy (2006) show how bubbles can raise domestic investment by providing stores of value that reduce capital outflows, while exposing the economy to crashes and capital flow reversals. These papers relate speculative valuations to investment through financing conditions. I focus on whether the expansion in productive capacity induced by optimism persists after a full correction, through the effects of AI deployment on production and investors' saving.

The multiplicity of steady states also relates to work on takeoff and coordination failure (Rosenstein-Rodan, 1943; Murphy, Shleifer, and Vishny, 1989; Cooper and John, 1988; Azariadis and Drazen, 1990; Matsuyama, 1991). Coordination-failure models can have multiple rational equilibrium paths from the same initial condition. The model has multiple steady states and a unique rational equilibrium path from each capital stock.

Section 2 presents the technology, household saving, investment, and beliefs. Section 3 establishes the steady states and characterizes the rational equilibrium paths. Section 4

[1] Andrews and Farboodi (2025) report declines in long-maturity Treasury and TIPS yields around major AI model releases.

derives the size and duration requirements for speculative investment, studies full and partial corrections, and examines interest rates and the post-correction yield curve. Section 5 examines distributional incidence and welfare. Section 6 concludes. The appendices provide the derivations, proofs, and numerical details.

2 Model

I consider a closed economy in continuous time with a single good used for consumption and investment. There are two types of households: workers, who supply labor and consume their wages, and investors, who own capital and derive utility from consumption and wealth. Competitive firms produce using conventional capital and labor services supplied by workers and AI capital. Investment is subject to convex adjustment costs. AI technology is available from the initial date; production and preference parameters remain fixed throughout the transition experiment.

2.1 Technology

AI capital supplies labor services that firms use together with conventional capital. Firms can therefore expand effective labor by installing capital, and investors receive the rental income from that capital. The allocation of capital between these two uses determines its marginal product and competitive rental rate.

Let L denote raw labor, K_c conventional capital, and N effective labor. With productivity $A > 0$, output is

$$Y = AK_c^{\alpha} N^{1-\alpha}, \qquad \alpha \in (0,1).$$

Firms allocate a homogeneous capital stock $K = K_c + K_\ell$ between conventional production and AI services. Each unit of AI capital K_ℓ supplies γ units of effective labor services, so $N = L + \gamma K_\ell$. AI deployment is bounded by $K_\ell \leq \bar{K}_\ell L$. The bound represents organizational capacity supplied with the workforce, with each worker supporting up to $\bar{K}_\ell$ units of AI capital. This scaling makes the aggregate technology constant-returns in (K, L), so competitive payments to raw labor and total capital exhaust output. I normalize $L = 1$ after taking the factor derivatives.

At low capital stocks, firms allocate all capital to conventional production because its marginal product exceeds that in AI use. As capital accumulates, the marginal product of conventional capital falls relative to that of AI capital. Deployment begins when the two become equal. During deployment, the optimal allocation satisfies

$$\alpha A K_c^{\alpha-1} N^{1-\alpha} = (1-\alpha) A K_c^{\alpha} N^{-\alpha} \gamma.$$

This condition gives $N = bK_c$, with $b \equiv (1-\alpha)\gamma/\alpha$. Combining it with $K = K_c + K_\ell$ and $N = 1 + \gamma K_\ell$ gives

$$N = bK_c, \qquad K_\ell = \frac{bK - 1}{\gamma + b}. \tag{1}$$

The first-order condition requires $K_c/N = 1/b$. Firms therefore allocate additional capital between conventional and AI use so that this ratio remains constant, keeping both marginal

products constant during deployment (Appendix A). Deployment begins at $K_{\mathrm{AI}} = 1/b$, where $K_\ell = 0$, and reaches the capacity constraint at $K_{\mathrm{sat}} = [1 + (\gamma + b)\bar{K}_\ell]/b$. Using $b = (1-\alpha)\gamma/\alpha$,

$$K_{\mathrm{sat}} - K_{\mathrm{AI}} = \frac{(\gamma + b)\bar{K}_\ell}{b} = \frac{\bar{K}_\ell}{1-\alpha} > 0.$$

Thus the length of the deployment interval increases with capacity $\bar{K}_\ell$.

Let $d(K)$ denote the common rental rate paid to capital in the two uses, and let $w(K)$ be the wage per unit of labor supplied by workers. Because the optimized aggregate technology is constant-returns, the envelope theorem and Euler's theorem give

$$d(K) = \left.\frac{\partial Y(K,L)}{\partial K}\right|_{L=1}, \qquad w(K) = \left.\frac{\partial Y(K,L)}{\partial L}\right|_{L=1}, \qquad Y(K) = Kd(K) + w(K). \quad (2)$$

The rental rate is

$$d(K) = \begin{cases} \alpha A K^{\alpha-1}, & K < K_{\mathrm{AI}}, \\ d_{\mathrm{flat}} \equiv \alpha A b^{1-\alpha}, & K_{\mathrm{AI}} \le K < K_{\mathrm{sat}}, \\ \alpha A (K - \bar{K}_\ell)^{\alpha-1}(1 + \gamma\bar{K}_\ell)^{1-\alpha}, & K \ge K_{\mathrm{sat}}. \end{cases} \quad (3)$$

Before deployment, all capital is conventional and its marginal product declines with K. During deployment, the allocation in (1) keeps conventional capital per unit of effective labor constant, so the marginal product remains at d_{flat}. After the deployment constraint binds, effective labor is fixed at $1 + \gamma\bar{K}_\ell$; further accumulation increases conventional capital and its marginal product declines again. Deployment capacity thus determines the interval over which additional capital can be installed without diminishing returns.

The constant marginal product of capital during deployment extends to other constant-returns technologies whose interior allocation fixes the input ratio. In such a region, the marginal products depend on that ratio, so both remain constant as firms expand the two inputs together. The interior allocation also implies $d_{\mathrm{flat}} = \gamma w_{\mathrm{flat}}$: the rental income from a unit of AI capital equals the wage value of the labor services it supplies. Under Cobb–Douglas, $w_{\mathrm{flat}} = (1-\alpha)Ab^{-\alpha}$.

Workers receive $w = Y_L$, while investors receive capital income $Kd(K)$. After saturation, the marginal product of AI capital exceeds $d(K)$; the difference is the rent on deployment capacity. Because deployment capacity is proportional to labor supply, hiring an additional worker also expands the capacity available for AI deployment. Once the constraint binds, the wage includes the value of this additional capacity as well as the worker's direct contribution to production.[2] The labor share is workers' wage income divided by output.

[2]If deployment capacity were fixed independently of labor and its rents accrued to households that consume them, the capital and interest-rate paths would be unchanged. Wages would still be higher and the labor share lower at the high-capital steady state, although the labor share would remain constant after saturation.

Applying (2) region by region gives

$$s_L(K) = \frac{w(K)}{Y(K)} = \begin{cases} 1-\alpha, & K < K_{\mathrm{AI}}, \\ \dfrac{1}{1+\gamma K}, & K_{\mathrm{AI}} \le K < K_{\mathrm{sat}}, \\ (1-\alpha) - \dfrac{\alpha \bar{K}_\ell}{K - \bar{K}_\ell}, & K \ge K_{\mathrm{sat}}. \end{cases} \tag{4}$$

Before deployment, workers receive the Cobb–Douglas share $1-\alpha$. During deployment, output increases while the wage is constant, and the income from the additional AI-provided labor services accrues to investors. The labor share therefore falls. After saturation, further conventional capital deepening raises the labor share toward $1-\alpha$, although it remains below its pre-deployment value. The wage level rises before deployment, remains constant during deployment, and rises again after saturation. A falling labor share during AI deployment is consistent with a higher wage at the high-capital steady state.

2.2 Households and the interest-rate schedule

Workers supply labor, hold no assets, and consume their wage.[3] Investors own the capital claim and can trade a riskless short-term bond. Its instantaneous interest rate is i_t. Investors' aggregate wealth is $W_t = q_t K_t$, where q_t is the price per unit of aggregate installed capital in units of the consumption good (Tobin's q). The bond is in zero net supply and therefore does not add to aggregate wealth.

Let c_t denote investors' consumption. Investors have wealth-in-utility preferences,

$$\int_0^\infty e^{-\rho t}\Big[\log c_t + \theta W_t\Big]\, dt, \tag{5}$$

where $\rho > 0$ is the rate of time preference and $\theta > 0$ is the weight on wealth in instantaneous utility. The Euler equation (Appendix B) is

$$\frac{\dot{c}_t}{c_t} = i_t - \rho + \theta c_t. \tag{6}$$

Because wealth provides utility directly, saving yields a benefit in addition to the financial return. The term θc_t in (6) measures this benefit relative to the marginal utility of consumption. For a given consumption level and growth rate, investors are therefore willing to hold wealth at a lower interest rate than when $\theta = 0$. Rearranging (6) gives

$$i_t = \rho - \theta c_t + \frac{\dot{c}_t}{c_t}. \tag{7}$$

The bond is riskless, and investors perceive no risk in the capital claim. Given their beliefs, the no-arbitrage condition equates the perceived return on capital to i_t.

[3] Investors bear the entire capital loss under this ownership assumption. For a given capital path, the wage results in Section 5 are independent of asset ownership.

Setting $\dot{c} = 0$ in (6) gives $i = \rho - \theta c$. In a rational steady state the capital claim and the bond have the same return, and constant aggregate wealth requires $c = iW$. Solving these two equations for c and i gives

$$c^{ss}(W) = \frac{\rho W}{1 + \theta W}, \qquad i^{ss}(W) = \frac{\rho}{1 + \theta W}, \qquad \frac{di^{ss}}{dW} = -\frac{\rho\theta}{(1 + \theta W)^2} < 0. \tag{8}$$

Consumption increases less than proportionately with wealth, so the consumption-wealth ratio falls. Investors are willing to hold a larger stock of wealth at a lower steady-state interest rate.[4] This non-homothetic saving response connects the distribution of AI income to the interest rate: capital installed during deployment and the income from its labor services belong to investors, whose propensity to save rises with wealth. These schedules describe steady-state household choices. During an investment expansion, investors' consumption also depends on the resources used for investment, and the interest rate includes the consumption-growth term in (7).

2.3 Investment, prices, and beliefs

Let I_t denote gross investment and $j_t = I_t/K_t$ the gross investment rate. Expenditure on new capital, including adjustment costs, is $K_t\Gamma(j_t)$ units of the consumption good, where

$$\Gamma(j) = \psi\left(e^{j/\psi} - 1\right), \qquad \psi > 0. \tag{9}$$

Adjustment costs per unit of existing capital are $\Gamma(j) - j$. The investment first-order condition is $q_t = \Gamma'(j_t) = e^{j_t/\psi}$. With depreciation rate $\delta > 0$,

$$j_t = \psi \log q_t, \qquad \frac{\dot{K}_t}{K_t} = \psi \log q_t - \delta. \tag{10}$$

The capital stock is continuous, while its price can jump when beliefs change. The gross investment rate increases with q_t and equals depreciation at the steady-state price $\bar{q} \equiv e^{\delta/\psi}$. The firm invests up to the point at which marginal investment expenditure, including adjustment costs, equals the price of installed capital. A higher price therefore increases investment, but convex adjustment costs limit how quickly capital accumulates. Net investment is positive when $q_t > \bar{q}$ and negative when $q_t < \bar{q}$.

The economy is closed, and workers consume their wages, so investors' saving funds all investment expenditure. Investors receive capital income $K_t d(K_t)$, pay for investment and adjustment costs, and consume the residual:

$$c_t = C(K_t, q_t) = K_t\left[d(K_t) - \psi(q_t - 1)\right]. \tag{11}$$

At a fixed capital stock, a higher q_t increases investment and the associated expenditure. Investors' consumption falls while wealth $W_t = q_t K_t$ rises. The decline in consumption

[4] With a general wealth-utility term $v(W)$, the rational steady-state interest rate is $i^{ss}(W) = \rho/[1 + Wv'(W)]$. The saving mechanism requires $Wv'(W)$ to increase with wealth, so that the interest-rate schedule is decreasing. The specification $v(W) = \theta W$ satisfies this condition.

represents the additional saving attracted by high perceived returns; in a broader setting, this saving could also come from other households or abroad.

The realized return on capital R_t^a includes the change in the value of the capital claim and the cash flow distributed to investors, both measured relative to $W_t = q_t K_t$:

$$R_t^a = \frac{\dot{q}_t}{q_t} + \psi \log q_t - \delta + \frac{d(K_t) - \psi(q_t - 1)}{q_t}, \qquad W_t = q_t K_t. \tag{12}$$

These definitions satisfy the aggregate wealth identity

$$\dot{W}_t = R_t^a W_t - c_t. \tag{13}$$

I represent optimism about future AI returns as an overestimate of the return on the aggregate capital claim. During the optimistic phase, this overestimate is a fixed amount x, the *belief wedge*, with $0 < x < \rho$. The upper bound ensures a positive realized return in a steady state with this wedge. Investors perceive the return on capital as $R_t^a + x$, so the no-arbitrage condition given their beliefs is

$$i_t = R_t^a + x. \tag{14}$$

The resource constraint (11) and the wealth identity (13) use realized income and expenditure. Optimism changes the return investors perceive, the price of installed capital, and firms' investment. Investors' consumption and investment expenditure, including adjustment costs, continue to sum to capital income.

At each date, investors observe current market wealth and use it as the initial condition for their forecasts. They revise forecast wealth as new observations arrive while maintaining the same return wedge x, which they regard as permanent until an unanticipated correction.

Investors choose consumption using the wealth path they anticipate, so their transversality condition applies to forecast wealth. For the same initial wealth and consumption path, the wealth forecast formed at a given planning date is at least e^{xt} times subsequent market wealth, where t measures elapsed time from that date. For a fixed wedge $0 \leq x < \rho$, Appendices B and F.1 establish that this condition is equivalent to

$$\lim_{t \to \infty} e^{-(\rho - x)t} \frac{q_t K_t}{c_t} = 0. \tag{15}$$

An equilibrium path for a fixed wedge satisfies equations (6), (10)–(12), and (14), has positive capital, price, and consumption, and satisfies the transversality condition (15). This condition excludes solutions on which consumption tends to zero and is needed to determine the initial equilibrium price.

The economy follows the equilibrium path for the current wedge until an unanticipated correction at τ, then follows the equilibrium path for the revised wedge. At a full correction, the wedge falls to zero and the no-arbitrage condition becomes $i_t = R_t^a$. Rational beliefs correspond to $x = 0$.

3 Steady States and Rational Equilibrium Paths

The technology and saving behavior determine the steady-state capital stocks. I first derive these steady states and then characterize the rational equilibrium paths that converge to them. The existence of several steady states leaves open whether a rational equilibrium path from the low-capital steady state can converge to the high-capital steady state.

3.1 Multiple steady states

A rational steady state requires $\dot{K} = 0$ and $\dot{q} = 0$. By (10), the first condition gives $q_t = \bar{q}$, at which gross investment just replaces depreciation. All rational steady states therefore have the same price per unit of capital. Investors' wealth $\bar{q}K$ is greater at the high-capital steady state because the capital stock is larger. A permanently larger stock is thus consistent with a temporary increase in the price of capital during the transition.

At a rational steady state, the return on capital equals the interest rate. The household Euler equation determines the interest rate at which investors are willing to hold wealth $W = \bar{q}K$:

$$i^{ss}(\bar{q}K) = \frac{\rho}{1 + \theta\bar{q}K}.$$

Replacement uses $\Gamma(\delta) = \psi(\bar{q} - 1)$ units of output per unit of capital. The net cash flow available for investors' consumption is therefore $d(K) - \Gamma(\delta)$ per unit of capital, and the return on a claim priced at $\bar{q}$ is

$$\frac{d(K) - \Gamma(\delta)}{\bar{q}}.$$

Thus K is a steady state only if this return equals the interest rate:

$$\frac{d(K) - \Gamma(\delta)}{\bar{q}} = \frac{\rho}{1 + \theta\bar{q}K}.$$

Define the steady-state user cost of capital $\Phi(K)$ and the difference between the rental rate and the user cost, $G(K)$, by

$$\Phi(K) \equiv \Gamma(\delta) + \frac{\bar{q}\rho}{1 + \theta\bar{q}K}, \qquad G(K) \equiv d(K) - \Phi(K). \tag{16}$$

The user cost $\Phi(K)$ is replacement expenditure, including adjustment costs, plus the return required on capital valued at $\bar{q}$. At a steady state, the rental rate equals the user cost:

$$G(K) = 0, \qquad \text{equivalently} \qquad d(K) = \Phi(K).$$

At the common steady-state price $\bar{q}$, a larger K implies greater wealth for investors and a lower interest rate. Replacement expenditure per unit is unchanged, so the user cost $\Phi(K)$ decreases with K. During AI deployment, firms expand conventional capital and effective labor together, keeping the rental rate constant. The difference $G(K)$ therefore increases with K. This interaction between AI deployment and the effect of wealth on saving allows the rental rate to cross the user cost from below in the deployment region.

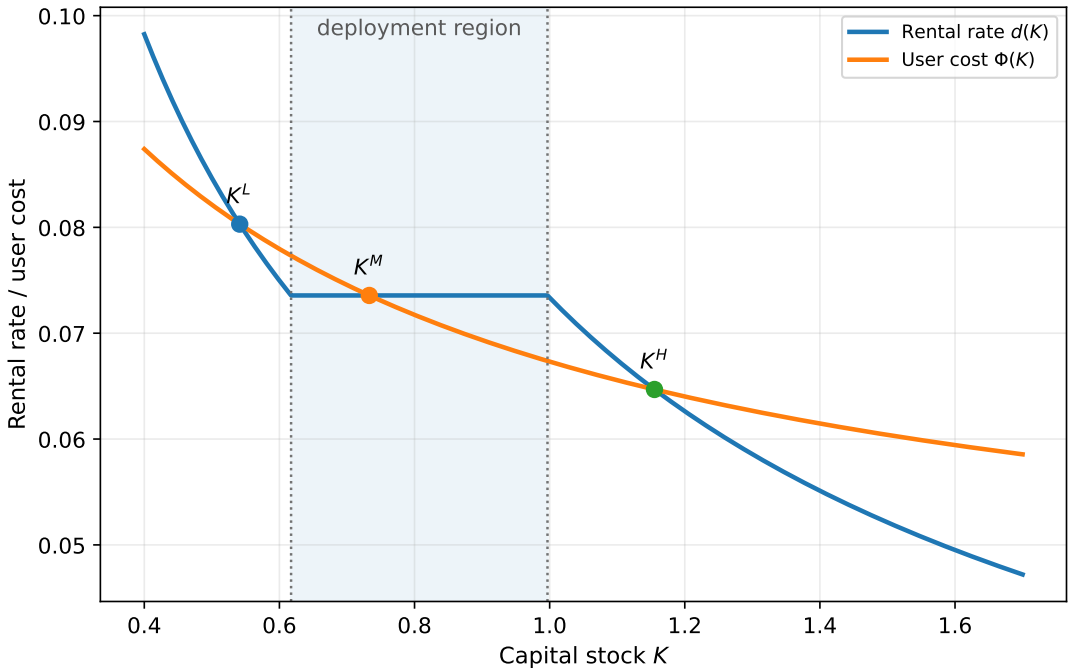


Figure 1: Rational steady states. The rental rate $d(K)$ equals the user cost $\Phi(K)$ below the deployment interval, within it, and above it. These intersections determine the low, middle, and high steady states.

The figures provide qualitative illustrations of the equilibrium mechanisms using the parameters in Appendix I. Figure 1 plots the rental rate $d(K)$ and the user cost $\Phi(K)$. The example has three intersections, one in each technology region.

If $\theta = 0$, the steady-state Euler equation gives $i = \rho$ at every level of wealth. The user cost is then constant at $\Phi(K) = \Gamma(\delta) + \bar{q}\rho$. Unless this level equals d_{flat}, the rental rate equals the user cost at exactly one capital stock. A user cost that decreases with capital is necessary for three isolated steady states.

After saturation, the rental rate decreases with capital. If $d(K)$ falls below $\Phi(K)$, there is a high-capital steady state in addition to the low and middle steady states. The following proposition gives sufficient conditions.

Proposition 1 (Deployment capacity and multiple steady states)**.** *Suppose*

$$\Phi(K_{\text{AI}}) > d_{\text{flat}} > \Gamma(\delta). \tag{17}$$

Then there exists a deployment capacity $\bar{K}_\ell^* > 0$ such that, for every $\bar{K}_\ell > \bar{K}_\ell^*$, the steady-state equation $G(K) = 0$ has at least one solution in $(0, K_{\text{AI}})$, a unique solution in $(K_{\text{AI}}, K_{\text{sat}})$, and at least one solution in (K_{sat}, ∞). If, in addition,

$$G'(K) < 0 \qquad \text{for } K \in (0, K_{\text{AI}}) \cup (K_{\text{sat}}, \infty), \tag{18}$$

then the low- and high-capital steady states are unique. At each, the rental rate crosses the user cost from above. The economy has exactly three steady states, denoted $K^L < K^M < K^H$.

Let i^L, i^M, i^H denote the interest rates at these three steady states.

The first inequality in (17) places the rental rate below the user cost at the start of deployment. During deployment, the rental rate remains constant while greater investor wealth lowers the steady-state interest rate and hence the user cost. Replacement expenditure per unit of capital is unchanged and sets a lower bound on the user cost. The condition $d_{\text{flat}} > \Gamma(\delta)$ places this lower bound below the rental rate. With enough deployment capacity, the user cost therefore falls below the rental rate before diminishing returns resume:

$$\Phi(K_{\text{AI}}) > d_{\text{flat}} > \Phi(K_{\text{sat}}). \tag{19}$$

The intersection during deployment is the middle steady state. After saturation, diminishing returns eventually reduce the rental rate below the user cost, yielding the high-capital steady state. Condition (18) ensures unique low and high steady states by requiring the rental rate to fall faster than the user cost outside the deployment region. Appendix D proves the proposition.

