Lotteries with Money Payoffs, continued

- Fix *u*, let *w* denote wealth, and set $u_w(z) = u(z+w)$: now we have a family of utility functions for wealth increments *z* indexed by initial wealth *w.*

(a) Recall from last time that the *coefficient of absolute risk aversion* at *z* is $A(z) := -u''(z)/u'(z)$

- Agent gets less risk averse as wealth increases iff she has decreasing absolute risk aversion.

- CARA (*constant absolute risk aversion*) utility $u(z) = -\exp(-\alpha z), A(z) = \alpha$.

- With CARA, the certain equivalent of a $N(\mu, \sigma^2)$ lottery is $\mu - \alpha\sigma^2/2$.

- And the same formula gives an approximation of the certain equivalent for small gambles under –any- (continuous, concave) utility function:

$$U(w+c) = \int U(w+x) f(x)dx$$

So if all realizations of for *x* are near $Ex =: \mu$ then

$$U(w+\mu) + (c-\mu)U'(w+\mu) \approx$$
$$\int \left[U(w+\mu) + (x-\mu)U'(w+\mu) + (x-\mu)^2 / 2U''(w+\mu) \right] f(x)dx$$

(*1st order approximation is just* $c = \mu$)

$$(c-\mu)U'(w+\mu) \approx (\sigma^2 / 2)U''(w+\mu)$$

$c \approx \mu - (\sigma^2 / 2)U''(w+\mu) / U'(w+\mu)$ (*remember here* σ *is small…*)

- With CARA preferences the utility of a gamble that is equally likely to give g or $-l$ is

  $$.5u(w-l)+.5u(w+g)=-.5\exp\left(-\alpha(w-l)\right)-.5\exp\left(-\alpha(w+g)\right)=$$
  $$-.5\exp(-\alpha w)\left(\exp(\alpha l)+\exp(-\alpha g)\right)$$

  So certain equivalent c satisfies
  $$\exp(-\alpha(w+c))=.5\exp(-\alpha w)\left(\exp(\alpha l)+\exp(-\alpha g)\right)$$
  $$c=-\ln\left(.5\left(\exp(\alpha l)+\exp(-\alpha g)\right)\right)/\alpha$$
  (*so the certain equivalent is independent of w*)

- Note that it doesn't much matter what g is unless αl is "small" (for example for $l=100$, $g>l$, and $\alpha>.1$, the $\exp(-\alpha g)$ term is negligible.)

- It is often argued that risk aversion is and "should be" decreasing in wealth.

- *Relative risk aversion* measures attitudes towards lotteries that are proportional to wealth.

- **Definition:** The *coefficient of relative risk aversion* at wealth *w* is $R(w) := \frac{-wu''(w)}{u'(w)}$.

- CARA utility $u(z) = \exp(-\alpha z)$ has relative risk aversion $R(w) = w\alpha$, which is increasing in *w*.

- An agent with increasing relative risk aversion gets more averse to proportional risks as he gets wealthier.

- The coefficient of relative risk aversion measures the agent's risk premium (as share of wealth) for a "small" gamble that is proportional to her wealth: set $U(w(1-\pi)) = E[U((1+y)w)]$ and do local approximations.

- CRRA (*constant relative risk aversion*): $u(w) = w^{1-\rho}, 0 < \rho < 1$ has $R = \rho$; $u(w) = \ln w$ has $R = 1$.

Aside: in static choice u''/u' *matters for risk averion. Looking ahead, in dynamic models things like* u''/u' *or* ρ *also influence present vs. future tradeoffs and savings- this has led to interest in non-EU models in macro and finance…*

Risk aversion and asset allocation (Arrow [1965], Pratt [1964) MWG Example 6.C.2: A risk averse agent divides portfolio between safe asset with return of 1 and a risky asset with random return *z*, cdf F.

- Pick investment $\alpha \in [0, w]$ to maximize $U(\alpha) = \int u(w + \alpha(z-1))dF$.

- U concave objective due to risk aversion.
- first order condition $U'(\alpha) = \int (z-1)u'(w + \alpha(z-1))dF$.

- So if $Ez \leq 1$, $U'(0) = u'(w)\int (z-1)dF \leq 0$ and the optimum is $\alpha = 0$.

- If $Ez > 1$, the optimal α^* is strictly positive.
- Note: both conclusions can fail with multiple risky assets- the correlation structure matters.

Claim: Suppose u_1 is more risk averse than u_2, that u_1, u_2 are concave and differentiable, and that the optimal investment levels α_1^*, α_2^* satisfy the FOC with equality. Then $\alpha_1^* \leq \alpha_2^*$

Proof: To show this show that $U_1'(\alpha) - U_2'(\alpha) \leq 0$ for all $\alpha \in [0,1]$.

Because u_1 is more risk averse than u_2 , $u_1 = g \circ u_2$ for some increasing concave *g*.

Normalize u_1 so that $g'(u_2(w)) = 1$, then since g' is decreasing, and $u_1'(w+\alpha(z-1)) = g'\left(u_2(w+\alpha(z-1))\right)u_2'(w+\alpha(z-1))$,

$$\left[u_1'(w+\alpha(z-1)) - u_2'(w+\alpha(z-1))\right](z-1) \leq 0 .$$

And since U_1, U_2 are concave, $\alpha_1^* \leq \alpha_2^*$.

- CARA agent invests constant amount regardless of wealth, agent with decreasing absolute risk aversion invests more as wealth increases.

- Can also show that a CRRA agent invests constant *share* of wealth.

- Note that if the return on the safe asset increases, so does the investor's "effective wealth." So she could invest less in the safe asset is her absolute risk aversion decreases fast enough. (Fishburn and Burr Porter, *J Man Science* [1976]).

- If the initial wealth w is stochastic, we need stronger condition than comparing risk aversion to conclude that agent 1 always invests less. Machina and Neilson *Ema* [1987] give a necessary and sufficient condition on the two utility functions.

Demand for Insurance

Simplest model: *Insurance against a purely monetary loss*

- Initial wealth *w*, may lose 1 unit with probability *p.*

- Can buy insurance against λ of the loss at cost of λq .

- $U(\lambda) = pu(w - q\lambda - (1-\lambda)) + (1-p)u(w - q\lambda)$.

- $U'(\lambda) = p(1-q)u'(w - q\lambda - (1-\lambda)) - q(1-p)u'(w - q\lambda)$

- If $q > p$, $U'(1) = p(1-q)u'(w-q) - q(1-p)u'(w-q) < 0$, agent buys less than full insurance: optimal to choose a deductible.

- If $q = p$ (actuarially fair insurance) the agent buys full insurance, $\lambda = 1$. Then whether or not the loss occurs agent has utility $u(w-p)$ and marginal utility $u'(w-p)$.

- But in a more general setting optimal purchase of actuarially fair insurance only equates the marginal utility in the various states.

Insurance with State-Dependent Utility

- Some accidents and illnesses can change utility at each wealth level.
- Model this with state-dependent utility functions u_H, u_I .

  (*see MWG 6E)*

- So suppose the agent's objective function is
  $U(\lambda) = pu_I(w - q\lambda - (1-\lambda)) + (1-p)u_H(w - q\lambda)$

- Then when $q = p$ if FOC holds the optimal purchase sets

  $u_I'(w - q\lambda - (1-\lambda)) = u_H'(w - q\lambda),$

  -this needn't equalize the utility levels

  -whether or not the agent buys full insurance e.g. whether $\lambda = 1$.

Risk Preference in the Lab

Problem: Given the wealth of most lab subjects, they should be almost risk neutral to typical lab gambles- but they're not.

- Holt and Laury *AER* [2002]: 2/3 of subjects exhibit non-trivial risk aversion to lotteries whose outcomes all range from [$0,$4]!

- Subjects asked to make 10 binary choices, 1 out of 10 paid (so same preferences as for a single choice if the independence axiom applies.)

Table 1. The Ten Paired Lottery-Choice Decisions with Low Payoffs

Option A	Option B	Expected Payoff Difference
1/10 of $2.00, 9/10 of $1.60	1/10 of $3.85, 9/10 of $0.10	$1.17
2/10 of $2.00, 8/10 of $1.60	2/10 of $3.85, 8/10 of $0.10	$0.83
3/10 of $2.00, 7/10 of $1.60	3/10 of $3.85, 7/10 of $0.10	$0.50
4/10 of $2.00, 6/10 of $1.60	4/10 of $3.85, 6/10 of $0.10	$0.16
5/10 of $2.00, 5/10 of $1.60	5/10 of $3.85, 5/10 of $0.10	-$0.18
6/10 of $2.00, 4/10 of $1.60	6/10 of $3.85, 4/10 of $0.10	-$0.51
7/10 of $2.00, 3/10 of $1.60	7/10 of $3.85, 3/10 of $0.10	-$0.85
8/10 of $2.00, 2/10 of $1.60	8/10 of $3.85, 2/10 of $0.10	-$1.18
9/10 of $2.00, 1/10 of $1.60	9/10 of $3.85, 1/10 of $0.10	-$1.52
10/10 of $2.00, 0/10 of $1.60	10/10 of $3.85, 0/10 of $0.10	-$1.85

- Risk neutral agents choose risky at (5/10,5/10); same is true for $u(x) = x^{1-r}$ for $r \in [-.15,15]$.