Three steady states can also arise when the rental rate declines during deployment. If $\Phi'(K) < d'(K) \leq 0$ over that interval, then $G(K)$ increases. Suppose also that $G(K_{\text{AI}}) < 0 < G(K_{\text{sat}})$, with $G(K) > 0$ for sufficiently small K and $G(K) < 0$ for sufficiently large K. Together with the outer-region slope condition (18), these restrictions give one steady state in each region. The global equilibrium and correction results below are established for the technology in Section 2.

At the low-capital steady state K^L, AI deployment has not begun. At the high-capital steady state K^H, deployment capacity has been exhausted, investors' wealth is greater, and the interest rate is lower. AI services expand effective labor and capital income, while investors are willing to hold their greater wealth at a lower interest rate. These changes allow the larger capital stock to be maintained under rational expectations.

3.2 Rational equilibrium paths

The initial price determines investment and subsequent accumulation. Convergence to a saddle steady state requires the capital-price pair to lie on its *saddle path*, the set of pairs whose equilibrium paths converge to that steady state. A rational transition from K^L to K^H requires the saddle path to K^H to be defined at K^L. The next two lemmas characterize local stability and the domain of each saddle path.

Capital accumulation satisfies

$$\dot{K}_t = K_t(\psi \log q_t - \delta),$$

so the $\dot{K} = 0$ locus is the horizontal line $q = \bar{q}$. Above that line investment exceeds depreciation and K_t rises; below it K_t falls.

Along a rational path, the return on the capital claim equals the interest rate, $R^a_t = i_t$; writing out both sides using (12) and (7),

$$\underbrace{\frac{\dot{q}_t}{q_t} + \psi \log q_t - \delta + \frac{d(K_t) - \psi(q_t - 1)}{q_t}}_{\text{realized return } R^a_t} = \underbrace{\rho - \theta c_t + \frac{\dot{c}_t}{c_t}}_{\text{interest rate } i_t}. \tag{20}$$

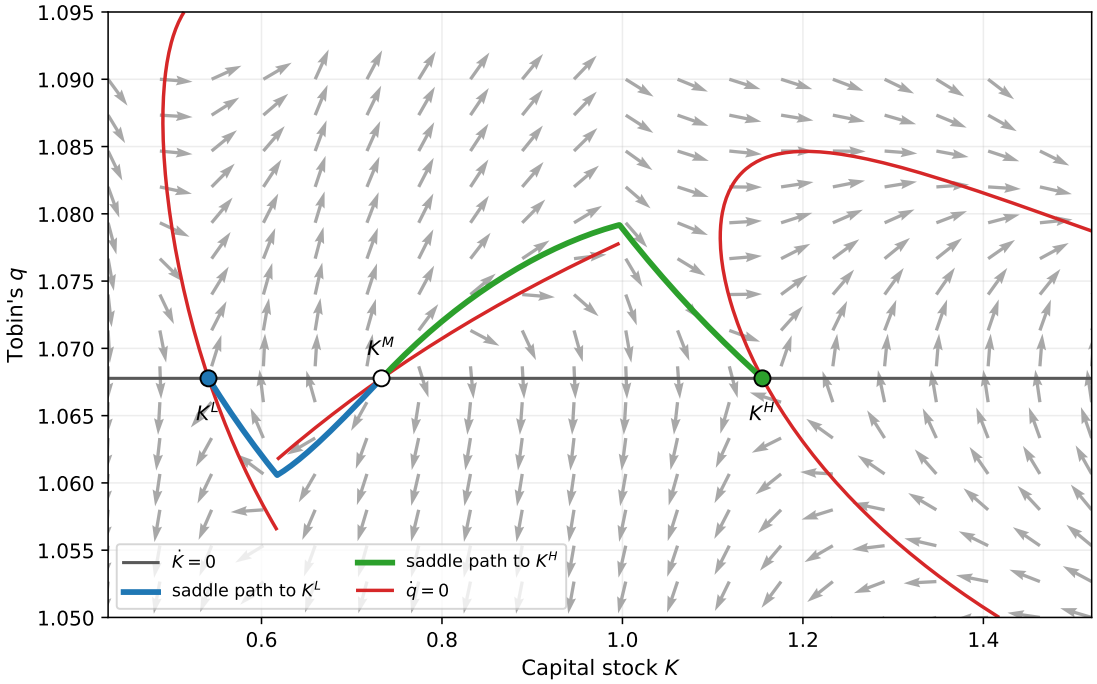


Figure 2: Rational phase diagram. The low- and high-capital steady states are saddles, and the middle steady state is an unstable node. The branches shown approach the middle steady state when followed backward in time. The horizontal line is $\dot{K} = 0$; the red curves are $\dot{q} = 0$. The breaks in the $\dot{q} = 0$ locus at K_{AI} and K_{sat} arise from the discontinuities in $d'(K)$.

The resource constraint (11) determines consumption given K and q. Rational equilibrium paths satisfy (15) with $x = 0$.

Substituting $c_t = C(K_t, q_t)$ and $\dot{c}_t = C_K \dot{K}_t + C_q \dot{q}_t$ into (20) gives an equation linear in $\dot{q}_t$. Solving it gives the law of motion for q. The $\dot{q} = 0$ locus equates the realized return excluding changes in q to the interest rate from the Euler equation, including the consumption growth generated by changing K. Appendix C reports the law of motion and the equation for this locus.

At $q = \bar{q}$, the sign of $\dot{q}$ is opposite to that of the rental rate minus the user cost:

$$\operatorname{sign} \dot{q}\Big|_{q=\bar{q}} = -\operatorname{sign} G(K). \tag{21}$$

At $q = \bar{q}$, gross investment replaces depreciation. Where the rental rate exceeds the user cost, the no-arbitrage condition under rational beliefs requires an anticipated price decline; where the rental rate falls short, it requires an anticipated price increase. Equation (21) gives the sign of $\dot{q}$ at the given capital-price pair. Figure 2 shows the $\dot{K} = 0$ and $\dot{q} = 0$ loci and the directions of motion.

Lemma 1 (Local stability under rational beliefs)**.** *At every steady state $(K^j, \bar{q})$ of the rational equilibrium system, the sign of the Jacobian determinant equals the sign of $G'(K^j)$. Consequently:*

(i) the middle steady state K^M is an unstable node. In the deployment region $d'(K^M) = 0$, which implies $G'(K^M) > 0$; the trace is positive and the eigenvalues are real there;

(ii) a low or high steady state K^j, $j \in \{L, H\}$, is a saddle whenever $G'(K^j) < 0$. In particular, both steady states are saddles under (18).

The Jacobian computation is in Appendix E.

The sign of $G'(K)$ determines local stability. Differentiating (16),

$$G'(K) = \underbrace{d'(K)}_{\leq 0} - \underbrace{\Phi'(K)}_{< 0}.$$

Diminishing returns lower $d(K)$, while investors' greater wealth lowers $\Phi(K)$. At the low and high steady states, the slope condition requires the rental rate to fall faster than the user cost. At the middle steady state, $d'(K^M) = 0$, so only the user cost falls. Hence $G'(K^M) > 0$, and the steady state is unstable.

The restrictions that give three steady states also determine the rational equilibrium path from each capital stock. Appendix F gives the proof.

Lemma 2 (Global rational equilibrium paths)**.** *Suppose (18) and (19) hold. There is exactly one rational equilibrium path from each $K_0 > 0$. It converges to K^L for $K_0 < K^M$, to K^H for $K_0 > K^M$, and is stationary at $K_0 = K^M$.*

On the saddle paths to the low- and high-capital steady states, the price is a function of capital defined on $(0, K^M)$ and (K^M, ∞), respectively, and consumption is positive. For $K^L < K < K^M$, the low saddle path has $q < \bar{q}$; for $K^M < K < K^H$, the high saddle path has $q > \bar{q}$. On these intervals, both saddle paths approach $(K^M, \bar{q})$ backward in time. Below K^L, the low saddle path has $q > \bar{q}$; above K^H, the high saddle path has $q < \bar{q}$. Capital converges monotonically along each saddle path.

The middle steady state K^M is the *capital threshold*. For $K^L < K < K^M$, the rational equilibrium price is below $\bar{q}$, so gross investment is less than depreciation and capital declines toward K^L. For $K^M < K < K^H$, the price exceeds $\bar{q}$, investment exceeds depreciation, and capital rises toward K^H.

Proposition 2 (Equilibrium path from the low-capital steady state)**.** *Under (18) and (19), the unique rational equilibrium path with $K(0) = K^L$ is stationary at $(K^L, \bar{q})$. There is no rational equilibrium path from K^L to K^H.*

The proof is in Appendix G.

The rational equilibrium price at K^L is $\bar{q}$, and gross investment equals depreciation. Starting at K^L with $q > \bar{q}$ would induce positive net investment. Once capital passes K^M, the resulting solution lies above the saddle path to K^H, so accumulation continues

beyond K^H. Further accumulation lowers the rental rate until, in the limit, capital income just covers replacement expenditure. Investors' consumption converges to zero, and the solution violates their transversality condition (Appendix F.4). The high-capital steady state is a rational equilibrium at a different capital stock. A transition to it requires optimism to persist until capital exceeds K^M. A full correction after that point is followed by convergence to K^H. Section 4 characterizes the size and duration requirements for this investment expansion.

4 Speculative Investment and Corrections

Suppose investors in an economy initially at K^L unexpectedly adopt the belief wedge $x > 0$ at date zero. Its size and duration determine whether the economy converges to K^H after a full correction.

4.1 The required wedge

This subsection determines how large a constant belief wedge must be for capital eventually to cross K^M. Section 4.2 determines how long that wedge must persist before a full correction.

Under the no-arbitrage condition (14), the perceived return on capital equals the interest rate, while the realized return is lower by x. For a given income stream, investors therefore pay a higher price, which increases investment through (10). Relative to Figure 2, the wedge changes the law of motion for q and shifts the $\dot{q} = 0$ locus.

With a permanent wedge x, the steady-state Euler equation gives $R^a + x = \rho - \theta c$. The wealth identity remains $c = R^a W$, so the realized return is

$$R^a = \frac{\rho - x}{1 + \theta W},$$

while the interest rate is $i = R^a + x$. Substituting $R^a = [d(K) - \Gamma(\delta)]/\bar{q}$ and $W = \bar{q}K$ gives the steady-state condition

$$d(K) = \Gamma(\delta) + \frac{\bar{q}(\rho - x)}{1 + \theta \bar{q} K}. \tag{22}$$

I refer to stationary allocations under a fixed belief wedge x as perceived steady states. Realized capital, consumption, and market wealth are constant at such an allocation; for $x > 0$, the wealth forecast from any planning date grows. Relative to Figure 1, the wedge lowers the user cost at each K by $\bar{q}x/(1 + \theta \bar{q} K)$. At a fixed wedge, investors' greater wealth lowers the steady-state interest rate and the realized return $i - x$, reducing the user cost.

In the deployment region, the rational middle steady state satisfies

$$K^M = \frac{1}{\theta \bar{q}} \left[\frac{\bar{q}\rho}{d_{\text{flat}} - \Gamma(\delta)} - 1 \right]. \tag{23}$$

Let $x^*(K)$ denote the required wedge: starting from $K \leq K^M$, capital crosses K^M under a constant wedge if and only if $x > x^*(K)$. This wedge is given by

$$x^*(K) = \theta\Big[d_{\text{flat}} - \Gamma(\delta)\Big]\Big(K^M - \max\{K, K_{\text{AI}}\}\Big) \qquad \text{for } K \leq K^M. \tag{24}$$

At the initial stock $K^L < K_{\text{AI}}$, the required wedge is

$$x^*(K^L) = \frac{1 + \theta\bar{q}K_{\text{AI}}}{\bar{q}}\Big[\Phi(K_{\text{AI}}) - d_{\text{flat}}\Big].$$

The required wedge is proportional to the difference between the user cost and the rental rate at the start of deployment. A larger difference means that the rental rate is further below the amount needed to cover replacement expenditure and the required return at $q = \bar{q}$. A larger wedge is therefore needed for net investment to remain positive at the start of AI deployment.

A larger wedge lowers the user cost, moving the perceived low steady state toward a higher capital stock and the middle steady state toward a lower one. They meet at K_{AI}. Any transition from below K_{AI} must pass this point, so the required wedge is the same for every initial stock below deployment.

During deployment, $x^*(K)$ is the wedge for which the perceived middle steady state has capital stock K. The rental rate remains constant while the user cost decreases with K, reducing the optimism needed for further accumulation. The required wedge therefore decreases with K, reaching zero at K^M, above which the rational equilibrium path converges to K^H.

Let $Q^H(K; x)$ denote the price on the saddle path of the perceived high steady state.

Lemma 3 (Perceived equilibrium paths)**.** *Suppose* (18) *and* (19) *hold.*

(i) For $x^*(K^L) < x < \rho$, there is a single steady state $K^H(x) > K_{\text{sat}}$ and a unique equilibrium path from every $K_0 > 0$. It converges to $K^H(x)$, with $Q^H(K; x) > \bar{q}$ below that stock and $Q^H(K; x) < \bar{q}$ above it.

(ii) For $0 \leq x < x^*(K^L)$, there are three steady states

$$K^L(x) < K_{\text{AI}} < K^M(x) < K_{\text{sat}} < K^H(x).$$

The unique equilibrium path converges to $K^L(x)$ below $K^M(x)$, to $K^H(x)$ above it, and is stationary at $K^M(x)$. Capital converges monotonically along the saddle paths.

(iii) At $x = x^*(K^L)$, capital below K_{AI} converges to K_{AI}; capital above it converges to $K^H(x)$. At K_{AI}, the only equilibrium path is stationary. The equilibrium initial price is jointly continuous in (K_0, x) for $K_0 > 0$ and $0 \leq x < \rho$, including the wedge at which the low and middle steady states merge.

A wedge above $x^*(K^L)$ keeps the price above $\bar{q}$ throughout $[K^L, K^M]$, so investment exceeds replacement until capital crosses the threshold. The wedge must persist long enough for that capital to be installed. Appendix F proves the lemma.

4.2 Correction timing and the duration requirement

Let the wedge remain at x until an unanticipated full correction at τ, when it falls to zero. Before the correction, the price is $Q^H(K;x)$, and capital evolves according to

$$q_t = Q^H(K_t;x), \qquad \dot{K}_t = K_t\Big[\psi \log Q^H(K_t;x) - \delta\Big], \qquad 0 \le t < \tau, \tag{25}$$

with consumption given by (11). The wedge is fixed throughout this phase; price and investment change as capital accumulates.

Take $x \in (x^*(K^L), \rho)$. Lemma 3(i) ensures that the perceived high saddle path is defined at the initial stock, with $Q^H(K;x) > \bar{q}$ from K^L through K^M. Capital therefore rises. The time needed to reach the capital threshold is

$$T(x) = \int_{K^L}^{K^M} \frac{dK}{K\Big[\psi \log Q^H(K;x) - \delta\Big]}. \tag{26}$$

Capital must be strictly above K^M at the full correction for the subsequent rational path to converge to K^H. This requires $\tau > T(x)$.

The denominator in (26) is net investment. A higher price along the path raises net investment and reduces $T(x)$. The duration $T(x)$ is strictly decreasing in the wedge, as established in Proposition 3. As x approaches $x^*(K^L)$ from above, the path approaches one that converges to K_{AI} without crossing it. Net investment near the start of deployment tends to zero, and the time needed to pass through this region grows without bound.

Figure 3 plots the required-wedge schedule, the duration requirement, and capital paths with early and late corrections. The two paths have the same wedge and differ only in the correction date.

Proposition 3 (Feasibility and duration)**.** *Under* (18) *and* (19)*, for* $x \in (x^*(K^L), \rho)$*:*

(i) The perceived high-capital saddle path is defined at the initial low stock, and capital reaches K^M *in finite time while the wedge remains constant.*

(ii) For any such x*, the economy converges to the high-capital steady state if* $\tau > T(x)$ *and to the low-capital steady state if* $\tau < T(x)$*, where* $T(x)$ *is given by* (26)*. If* $\tau = T(x)$*, the post-correction path is stationary at the middle steady state.*

(iii) The duration $T(x)$ *is strictly decreasing on* $(x^*(K^L), \rho)$*, and* $T(x) \to \infty$ *as* $x \downarrow x^*(K^L)$*.*

The size requirement ensures that investment exceeds replacement throughout $[K^L, K^M]$. The duration requirement determines whether capital has crossed K^M by the full correction. Prices can adjust immediately when investors become optimistic, but convex adjustment costs make capital accumulate gradually. Even with a wedge above the required level, a full correction can occur before enough capital has been installed. A larger wedge raises the price and speeds accumulation along the optimistic path, reducing the time needed to cross the threshold. Appendix G contains the proofs.

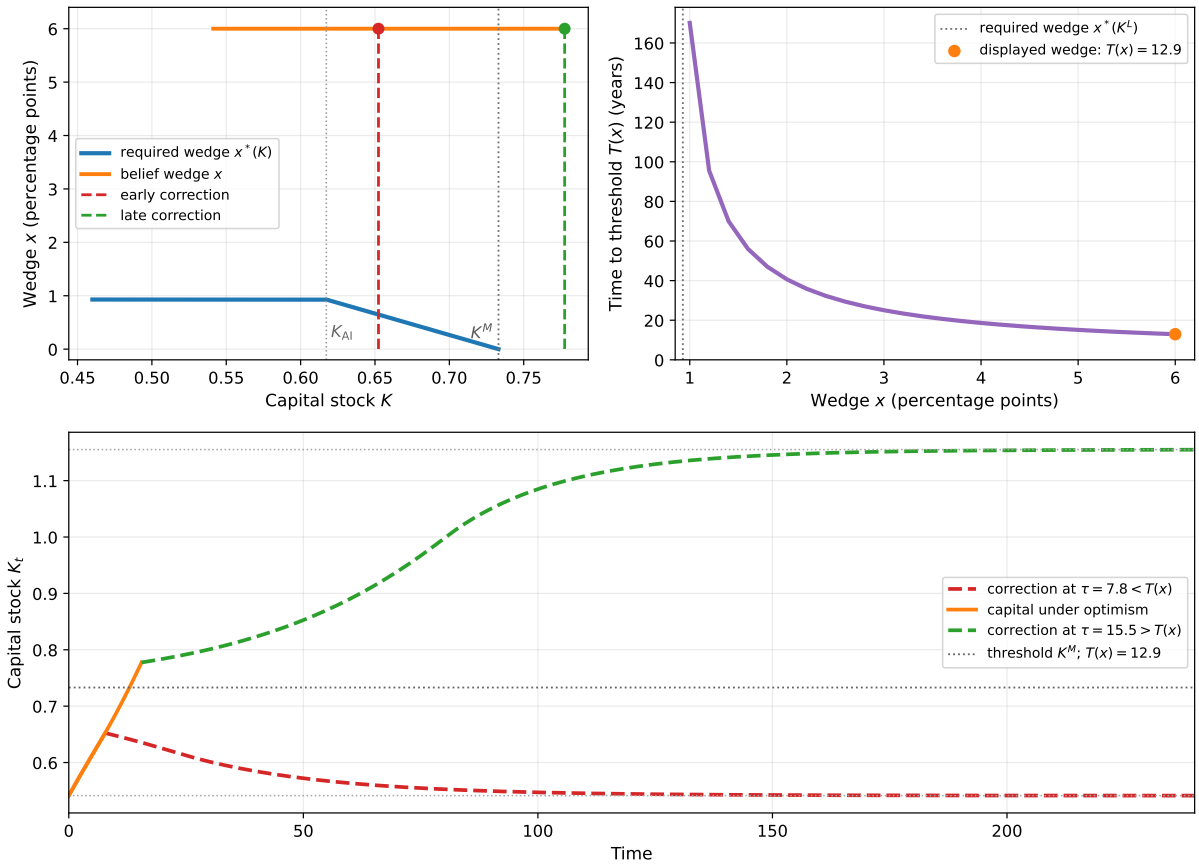


Figure 3: Required optimism and correction timing. The upper-left panel shows $x^*(K)$ and a common wedge ending at two dates. The upper-right panel plots the time $T(x)$ to reach the capital threshold. In the lower panel, capital converges to K^L after the early correction and to K^H after the late correction. The upper-left and lower panels use $x = 0.06$; all panels use the illustrative parameters in Appendix I.

4.3 Corrections and interest rates

Corollary 1 (Price after a full correction)**.** *Along any pre-correction path governed by* (25)*, if $\tau > T(x)$, the price jumps at the correction to $Q^H(K_\tau; 0)$ on the rational high saddle path, and the economy converges to K^H. If $\tau < T(x)$, the price jumps to the value on the rational low saddle path at K_τ, and the economy returns to K^L.*

A full correction lowers the price. Its effect on subsequent investment depends on whether capital has crossed K^M. After an early full correction, $K_\tau < K^M$, and the rational price lies on the low saddle path with $q < \bar{q}$. Gross investment falls below depreciation, so capital gradually returns to K^L. After a late correction with $K_\tau \in (K^M, K^H)$, the price falls to the rational high saddle path but remains above $\bar{q}$. Net investment remains positive and capital converges to K^H.

Interest rates during the transition. Optimism raises the price of installed capital and firms' investment demand, putting upward pressure on the interest rate during the expansion. The capital installed during the expansion and the income from its AI services belong to investors, whose propensity to save rises with wealth. At the high-capital steady state, investors are willing to hold their greater wealth at a lower interest rate, with $i^H = \rho/(1 + \theta\bar{q}K^H) < i^L$. Their saving response supports the larger capital stock after optimism ends.

In the closed economy, investors supply the saving required for installation by reducing consumption at a given capital stock, as in (11). Equation (7) implies that the interest rate exceeds its initial level whenever investors' consumption is below its initial level and is nondecreasing.[5]

Figures 4 and 5 illustrate a transition with a late full correction. The interest rate is above its initial level during the optimistic phase and converges to a lower level after the correction.

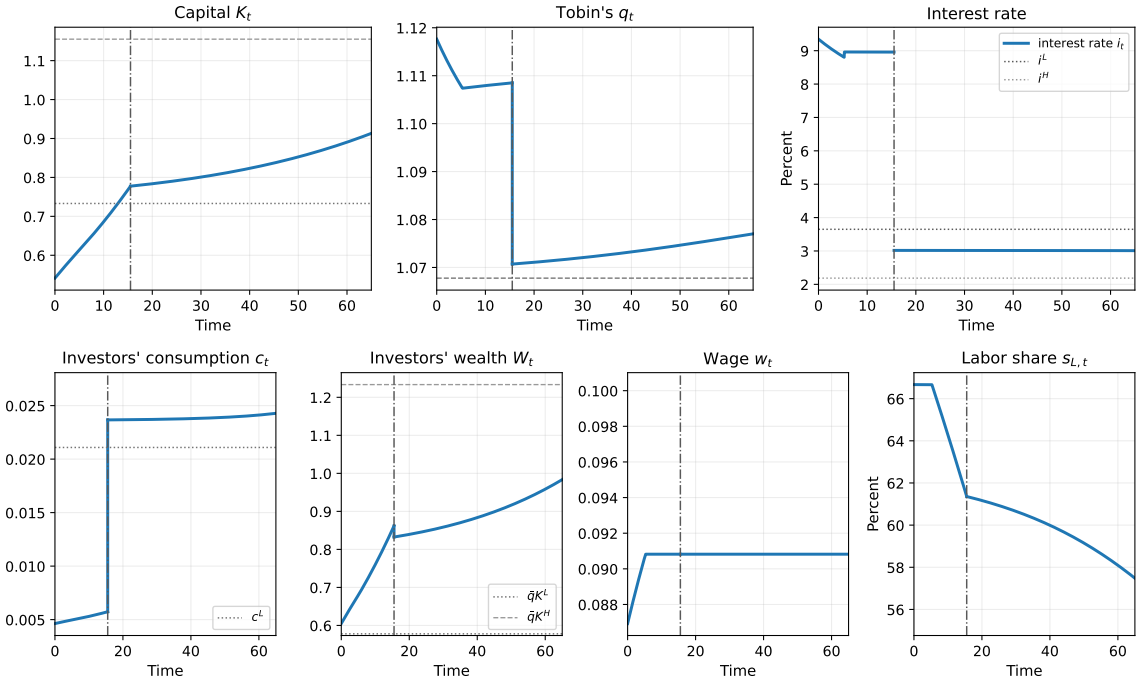


Figure 4: Transition with a late correction. Capital, price, the interest rate, investors' consumption and wealth, the wage, and the labor share for $\tau > T(x)$. Before the correction, investors' consumption is below its initial level and increasing, so the interest rate exceeds its initial level. Investors' wealth falls at the correction and converges to a level above its initial value. The vertical line marks the correction date. The figure uses $x = 0.06$ and the illustrative parameters in Appendix I.

[5]Let c^L denote investors' consumption at the initial steady state. Subtracting its Euler equation from (7) gives $i_t - i^L = \theta(c^L - c_t) + \dot{c}_t/c_t$. Appendix H gives the comparison with stationary consumption at the current wealth and belief wedge.

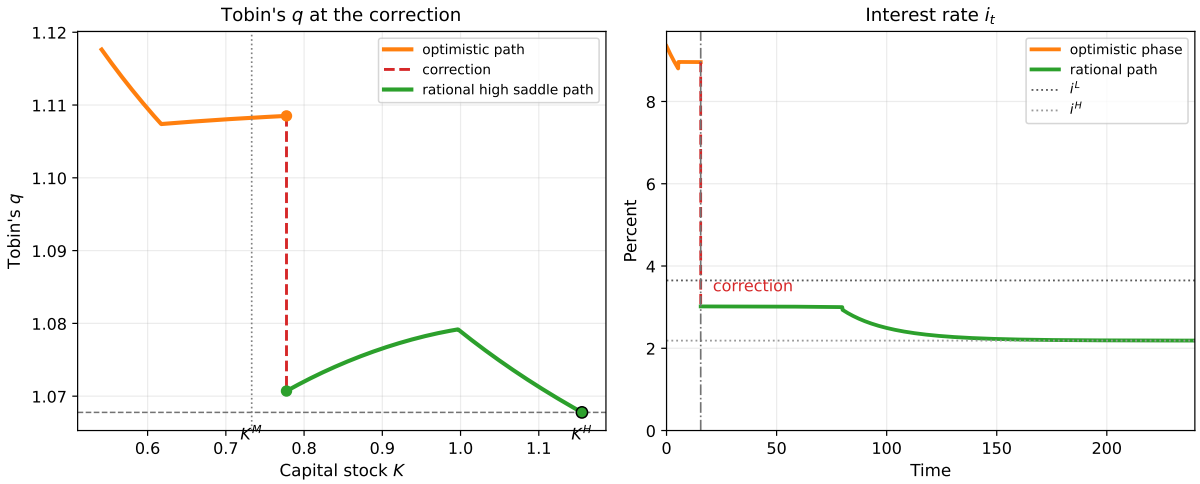


Figure 5: Price and interest rates after a late correction. At the correction, $K > K^M$ and the price falls to $Q^H(K;0)$. Capital then converges to K^H. The interest rate converges to a value below its initial level. The figure uses $x = 0.06$ and the illustrative parameters in Appendix I.

The yield curve after a correction. After a full correction near the capital threshold, yields at sufficiently long maturities exceed the short rate if accumulation reverses and lie below it if accumulation continues. Along the rational equilibrium path, a zero-coupon bond of remaining maturity $h > 0$ has continuously compounded yield $y_\tau(h) = h^{-1} \int_0^h i_{\tau+s}\, ds$. Hence

$$\lim_{h\to\infty} y_\tau(h) = \begin{cases} i^L, & K_\tau < K^M, \\ i^H, & K_\tau > K^M. \end{cases} \tag{27}$$

As capital at the correction approaches K^M from either side, the short rate immediately after the correction approaches i^M, with $i^H < i^M < i^L$. The two limiting yields differ because investors' wealth is higher, and the steady-state interest rate lower, at the high-capital steady state. A correction just below the threshold is followed by a return toward the low-capital steady state and its higher interest rate, so yields at sufficiently long maturities exceed the current short rate. Just above the threshold, further accumulation leads to the lower rate at K^H, so yields at sufficiently long maturities lie below the current short rate.

As capital at the correction approaches the threshold, the subsequent path remains close to the middle steady state for longer. At each fixed maturity, the yield and the short rate both converge to i^M, so their spread approaches zero (Appendix G).

4.4 Partial corrections and investment

A partial correction can leave net investment positive while slowing accumulation enough that capital remains below K^M at the subsequent full correction.

Suppose the wedge falls unexpectedly from $x_0 \in (x^*(K^L), \rho)$ to $x_1 \in [0, x_0)$. Investors treat the remaining wedge as permanent until another unanticipated correction. A full correction is the case $x_1 = 0$.

By Lemma 3, the sign of post-correction net investment is determined by the remaining wedge relative to $x^*(K)$. For $K_{\text{AI}} < K < K^M$, investment exceeds replacement if $x_1 > x^*(K)$ and falls below it if $x_1 < x^*(K)$. When $x_1 = x^*(K)$, the economy remains at the perceived middle steady state. Once $K > K^M$, the equilibrium path converges to K^H with $x_1 = 0$. During deployment below K^M, positive net investment requires a positive remaining wedge; as capital approaches K^M, the required wedge falls to zero.

I express correction size as a proportional price decline to compare corrections at different capital stocks. Write the pre-correction price as $Q_0(K) = Q^H(K; x_0)$, the post-correction equilibrium price as q^+, and the proportional price decline as $D = 1 - q^+/Q_0(K)$. By (10), capital growth immediately after the correction is

$$\frac{\dot{K}^+}{K} = \psi \log Q_0(K) - \delta + \psi \log(1 - D). \tag{28}$$

At $x_1 = x^*(K)$, the post-correction price is $\bar{q}$ and net investment is zero. The corresponding proportional price decline is the *correction boundary*,

$$D^*(K) = 1 - \frac{\bar{q}}{Q_0(K)}. \tag{29}$$

Proposition 4 (Partial corrections and investment)**.** *Under (18) and (19), suppose $d(K) + \psi + Kd'(K) > 0$ along the saddle path as capital increases toward the perceived high steady state. For $K \in (K_{\text{AI}}, K^M)$:*

(i) A partial correction $D < D^*(K)$ lowers investment, with net investment remaining positive. A correction $D > D^*(K)$ implies negative net investment along the path toward the perceived low steady state. When $D = D^*(K)$, net investment is zero and the economy remains at the perceived middle steady state.

(ii) The correction boundary is strictly increasing:

$$\frac{dD^*(K)}{dK} = \frac{\bar{q}\, Q_0'(K)}{Q_0(K)^2} > 0.$$

There exist two stocks $K_{\text{early}} < K_{\text{late}}$ and equilibrium corrections $D_{\text{early}} < D_{\text{late}}$ for which net investment is negative after the smaller early correction and positive after the larger late correction.

During deployment, the rental rate is constant and the user cost decreases with capital. Under the proposition's conditions, the price rises along the high saddle path (Appendix H). This price increase depends on the effect of wealth on saving. With $\theta = 0$, the price would remain constant during deployment. Because $\bar{q}$ is fixed, a larger proportional price decline is needed to reduce net investment to zero.[6]

[6]The result concerns the correction boundary $D^*(K)$. The price decline caused by a fixed reduction in the wedge need not increase with K.

The equilibrium corrections in Figure 6 satisfy $D^*(K_{\text{early}}) < D_{\text{early}} < D_{\text{late}} < D^*(K_{\text{late}})$. The later price decline is larger in both proportional and absolute terms. Net investment is negative after the early correction and positive after the late correction.

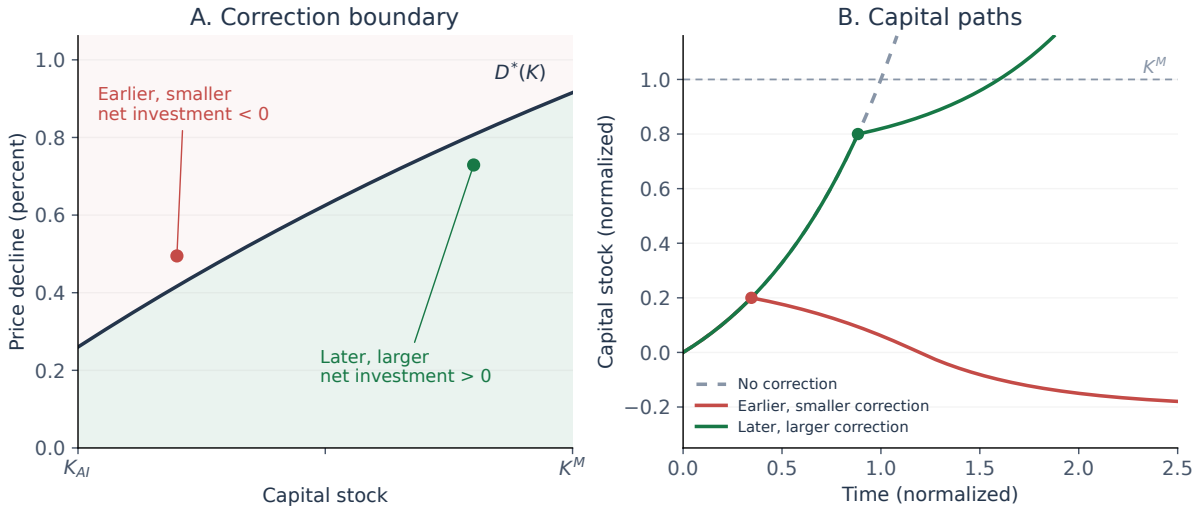


Figure 6: Partial corrections and net investment. The early correction exceeds $D^*(K)$, making net investment negative. The larger late correction is below $D^*(K)$, leaving net investment positive. The left panel reports price declines in percent. In the right panel, capital is normalized as $(K - K_{\text{AI}})/(K^M - K_{\text{AI}})$, and time is divided by the time needed for capital to increase from K_{AI} to K^M without a correction. Both post-correction capital-price pairs lie on equilibrium saddle paths. The figure uses the illustrative parameters in Appendix I and $x_0 = 0.012$.

For $x_1 > x^*(K)$, the time from the partial correction to the capital threshold is

$$\mathcal{T}(K, x_1) = \int_K^{K^M} \frac{dk}{k[\psi \log Q^H(k; x_1) - \delta]}. \tag{30}$$

A lower x_1 reduces the price along the optimistic path and increases $\mathcal{T}(K, x_1)$. As $x_1 \downarrow x^*(K)$, the post-correction point approaches the perceived middle steady state and $\mathcal{T}(K, x_1) \to \infty$. Near that state, the price is only slightly above $\bar{q}$, so gross investment barely exceeds depreciation. Capital therefore remains close to its level at the correction for a long time. For any fixed remaining duration of optimism, a sufficiently large correction below $D^*(K)$ leaves capital below K^M at the full correction. Net investment remains positive until the full correction, but the subsequent rational path returns to K^L.

Financing costs. Partial corrections are one way the gap between the actual and required belief wedge can narrow. Higher financing costs raise the user cost of capital and narrow this gap by raising the wedge needed to sustain a given capital stock. With the belief wedge unchanged, they can reduce the price investors are willing to pay for installed capital and thereby reduce investment. The immediate investment response depends on the resulting price decline, as in Proposition 4.

A permanent upward shift in the user-cost schedule $\Phi(K)$, such as a permanent charge per unit of installed capital, also changes the rational steady states. Under the crossing conditions of Proposition 1, it raises the middle steady state K^M and lowers the high-capital steady state K^H. Appendix J.1 shows that, for this charge and under an additional positivity condition, K^M remains the capital threshold. Capital must then exceed a higher threshold before optimism ends for the post-correction path to converge to the high-capital steady state. The capital stock at that steady state is smaller. A sufficiently large upward shift can eliminate the high-capital steady state.

Appendix J studies a temporary increase in financing costs caused by lenders assigning a higher probability to a full correction. Lenders are a separate group of investors with the preferences of Section 2.2 whose asset holdings are restricted to loans. Equity investors continue to regard their belief wedge as permanent. The loan contracts write principal down by the percentage decline in the price of capital. Before capital crosses the threshold, a full correction reduces lenders' wealth through losses on their loans and the subsequent contraction of lending. Higher perceived correction risk therefore lowers the wealth lenders expect to hold and, through the wealth motive of Section 2.2, raises the loan rate needed to sustain a given stock of lending, and hence the user cost of capital. Once the full correction occurs, lenders' assessment of correction risk no longer enters the user-cost schedule, so the reassessment leaves the post-correction capital threshold unchanged. For a small lender sector, a belief wedge close to $x^*(K^L)$, and capital stocks just above K_{AI}, the appendix establishes equilibrium reversals of accumulation, delays in crossing the threshold, and larger later price declines without reversal.

5 Distributional Incidence and Welfare

At the high-capital steady state, investors are wealthier and receive a larger share of income. These comparisons do not establish whether the expansion benefits them. Workers gain from accumulation without funding installation. Investors fund it by forgoing consumption, and the utility they gain from consumption and market wealth need not compensate for that cost. Section 5.1 characterizes wages and workers' consumption. Section 5.2 evaluates investors' ex post welfare over the entire path and gives conditions under which it is lower than if the economy had remained at K^L, despite higher wealth when the expansion is permanent.

5.1 Workers' wages and consumption

For workers, the marginal product of labor in (2) gives

$$w(K) = \begin{cases} (1-\alpha)AK^{\alpha}, & K < K_{\text{AI}}, \\ w_{\text{flat}}, & K_{\text{AI}} \le K < K_{\text{sat}}, \\ Y(K)\left[(1-\alpha) - \dfrac{\alpha \bar{K}_{\ell}}{K - \bar{K}_{\ell}}\right], & K \ge K_{\text{sat}}. \end{cases} \tag{31}$$

The wage increases with capital before deployment. During deployment, $N = bK_c$ keeps conventional capital per unit of effective labor constant, so the wage is flat. After saturation, an additional unit of raw labor supplies labor directly and expands AI deployment capacity. The wage can be written as

$$w = \frac{(1-\alpha)Y}{N} + \bar{K}_\ell \left[\frac{\gamma(1-\alpha)Y}{N} - \frac{\alpha Y}{K_c} \right].$$

The first term is the marginal product of labor supplied directly by workers. The bracketed term is the scarcity rent per unit of deployment capacity, which is zero at K_{sat} and positive above it. Each worker provides $\bar{K}_\ell$ units of this capacity. The resulting wage is increasing in K, and the pieces meet at the two technology boundaries.

Corollary 2 (Wages and the labor share)**.** *The wage $w(K)$ is weakly increasing in K. Along the paths with early and late corrections in Proposition 3 and Corollary 1, it never falls below its initial level $w(K^L)$. At the high-capital steady state, the wage is higher and the labor share lower than at the low-capital steady state.*

Capital rises during the optimistic phase. After an early full correction, it converges to K^L from above; after a late correction, it converges to K^H. Since capital never falls below K^L and the wage is weakly increasing in capital, the wage never falls below its initial level. By (4), the labor share falls during deployment and rises after saturation, but remains below its pre-deployment value $1-\alpha$. Appendix G proves the corollary.

Workers consume their wages and do not finance investment, so the same comparison applies to their consumption. It never falls below its initial level and is strictly higher whenever capital exceeds K^L. Any welfare criterion increasing in consumption at each date therefore ranks both paths above remaining at K^L. An early correction makes the gain temporary; a late correction makes it permanent.

5.2 Investors' wealth, consumption, and welfare

Investors fund installation, including adjustment costs, out of capital income. A full correction lowers the price of capital and hence their market wealth. It also lowers investment expenditure, which releases resources for consumption. Since capital is continuous at the correction, (11) gives

$$c_{\tau^+} - c_{\tau^-} = \psi K_\tau (q_{\tau^-} - q_{\tau^+}) = -\psi (W_{\tau^+} - W_{\tau^-}) > 0. \tag{32}$$

After an early correction, investors' wealth and consumption return to their initial levels. After a late correction, wealth converges to $\bar{q}K^H$, and consumption converges to a higher level because the steady-state consumption rule in (8) is increasing in wealth. Investors' steady-state wealth is greater at K^H because they hold more capital; the price per unit is the same in both steady states. A welfare comparison must also account for the resources used to install that capital and the timing of its returns.

I evaluate welfare ex post, applying the preferences in (5) to realized consumption and market wealth over the entire path, including the expansion and the correction.

The benchmark is the rational equilibrium that remains at K^L, with $W^L = \bar{q}K^L$ and $c^L = c^{ss}(W^L)$. The welfare difference is

$$\Delta U = \int_0^\infty e^{-\rho t} \left[\log \frac{c_t}{c^L} + \theta(W_t - W^L) \right] dt. \tag{33}$$

The first term measures the change in discounted utility from consumption; the second measures the direct utility from the change in market wealth. The capital loss at the correction enters through the wealth and consumption paths rather than as a separate subtraction from utility.[7]

The conditions below use the stock $K_\delta > K_{\text{sat}}$ at which rental income just covers replacement expenditure, $d(K_\delta) = \Gamma(\delta)$.

Proposition 5 (Investors' ex post welfare)**.** *Under (18) and (19), for every initial belief wedge $x \in (0, \rho)$ and full-correction date $\tau > 0$, consider the equilibrium path starting at K^L, possibly with a partial correction as in Proposition 4 before τ.*

(i) Investors' discounted utility from consumption is lower than if the economy had remained at K^L.

(ii) If $\psi > \theta c^L$ and

$$\frac{\rho\theta\bar{q}c^L}{\psi} < \min\left\{ (1-\alpha)d(K^L),\ \frac{K_\delta}{K_\delta - K^L}\left[d(K^L) - d_{\text{flat}} \right] \right\}, \tag{34}$$

then their ex post welfare is also lower: $\Delta U < 0$.

At K^L, the financial return i^L is below the rate of time preference ρ. The direct utility from wealth sustains saving at this lower financial return. Investors fund additional installation, including convex adjustment costs, while accumulation does not raise the rental rate. Discounted at ρ, the additional capital income cannot cover the expenditure on accumulating and maintaining the additional stock. Appendix G shows that the present value of investors' consumption falls, which also lowers their discounted utility from consumption. The proof establishes both welfare comparisons, under their respective conditions, for any feasible path that starts at K^L, remains at or above it with capital strictly higher over a positive interval, and satisfies investors' resource constraint (11) and the investment equation (10).

The additional conditions in part (ii) ensure that utility from market wealth does not offset this loss. In the welfare bound, consumption costs are valued at the initial marginal utility $1/c^L$. At a given capital stock, a higher price increases both market wealth and investment expenditure. The condition $\psi > \theta c^L$ ensures that, valued this way, the consumption cost exceeds the direct wealth benefit.

Condition (34) requires the capital-income loss from the lower rental rate to exceed an upper bound on the benefit of additional wealth after accounting for investment

[7] For fixed $x \in (x^*(K^L), \rho)$, write $\Delta U(x, \tau)$ for (33) along the path with a full correction at τ. Although capital converges to K^L for $\tau < T(x)$ and to K^H for $\tau > T(x)$, $\Delta U(x, \tau)$ is continuous at $\tau = T(x)$. Appendix G gives the proof.

expenditure. Both sides are expressed in consumption units per unit of capital added above K^L. The two terms on the right provide lower bounds on capital income lost because the rental rate is below $d(K^L)$. The first applies before deployment; the second applies during and after deployment. Appendix G derives these bounds, and Appendix I verifies the conditions at the illustrative parameters.

The higher wages generated by accumulation benefit workers, who do not fund installation. Investors fund the additional capital, and under the conditions in Proposition 5(ii), their gains from consumption and market wealth do not compensate for the consumption forgone. When capital converges to K^H, investors therefore have lower ex post welfare despite higher wealth and a larger share of income, while workers consume more despite a lower labor share.