- 2/3 the subjects switched to risky at or after 5/10 so (if CRRA) their R>.15; average switch point of 5.2 corresponds to CARA of $\alpha = .2$.

- Rabin *Ema* [2000]: (*Under EU) "approximate risk neutrality holds not just for negligible stakes but for quite sizable and economically important stakes. Economists often invoke expected utility to explain substantial (observed or posited) risk aversion over stakes where the theory actually predicts virtual risk neutrality."*

- That is, an agent who rejects small gambles with positive expected value <u>over a range of wealth levels</u> and has a concave utility function will reject very favorable large gambles.

  - He shows this in a few related results, this one is the simplest to paraphrase:

Corollary (Rabin [2000]) Suppose that u is strictly increasing and weakly concave, and that there are $g > l > 0$ such that for all w,

$$.5u(w-l)+.5u(w+g)<u(w) .$$

Then for a function *m* (*defined in the paper*), for all integers *k*, for all $m < m(k)$,

$$.5u(w-2kl)+.5u(w+mg)<u(w) ,$$

where $m(k)$ can be “implausibly big” or even infinite: The agent must reject very favorable gambles.

Intuition: If reject (1/2 chance -100, ½ chance 110), then

$$.5u(w+110)+.5u(w-100)\le u(w)$$

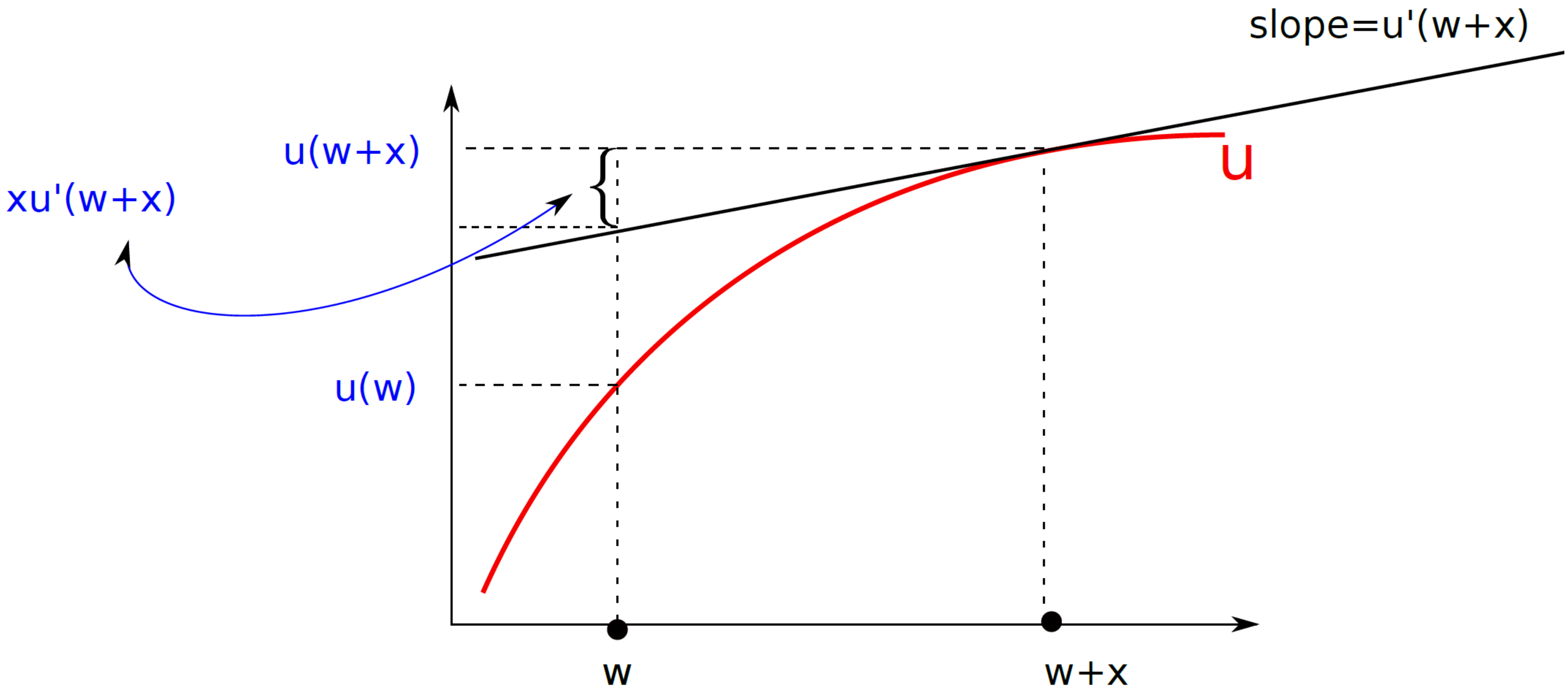
$$u(w+110)-u(w)\le u(w)-u(w-100)$$

And since *u* is concave,

$$110u'(w+110)\le u(w+110)-u(w)$$

$$\le u(w)-u(w-100)\le 100u'(w-100)$$

So $\dfrac{u'(w+110)}{u'(w-100)}\le\dfrac{10}{11}$: Marginal utility of wealth can't decrease too slowly.



$$x \cdot u'(w+x) \leq u(w+x) - u(w)$$

By same argument, if the agent rejects the gamble when $w' = w + 210$, then

$$\frac{u'(w+210+110)}{u'(w+210-100)} = \frac{u'(w+320)}{u'(w+110)} \le \frac{10}{11} .$$

So $\dfrac{u'(w+320)}{u'(w-100)} = \dfrac{u'(w+320)}{u'(w+110)} \dfrac{u'(w+110)}{u'(w-100)} \le \left(\dfrac{10}{11}\right)^2 .$

If can iterate 100 times, then because $\left(\dfrac{10}{11}\right)^{100} \approx 7 \times 10^{-5}$, the marginal utility of wealth is very low for high wealth.

So there is *no* value of *g* that would make the agent accept ½ chance of losing $1000.

This uses the "reject at all w" condition.

But can get similar bounds if we only know the agent is risk averse and turns down the l,g gamble for all wealth levels less than some $\bar{w}$: if $.5u(w-100)+.5u(w+110)<u(w)$ for all $w\leq 300{,}000$ then at $w=290{,}000$ the agent refuses l=-1000,g= 718,000!

Intuition: concavity says $u'(x)\leq u'(300{,}000)$ for all $x>300{,}000$.

Nothing special about ½- ½ bets, they are just used for convenience.

What to make of this?

- Rabin suggests loss aversion as the key.
- But we also see small stakes risk aversion when gambles are entirely in the gains domain as in Holt and Laury.

- An alternative explanation combines the idea that the value of money comes when you spend it and the idea of "narrow bracketing" based on "mental accounts", as in Shefrin and Thaler *Econ Inquiry* [1988])

  > *"..self-control is costly…some mental accounts, those which are labelled 'wealth,' are less tempting than those labelled 'income.'"*

- This leaves open the question of how these brackets or accounts are set.

- Fudenberg-Levine *AER* [2006] propose that the mental account corresponds to daily or weekly consumption expenditures, and argue that small stakes risk aversion comes from treating lab payments as windfall gains.

Idea: Set mental account when not tempted: “pocket cash.”

- Absent windfalls, either impossible or costly to spend more than in mental account.
- Perfect foresight: implement first-best consumption by appropriate choice of cash=desired spending.
- After account is set, the cost of resisting temptation acts as tax on savings out of unplanned-for windfalls: planned consumption didn’t take into account this cost.
- So if win $10 spend it all, but if win $10,000 save some; thus high risk aversion for small winnings (as in experiments) but non-crazy risk aversion for large winnings due to income smoothing.
- *More on the FL model later if time permits..*

Stochastic Dominance

Definition: For any $p \in \Delta(\mathbb{R})$ let

- F_p be its *cumulative distribution function* (*c.d.f)*:

  $F_p(x) = \Pr[z \leq x] = \int_{-\infty}^{x} p(x)dx$.

- $F_p^{-1} : (0,1) \to \mathbb{R}$ be its *quantile function*

  $F_p^{-1}(u) = \inf\left\{x : F_p(x) \geq u\right\}$.

Definition: For $p,q \in \Delta(\mathbb{R})$, *p* first-order stochastically dominates *q,* written $p \geq^{fosd} q$, if $F_p(x) \leq F_q(x) \forall x.$

Note: In the statistics literature this is simply called "stochastic dominance" and *p* is said to be "stochastically larger" than *q.* (e.g. Mann and Whitney *Ann. Math. Stat.* [1947])

FOSD doesn't let us compare the lotteries realization by realization as they might be independent. But it does say we can represent them with a pair of perfectly correlated lotteries where one is always as large as the other:

Fix $p, q \in \Delta(\mathbb{R})$. For $u \in (0,1)$ set $x = F_p^{-1}(u)$ and $y = F_q^{-1}(u)$, e.g. if $x = .5$ these are the medians of *p* and *q*.