6 Conclusion

Temporary optimism about AI returns can induce investment that permanently increases productive capacity. During deployment, firms expand conventional capital and AI-provided labor services together, keeping the marginal product of capital constant. Income from those services accrues to investors. As their wealth increases, they are willing to hold it at a lower interest rate, allowing the larger capital stock to be sustained under rational expectations. The rational equilibrium path from the low-capital steady state remains there. Convergence to the high-capital steady state requires optimism large enough to raise investment above replacement and persistent enough for capital to cross the threshold.

Investment demand puts upward pressure on the interest rate during the expansion. At the high-capital steady state, investors' greater wealth lowers the interest rate, wages are higher, and the labor share is lower. Investors bear the capital loss at the correction. Workers consume more even when the expansion is temporary. Under the sufficient conditions in Proposition 5, investors who finance a permanent expansion have lower ex post welfare despite higher wealth.

The outcome of a correction depends on capital relative to the threshold and on the remaining belief wedge. A full correction before capital crosses the threshold returns the economy to the low-capital steady state; a full correction afterward is followed by convergence to the high-capital steady state. Partial corrections reduce investment. During deployment, the price along the optimistic path rises, so later corrections can be larger without reversing accumulation. Slower accumulation can leave capital below the threshold at the full correction, so the subsequent rational path returns to the low-capital steady state. A fall in AI valuations is consistent with either a temporary or a permanent increase in productive capacity.

A Technology Details

Before normalizing raw labor, firms choose the split $K = K_c + K_\ell$, with $0 \le K_\ell \le \bar{K}_\ell L$, to maximize $Y = AK_c^\alpha N^{1-\alpha}$ with $N = L + \gamma K_\ell$. If (K, L, K_ℓ) is feasible, then scaling all three by the same factor remains feasible and scales output by that factor. The maximized technology is therefore constant-returns in (K, L).

The interior first-order condition equates the marginal products of capital in its two uses,

$$\alpha A K_c^{\alpha-1} N^{1-\alpha} = (1-\alpha) A K_c^\alpha N^{-\alpha} \gamma,$$

which gives $N = bK_c$ with $b = (1-\alpha)\gamma/\alpha$. Setting $L = 1$ gives (1). By the envelope theorem, the rental rate equals the marginal product of capital in conventional use. During interior deployment, it also equals the marginal product in AI use, $\alpha A b^{1-\alpha}$. Below $K_{\text{AI}} = 1/b$, the constraint $K_\ell \ge 0$ binds and all capital is conventional. Above $K_{\text{sat}} = [1 + (\gamma + b)\bar{K}_\ell]/b$, the capacity constraint binds. Substituting the relevant allocation into the envelope derivative gives the three pieces of (3).

Before deployment, the marginal product of labor is the standard Cobb–Douglas wage. In the interior region, the optimized technology is linear in K and L, with derivatives d_{flat} and w_{flat}. After saturation,

$$Y(K, L) = A(K - \bar{K}_\ell L)^\alpha \Big[L(1 + \gamma \bar{K}_\ell)\Big]^{1-\alpha}.$$

Differentiating with respect to L and then setting $L = 1$ gives the last line of (31) and, after division by output, the last line of (4). Since the maximized technology is constant-returns, Euler's theorem gives $Y = Kd + w$, which proves (2).

B Wealth-in-Utility and the Consumption Rule

At a planning date, investors evaluate consumption and wealth along the path they anticipate. Let $\widetilde{W}_t$ denote forecast wealth, initialized at market wealth $\widetilde{W}_0 = W_0$. With a fixed belief wedge, the perceived budget constraint is

$$\dot{\widetilde{W}}_t = i_t \widetilde{W}_t - c_t, \tag{35}$$

because the perceived capital return equals the interest rate. Investors evaluate the utility in (5) using forecast wealth. The physical resource constraint and the accounting identity for $W = qK$ continue to use realized income and expenditure.

Investors can allocate wealth between the capital claim and a bond paying i_t. Let λ_t be the current-value costate, the marginal value of wealth in the household optimization problem. The consumption first-order condition is $1/c_t = \lambda_t$, and the costate equation is

$$\dot{\lambda}_t = (\rho - i_t)\lambda_t - \theta.$$

Using $\lambda_t = 1/c_t$ gives

$$\frac{\dot{c}_t}{c_t} = i_t - \rho + \theta c_t,$$

which is (6). Linearity of the wealth term makes its marginal utility equal to θ, independently of forecast wealth. Reinitializing the forecast at observed market wealth therefore leaves the marginal wealth term in the Euler equation unchanged. This property allows the equilibrium conditions to be written in terms of realized consumption, market wealth, and the current belief wedge.

Because the bond is in zero aggregate supply, investors' aggregate wealth equals the value of the capital claim, $W = qK$. The interest rate enters the Euler equation through the household's marginal saving decision. In a rational steady state, the no-arbitrage condition gives $i = R^a$, while constant aggregate wealth in (13) gives $c = iW$. Combining these conditions with the steady-state Euler equation yields $c^{ss}(W) = \rho W/(1+\theta W)$ and $i^{ss}(W) = \rho/(1+\theta W)$, with $di^{ss}/dW < 0$ as in (8).

The investors' transversality condition. The transversality condition is

$$\lim_{t\to\infty} e^{-\rho t}\lambda_t \widetilde{W}_t = \lim_{t\to\infty} e^{-\rho t}\frac{\widetilde{W}_t}{c_t} = 0. \tag{36}$$

Market wealth obeys $\dot{W} = (i-x)W - c$. Subtracting this identity from (35), with equal initial wealth, gives $\widetilde{W}_t \geq W_t > 0$. Moreover,

$$\frac{d}{dt}\log\frac{\widetilde{W}_t}{W_t} = x + c_t\left(\frac{1}{W_t} - \frac{1}{\widetilde{W}_t}\right) \geq x.$$

Hence $\widetilde{W}_t \geq e^{xt}W_t$, and (36) implies (15). Appendix F.1 proves the converse from the equilibrium equations, positivity, and (15).

Applying the transversality condition for rational beliefs directly to market wealth imposes no additional restriction when $x > 0$. In fact,

$$\frac{d}{dt}\left(e^{-\rho t}\frac{W_t}{c_t}\right) = -xe^{-\rho t}\frac{W_t}{c_t} - e^{-\rho t}(1+\theta W_t),$$

so that condition holds on every solution with positive consumption at all dates. Investors' transversality condition uses forecast wealth under the belief that the current wedge is permanent. After a correction, investors evaluate their plans using the new wedge.

C Equilibrium Dynamics

The resource constraint. Gross investment is $I = K\psi\log q$, and total investment expenditure, including adjustment costs, is $K\Gamma(I/K) = K\psi(q-1)$. Investors receive capital income $Kd(K)$, pay this expenditure, and consume the difference. Hence

$$C(K,q) = K\Big[d(K) - \psi(q-1)\Big], \qquad C_K = d(K) - \psi(q-1) + Kd'(K), \qquad C_q = -\psi K.$$

These expressions imply (11). Substituting (12) into $R^aW - c$, and using $W = qK$ and (10), gives $\dot{q}K + q\dot{K} = \dot{W}$.

The law of motion for q. Substituting $c = C(K,q)$, $\dot{c} = C_K\dot{K} + C_q\dot{q}$, and $\dot{K} = K(\psi \log q - \delta)$ into the Euler equation and the no-arbitrage condition $i = R^a + x$ gives

$$\dot{q} = \frac{\rho - \theta C + \dfrac{C_K}{C}K(\psi \log q - \delta) - (\psi \log q - \delta) - \dfrac{C}{qK} - x}{\dfrac{1}{q} - \dfrac{C_q}{C}}. \tag{37}$$

The denominator is strictly positive whenever $C > 0$, because $C_q = -\psi K < 0$. The rational system sets $x = 0$. Its $\dot{q} = 0$ locus satisfies

$$\psi \log q - \delta + \frac{C(K,q)}{qK} = \rho - \theta C(K,q) + \frac{C_K(K,q)}{C(K,q)}K(\psi \log q - \delta). \tag{38}$$

The left side is the realized return excluding changes in q; the right side is the interest rate implied by the Euler equation, including the consumption growth generated by changes in K. At $\dot{q} = 0$, capital can still accumulate and change consumption, which is why the consumption-growth term remains in the nullcline condition. Equation (37) together with (10) defines the vector field $F^x(K,q)$; F^0 is the rational benchmark shown in Figure 2.

The perceived system. A fixed belief wedge enters (37) through the term $-x$. The high-capital saddle path of this vector field is $Q^H(K;x)$. When $x > x^*(K^L)$, Lemma 3 establishes that the unique equilibrium path follows this saddle path before a full correction. The post-correction price is the value on the rational saddle path at the capital stock reached.

D Proof of Proposition 1

Recall that

$$G(K) = d(K) - \Phi(K), \qquad \Phi(K) = \Gamma(\delta) + \frac{\bar{q}\rho}{1 + \theta\bar{q}K}.$$

Since $\Phi(K) \downarrow \Gamma(\delta)$ as $K \to \infty$, the condition $d_{\text{flat}} > \Gamma(\delta)$ implies that there exists $\bar{K}_\ell^* > 0$ such that, for every $\bar{K}_\ell > \bar{K}_\ell^*$,

$$\Phi(K_{\text{sat}}) < d_{\text{flat}}.$$

Together with $\Phi(K_{\text{AI}}) > d_{\text{flat}}$, this gives the inequalities in (19). After saturation, $d'(K) = (\alpha - 1)d(K)/(K - \bar{K}_\ell) < 0$. The slope condition (18) is still needed because $G'(K) = d'(K) - \Phi'(K)$, and greater wealth lowers the user cost.

The existence of the three roots follows from continuity. As $K \to 0^+$, $d(K) \to \infty$ while $\Phi(K)$ is bounded, so $G(K) > 0$. At K_{AI}, the left inequality in (19) gives $G(K_{\text{AI}}) < 0$, so a low root exists in $(0, K_{\text{AI}})$. On the flat region, $d(K) = d_{\text{flat}}$ and Φ is strictly decreasing, so G is strictly increasing. The right inequality in (19) gives $G(K_{\text{sat}}) > 0$, so the middle steady state exists and is unique. Beyond saturation, $d(K) \to 0$ while $\Phi(K) \to \Gamma(\delta) > 0$, so $G(K) \to -\Gamma(\delta) < 0$; with $G(K_{\text{sat}}) > 0$, a high root exists. If (18) holds, G is strictly decreasing in both outer regions, so each outer root is unique. □

E Local Stability: Proof of Lemma 1

Write the rational vector field as $F^0 = (f_1, f_2)$, with

$$f_1(K, q) = K(\psi \log q - \delta)$$

and $f_2(K, q)$ equal to the right-hand side of (37) at $x = 0$. Its denominator is positive when consumption is positive. On the horizontal nullcline $q = \bar{q}$, capital growth vanishes and $C(K, \bar{q}) = K[d(K) - \Gamma(\delta)]$. The numerator of f_2 then becomes

$$\rho - \theta C(K, \bar{q}) - \frac{C(K, \bar{q})}{\bar{q}K} = -\frac{1 + \theta \bar{q} K}{\bar{q}} G(K).$$

This proves the sign identity (21): the price rises where the rental rate of capital falls short of its user cost and falls where it exceeds that cost.

At a steady state, $f_{1,K} = 0$ and $f_{1,q} = K\psi/\bar{q} > 0$. Because the numerator of f_2 is zero at the root, differentiating it with respect to K gives

$$f_{2,K}(K^j, \bar{q}) = -\frac{1 + \theta \bar{q} K^j}{\bar{q}} \frac{G'(K^j)}{\dfrac{1}{\bar{q}} - \dfrac{C_q}{C}}.$$

It follows that

$$\operatorname{sign} \det DF^0(K^j, \bar{q}) = \operatorname{sign} G'(K^j).$$

At a steady state with $G'(K^j) < 0$, the determinant is negative, so the steady state is a saddle.

At the middle steady state, $d' = 0$. In that region $C = K[d_{\text{flat}} - \psi(q-1)]$, so $C_K K/C = 1$ and the two terms involving capital growth cancel in the numerator of (37). Differentiating the remaining numerator with respect to q, and then evaluating at $(K^M, \bar{q})$, gives

$$\theta \psi K^M + \frac{\psi}{\bar{q}} + \frac{d_{\text{flat}} - \Gamma(\delta)}{\bar{q}^2} > 0.$$

The denominator of (37) is positive, so the trace is positive. Since $G'(K^M) = -\Phi'(K^M) > 0$, the determinant is also positive. Direct substitution of the flat-region expressions into the characteristic discriminant reduces it to a perfect square divided by a positive denominator. The eigenvalues are therefore real and positive: K^M is an unstable node. This proves Lemma 1. For a fixed wedge $0 \le x < \rho$, the same calculation applies with $\rho - x$ in place of ρ. The saddle paths for the numerical example are shown in Figure 2. □

F Global Equilibrium Paths

This appendix proves Lemmas 2 and 3. Throughout, maintain (18) and (19) and hold the wedge fixed along each path. I first use transversality to establish compactness of equilibrium paths for $0 \le x < \rho$, then construct the rational saddle paths.

Under rational beliefs, the signs of $\dot{q}$ on $q = \bar{q}$ keep the high saddle path above that line on (K^M, K^H) and the low saddle path below it on (K^L, K^M) as they are followed backward. A consumption bound for the high saddle path and an invariant region for the low saddle path establish that both approach $(K^M, \bar{q})$. These connections exclude periodic or other nonconvergent equilibrium paths. The arguments extend to positive wedges, with separate treatment of the case in which the low and middle steady states merge.

F.1 Transversality and compactness

For the proof, define

$$\begin{aligned} S(K) &= d(K) + \psi, \quad B(K,q) = S(K) - \psi q, \\ g(q) &= \psi \log(q/\bar{q}), \qquad\quad M = \rho - x - \theta K B - B/q. \end{aligned}$$

Thus $c = KB$, $W = qK$, and positive consumption requires $K > 0$ and $0 < q < S(K)/\psi$. Away from the technology boundaries, the equilibrium equations are

$$\dot{K} = Kg, \qquad \dot{q} = \frac{qB}{S}\left(M + \frac{Kd'}{B}g\right). \tag{39}$$

The wealth–consumption ratio $Z = W/c = q/B$ satisfies

$$\dot{Z} = (\rho - x)Z - 1 - \theta W. \tag{40}$$

Solving this equation with (15) gives

$$Z_t = \int_t^\infty e^{-(\rho-x)(s-t)}(1 + \theta W_s)\, ds \geq \frac{1}{\rho - x}. \tag{41}$$

The transformation is invertible:

$$q = \frac{S(K)Z}{1 + \psi Z}, \qquad c = \frac{KS(K)}{1 + \psi Z}. \tag{42}$$

In (K, Z) coordinates the field is

$$\begin{aligned} \dot{K} &= K\left[\psi \log\left(\frac{S(K)Z}{1 + \psi Z}\right) - \delta\right], \\ \dot{Z} &= (\rho - x)Z - 1 - \theta \frac{KS(K)Z}{1 + \psi Z}. \end{aligned} \tag{43}$$

It is continuous and locally Lipschitz on $(0, \infty)^2$, including the technology boundaries, and depends continuously on x.

Compactness. Every equilibrium path has its forward orbit in a compact subset of the positive-consumption region. Let $K_\delta > K_{\text{sat}}$ solve $d(K_\delta) = \Gamma(\delta)$. Positive consumption requires $q < 1 + d(K)/\psi$. Hence $K > K_\delta$ implies $q < \bar{q}$ and $\dot{K} < 0$, so

$$K_t \leq \overline{K} \equiv \max\{K_0, K_\delta\}.$$

Also $W = qK < KS(K)/\psi$. Since $Kd(K) \to 0$ as $K \downarrow 0$, wealth has a uniform upper bound $\overline{W}$ for $0 < K \leq \overline{K}$. Equation (41) gives

$$\frac{1}{\rho - x} \leq Z_t \leq \frac{1 + \theta\overline{W}}{\rho - x}. \tag{44}$$

The lower bound implies $q_t \geq S(K_t)/(\psi + \rho - x)$. Since $d(K) \to \infty$ as $K \downarrow 0$, this price exceeds $\bar{q}$ at sufficiently small stocks, where capital must increase. Capital is bounded away from zero. Equations (42) and (44) then bound price and consumption away from zero and price away from the zero-consumption boundary. These bounds are uniform when K_0 ranges over a compact subset of $(0, \infty)$ and x over a compact subset of $[0, \rho)$.

The bounds also establish the converse of the transversality implication in Appendix B. Along a solution of the equilibrium equations satisfying positivity and (15), consumption has a positive lower bound $\underline{c}$. Forecast wealth remains positive because $\widetilde{W} \geq W$. By the Euler equation,

$$\frac{d}{dt}\left(e^{-\rho t}\frac{\widetilde{W}_t}{c_t}\right) = -\theta c_t e^{-\rho t}\frac{\widetilde{W}_t}{c_t} - e^{-\rho t}.$$

Thus $0 \leq e^{-\rho t}\widetilde{W}_t/c_t \leq (W_0/c_0)e^{-\theta \underline{c} t} \to 0$. Given the equilibrium equations and positivity, condition (15) is therefore equivalent to the transversality condition on forecast wealth.

F.2 The rational high-capital saddle path

Set $x = 0$. Proposition 1 gives

$$0 < K^L < K_{\text{AI}} < K^M < K_{\text{sat}} < K^H.$$

The function G is positive on $(0, K^L)$ and (K^M, K^H), and negative on (K^L, K^M) and (K^H, ∞). The low- and high-capital steady states are saddles and the middle steady state is a source.

Set

$$\bar{B} = d_{\text{flat}} - \Gamma(\delta) > 0, \qquad c^M = K^M\bar{B}.$$

At the middle steady state,

$$\rho - \theta c^M = \bar{B}/\bar{q}, \qquad \psi + \rho - \theta c^M = (d_{\text{flat}} + \psi)/\bar{q}. \tag{45}$$

For $K^M < K \leq K^H$, the sign of G and the increasing steady-state consumption schedule imply

$$K[d(K) - \Gamma(\delta)] \geq \frac{\rho\bar{q}K}{1 + \theta\bar{q}K} > c^M. \tag{46}$$

Differentiating $c = KB$ along the rational path gives

$$\frac{\dot{c}}{c} = \left(1 + \frac{Kd'}{S}\right) g + \psi - \frac{\psi q}{S}(\psi + \rho - \theta c). \tag{47}$$

On the boundary $c = c^M$ with $K^M < K < K^H$, (46) implies $q > \bar{q}$. Since $d' \leq 0$ and $S(K) \leq d_{\text{flat}} + \psi$,

$$\frac{\dot{c}}{c} \leq \psi \left[\log(q/\bar{q}) + 1 - q/\bar{q}\right] < 0. \tag{48}$$

The inequality holds on each side of the saturation boundary.

The local saddle path to the left of K^H has $q > \bar{q}$ and, close to the saddle, $c > c^M$. Follow it backward. At $q = \bar{q}$, $G > 0$ implies $\dot{q} < 0$, so backward flow points into $q > \bar{q}$. At $c = c^M$, (48) makes backward flow point into $c > c^M$. The saddle path remains in

$$K^M < K < K^H, \qquad q > \bar{q}, \qquad c > c^M.$$

Capital decreases backward. This region closes to $(K^M, \bar{q})$ at $K = K^M$; the only steady states in its closure are its endpoints. Its other boundaries cannot terminate the orbit. Monotonicity of K therefore gives the backward limit $(K^M, \bar{q})$. The saddle path is defined on (K^M, K^H), with a unique price at each capital stock.

The local saddle path to the right of K^H has $q < \bar{q}$. Where $q = \bar{q}$ lies in the positive-consumption region, $G < 0$ makes backward flow point into $q < \bar{q}$. Once $d(K) \leq \Gamma(\delta)$, positive consumption itself requires $q < \bar{q}$. Capital increases backward. The saddle path cannot terminate at a finite stock: bounded stocks on a backward interval give a uniform wealth bound on the subsequent path, and (41) keeps price and consumption in the interior. A finite limiting stock would have to be another steady state above K^H, which does not exist. The saddle path is defined for every $K > K^H$. Convergence to the interior high steady state guarantees transversality on both segments.

F.3 The rational low-capital saddle path

The low saddle path can be constructed without additional parameter restrictions. Differentiating M along (39) yields

$$\dot{M} = a_M M - b_M, \qquad a_M = \frac{B}{q} + \frac{\theta K \psi q B}{S} > 0, \qquad b_M = \theta K B g \left(1 + \frac{Kd'}{S}\right). \tag{49}$$

This identity holds for every fixed wedge. On the interval (K^L, K^M),

$$S + Kd' = \begin{cases} \psi + \alpha d > 0, & K < K_{\text{AI}}, \\ \psi + d > 0, & K > K_{\text{AI}}. \end{cases}$$

Define $\Xi = M - \theta K q g$ and consider the region

$$K^L < K < K^M, \qquad \theta K q g < M < 0. \tag{50}$$

The inequalities imply $g < 0$, so $q < \bar{q}$. Consumption is positive because $d(K) \geq d_{\text{flat}} > \Gamma(\delta)$ on this interval.

Both boundaries make the region invariant backward in time. At $M = 0$, (49) gives $\dot{M} = -b_M > 0$. At $\Xi = 0$, direct differentiation gives

$$\dot{\Xi} = -\theta K q \left[\left(1 + \frac{K d' + \theta K q B}{S} \right) g^2 + \frac{K d'}{q} g \right] < 0. \tag{51}$$

The first coefficient is positive, while $d' \leq 0$ and $g < 0$ make the second term nonnegative. The directional inequalities hold on both sides of the technology boundary.

The local saddle path to the right of K^L enters this region. Let $\lambda_- < 0$ be the stable eigenvalue at the low saddle. Along that saddle path, linearizing (49) gives

$$\frac{M}{\theta K q g} \longrightarrow \frac{(B/q)(1 + K d'/S)}{a_M - \lambda_-} \in (0, 1),$$

where the quantities on the right are evaluated at $(K^L, \bar{q})$. Here $0 < 1 + K d'/S < 1$ and $a_M > B/q$. Since $g < 0$ on the right saddle path, the ratio establishes $\theta K q g < M < 0$ locally.

The region closes to $q = \bar{q}$ at both K^L and K^M. To see this, for $q < \bar{q}$,

$$M_q = \theta K \psi + S/q^2 > 0, \qquad \Xi_q = S/q^2 - \theta K g > 0.$$

At either root, $M = \Xi = 0$ when $q = \bar{q}$, so no lower price satisfies $\Xi > 0$. At an interior stock, $G < 0$ gives $M(K, \bar{q}) > 0$. The zero of M lies below $\bar{q}$, and the zero of Ξ lies below it; these two curves bound the region in (50). They stay away from $q = 0$, where M and Ξ tend to minus infinity, and consumption has a positive lower bound. Following the saddle path backward therefore gives increasing capital until its limit at $(K^M, \bar{q})$. The saddle path is defined on (K^L, K^M), with capital increasing monotonically backward in time.

The saddle path to the left of K^L has $q > \bar{q}$. Since $G > 0$ below K^L, backward flow at $q = \bar{q}$ points into $q > \bar{q}$, and capital decreases backward. The bounds from (41) prevent termination at an interior positive stock. There is no further steady state below K^L, so the saddle path is defined for every $0 < K < K^L$.

F.4 Rational uniqueness

The saddle paths and the middle stationary path establish existence. Every point on a saddle path converges to an interior steady state and satisfies transversality.

For uniqueness, use the continuous locally Lipschitz field (43). A periodic orbit has index $+1$, equal to the sum of the indices of the steady states it encloses. The middle source has index $+1$ and each saddle has index -1. A periodic orbit would have to enclose the source and neither saddle. But the construction of the high saddle path gives a trajectory from the source to the high saddle. It could not cross a periodic orbit, so an orbit enclosing the source would also enclose the high saddle. This contradicts the index requirement.

Compactness leaves a steady state or a collection of steady states and connecting trajectories as the possible omega-limit set.[8] The latter possibility is also excluded. The source has a neighborhood with strictly outward flow and cannot belong to the omega-limit set of another orbit. Every trajectory entering either saddle lies on a saddle path already constructed. Backward, these paths approach the source or leave every compact subset of the domain toward $K = 0$ or $K = \infty$. None can be a connection in a compact omega-limit set that excludes the source. Every equilibrium path therefore converges to one steady state.

A path converging to a saddle lies on its saddle path. The two saddle paths express the price as a function of capital on disjoint intervals, $(0, K^M)$ and (K^M, ∞). Only the stationary path converges to the source. This proves Lemma 2.

A price above $\bar{q}$ at K^L. Consider a solution of the rational equilibrium equations starting at $K_0 = K^L$ with $q_0 > \bar{q}$ and positive consumption, without imposing transversality. The solution cannot cross below $\bar{q}$. On (K^L, K^M) and (K^H, K_δ), $G < 0$, so (21) gives $\dot{q} > 0$ at $q = \bar{q}$. On (K^M, K^H), the solution lies above the high saddle path, whose limiting price at K^M is $\bar{q}$, and cannot cross it. Nor can it reach a steady state in finite time. Thus $q > \bar{q}$ and capital increases for as long as consumption is positive.

Positive consumption and $q > \bar{q}$ imply $K < K_\delta$. Price and wealth are bounded because $q < S(K)/\psi$ and $K \geq K^L$. Moreover, (40) gives $\dot{Z} \leq \rho Z$, so Z cannot diverge in finite time. By (42), consumption remains positive at every finite date, and the solution extends to all $t \geq 0$. Since capital increases and is bounded, it has a limit. The law of motion for q is bounded on this region, so integrating the capital-accumulation equation gives $q_t \to \bar{q}$. If the limiting capital stock were below K_δ, the limiting point would have positive consumption and would have to be a steady state. Capital cannot return to K^L; the middle source cannot be approached by a nonstationary solution; and convergence to K^H would require the high saddle path, which is undefined at K^L. It follows that $K_t \to K_\delta$, $q_t \to \bar{q}$, and $c_t \to 0$.

At this limit, capital income just covers replacement expenditure, since $d(K_\delta) = \Gamma(\delta)$. The wealth–consumption ratio Z_t diverges while wealth remains bounded. Transversality would imply (41) and hence the uniform upper bound in (44), a contradiction. This solution therefore cannot be a rational equilibrium path.

F.5 Equilibrium paths with positive wedges

For a fixed wedge, define the rental rate minus the user cost as

$$G_x(K) = d(K) - \Gamma(\delta) - \frac{\bar{q}(\rho - x)}{1 + \theta \bar{q} K} = G(K) + \frac{\bar{q} x}{1 + \theta \bar{q} K}. \tag{52}$$

[8] The Poincaré–Bendixson theorem for locally Lipschitz planar fields is discussed in Hanche-Olsen (2017, Chapter 6); Deconinck (2009, Section 6.4) states the finite-equilibrium limit-set classification. The index formula is presented in Pernarowski (n.d.).

On either outer region,

$$G'_x(K) = G'(K) - \frac{\theta\bar{q}^2 x}{(1+\theta\bar{q}K)^2} < 0.$$

During deployment, $G'_x > 0$ for $x < \rho$. Also $G_x(K_{\text{sat}}) > 0$, and $G_x(K_{\text{AI}})$ is negative, zero, or positive according as x is below, equal to, or above $x^*(K^L)$. The required wedge satisfies $0 < x^*(K^L) < \rho$, since

$$\rho - x^*(K^L) = \frac{1+\theta\bar{q}K_{\text{AI}}}{\bar{q}}\Big[d_{\text{flat}} - \Gamma(\delta)\Big] > 0.$$

Wedges below $x^*(K^L)$. For $0 \le x < x^*(K^L)$, the perceived system satisfies (18) and (19) with $\rho - x$ in place of ρ. The compactness, saddle-path, and uniqueness arguments apply with that substitution. They give the steady states and equilibrium paths in Lemma 3(ii).

Wedges above $x^*(K^L)$. For $x^*(K^L) < x < \rho$, $G_x > 0$ at every stock below its unique root $K^H(x) > K_{\text{sat}}$, and $G_x < 0$ above it. This steady state is a saddle. To its left, the saddle path has $q > \bar{q}$; backward flow at $q = \bar{q}$ points into this region. Capital decreases backward, and the bounds in (41) prevent termination at a positive stock. There is no lower steady state, so the saddle path is defined on $(0, K^H(x))$. The segment to the right is defined for all $K > K^H(x)$ by the same argument used for the rational high saddle.

Compactness and the horizontal-nullcline signs give uniqueness directly. Below $K^H(x)$, a path at or below $\bar{q}$ enters $q < \bar{q}$ and has declining capital, with no lower steady state to approach. Above $K^H(x)$, a path at or above $\bar{q}$ has increasing capital, with no higher steady state to approach. Neither can remain in a compact interior set. A nonstationary equilibrium path must therefore have $q > \bar{q}$ below $K^H(x)$ or $q < \bar{q}$ above it, and cannot cross the steady-state stock. Its capital converges monotonically to $K^H(x)$, so it belongs to the saddle path. This proves part (i).

The case $x = x^*(K^L)$. At $x = x^*(K^L)$, the low and middle steady states coincide at K_{AI}. The other steady state is $K^H(x)$, and $G_x > 0$ below it except at K_{AI}. The consumption bound for the high saddle path remains valid with $\rho - x$ replacing ρ, K_{AI} replacing K^M, and $c = K_{\text{AI}}[d_{\text{flat}} - \Gamma(\delta)]$ at the lower endpoint. It gives a high saddle path defined for $K > K_{\text{AI}}$ whose left endpoint is $(K_{\text{AI}}, \bar{q})$.

Extending the pre-deployment technology smoothly across the boundary gives a saddle at $(K_{\text{AI}}, \bar{q})$, because the left derivative of G_x is negative. The saddle path on $K < K_{\text{AI}}$ has $q > \bar{q}$. Backward flow keeps it there, and compactness of the forward path prevents an endpoint at a positive stock. The saddle path is defined for all $0 < K < K_{\text{AI}}$. Extending the deployment technology to the left instead gives a source: its two eigenvalues are positive by the calculation in Appendix E. A nonstationary path cannot approach $(K_{\text{AI}}, \bar{q})$ while remaining to its right.

These facts also establish uniqueness. A path with $q \le \bar{q}$ below the high steady state, other than the stationary path at K_{AI}, enters $q < \bar{q}$ and has decreasing capital.

Compactness would require convergence to a lower steady state. The only candidate is K_{AI}, approached from its right, which the source calculation excludes. A path with $q \geq \bar{q}$ above the high steady state has increasing capital and no higher steady state to approach. The remaining equilibrium paths have monotone capital and converge to one of the two steady states. Convergence to the high saddle requires its saddle path, which is undefined at or below K_{AI}. Convergence to K_{AI} must occur from the left on the unique saddle path. At $K = K_{\text{AI}}$, only the stationary path is an equilibrium path.

F.6 Continuity of the equilibrium price

Let $Q(K; x)$ denote the unique equilibrium initial price. Consider $(K_n, x_n) \to (K_0, x_0)$, where $K_0 > 0$ and $0 \leq x_0 < \rho$. The bounds in Section F.1 place all the corresponding forward paths in a common compact subset of the positive-consumption region. Every subsequence of initial prices has a convergent further subsequence. In (K, Z) coordinates, continuous dependence on initial data and parameters gives convergence of the associated paths on every bounded time interval.[9]

The limiting path stays in the same compact interior set and has bounded Z. Since $\rho - x_0 > 0$, it satisfies (15) and is an equilibrium path. Uniqueness at (K_0, x_0), including the case $x_0 = x^*(K^L)$ proved above, implies that its initial price is $Q(K_0; x_0)$. Every convergent subsequence has this limit, proving joint continuity of Q. This completes Lemma 3. □

F.7 Prices as a function of the wedge

At a common (K, q) away from $q = \bar{q}$,

$$\frac{\partial}{\partial x}\left(\frac{dq}{dK}\right) = -\frac{1}{\dot{K}[1/q - C_q/C]}.$$

The derivative is negative on saddle paths with increasing capital and positive on saddle paths with decreasing capital. The low and high steady-state capital stocks increase with x, because $\partial_x G_x > 0$ and $G'_x < 0$ at those roots.

Consider $x_2 > x_1$. At $K^H(x_1)$, the x_2 high saddle path lies above $\bar{q}$. On the common growing region, a crossing of the two saddle paths could only take their price difference from positive to negative as K increases. This would contradict their ordering at $K^H(x_1)$. Hence $Q^H(K; x_2) > Q^H(K; x_1)$ throughout their common growth region.

For two low saddle paths with decreasing capital, $K^L(x_2) > K^L(x_1)$. At $K^L(x_2)$, the x_2 price equals $\bar{q}$, while the x_1 price is below it. On the common declining region, a crossing could only take the difference from negative to positive as K increases, so the initial positive ordering is preserved. The price on a low saddle path with declining capital is therefore also strictly increasing in the wedge.

[9] See Hanche-Olsen (2017, Chapter 1) for continuous dependence of locally Lipschitz flows on initial data and parameters.

G Proofs for Sections 3–5

Closed-form required wedge. In the flat deployment region, a permanent belief wedge changes the steady-state condition to

$$d_{\text{flat}} = \Gamma(\delta) + \frac{\bar{q}(\rho - x)}{1 + \theta \bar{q} K}.$$

Solving for the perceived middle steady state gives

$$K^M(x) = \frac{1}{\theta \bar{q}} \left[\frac{\bar{q}(\rho - x)}{d_{\text{flat}} - \Gamma(\delta)} - 1 \right] = K^M - \frac{x}{\theta [d_{\text{flat}} - \Gamma(\delta)]}.$$

For $K \in [K_{\text{AI}}, K^M]$, the wedge that moves this steady state to K solves $K = K^M(x)$:

$$x^*(K) = \theta [d_{\text{flat}} - \Gamma(\delta)](K^M - K).$$

For $K < K_{\text{AI}}$, the low and middle steady states merge at the start of deployment, so the required wedge remains $x^*(K_{\text{AI}})$. Combining the two cases yields (24). Lemma 3(i) establishes a unique equilibrium path from K^L along the saddle path to $K^H(x)$, with positive net investment through K^M.

Proposition 2. By Lemma 2, the unique rational equilibrium path from K^L lies on the low saddle path. At that stock the saddle path contains the stationary point $(K^L, \bar{q})$. The path remains there and cannot converge to K^H. □

Proposition 3(i). Let $x \in (x^*(K^L), \rho)$. By Lemma 3(i), $Q^H(K; x)$ is continuous and strictly above $\bar{q}$ on the compact interval $[K^L, K^M]$, and consumption is positive along the path. Hence $\psi \log Q^H(K; x) - \delta$ is bounded below by a positive number. Capital rises from K^L to K^M in finite time. □

Proposition 3(ii). Part (i) makes the denominator in (26) positive and bounded away from zero, so $0 < T(x) < \infty$. The solution reaches K^M exactly at $T(x)$. Since capital is continuous at the correction, $\tau < T(x)$ implies $K_\tau < K^M$, while $\tau > T(x)$ implies $K_\tau > K^M$. By Lemma 2, the rational path converges to K^L in the first case and to K^H in the second. When $\tau = T(x)$, the post-correction state is the middle steady state. □

Proposition 3(iii). Section F.7 establishes $Q^H(K; x_2) > Q^H(K; x_1)$ on the common growth region when $x_2 > x_1$. For $x_2 > x_1 > x^*(K^L)$, this region contains $[K^L, K^M]$. The denominator in (26) is therefore strictly larger at the higher wedge, making $T(x)$ strictly decreasing.

As $x \downarrow x^*(K^L)$, joint continuity of the equilibrium price and continuous dependence of the flow give convergence on each bounded time interval to the equilibrium path from K^L with wedge $x^*(K^L)$. That path converges to $K_{\text{AI}} < K^M$ and never reaches K^M. A bounded sequence of hitting times would give a finite hitting time on the limiting path, a contradiction. Thus $T(x) \to \infty$. □

Corollary 1. At the full correction, the wedge falls from x to zero and the price adjusts to the rational equilibrium path from K_τ. If $\tau > T(x)$, then $K_\tau > K^M$, so the price lies on the rational high saddle path and capital converges to K^H. If $\tau < T(x)$, then $K_\tau \in (K^L, K^M)$, so the price lies on the rational low saddle path and capital converges to K^L. □

The post-correction yield curve. No arbitrage gives the price $\exp[-\int_0^h i_{\tau+s}\,ds]$ for a unit payoff at $\tau + h$, hence the yield formula. Corollary 1 and the Euler equation imply $i_{\tau+s} \to i^L$ below K^M and $i_{\tau+s} \to i^H$ above it. Time averaging gives (27).

As $K_\tau \to K^M$ along either rational saddle path, the price approaches $\bar{q}$. Continuity of the short rate then gives $i_{\tau+} \to i^M$. Since $i^H < i^M < i^L$ by (8), nearby stocks have $i^H < i_{\tau+} < i^L$; the long-yield limits then give the stated spread signs for all sufficiently large h.

On each fixed finite horizon, continuous dependence on initial conditions gives uniform convergence to the stationary middle path as $K_\tau \to K^M$. Thus $y_\tau(h) \to i^M$ and the spread vanishes at every fixed maturity. □

Corollary 2. The first two pieces of (31) are increasing and flat. In the saturation region, differentiating the last piece yields a positive derivative because $\alpha \in (0, 1)$ and $K > \bar{K}_\ell$. The pieces coincide at both technology boundaries, so $w(K)$ is weakly increasing everywhere and $w(K^H) > w(K^L)$. On the perceived path capital rises. After an early correction, the rational equilibrium path lies on the low saddle path with $q < \bar{q}$ (Lemma 2), so capital converges monotonically to K^L from above. Capital remains at or above K^L after either correction, so the wage remains at or above its initial value. □

Proposition 5. I prove both parts for any path with positive price and consumption, locally absolutely continuous capital, and (10) and (11) holding almost everywhere, with $K_0 = K^L$ and $K_t \geq K^L$, strictly above K^L on a positive time interval. I then verify these restrictions for the equilibrium paths in the proposition.

Positive consumption requires $q < 1 + d(K)/\psi$. Thus $q < \bar{q}$ whenever $K \geq K_\delta$, and a path starting at K^L cannot cross K_δ from below. Together with the assumed lower bound, this gives $K^L \leq K_t < K_\delta$. Since $d(K_t) \leq d(K^L)$, price, consumption, and wealth are bounded above. The positive parts of both welfare integrands are therefore bounded, so the discounted integrals are well defined, allowing the value $-\infty$.

The low steady state satisfies

$$c^L = K^L[d(K^L) - \Gamma(\delta)], \qquad \bar{q}i^L = \frac{c^L}{K^L}, \qquad \rho - i^L = \theta c^L > 0.$$

The resource constraint (11) gives the exact identity

$$c_t - c^L = K_t[d(K_t) - d(K^L)] + \frac{c^L}{K^L}(K_t - K^L) - \psi K_t(q_t - \bar{q}).$$

Convexity of the exponential and the investment equation imply

$$K_t(q_t - \bar{q}) \geq \bar{q} K_t \log(q_t/\bar{q}) = \frac{\bar{q}}{\psi} \dot{K}_t.$$

Capital is continuous, $K_0 = K^L$, and K_t is bounded, so integration by parts gives

$$\int_0^\infty e^{-\rho t} \dot{K}_t \, dt = \rho \int_0^\infty e^{-\rho t} (K_t - K^L) \, dt.$$

Substitution into the resource identity and integration therefore yield

$$\begin{aligned} \int_0^\infty e^{-\rho t}(c_t - c^L) \, dt &\leq -\int_0^\infty e^{-\rho t} \left[\bar{q}(\rho - i^L)(K_t - K^L) + K_t \Big[d(K^L) - d(K_t) \Big] \right] dt \\ &< 0. \end{aligned}$$

The strict inequality follows from $\rho > i^L$, $d(K_t) \leq d(K^L)$, and $K_t > K^L$ on a positive time interval. Since $c^L \log(c_t/c^L) \leq c_t - c^L$, discounted utility from consumption is also lower. This proves part (i).

For part (ii), add the wealth-utility term to the same concavity bound and use the low-steady-state identities to obtain

$$\begin{aligned} c^L \left[\log \frac{c_t}{c^L} + \theta(W_t - W^L) \right] \leq K_t[d(K_t) - d(K^L)] + \rho \bar{q}(K_t - K^L) \\ - (\psi - \theta c^L) K_t (q_t - \bar{q}). \end{aligned}$$

Under $\psi > \theta c^L$, substitution of the investment bound preserves the upper bound. The same integration by parts then gives

$$c^L \Delta U \leq \int_0^\infty e^{-\rho t} \left[\frac{\rho \theta \bar{q} c^L}{\psi} (K_t - K^L) - K_t \Big[d(K^L) - d(K_t) \Big] \right] dt. \tag{53}$$

For $K^L < K \leq K_{\text{AI}}$, the Cobb–Douglas rental rate and concavity of K^α give

$$\begin{aligned} K[d(K^L) - d(K)] &= d(K^L) \left[K - (K^L)^{1-\alpha} K^\alpha \right] \\ &\geq (1 - \alpha) d(K^L)(K - K^L). \end{aligned}$$

For $K_{\text{AI}} \leq K \leq K_\delta$, $d(K) \leq d_{\text{flat}}$, and $K/(K - K^L)$ is decreasing. Hence

$$K[d(K^L) - d(K)] \geq [d(K^L) - d_{\text{flat}}] K \geq \frac{K_\delta}{K_\delta - K^L} [d(K^L) - d_{\text{flat}}](K - K^L).$$

Condition (34) therefore makes the integrand in (53) nonpositive everywhere and strictly negative whenever $K_t > K^L$. Hence $\Delta U < 0$, proving part (ii).

For $0 < x < x^*(K^L)$, the lower user cost in (22) places the perceived low steady state at $K^L < K^L(x) < K_{\text{AI}}$. At $x = x^*(K^L)$, capital starting at K^L instead converges to K_{AI}. Lemmas 2 and 3 therefore imply that, for every $0 < x < \rho$, the initial expansion has $K_t > K^L$ on a positive time interval. The paper's expansions and their post-correction paths remain at or above K^L, including after partial corrections with nonnegative remaining wedges. Both parts apply regardless of the full-correction date or the steady state to which capital subsequently converges. □

Welfare at the capital threshold. Fix $x \in (x^*(K^L), \rho)$, and let correction dates τ_n converge to $T(x)$. The pre-correction path is the same for every τ_n, and $K_{\tau_n} \to K^M$. Continuity of the rational equilibrium price gives $q_{\tau_n^+} \to \bar{q}$. Continuous dependence of the rational flow then implies that, on each bounded interval after the correction, the path converges uniformly to the stationary allocation $(K^M, \bar{q})$.

For all sufficiently large n, the rational continuations lie on the portions of the two saddle paths with capital in $[K^L, K^H]$. Including the middle steady state, these paths form a compact set on which consumption is continuous and strictly positive. Consumption is therefore bounded away from zero, and both $\log(c/c^L)$ and $W - W^L$ are uniformly bounded. The same bounds hold on the pre-correction paths up to a common finite horizon containing all τ_n. Discounting supplies an integrable bound for the welfare integrand. Dominated convergence in (33) gives

$$\lim_{\tau \to T(x)^-} \Delta U(x, \tau) = \Delta U(x, T(x)) = \lim_{\tau \to T(x)^+} \Delta U(x, \tau).$$

□

H Partial Corrections and Investment

Lemma 3 determines the equilibrium path for any remaining wedge $0 \leq x_1 < x_0 < \rho$. For $x_1 < x^*(K^L)$, capital converges to the low steady state below $K^M(x_1)$ and to the high steady state above it; at $K^M(x_1)$, the equilibrium path is stationary. For $x_1 > x^*(K^L)$, every equilibrium path converges to the high steady state. Capital increases toward the limiting steady state from below and decreases toward it from above. The equilibrium price is continuous in capital and the wedge, including at $x_1 = x^*(K^L)$, where the low and middle steady states merge.