Then

$$F_p(x) \le F_q(x) \forall x \text{ iff } x(u) \ge y(u) \forall u \in (0,1) \text{ ,}$$

and if we let *u* be uniformly distributed on (0,1) we have *x* pointwise at least as big as *y*.

This may help give intuition for the following result:

Theorem: $p \geq^{fosd} q$ iff $\int udp \geq \int udq$ for every weakly increasing function $u:\mathbb{R} \to \mathbb{R}$.

Proof:

(i) Suppose there is *y* s.t. $F_p(y) > F_q(y)$. Then if $u(x) = 1(x > y)$,
$\int udp = 1 - F_p(y) < 1 - F_q(y) = \int udq$.

(ii) Conversely suppose $F_p(x) \leq F_q(x) \forall x$. And to simplify assume the c.d.f.'s are strictly increasing (no gaps in the support) and continuous (no atoms) on a common support [a,b] and that *u* is continuous (*can extend by approximating the increasing function u with polynomials*) .

Then we can integrate by parts:

$$\int_a^b u(x)dF_p(x) = u(x)F_p(x)|_a^b - \int_a^b F_p(x)du(x) = u(b) - \int_a^b F_p(x)du(x)$$

$$\geq u(b) - \int_a^b F_q(x)du(x) = \int_a^b u(x)dF_q(x).$$

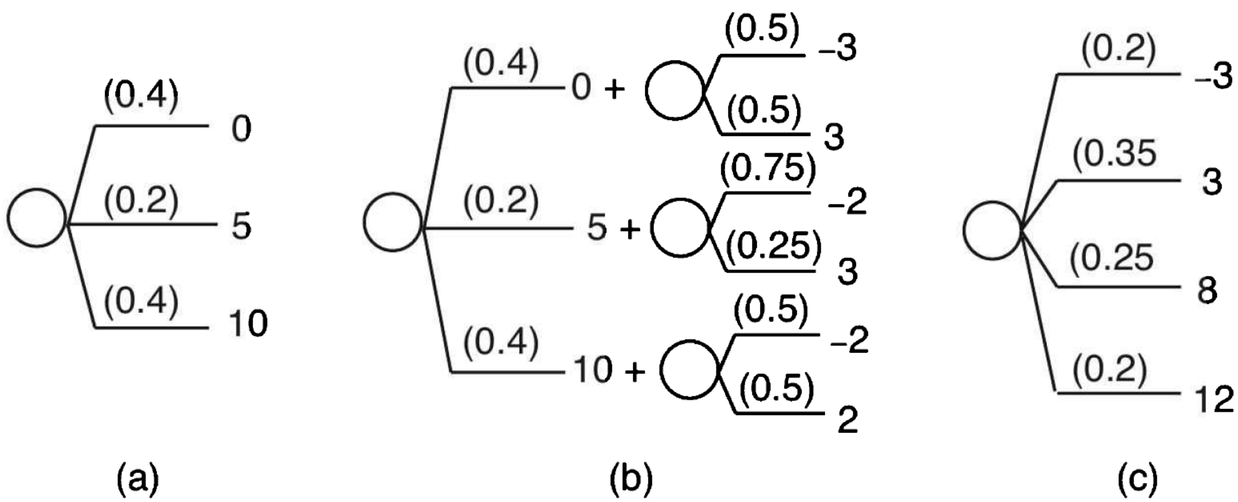
The characterization of FOSD as ranking the expected utility of lotteries applies for all non-decreasing utility functions. What if we add “concave”?

Definition: If the supports of *p* and *q* lie in [a,b], we say *p* second-order stochastically dominates *q* , $p \succsim^{sosd} q$, if $\int_a^c F_q(x)dx \geq \int_a^c F_p(x)dx \, \forall c \in [a,b]$.

Theorem: Assume $Ep = Eq$. Then $p \succsim^{sosd} q$ iff for all non-decreasing concave *u* on [a,b], $\int_a^b udF_p \geq \int_a^b udF_q$.

Proof omitted, use integration by parts.

The intuition comes from a related result: p second-order stochastically dominates q only if we can write p and q as the marginal distributions of random variables z_p *and* z_q *with* $z_q = z_p + \varepsilon$ *and* $E(\varepsilon \mid z_p) \le 0$.



Here lottery (a) is better than lottery (c) for any risk-averse utility function.

Reading for next time: Strzalecki 5.8, 5.9.1-5.9.4, Neilson and Stowe *J Risk Uncertainty* [2002], Epper and Fehr-Duda *Annual Review of Economics* [2012], Bruhin, Epper and Fehr-Duda *Ema* [2015].