The equilibrium correction boundary. Let $\bar{B} = d_{\text{flat}} - \Gamma(\delta) > 0$. Equation (22) gives

$$K^M(x) = K^M - \frac{x}{\theta \bar{B}}.$$

At $K \in (K_{\text{AI}}, K^M)$, the wedge that places the middle steady state at K is $x^*(K)$. By Lemma 3, the post-correction price is above $\bar{q}$ when $x_1 > x^*(K)$, below $\bar{q}$ when $x_1 < x^*(K)$, and equal to $\bar{q}$ at the source. As the remaining wedge approaches the required wedge from either side, the post-correction price converges to $\bar{q}$. Substituting into $D = 1 - q^+/Q_0(K)$ proves (29) and part (i) of Proposition 4. Thus $D^*(K)$ is the supremum of corrections consistent with positive net investment. At $D = D^*(K)$, net investment is zero and capital remains constant.

The price during deployment. Fix x and define

$$B = d(K) - \psi(q - 1), \qquad S = d(K) + \psi, \qquad g = \psi \log q - \delta,$$

and $M = \rho - x - \theta KB - B/q$. Consumption is KB, and the vector field (37) becomes

$$\dot{K} = Kg, \qquad \dot{q} = \frac{qB}{S}\left(M + \frac{Kd'(K)}{B}g\right). \tag{54}$$

Since

$$M_K = -\theta(B + Kd') - \frac{d'}{q}, \qquad M_q = \theta K\psi + \frac{S}{q^2},$$

the time derivative of M satisfies

$$\dot{M} = a_M M - b_M, \qquad a_M = \frac{B}{q} + \frac{\theta K\psi qB}{S} > 0, \qquad b_M = \theta KBg\left(1 + \frac{Kd'}{S}\right). \tag{55}$$

Along a growing high saddle path, $B > 0$ and $g > 0$. If $S + Kd' > 0$, then $b_M > 0$. At the limiting high steady state, $M = 0$ and a_M converges to a positive constant. Solving (55) with this bounded terminal condition yields

$$M_t = \int_t^\infty \exp\Big[-\int_t^s a_{M,u}\,du\Big]\, b_{M,s}\,ds > 0. \tag{56}$$

The identity applies piecewise across the technology kinks, where M is continuous. Wherever $d' = 0$, equations (54) and (56) imply $\dot{q} > 0$. Since $\dot{K} > 0$, the saddle path has $\partial_K Q^H(K; x) > 0$. In particular, $Q_0'(K) > 0$ during deployment, which proves the derivative in part (ii) of Proposition 4.

With $\theta = 0$, the forcing term b_M vanishes, the bounded terminal solution has $M = 0$, and q is constant wherever the rental rate is constant along an equilibrium path with increasing capital. The strict increase in the correction boundary therefore depends on the effect of wealth on saving.

The regularity condition is automatic before and during deployment. After saturation,

$$S + Kd' = \psi + d(K)\frac{\alpha K - \bar{K}_\ell}{K - \bar{K}_\ell}.$$

A sufficient restriction in primitives is

$$\gamma\bar{K}_\ell(1 - 2\alpha) \le \alpha^2. \tag{57}$$

It is equivalent to $K_{\text{sat}} \ge \bar{K}_\ell/\alpha$, which makes $d(K) + Kd'(K)$ nonnegative after saturation. The illustrative parameters satisfy this restriction. It is stronger than needed for $S + Kd' > 0$, because the positive term ψ can offset a negative derivative of capital income.

Consumption compression and interest rates. For a fixed belief wedge x, the steady-state household schedules are

$$c_x^{ss}(W) = \frac{(\rho - x)W}{1 + \theta W}, \qquad i_x^{ss}(W) = \frac{\rho + x\theta W}{1 + \theta W}. \tag{58}$$

The first expression follows from the steady-state Euler equation and wealth identity; the second uses the no-arbitrage condition $i = R^a + x$. They give consumption and the interest rate consistent with constant household wealth at a given W and belief wedge. At a given wealth level, a larger belief wedge raises the benchmark interest rate. Since $i_x^{ss}(W) + \theta c_x^{ss}(W) = \rho$, the Euler equation gives

$$i_t = i_x^{ss}(W_t) + \theta\Big[c_x^{ss}(W_t) - c_t\Big] + \frac{\dot{c}_t}{c_t}. \tag{59}$$

Under the regularity condition $d(K) + \psi + Kd'(K) > 0$ along the growing high saddle path, consumption satisfies

$$c_t < c_x^{ss}(W_t) = \left(1 - \frac{x}{\rho}\right) c^{ss}(W_t). \tag{60}$$

The comparison evaluates stationary consumption at the same wealth and belief wedge as the transition.

By the definition of M_t, $M_t = \rho - x - (\theta + 1/W_t)c_t$. Hence

$$c_x^{ss}(W_t) - c_t = \frac{W_t}{1 + \theta W_t} M_t > 0, \tag{61}$$

where the inequality follows from (56). This proves (60). Substituting (61) into (59) gives the equivalent expression

$$i_t = \frac{\rho + x\theta W_t}{1 + \theta W_t} + \frac{\theta W_t}{1 + \theta W_t} M_t + \frac{\dot{c}_t}{c_t}.$$

The consumption-compression term is positive along the high saddle path when capital is increasing and converges to zero with M_t at the steady state for the fixed wedge. Moreover, $\partial_W i_x^{ss}(W) = -\theta(\rho - x)/(1 + \theta W)^2 < 0$, while $i_0^{ss}(W) = i^{ss}(W)$ recovers the rational steady-state comparison.

A necessary condition for (60) is

$$d(K^H(x)) + \psi + K^H(x)d'(K^H(x)) \geq 0.$$

If this expression is negative, $b_{M,t} < 0$ for all sufficiently large t along the growing high saddle path. The equalities in (56) and (61) hold without the regularity restriction, so $M_t < 0$ and $c_t > c_x^{ss}(W_t)$ near the steady state.

Price as a function of the wedge. Section F.7 proves that the price is strictly increasing in the wedge on the common portions of the high saddle paths with increasing capital and of the low saddle paths with decreasing capital. At the wedge for which $K = K^M(x)$, the equilibrium price is $\bar{q}$. By continuity in x (Lemma 3), the prices on the low and high saddle paths converge to this value. Price declines sufficiently close to $D^*(K)$ on either side can therefore be generated by remaining wedges on the corresponding side of $x^*(K)$.

For $K_{\text{AI}} < K_{\text{early}} < K_{\text{late}} < K^M$, choose a sufficiently small $\varepsilon > 0$ and set

$$D_{\text{early}} = D^*(K_{\text{early}}) + \varepsilon, \qquad D_{\text{late}} = D^*(K_{\text{late}}) - \varepsilon.$$

Taking $2\varepsilon < D^*(K_{\text{late}}) - D^*(K_{\text{early}})$ gives $D_{\text{early}} < D_{\text{late}}$, with the early post-correction price on the declining low saddle path and the late post-correction price on the growing high saddle path. Both are attainable by the preceding continuity argument. Since $Q_0(K_{\text{early}}) < Q_0(K_{\text{late}})$, the late absolute price decline is also larger. This proves the remaining statement of Proposition 4.

Delay near the correction boundary. Fix $K_\circ \in (K_{\text{AI}}, K^M)$ and let $\mu = x_1 - x^*(K_\circ) \downarrow 0$. The post-correction source is

$$K^M(x_1) = K_\circ - \frac{\mu}{\theta \bar{B}}.$$

At $x_1 = x^*(K_\circ)$, the eigenvalues at the middle steady state are

$$\lambda_1 = \frac{\bar{B}}{\bar{q}}, \qquad \lambda_2 = \frac{\theta K_\circ \psi \bar{q} \bar{B}}{d_{\text{flat}} + \psi}.$$

Both are positive. Assume they are distinct, and let $\lambda_\circ$ be the eigenvalue associated with the eigenvector tangent to the saddle path; generically it is the smaller one. The local expansions are

$$\dot{K} = \lambda_\circ (K - K^M(x_1)) + \text{higher-order terms},$$
$$q - \bar{q} = \frac{\lambda_\circ \bar{q}}{K_\circ \psi}(K - K^M(x_1)) + \text{higher-order terms}.$$

The time to exit a fixed small neighborhood of the source is $\lambda_\circ^{-1} \log(1/\mu) + O(1)$, while the time from that neighborhood to K^M remains bounded. Because $q^+ - \bar{q} = Q_0(K_\circ)[D^*(K_\circ) - D]$ is proportional to μ to first order,

$$\mathcal{T}(K_\circ, x_1) = \frac{1}{\lambda_\circ} \log\left(\frac{1}{D^*(K_\circ) - D}\right) + O(1) \longrightarrow \infty. \tag{62}$$

This proves the duration claim near the correction boundary in Section 4.4.

Other convex adjustment costs. The arguments for the correction boundary and price monotonicity extend to other investment expenditure functions Γ that are smooth and strictly convex. Set $j(q) = (\Gamma')^{-1}(q)$, $B = d - \Gamma(j)$, $h = q/\Gamma''(j)$, $S = B + qh$, and $g = j - \delta$. Equations (54)–(56) hold with $a_M = B/q + \theta K h q B/S$ and the regularity condition $S + Kd' > 0$. The steady-state price is $\bar{q} = \Gamma'(\delta)$, so $D^*(K) = 1 - \bar{q}/Q_0(K)$ remains the correction boundary.

I Illustrative Numerical Example

The figures use the following parameters:

$$\alpha = \tfrac{1}{3}, \quad A = 0.16, \quad \rho = 0.089, \quad \delta = 0.040,$$
$$\theta = 2.49, \quad \gamma = 0.81, \quad \bar{K}_\ell = 0.253, \quad \psi = 0.61.$$

These values imply $\bar{q} = e^{\delta/\psi} = 1.0678$, $b = (1-\alpha)\gamma/\alpha$, $K_{\text{AI}} = 1/b = 0.6173$, and $K_{\text{sat}} = K_{\text{AI}} + \bar{K}_\ell/(1-\alpha) = 0.9968$. The rate of time preference is ρ; the steady-state interest rate is $i^{ss} = \rho/(1+\theta W)$ by (8), which is below ρ at all three steady states. The inequalities in (19) hold, $G' < 0$ on both outer regions, and the three steady states are $K^L = 0.5413$, $K^M = 0.7330$, and $K^H = 1.1551$. The associated steady-state interest rates are $i^L = 3.65$ percent, $i^M = 3.02$ percent, and $i^H = 2.19$ percent. The required wedge is $x^*(K^L) = 0.0093$.

The slope condition can also be verified analytically. At $\alpha = 1/3$, write $t = \theta\bar{q}$. Before deployment, the ratio $(-\Phi')/(-d')$ is proportional to $K^{5/3}/(1+tK)^2$, which increases until $K = 5/t$. Since $tK_{\text{AI}} < 5$, its maximum over that region is at K_{AI}, where the ratio is 0.455872. After saturation, the ratio is proportional to $(K - \bar{K}_\ell)^{5/3}/(1+tK)^2$, whose maximum is at $K = 5/t + 6\bar{K}_\ell$, where the ratio is approximately 0.420771. Both are below one, giving $G' < 0$ throughout the outer regions.

Lemmas 2 and 3 establish the rational and optimistic equilibrium paths at these parameters. Numerical integration supplies the paths plotted in the figures.

The parameters are illustrative, rather than calibrated. Tobin's q is the price per unit of installed capital. Rate parameters are annual, so transition times are in years.

Before the late correction, investors' consumption lies between 0.0046 and 0.0057, below its value $c^L = 0.0211$ at the low-capital steady state, and it is nondecreasing. The Euler equation (7) then implies an interest rate above its initial level throughout the optimistic phase in this illustration. The interest rate ranges from 8.80 to 9.35 percent. The realized return in (14) is six percentage points lower; the correction causes the discrete capital loss described in Section 5.

The full-correction figures use $x = 0.06$, about 6.5 times the required wedge $x^*(K^L)$. At this wedge, the time needed to reach K^M is $T(x) = 12.93$. For smaller wedges, the corresponding times are $T(0.01) = 170.13$, $T(0.02) = 40.65$, $T(0.03) = 25.04$, $T(0.04) = 18.62$, and $T(0.05) = 15.12$. The required duration rises without bound as the wedge approaches $x^*(K^L)$ from above. The early correction is set at $\tau = 0.60T(x) = 7.76$, when $K_\tau = 0.6523 < K^M$; the late correction is set at $\tau = 1.20T(x) = 15.52$, when $K_\tau = 0.7775 > K^M$. The two correction dates illustrate convergence to different rational steady states. At the late correction, the price falls from 1.1085 to 1.0707. The post-correction price lies on the rational high saddle path.

Figure 6 uses $x_0 = 0.012$ and corrections at 20 and 80 percent of the interval from K_{AI} to K^M. The remaining wedges imply price declines on opposite sides of the correction boundary, with the larger decline occurring later.

Welfare condition. The parameters give $c^L = 0.02109$ and $K_\delta = 2.01861$. The additional conditions for part (ii) of Proposition 5 hold:

$$\theta c^L = 0.05251 < \psi, \qquad \frac{\rho\theta\bar{q}c^L}{\psi} = 0.008181 < \min\{0.053535,\ 0.009206\}.$$

The full welfare loss therefore applies to every belief wedge and correction date in the proposition. Part (i) requires no additional conditions.

J Financing Costs and Investment

This appendix supplies the derivations and equilibrium construction for the financing-cost remark in Section 4.4.

J.1 Permanent shifts in the user cost

Use Φ and G from (16), and maintain the crossing conditions

$$\Phi(K_{\mathrm{AI}}) > d_{\mathrm{flat}} > \Phi(K_{\mathrm{sat}}), \qquad G'(K) < 0 \quad \text{for } K \in (0, K_{\mathrm{AI}}) \cup (K_{\mathrm{sat}}, \infty). \tag{J.1}$$

Let investors pay a permanent flow charge $\bar{q}sK$, $s \geq 0$, to workers, who consume it. Investors' consumption becomes

$$c = K[d(K) - \bar{q}s - \psi(q-1)]. \tag{J.2}$$

Equating the net steady-state return $[d(K) - \Gamma(\delta)]/\bar{q} - s$ to $\rho/(1+\theta\bar{q}K)$ gives

$$\Phi_s(K) = \Phi(K) + \bar{q}s, \qquad G_s(K) = G(K) - \bar{q}s. \tag{J.3}$$

Write $K^j(s)$, $j \in \{L, M, H\}$, for the corresponding steady states while they remain in their respective technology regions. Implicit differentiation gives

$$\frac{\mathrm{d}K^j(s)}{\mathrm{d}s} = \frac{\bar{q}}{G'(K^j(s))}. \tag{J.4}$$

Since $G' > 0$ during deployment and $G' < 0$ outside it, $K^M(s)$ increases and both outer stocks decrease. The middle and high roots meet at K_{sat} when

$$s = s^\dagger \equiv \frac{d_{\mathrm{flat}} - \Phi(K_{\mathrm{sat}})}{\bar{q}} > 0. \tag{J.5}$$

The boundary signs give three roots for $s < s^\dagger$ and no middle or high root for $s > s^\dagger$. The low root persists: $G_s(K) \to \infty$ as $K \downarrow 0$, $G_s(K_{\mathrm{AI}}) < 0$, and $G_s' < 0$ below deployment.

For the threshold property, also impose

$$d_{\mathrm{flat}} < \psi\bar{q}. \tag{J.6}$$

This keeps the positive-consumption domain nonempty at every stock for $0 \leq s < s^\dagger$.

Lemma J.1 (Rational threshold with a permanent charge)**.** *Under* (J.1) *and* (J.6)*, for every* $0 \le s < s^\dagger$ *and every initial stock* $K_0 > 0$*, there is a unique rational equilibrium path in the economy with charge* $\bar{q}sK$*. It converges to* $K^L(s)$ *if* $K_0 < K^M(s)$ *and to* $K^H(s)$ *if* $K_0 > K^M(s)$*, and is stationary at* $K_0 = K^M(s)$*.*

Proof. Replace $d(K)$ by $d(K) - \bar{q}s$ throughout Appendix F. The investment equation, $\bar{q}$, and the wealth–consumption ratio equation retain their form. The equilibrium price and resource constraint become

$$q = \frac{[d(K) + \psi - \bar{q}s]Z}{1 + \psi Z}, \qquad c = \frac{K[d(K) + \psi - \bar{q}s]}{1 + \psi Z}, \qquad \dot{Z} = \rho Z - 1 - \theta q K.$$

The transformed crossing conditions hold for every $s < s^\dagger$: $G_s(K_{\mathrm{AI}}) < 0$, $G_s(K_{\mathrm{sat}}) > 0$, and $G'_s = G'$. Hence the low and high steady states remain saddles and the middle steady state remains a source.

For the positivity and compactness arguments, use

$$\bar{q}s < d_{\mathrm{flat}} - \Phi(K_{\mathrm{sat}}) < d_{\mathrm{flat}} - \Gamma(\delta), \qquad \psi - \bar{q}s > \psi\bar{q} - d_{\mathrm{flat}} > 0.$$

Thus $d(K) + \psi - \bar{q}s > 0$ at every $K > 0$. Let $K_\delta(s) > K_{\mathrm{sat}}$ solve $d(K_\delta(s)) - \bar{q}s = \Gamma(\delta)$. Positive consumption implies $q < \bar{q}$ above this stock, so every forward path satisfies $K_t \le \max\{K_0, K_\delta(s)\}$. On bounded stock intervals, $qK < K[d(K) + \psi - \bar{q}s]/\psi$ bounds wealth above. The transversality representation

$$Z_t = \int_t^\infty e^{-\rho(u-t)}(1 + \theta W_u)\,\mathrm{d}u$$

then bounds Z above and below. Since $d(K) - \bar{q}s \to \infty$ as $K \downarrow 0$, the price bound excludes convergence to zero capital. These bounds give compactness in the positive-consumption region, as in Appendix F.1.

For the high branch on $(K^M(s), K^H(s))$, net rental income at $q = \bar{q}$ satisfies

$$K[d(K) - \bar{q}s - \Gamma(\delta)] \ge \frac{\rho\bar{q}K}{1 + \theta\bar{q}K} > K^M(s)[d_{\mathrm{flat}} - \bar{q}s - \Gamma(\delta)].$$

The last expression is consumption at the middle steady state. The consumption barrier in Appendix F.2 therefore applies: replacing d by $d - \bar{q}s$ leaves $d' \le 0$ unchanged and preserves $d(K) + \psi - \bar{q}s \le d_{\mathrm{flat}} + \psi - \bar{q}s$ on this interval. Backward flow keeps the high branch above $q = \bar{q}$ and above middle-state consumption, and the branch approaches the middle source.

For the low branch, the coefficient used in Appendix F.3 is positive on $(K^L(s), K^M(s))$. Below deployment it is

$$\psi + \alpha d(K) - \bar{q}s > \psi\bar{q} - (1 - \alpha)d_{\mathrm{flat}} > 0;$$

during deployment it is $\psi + d_{\mathrm{flat}} - \bar{q}s > \psi\bar{q} > 0$. Also $d(K) - \bar{q}s > \Gamma(\delta)$ on this interval. The two invariant-region inequalities in Appendix F.3 retain their signs, so the low branch approaches the same middle source backward. The outer extensions use the unchanged boundary signs and the compactness bounds above. Both saddle paths satisfy transversality. Finally, the source-to-saddle connections exclude periodic or other nonstationary limit sets by the argument in Appendix F.4. The two saddle paths and the middle stationary path give the stated existence and uniqueness. □

J.2 Debt-financing equations

Contract and beliefs. Equity investors have mass $1-m$ and lenders mass $m \in (0,1)$, with the preferences of Section 2.2 applied per person. Lenders' debt-only investment mandates exclude ownership of productive capital and equity, including after a full correction. Equity investors hold capital and borrow through instantaneous loans. Only a fraction χ of the capital value is pledgeable to lenders, so borrowing is subject to

$$B \leq \chi q K, \qquad 0 < \chi < 1. \tag{J.7}$$

On a binding path, total wealth is $W = qK$, equity wealth is $E = (1-\chi)W$, and lenders' wealth is $B = \chi W$. The experiment begins on such a path during an expansion.

The contract caps repayment at the current pledgeable value. At a price decline from q^- to q^+, it writes existing principal down to the new limit:

$$B^+ = \min\{B^-, \chi q^+ K\} = \chi q^+ K. \tag{J.8}$$

The equality uses the initially binding limit and continuity of capital. Because outstanding debt equals the pledgeable value before the decline, any fall in price reduces contractual repayment. Both claims fall by the percentage price decline; between jumps, debt adjusts through lending and repayment. Capital is therefore the only predetermined state on the binding paths.

Equity investors regard x as permanent; lenders assign intensity λ to a full correction, after which $x = 0$ and both groups are rational. A reassessment unexpectedly raises λ with x and χ unchanged. The write-down rule applies at either event.

Define $\kappa = \chi/m$. Wealth per lender is $\kappa q K$, and wealth per equity investor is $(1-\chi)qK/(1-m)$; lenders are wealthier precisely when $\kappa > 1$. The construction fixes a sufficiently large κ, bounds λ in a small interval, and takes m small with $\chi = \kappa m$. Lemma J.2 verifies that borrowing to the limit is strictly optimal before and after a full correction. Small aggregate debt is consistent with substantial wealth per lender because κ remains fixed.

For a borrower with capital k and debt $b \leq \chi q k$, recovery is $\min\{b, \chi q^c k\}$, where $q^c = Q^c(K)$ is the rational price after a full correction. Let c^e, c^b denote group consumption and superscript c a post-correction quantity. The competitive payment schedule is

$$\mathcal{P}(k,b) = ib + \lambda \frac{c^b}{c^{b,c}} (b - \chi q^c k)_+. \tag{J.9}$$

The rate i on an instantaneous riskless claim is priced by lenders:

$$i = \rho - \frac{\theta c^b}{m} + \frac{\dot{c}^b}{c^b} - \lambda \left(\frac{c^b}{c^{b,c}} - 1 \right).$$

For binding debt with $q^c < q$, the loan rate and spread satisfy

$$r^b = \mathcal{P}(K,B)/B, \qquad r^b - i = \lambda \frac{c^b}{c^{b,c}} \left(1 - \frac{q^c}{q} \right). \tag{J.10}$$

Equilibrium dynamics. Let a denote equity investors' marginal perceived return. The group budgets and Euler equations are

$$\dot{E} = aE - xW - c^e, \qquad \dot{c}^e/c^e = a - \rho + \theta c^e/(1-m), \tag{J.11}$$

$$\dot{B} = r^b B - c^b, \qquad \dot{c}^b/c^b = r^b - \rho + \theta c^b/m + \lambda\left[Z^c(K)/Z_b - 1\right], \tag{J.12}$$

where $Z_e = E/c^e$, $Z_b = B/c^b$, and $Z^c(K) = \chi Q^c(K)K/c^{b,c}(K)$. The resource constraint and capital-growth rate are

$$C(K,q) = K[d(K) - \psi(q-1)] = c^e + c^b, \qquad g(q) = \psi \log q - \delta. \tag{J.13}$$

Using $E = (1-\chi)qK$ and $B = \chi qK$ gives

$$q = \frac{d(K) + \psi}{\psi + (1-\chi)/Z_e + \chi/Z_b}. \tag{J.14}$$

The dynamic system is

$$\dot{K} = Kg(q), \tag{J.15}$$

$$\dot{Z}_e = \left(\rho - \frac{x}{1-\chi}\right) Z_e - 1 - \frac{\theta(1-\chi)qK}{1-m}, \tag{J.16}$$

$$\dot{Z}_b = (\rho + \lambda) Z_b - 1 - \frac{\theta\chi qK}{m} - \lambda Z^c(K). \tag{J.17}$$

The constructed equilibria have $0 < x < (1-\chi)\rho$. The rates along a smooth segment satisfy

$$r^b = g + \dot{q}/q + 1/Z_b, \qquad a - r^b = 1/Z_e - 1/Z_b + x/(1-\chi). \tag{J.18}$$

Convexity and homogeneity of $\mathcal{P}$ make borrowing to the limit optimal when $a \geq \mathcal{P}_b$. At the borrowing limit with $q^c < q$, this condition becomes

$$\frac{1}{Z_e} + \frac{x}{1-\chi} - \frac{1}{Z_b} \geq \lambda \frac{Z^c(K)}{Z_b}. \tag{J.19}$$

The construction gives a strict inequality. After a full correction $x = \lambda = 0$. Homogeneity also gives

$$R^a + x = (1-\chi)a + \chi r^b, \qquad R^a = g + \dot{q}/q + C/(qK).$$

Together with the budgets, this identity is equivalent to (J.13) and (J.18).

Rational continuation. Write $\theta_e = \theta(1-\chi)/(1-m)$ and $\theta_b = \theta\chi/m$. In the integrals below, $t' \geq 0$ denotes elapsed time. After a full correction, bounded ratios satisfy

$$Z_e^c(t) = \frac{1}{\rho} + \theta_e \int_0^\infty e^{-\rho t'} W^c_{t+t'}\, \mathrm{d}t', \qquad Z_b^c(t) = \frac{1}{\rho} + \theta_b \int_0^\infty e^{-\rho t'} W^c_{t+t'}\, \mathrm{d}t'. \tag{J.20}$$

Consequently,

$$Z_b^c - 1/\rho = \frac{\chi(1-m)}{m(1-\chi)}(Z_e^c - 1/\rho). \tag{J.21}$$

Substituting this relation into (J.14) reduces the rational continuation to a two-dimensional system in (K, Z_e). If $\chi > m$, then $Z_b^c > Z_e^c$, so (J.19) holds strictly on every positive rational continuation constructed below. The capital return exceeds the loan rate: $R^a - r^b = (1-\chi)(1/Z_e^c - 1/Z_b^c) > 0$. Lenders' debt-only mandates allow this return difference to persist after the correction.

The rational steady states solve

$$d(K) = \Gamma(\delta) + \bar{q}\rho\left[\frac{1-\chi}{1+\theta_e\bar{q}K} + \frac{\chi}{1+\theta_b\bar{q}K}\right]. \tag{J.22}$$

For small m at fixed κ, Lemma J.2 gives three steady states $\widetilde{K}^L < \widetilde{K}^M < \widetilde{K}^H$ and the corresponding rational branches. Splitting wealth between groups changes the threshold, but the prior intensity λ does not enter (J.22). For $\widetilde{K}^L < K < \widetilde{K}^M$, the correction has $Q^c(K) < \bar{q}$; thereafter $K_{t'}^c < K$ for $t' > 0$ and $q_{t'}^c \le \bar{q}$.

J.3 Lender reassessments

Stationary loan rate. At a stationary allocation before a full correction, $q = \bar{q}$. Write $V = \bar{q}K$, $B = \chi V$, and $E = (1-\chi)V$. Equations (J.11)–(J.12) give

$$c^{b,s} = \frac{(\rho+\lambda)B}{1+\theta B/m + \lambda Z^c(K)}, \qquad c^{e,s} = \frac{\rho E - xV}{1+\theta E/(1-m)}. \tag{J.23}$$

Since $c^{b,s} = r^{b,s}B$, the contractual loan rate is

$$\begin{aligned} r^{b,s}(K,\lambda) &= \frac{\rho+\lambda}{1+\theta B/m + \lambda Z^c(K)}, \\ Z^c(K) &= \frac{1}{\rho} + \frac{\theta}{m}\int_0^\infty e^{-\rho t'} B_{t'}^c \,\mathrm{d}t', \quad B_{t'}^c = \chi q_{t'}^c K_{t'}^c. \end{aligned} \tag{J.24}$$

Equivalently,

$$r^{b,s}(K,\lambda) = \frac{\rho}{1+\theta\overline{B}(K,\lambda)/m}, \qquad \overline{B}(K,\lambda) = \rho\,\mathbb{E}_\lambda\left[\int_0^\infty e^{-\rho t} B_t \,\mathrm{d}t\right]. \tag{J.25}$$

Under lenders' beliefs, debt stays at B until an exponentially distributed correction, is written down to $B_0^c = \chi Q^c(K)K$, then follows the rational path. Thus

$$\overline{B}(K,\lambda) = \frac{\rho}{\rho+\lambda}B + \frac{\lambda}{\rho+\lambda}\rho\int_0^\infty e^{-\rho t'} B_{t'}^c \,\mathrm{d}t'. \tag{J.26}$$

For $\widetilde{K}^L < K < \widetilde{K}^M$, $B_{t'}^c < B$, so

$$\begin{aligned} 1+\theta B/m - \rho Z^c(K) &= \frac{\theta}{m}\left[B - \rho\int_0^\infty e^{-\rho t'} B_{t'}^c \,\mathrm{d}t'\right] > 0, \\ \frac{\partial r^{b,s}}{\partial \lambda} &= \frac{1+\theta B/m - \rho Z^c(K)}{[1+\theta B/m + \lambda Z^c(K)]^2} > 0. \end{aligned} \tag{J.27}$$

Because corrected debt and consumption are independent of the prior intensity, the stationary spread satisfies

$$r^{b,s} - i^s = \lambda r^{b,s} \frac{B - B_0^c}{c^{b,c}}, \qquad \frac{\partial (r^{b,s} - i^s)}{\partial \lambda} = \frac{B - B_0^c}{c^{b,c}} \left(r^{b,s} + \lambda \frac{\partial r^{b,s}}{\partial \lambda} \right) > 0. \tag{J.28}$$

The loan-rate response depends on the wealth motive: if $\theta = 0$, then $Z^c = 1/\rho$ and $r^{b,s} = \rho$ for every λ, with a higher spread offset by a lower safe rate.

Required belief wedge. Substituting (J.23) into the resource constraint at $q = \bar{q}$ gives the user-cost condition

$$d(K) = \Gamma(\delta) + \bar{q} \left[\chi r^{b,s}(K, \lambda) + \frac{(1-\chi)\rho - x}{1 + \theta(1-\chi)\bar{q}K/(1-m)} \right]. \tag{J.29}$$

The bracket contains the debt- and equity-weighted returns. The wedge per unit of equity is $x/(1-\chi)$, so weighting by the equity share leaves x unscaled. Solving for the required belief wedge gives

$$\mathcal{X}(K, \lambda) = \frac{\rho E - [1 + \theta E/(1-m)]\{C(K, \bar{q}) - r^{b,s}(K, \lambda) B\}}{V}. \tag{J.30}$$

Its intensity derivative is

$$\mathcal{X}_\lambda = \left[1 + \frac{\theta E}{1-m} \right] \frac{B}{V} \frac{\partial r^{b,s}}{\partial \lambda} > 0. \tag{J.31}$$

Equilibrium result. Maintain (J.1). Let $x_* = x^*(K^L)$, as defined in Section 4.1, and write

$$H = d_{\text{flat}} - \Gamma(\delta) > 0, \qquad S = d_{\text{flat}} + \psi = H + \psi \bar{q}.$$

Impose the additional restriction

$$\mu_1 = H/\bar{q} < \mu_2 = \theta K_{\text{AI}} \psi \bar{q} H / S. \tag{J.32}$$

These are the deployment-region eigenvalues at the critical zero-debt allocation. Section J.4.5 establishes a nonempty open primitive domain. The result is local: at fixed κ, the admissible deviations from x_* and K_{AI} are of order m, as is the change in the required wedge. The proof derives the admissible wedge interval from (J.31) and constructs the equilibrium branches.

Proposition J.1 (Lender reassessments and investment)**.** *Under* (J.1) *and* (J.32)*, there is a finite constant $\kappa_0 > 1$ such that, for each fixed $\kappa > \kappa_0$, there exists $\bar{\lambda}(\kappa) > 0$ with the following property. Fix $0 \le \lambda_0 < \lambda_1 \le \bar{\lambda}(\kappa)$. For every sufficiently small $m > 0$, with $\chi = \kappa m$, there exist a belief wedge $x_m = x_* + O(m)$ and an interval $I_m \subset (K_{\text{AI}}, \widetilde{K}^M)$ of positive width $O(m)$, immediately above K_{AI}, such that:*

(i) There is an equilibrium expansion under λ_0 through I_m. Suppose the economy is on this path, with the borrowing limit binding, at a stock $K \in I_m$. There is a critical intensity $\lambda^*(K) \in (\lambda_0, \lambda_1)$. Reassessments to intensities in $(\lambda^*(K), \lambda_1)$ close to $\lambda^*(K)$ lower the price below $\bar{q}$ and reverse accumulation. Reassessments to intensities in $(\lambda_0, \lambda^*(K))$ close to $\lambda^*(K)$ lower the price while leaving it above $\bar{q}$ and accumulation positive. The former paths converge to a low steady state and the latter to a high steady state, conditional on the unchanged x_m, the revised λ, and no full correction.

(ii) Alternative reassessments at two stocks $K_e < K_l$ in I_m along the initial expansion yield proportional price declines $D_e < D_l$, with the earlier decline reversing accumulation and the later decline leaving it positive.

(iii) At any fixed $K \in I_m$, reassessments to intensities approaching $\lambda^*(K)$ from below leave accumulation positive but delay reaching $\widetilde{K}^M$ arbitrarily long, conditional on no full correction. If a full correction occurs before that crossing, capital converges to $\widetilde{K}^L$.

Borrowing up to the limit is strictly optimal along all these paths, including their rational continuations.

J.4 Proof of Proposition J.1

Fix $\kappa > 1$ and write $\chi = \kappa m$, with $0 < m < 1/\kappa$. Subscripts m on $\mathcal{X}_m$ and Z_m^c record their dependence on lender mass under this restriction. At $m = 0$, the aggregate block becomes the main model in (K, Z) coordinates:

$$q = \frac{[d(K) + \psi]Z}{1 + \psi Z}, \qquad \dot{K} = Kg(q), \qquad \dot{Z} = (\rho - x)Z - 1 - \theta qK. \tag{J.33}$$

The lender equation is triangular at this limit: its own-variable derivative is $\rho + \lambda$, and it does not affect (K, Z). The proof first establishes the rational continuations and borrowing incentives, then selects the belief wedge and constructs the equilibrium branches on both sides of the deployment boundary.

J.4.1 Rational paths and the binding borrowing limit

The next lemma constructs the intensity bound $\bar{\lambda}(\kappa)$; write $\Lambda = [0, \bar{\lambda}(\kappa)]$ for the resulting compact interval.

Lemma J.2. *There is a constant $\kappa_0 > 1$ such that, for each fixed $\kappa > \kappa_0$, there exists $\bar{\lambda}(\kappa) > 0$ with the following properties for all $\lambda \in [0, \bar{\lambda}(\kappa)]$ and sufficiently small $m > 0$. The rational steady states and stable branches persist. The low and high branches meet the rational middle stationary allocation continuously, and capital moves monotonically toward the corresponding outer steady state. Writing $(K_{t'}^r, q_{t'}^r)$ for the rational path of the zero-debt economy from K, the corrected lender ratio $Z_m^c(K)$ converges in C^1 near K_{AI} to*

$$Z_0^c(K) = \frac{1}{\rho} + \theta\kappa \int_0^\infty e^{-\rho t'} q_{t'}^r K_{t'}^r \, \mathrm{d}t'. \tag{J.34}$$

The borrowing inequality is strict on the forward paths from I_m constructed below and on their rational continuations.

Proof. Use (J.21) to reduce the rational field to (K, Z_e). On compact positive sets this is a piecewise smooth perturbation of the main rational field; its first derivatives converge on each technology region. The three steady states persist by their transverse crossings, and the low and high stable manifolds persist with smooth parameter dependence.

The connections to the middle source are preserved by strict entry directions. At the rational reference middle stock the eigenvalues satisfy $\mu_1^r = H/\bar{q} < \mu_2^r = \theta K^M \psi \bar{q} H/S < \rho$. On the high branch Appendix F.2 establishes $C > c^M$. The stronger source direction would instead give $C - c^M = [H - \bar{q}\mu_2^r](K - K^M) + o(|K - K^M|) < 0$. Thus the weak-mode capital coefficient is positive.

For the low branch, the invariant-region argument in Appendix F.3 gives

$$\Xi = \rho - \theta C - \frac{d(K) - \psi(q-1)}{q} - \theta K q g(q) > 0.$$

At the middle source its linearized direction has eigenvalue μ_1^r, while on $\Xi = 0$ in the deployment region,

$$\dot{\Xi} = -\theta K q \left[1 + \frac{\theta K q [d_{\text{flat}} - \psi(q-1)]}{S}\right] g(q)^2 < 0$$

away from stationarity. The strong source manifold on the left consequently has

$$\Xi = -\frac{M_\Xi}{2\mu_2^r - \mu_1^r}(q - \bar{q})^2 + o((q - \bar{q})^2) < 0, \qquad M_\Xi > 0.$$

The low branch cannot coincide with it, so its weak-mode capital coefficient is negative. The two strict source-entry signs, terminal stable directions, and investment signs on the intervening compact segments persist. Technology boundaries are crossed with nonzero capital growth wherever the branches pass between regions. These facts give the two rational graphs with their investment signs. On the compact stock intervals used below, the graphs converge uniformly and have a common Lipschitz bound. Near the rational source, their nonzero weak-mode coefficients give the same limiting tangent from either side; away from the source, this bound follows from the graph equations and finite-time continuation.

Near K_{AI}, the correction follows the declining rational branch and $Kg(q)$ is bounded away from zero. There

$$\frac{\mathrm{d}Z_b^c}{\mathrm{d}K} = \frac{\rho Z_b^c - 1 - \theta\kappa q K}{Kg(q)}.$$

The expression is continuous at the technology boundary. Smooth dependence on each side and continuity of this derivative give the stated C^1 convergence, with uniformly bounded one-sided higher derivatives. The integral representation follows from (J.20).

The rational borrowing inequality is strict because $\chi > m$ implies $Z_b^c > Z_e^c$. For the optimistic reference high branch and its limiting stationary allocation at K_{AI}, the bounded lender solution at zero mass is

$$Z_b(t) = \frac{1}{\rho+\lambda} + \int_0^\infty e^{-(\rho+\lambda)t'}\left[\theta\kappa q_{t+t'}K_{t+t'} + \lambda Z_0^c(K_{t+t'})\right]\mathrm{d}t'.$$

On these compact reference continuations, Z_b and Z_0^c are of order κ, their ratio is uniformly bounded, and $1/Z_e + x_*$ has a positive lower bound. There is a $\kappa_0 > 1$ such that $1/Z_b$ is smaller than half that bound at $\lambda = 0$ whenever $\kappa > \kappa_0$. For each fixed such κ, choose $\bar{\lambda}(\kappa) > 0$ small enough that (J.19) is strict uniformly on Λ. Continuity preserves this margin at positive m on the forward high paths established below and on the local low connection, which remains near the reference stationary allocation at K_{AI}. The full-correction price is strictly below the optimistic reference price on the relevant compact paths, by the monotonicity in the belief wedge established in Appendix F.7. This strict inequality also persists, so the recovery branch used in (J.19) is the appropriate one. □

J.4.2 The admissible belief wedge

Let $V_* = \bar{q}K_{\mathrm{AI}}$, $C_* = K_{\mathrm{AI}}H$, and $b_* = \kappa V_*$. The rational correction from K_{AI} has $K_{t'}^r < K_{\mathrm{AI}}$ for $t' > 0$ and $q_{t'}^r \le \bar{q}$. Therefore

$$1 + \theta b_* - \rho Z_0^c(K_{\mathrm{AI}}) > 0.$$

Define

$$z_\lambda = \frac{(\rho+\lambda)b_*}{1 + \theta b_* + \lambda Z_0^c(K_{\mathrm{AI}})}. \tag{J.35}$$

This is the limiting stationary lender consumption per person at the critical capital stock, and $z_\lambda' > 0$. Expanding (J.30) gives

$$\mathcal{X}_m(K_{\mathrm{AI}}, \lambda) = x_* + mA(\lambda) + O(m^2), \tag{J.36}$$

$$A(\lambda) = -\rho\kappa - \theta(1-\kappa)C_* + \frac{1+\theta V_*}{V_*}z_\lambda, \qquad A'(\lambda) > 0. \tag{J.37}$$

Fix $\lambda_0 < \lambda_1$ in the interval Λ supplied by Lemma J.2. Since A is strictly increasing, choose a_* in the nonempty interval

$$A(\lambda_0) < a_* < A(\lambda_1), \qquad x_m = x_* + ma_*. \tag{J.38}$$

For every fixed interior choice of a_*, (J.36) implies, at sufficiently small m,

$$\mathcal{X}_m(K_{\mathrm{AI}}, \lambda_0) < x_m < \mathcal{X}_m(K_{\mathrm{AI}}, \lambda_1). \tag{J.39}$$

The positive loan-rate response therefore yields a nonempty interval between the two required wedges at K_{AI}. Varying a_* over any compact subinterval of $(A(\lambda_0), A(\lambda_1))$ yields an interval of admissible belief wedges of positive width proportional to m. For small m, $0 < x_m < (1-\chi)\rho$.

During deployment the reference derivative is $\mathcal{X}_0'(K) = -\theta H$. Lemma J.2 and (J.31) imply

$$\mathcal{X}_{m,K} = -\theta H + O(m) < 0, \qquad \mathcal{X}_{m,\lambda} = mA'(\lambda) + O(m^2) > 0 \tag{J.40}$$

uniformly for $K = K_{\mathrm{AI}} + O(m)$ on the deployment side. For fixed (x_m, λ), let $K_M^m(\lambda)$ denote the middle stationary stock of the smooth deployment extension. It satisfies

$$K_M^m(\lambda) = K_{\mathrm{AI}} + m\frac{A(\lambda) - a_*}{\theta H} + O(m^2). \tag{J.41}$$

When $\mathcal{X}_m(K_{\mathrm{AI}}, \lambda) < x_m$, this stationary point lies in the extension to $K < K_{\mathrm{AI}}$. When the inequality is reversed, a low saddle lies to the left of K_{AI} and a middle source lies to its right, both in their actual technology regions. Their local existence follows from the implicit-function theorem on the two technology extensions.

For the choice (J.38), the comparison interval is

$$I_m = (K_{\mathrm{AI}}, K_M^m(\lambda_1)).$$

Its width is positive and of order m by (J.41), and it lies below $\widetilde{K}^M$ for sufficiently small m. At each $K \in I_m$, the required wedge under λ_0 is below x_m, while that under λ_1 is above it. These comparisons select the stocks and intensities used below. The matching argument below establishes the increasing and declining equilibrium branches throughout I_m, including stocks arbitrarily close to K_{AI}.

J.4.3 Uniform construction of the equilibrium branches

Fix the reference primitives, κ, and the compact interval Λ before taking $m \downarrow 0$. All constants below are uniform in $\lambda \in \Lambda$ and sufficiently small m and $|\nu|$. The parameter ν, defined below, measures the price drift at the deployment boundary. Smooth extensions are used only for local analysis; each constructed path follows the actual technology on its side of $K = K_{\mathrm{AI}}$.

Reference eigenvalues. At the critical zero-debt allocation, let $u = K - K_{\mathrm{AI}}$, $v = q - \bar{q}$, and $f_q = K_{\mathrm{AI}}\psi/\bar{q}$. Write $d_L' = d'(K_{\mathrm{AI}}^-)$ for the rental-rate slope just below deployment. The aggregate linearizations on the two technology extensions are

$$J_j = \begin{pmatrix} 0 & f_q \\ \beta_j & \tau_j \end{pmatrix}, \qquad j = L, R, \tag{J.42}$$

where

$$\beta_R = -\theta\bar{q}H^2/S < 0, \qquad \tau_R = H/\bar{q} + \theta K_{\mathrm{AI}}\psi\bar{q}H/S,$$
$$\beta_L = \beta_R - \frac{\bar{q}H}{S}(\theta K_{\mathrm{AI}} + 1/\bar{q})d_L' > 0, \qquad \tau_L = \tau_R + K_{\mathrm{AI}}\psi d_L'/S.$$

Write $\sigma_- < 0 < \sigma_+$ for the left eigenvalues. For either right root μ_i in (J.32),

$$\det(\mu_i I - J_L) = \frac{K_{\mathrm{AI}}\psi d_L'}{S}(\rho - x_* - \mu_i) < 0. \tag{J.43}$$

Moreover the left characteristic polynomial at $\rho - x_*$ equals the right one there and is positive. Hence

$$0 < \mu_1 < \mu_2 < \sigma_+ < \rho - x_* < \rho + \lambda. \tag{J.44}$$

These eigenvalues have uniform positive separations on Λ.

Uniform remainder bounds. Choose a closed neighborhood of K_{AI} separated from the rational low and middle stocks. On the rational correction path through this neighborhood, $Kg(Q^c(K)) \le -v_{\min} < 0$ for a uniform $v_{\min} > 0$ and small m. The bounded rational ratios in Lemma J.2 satisfy, in particular,

$$(Z_m^c)'(K) = \frac{\rho Z_m^c(K) - 1 - \theta\kappa Q^c(K)K}{Kg(Q^c(K))}.$$

The rational equity ratio has the analogous equation. The denominator bound and the resource constraint give uniform first-derivative bounds. Differentiating these equations on either production region gives uniform one-sided second-derivative bounds. Both ratios are continuously differentiable across K_{AI}, because their drift equations and Q^c are continuous there. Extend the ratios from each side with these endpoint derivatives and bounded second derivatives. The production functions on the two extensions are smooth with bounded derivatives near the strictly positive stock K_{AI}.

On this neighborhood, total consumption, Z_e, Z_b, and equity wealth are bounded away from zero. The change from (K, Z_e, Z_b) to (K, q, Z_b) is therefore invertible with uniformly bounded derivatives of the required orders; explicitly,

$$Z_e = \frac{1-\chi}{[d(K)+\psi]/q - \psi - \chi/Z_b}.$$

Consequently the two extended vector fields in these coordinates have uniformly bounded second derivatives. This establishes the local regularity needed below, independently of the distance to the critical wedge. The price drift can have different one-sided values at the technology boundary, since it contains d'. The original vector field in (K, Z_e, Z_b) is continuous and locally Lipschitz, which determines the switched flow uniquely.

Centering at the exact boundary allocation. For each (m, λ), set $K = K_{\mathrm{AI}}$, $q = \bar{q}$, and

$$Z_{b,0} = \frac{1 + \theta\kappa\bar{q}K_{\mathrm{AI}} + \lambda Z_m^c(K_{\mathrm{AI}})}{\rho + \lambda}.$$

Choose $Z_{e,0}$ from the resource constraint and set $c_0^e = (1-\chi)\bar{q}K_{\mathrm{AI}}/Z_{e,0}$. At this allocation $\dot{K} = \dot{Z}_b = 0$. Define the exact price drift by

$$\nu = \left.\frac{\partial q}{\partial Z_e}\right|_0 \frac{\bar{q}K_{\mathrm{AI}}}{c_0^e}[\mathcal{X}_m(K_{\mathrm{AI}}, \lambda) - x_m]. \tag{J.45}$$

The coefficient is positive and bounded away from zero, so ν has the sign of the required-wedge gap even when that gap is arbitrarily small. Since $\dot{K} = 0$ at the centering allocation, this drift is the same on both extensions.

Let $y = (u, v, w) = (K - K_{\mathrm{AI}}, q - \bar{q}, Z_b - Z_{b,0})$. Taylor's formula and the derivative bounds above give, for a uniform constant $M_R > 0$,

$$\dot{y} = A_j^{m,\lambda} y + \nu e_2 + R_j(y), \qquad |R_j(y)| \le M_R |y|^2, \quad |DR_j(y)| \le M_R |y|, \tag{J.46}$$

where $e_2 = (0, 1, 0)^\top$ and

$$A_j^{m,\lambda} = \begin{pmatrix} 0 & f_q & 0 \\ \beta_j & \tau_j & 0 \\ -\theta\kappa\bar{q} - \lambda(Z_0^c)'(K_{\mathrm{AI}}) & -\theta\kappa K_{\mathrm{AI}} & \rho + \lambda \end{pmatrix} + O(m + |\nu|). \tag{J.47}$$

Here Z_0^c is the limiting rational lender ratio at fixed κ. The aggregate entries in the third column are $O(m)$. In particular, the remainder constant in (J.46) is uniform as $\nu \to 0$; it does not involve dividing by the speed of a pre-correction path at the boundary.

Lemma J.3 (Uniform matching at the deployment boundary)**.** *There are $m_0 > 0$ and $\nu_0 > 0$ such that the following conclusions hold for $0 < m < m_0$ and $0 < |\nu| < \nu_0$.*

1. *If $\nu > 0$, the right stable branch of the low saddle crosses $K = K_{\mathrm{AI}}$ with $q < \bar{q}$ and converges backward to the middle source. Capital decreases forward on this connection. The high stable branch converges backward to the same source from above, with increasing capital forward.*

2. *If $\nu < 0$, the high stable branch reaches $K = K_{\mathrm{AI}}$ with $q > \bar{q}$. Capital increases forward on its portion to the right of this boundary.*

The high branch has a strictly increasing price as a function of capital in a fixed neighborhood to the right of its middle stationary point. For $\nu < 0$ that stationary point is on the smooth extension to the left of K_{AI}; only the path at $K \ge K_{\mathrm{AI}}$ is used. The local neighborhoods and all bounds are uniform in ν.

Proof. The implicit-function theorem on each smooth extension gives stationary points with $v = 0$ and

$$u_L = -\nu/\beta_L + O(m|\nu| + \nu^2), \qquad u_M = -\nu/\beta_R + O(m|\nu| + \nu^2), \qquad w_j = O(|\nu|). \tag{J.48}$$

The uniform derivative bounds and spectral gaps give local stable manifolds with uniform quadratic graph bounds and continuous parameter dependence (Teschl, 2012, Theorem 9.6). This theorem is applied separately to the smooth extensions.

Entry of the declining branch. For $\nu > 0$, the low saddle lies to the left of K_{AI}. Its right stable branch, normalized by its capital coordinate, reaches $u = 0$ at

$$v_0 = \frac{\sigma_- \nu}{f_q \beta_L} + O(m\nu + \nu^2) < 0, \qquad w_0 = O(\nu). \tag{J.49}$$

The stable-manifold graph is evaluated over a distance $O(\nu)$, which accounts for the uniform quadratic error. Relative to the middle source, the leading entry slope is

$$p_0 = \frac{-\beta_R}{\sigma_+} = \frac{\mu_1 \mu_2}{f_q \sigma_+} < \frac{\mu_1}{f_q}. \tag{J.50}$$

In particular, $v_0/(-u_M) = p_0 + O(m+\nu)$, uniformly as $\nu \downarrow 0$.

Diagonalize the Jacobian at the extended middle stationary point. Denote its eigen-coordinates by ξ_1, ξ_2, ξ_3, normalizing the first two eigenvectors to have unit capital component and the third to have unit lender component. This change of coordinates and its inverse are uniformly bounded. The third eigenvector has capital and price components $O(m)$. For a uniform $\underline{\mu} > 0$, the eigenvalues obey

$$0 < \mu_{1,m} < \mu_{2,m} < \mu_{3,m}, \qquad \mu_{1,m} \geq 2\underline{\mu}, \qquad \mu_{i,m} - \mu_{1,m} \geq 2\underline{\mu} \quad (i = 2, 3).$$

Their dependence on λ and ν is suppressed in this notation. At the declining branch's entry, $\xi_1 < 0$ and, for a uniform $M_{\text{ent}} > 0$, $-\xi_1 \geq M_{\text{ent}}\nu$. All coordinates are $O(\nu)$. With $\eta = -\xi_1$ and $r_i = \xi_i/\eta$, the aggregate projection gives

$$r_2 = r_* + O(m+\nu), \qquad r_* = \frac{\mu_1 - f_q p_0}{\mu_2 - f_q p_0}, \qquad 0 < r_* < \frac{\mu_1}{\mu_2}, \qquad |r_3| \leq M_*.$$

The bound M_* is uniform on Λ, since the lender entry is $O(\nu)$ and $\eta \geq M_{\text{ent}}\nu$. Choose constants

$$r_* < b_2 < \mu_1/\mu_2, \qquad b_3 > M_* + 1. \tag{J.51}$$

Reducing m_0, ν_0 puts every entry strictly inside $|r_2| \leq b_2$, $|r_3| \leq b_3$, including entries with arbitrarily small $\nu > 0$.

Backward invariance. In eigen-coordinates the forward system is

$$\dot{\xi}_i = \mu_{i,m}\xi_i + N_i(\xi), \qquad |N(\xi)| \leq M_R|\xi|^2.$$

For backward time $\tau = -t$, direct differentiation on the negative-ξ_1 cone gives

$$\eta' = -\mu_{1,m}\eta + N_1, \qquad r_i' = -(\mu_{i,m} - \mu_{1,m})r_i - \frac{N_i + r_i N_1}{\eta}, \quad i = 2, 3.$$

On the cone $|\xi| \leq C_1\eta$. Hence, for a uniform L,

$$|\eta' + \mu_{1,m}\eta| \leq L\eta^2, \tag{J.52}$$

$$D^+|r_i| \leq -2\underline{\mu}|r_i| + L\eta, \qquad i = 2, 3. \tag{J.53}$$

Here D^+ denotes the upper right derivative. Choose a radius $r > 0$ such that

$$Lr < \underline{\mu} \min\{1, b_2, b_3\}. \tag{J.54}$$

Then $\eta' \leq -\underline{\mu}\eta$ and both ratio boundaries point strictly inward. The cone is backward invariant for $0 < \eta < r$. Since the entry amplitude is $O(\nu)$, the same radius works for every $0 < \nu < \nu_0$.

The coordinate normalization gives

$$\frac{u - u_M}{\eta} = -1 + r_2 + O(m)(1 + |r_3|), \qquad \frac{v}{\eta} = -\frac{\mu_1}{f_q} + \frac{\mu_2}{f_q}r_2 + O(m + |\nu|)(1 + |r_3|).$$

Both expressions are negative by (J.51). Starting at $u = 0$, capital therefore increases backward, remains to the right of the boundary, and approaches the middle source. The amplitude tends to zero and the ratios remain bounded, so all three coordinates converge. This proves the declining connection.

Entry and invariance of the increasing branch. First consider the zero-debt critical high branch. Its aggregate path is independent of λ. The consumption bound in Appendix F.5 gives $C > C_*$, where $C_* = K_{\mathrm{AI}}H$ is consumption at the critical allocation. The stronger aggregate direction instead has

$$C - C_* = [H - \bar{q}\mu_2](K - K_{\mathrm{AI}}) + o(|K - K_{\mathrm{AI}}|) < 0.$$

It cannot be the limiting direction of this branch. Thus the weak-mode capital coefficient is strictly positive: as $t \to -\infty$, $(u, v) = A_\xi e^{\mu_1 t}(1, \mu_1/f_q) + o(e^{\mu_1 t})$ for some $A_\xi > 0$.

The bounded reference lender ratio along this path is

$$Z_b(t) = \frac{1}{\rho + \lambda} + \int_0^\infty e^{-(\rho+\lambda)t'}[\theta\kappa W(t + t') + \lambda Z_0^c(K(t + t'))]\,\mathrm{d}t'.$$

Expand the integrand about the critical allocation up to the time the path exits a fixed source neighborhood. Its first-order term is precisely the lender component of the weak eigenvector. The remaining local term is $o(e^{\mu_1 t})$ after integration: the uniform gap $\rho + \lambda - \mu_1 > 0$ bounds the integral of the normalized remainder. The bounded tail after exit is $O(e^{(\rho+\lambda)t}) = o(e^{\mu_1 t})$, uniformly on Λ. Consequently both nonweak eigen-coordinates divided by the positive weak coordinate tend to zero uniformly on Λ.

Choose

$$0 < b_2^+ < \min\{b_2, \mu_1^2/(2\mu_2^2)\}. \tag{J.55}$$

A fixed section $K = K_{\mathrm{AI}} + h$, with $h > 0$ sufficiently small, then places every reference high branch strictly inside the positive cone $\eta = \xi_1 > 0$, $|\xi_2/\eta| < b_2^+$, $|\xi_3/\eta| < b_3$, at amplitude below $r/2$. At the terminal high saddle the stable eigenvalue is uniformly separated from the two unstable eigenvalues. Its local stable branch therefore varies continuously with parameters. The reference segment from that neighborhood to the fixed section takes finite backward time; its technology-boundary crossings are transversal. Continuous dependence in (K, Z_e, Z_b) preserves the strict cone-entry margins for small $m, |\nu|$, uniformly on Λ. This argument uses uniform convergence and the common local Lipschitz bound for the rational continuation; it does not require twice differentiability at its middle stock.

For $\eta = \xi_1$, the signs of the nonlinear terms in the ratio equations change, but the bounds (J.52)–(J.53) are unchanged. Reduce r to make them inward pointing also at b_2^+. In this narrower cone, $u - u_M > 0$ and $v > 0$. Moreover,

$$\frac{\dot{q}}{\eta} = \frac{\mu_1^2 + \mu_2^2 r_2}{f_q} + O(m + |\nu| + \eta)(1 + |r_3|) > 0. \tag{J.56}$$

The strict inequality follows from (J.55) after one further uniform reduction of the neighborhood. Thus $\dot{K} > 0$ and $\mathrm{d}q/\mathrm{d}K > 0$ on the local high branch. If $\nu > 0$, backward invariance gives convergence to the physical middle source. If $\nu < 0$, the extended source

has $u_M < 0$. The high branch must reach $u = 0$ in finite backward time, with η and v bounded above and below by positive multiples of $|\nu|$. Only its portion at $u \geq 0$ is retained.

Transversality at the switch. At the declining crossing, (J.49) gives

$$\dot{K} = \frac{\sigma_- \nu}{\beta_L} + O(m\nu + \nu^2) < 0.$$

The declining crossing is transversal for each $\nu > 0$, although its speed tends to zero as $\nu \downarrow 0$. The scaled entry estimates provide the uniform control. At this crossing, states and the reconstructed Z_e agree at the boundary, so uniqueness in the original coordinates joins the two smooth pieces into a path of the actual switched system. For $\nu < 0$, the increasing branch reaches $K = K_{\mathrm{AI}}$ with positive forward capital growth comparable to $|\nu|$; only its portion at $K \geq K_{\mathrm{AI}}$ is used. The rational continuation crosses the boundary with the uniform negative speed used in the remainder-bound argument. This completes the proof. □

Completion of the equilibrium construction. Use the wedge x_m selected in (J.38). Under λ_0, (J.45) is negative. The high branch in Lemma J.3 passes through every stock of I_m with $q > \bar{q}$ and continues forward to the high conditional steady state. Start the experiment on this binding expansion at the stated stock in I_m. For revised intensities with positive (J.45), the lemma supplies both the declining and increasing connections to the physical middle source. It also covers increasing paths when that source remains on the left extension. On the remaining compact segments and at the terminal saddles, continuity preserves the reference investment signs. No continuation of the initial expansion below K_{AI} is needed.

At each fixed $m > 0$, the constructed paths have positive consumption and equity wealth, bounded away from zero on their forward continuations. Lenders' wealth–consumption ratios are bounded before and after the anticipated correction, so their expected discounted terminal ratio tends to zero. Equity investors forecast wealth $\widetilde{E}$ with $\dot{\widetilde{E}} = a\widetilde{E} - c^e$. Their actual budget is $\dot{E} = aE - xqK - c^e$, so $\widetilde{E} \geq E > 0$. Their Euler equation gives

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(e^{-\rho t}\frac{\widetilde{E}}{c^e}\right) = -\frac{\theta c^e}{1-m}\left(e^{-\rho t}\frac{\widetilde{E}}{c^e}\right) - e^{-\rho t}.$$

The positive lower bound for consumption implies transversality. Together with the convex payment schedule and the borrowing inequalities in Lemma J.2, these conditions support household optimality.

J.4.4 Price declines and delay

Let $Q_0(K)$ be the pre-reassessment price under λ_0. At a deployment stock in the interval swept out by (J.41), the critical intensity solves

$$\mathcal{X}_m(K, \lambda^*(K)) = x_m, \qquad \frac{\mathrm{d}\lambda^*}{\mathrm{d}K} = -\frac{\mathcal{X}_{m,K}}{\mathcal{X}_{m,\lambda}} > 0. \tag{J.57}$$

At that intensity, $K = K_M^m(\lambda^*)$ and $q = \bar{q}$. The low and high stable branches converge continuously to this point. Nearby higher intensities put the current stock below the middle source, with $q < \bar{q}$ and declining capital. Nearby lower intensities put it above the source, with $q > \bar{q}$ and increasing capital. These are equilibrium paths by Lemma J.3 and the completion argument above. Their prices can be chosen below $Q_0(K)$, so the jump is implemented by (J.8).

The correction boundary of Section 4.4, evaluated at the debt economy's initial price, is $D^*(K) = 1 - \bar{q}/Q_0(K)$. The positive-cone argument in Lemma J.3 gives $Q_0'(K) > 0$ on I_m. Thus D^* increases with K. Choose $K_e < K_l$ and intensities just above $\lambda^*(K_e)$ and just below $\lambda^*(K_l)$. By continuity, the associated price declines can satisfy

$$D^*(K_e) < D_e < D_l < D^*(K_l).$$

Both are lender reassessments relative to λ_0; their intensities can also be chosen with $\lambda_l > \lambda_e$. The smaller early decline reverses accumulation, while the larger later decline leaves it positive.

For delay, hold m and $K \in I_m$ fixed and let the revised intensity approach $\lambda^*(K)$ from below. Put $\Delta = K - K_M^m(\lambda) > 0$. The positive cone gives constants $0 < M_- < M_+$, uniform for λ close to $\lambda^*(K)$, such that

$$M_-[K_t - K_M^m(\lambda)] \leq \dot{K}_t \leq M_+[K_t - K_M^m(\lambda)] \tag{J.58}$$

until the path reaches a fixed nearby stock above K. Integrating these bounds shows that the passage time diverges logarithmically as $\Delta \downarrow 0$. That stock can be chosen below $\widetilde{K}^M$, which remains a positive distance above K_{AI} for small m. The time to reach the rational threshold therefore also diverges. A full correction before the crossing sends the economy to the rational low branch.

This delay limit is distinct from the deployment-boundary limit in Lemma J.3. At the fixed stock $K > K_{\mathrm{AI}}$,

$$\mathcal{X}_m(K_{\mathrm{AI}}, \lambda^*(K)) - x_m = -\int_{K_{\mathrm{AI}}}^{K} \mathcal{X}_{m,K}(u, \lambda^*(K))\,\mathrm{d}u > 0.$$

Hence ν approaches a positive value as $\lambda \uparrow \lambda^*(K)$. Uniformity as $\nu \to 0$ ensures that the construction covers stocks arbitrarily close to K_{AI}; the fixed-K delay follows from approaching the middle source within that family. This proves Proposition J.1.

J.4.5 A nonempty primitive domain

Fix production primitives, $\psi > 0$, and $\theta K_{\mathrm{AI}} > 1$. Choose $H > 0$ small, and set

$$\bar{q} = 1 + (d_{\mathrm{flat}} - H)/\psi, \qquad \delta = \psi \log \bar{q},$$
$$\rho = \frac{H(1 + \theta \bar{q} K_{\mathrm{AI}})}{\bar{q}} + \zeta, \qquad 0 < \zeta < \theta H(K_{\mathrm{sat}} - K_{\mathrm{AI}}).$$

For $H < d_{\text{flat}}$, depreciation is positive. The rational steady-state gap is negative at K_{AI} and positive at K_{sat}, and its middle root is $K_{\text{AI}} + \zeta/(\theta H)$. On each Cobb–Douglas outer region,

$$-d'(K)(1 + \theta\bar{q}K)^2/(\theta\bar{q}^2)$$

has a strictly positive lower bound. It diverges as $K \downarrow 0$ in the low region and as $K \to \infty$ in the high region, and is positive at the technology boundaries. Since $\rho = O(H)$, small H makes the steady-state gap strictly decreasing on both outer regions, giving the required low and high roots and strict outer crossings. Also

$$\frac{\mu_2}{\mu_1} = \frac{\theta K_{\text{AI}} \psi \bar{q}^2}{H + \psi\bar{q}} \longrightarrow \theta K_{\text{AI}}(1 + d_{\text{flat}}/\psi) > 1 \quad \text{as } H \downarrow 0.$$

The spectral restriction follows. Here $x_* = \zeta > 0$. All inequalities remain strict in a neighborhood of the chosen finite primitives. For each fixed $\kappa > \kappa_0$, choose $\bar{\lambda}(\kappa)$ from Lemma J.2 and fix the two intensities and an interior a_* as in (J.38). The borrowing margins, Lemma J.3, and the continuation arguments then hold for every sufficiently small positive m. Reduce this upper bound on m below $1/\kappa$ and, if necessary, to ensure positive consumption, (J.40), and the local bounds used above. The admissible belief-wedge interval and I_m are nonempty for such m. This establishes the parameter choices asserted in Proposition J.1.

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