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A CURIOUS PROPERTY OF BELIEF OPERATORS

by

Stephen Morris

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A Curious Property of Belief Operators

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Abstract

When two individuals have non-trivial information and a common prior, there exists a non-empty strict subset of the state space which both believe with probability at least $\frac{1}{2}$ whenever it is true (a "$\frac{1}{2}$-evident" event). If they have different priors or there are more than two individuals, then there may be no p-evident event, for any $p > 0$.

1 Introduction

An event is p-evident [Monderer and Samet (1989), henceforth MS] if, whenever it is true, it is believed by everyone with probability at least p. This note shows that, in a finite state space, if two individuals share a common prior and have some non-trivial information, there always exists a non-empty event (other than the whole state space) which is $\frac{1}{2}$-evident.

MS showed that the notion of p-evident events is key to characterizing "common p-belief", a natural weakening of common knowledge. Although they assume the common prior assumption [CPA], they point out that none of their results depend on it. But without the CPA, for any given $p > 0$, there need not exist any p-evident event. Even with the CPA, there need not exist any p-evident event if there are many individuals.

The note is organized as follows. The next section reviews the MS result and gives a corollary about global existence of non-trivial p-evident events. Section 3 gives the result about common $\frac{1}{2}$-belief. Sections 4 and 5 respectively consider what happen without the common prior

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assumption and with many individuals. Section 6 gives a version of a result of Morris, Rob and Shin (1993) which illustrates the significance of the existence of p-evident events. Section 7 concludes.

2 Common p-belief

Fix a finite state space Ω and a finite collection of individuals, $\mathcal{I}$. Assume there are at least two individuals. Each individual $i \in \mathcal{I}$ observes a partition $\mathcal{P}_i$ of the state space; write $P_i(\omega)$ for the element of $\mathcal{P}_i$ containing ω. Let π_i be individual i's prior probability distribution on Ω; assume each π_i assigns strictly positive probability to each element of $\mathcal{P}_i$. Define a belief operator, for individual i, on subsets of Ω as follows:-

$$B_i^p(E) = \{\omega \in \Omega | \pi_i[E|P_i(\omega)] \geq p\}$$

Define an "everyone believes" operator by

$$B_*^p(E) = \bigcap_{i \in \mathcal{I}} B_i^p(E)$$

Definition 1 *An event E is common p-belief at state ω if, for each $n \geq 1$, $\omega \in [B_*^p]^n(E)$.*

Definition 2 *An event E is p-evident if $E \subseteq B_*^p(E)$.*

MS introduced these notions and related them as follows.

Theorem 1 *An event F is common p-belief at state ω if and only if there exists an p-evident event E such that $\omega \in E \subseteq B_*^p(F)$.*

The proof (for the finite state space case) uses only the following properties of the belief operator: for all events E and F, $B_i^p(B_i^p(E)) = B_i^p(E)$ and $E \subseteq F \Longrightarrow B_i^p(E) \subseteq B_i^p(F)$. In what follows, these properties will be used (without comment), as will the properties that $B_i^p(\emptyset) = \emptyset$ and $B_i^p(\Omega) = \Omega$, all of which follow directly from the definitions. All of these properties are true whether or not individuals share the same prior. The following example illustrates the correctness on the theorem even when individuals' priors have disjoint supports.

Example 1 $\Omega = \{a,b,c,d,e,f\}$; $\mathcal{I} = \{1,2\}$; $\mathcal{P}_1 = (\{a,b,c,d\},\{c,f\})$; $\mathcal{P}_2 = (\{a,b\},\{c,d,e,f\})$; $\pi_1 = (\frac{1}{10},0,\frac{4}{5},0,\frac{1}{10},0)$; $\pi_1 = (0,\frac{1}{10},0,\frac{4}{5},0,\frac{1}{10})$.

Observe that event $\{c,d\}$ is $\frac{8}{9}$-evident since $B_1^{\frac{8}{9}}(\{c,d\}) = \{a,b,c,d\}$ and $B_2^{\frac{8}{9}}(\{c,d\}) = \{c,d,e,f\}$. Any event containing both c and d is common $\frac{8}{9}$-belief at both c and d.

MS's theorem 1 gives a relation between common p-belief and the p-evident property for particular events. I give a global corollary about the existence of common p-belief and p-evident events. Say that event E is *non-trivial* if $E \notin \{\emptyset,\Omega\}$; say that E is *contingent p-evident* if E is p-evident and $B_i^p(E)$ is non-trivial for some $i \in \mathcal{I}$.

Corollary 1 *The following three statements are equivalent:- [1] There exists a contingent p-evident event. [2] There exists an event which is sometimes but not always common p-belief, i.e. there exists E such that $\bigcap_{n\geq 1} [B_*^p]^n(E) \notin \{\emptyset,\Omega\}$. [3] There exists an event which is measurable with respect to some individual's partition and is sometimes but not always common p-belief.*

Proof. ([1] $\Rightarrow$ [3]) Suppose E is contingent p-evident. Then there exists an individual i and an event F with $E \subseteq F = B_i^p(E) \notin \{\emptyset,\Omega\}$; now

$$E \subseteq \bigcap_{n\geq 1} [B_*^p]^n(E) \subseteq \bigcap_{n\geq 1} [B_*^p]^n(F) \subseteq B_*^p(B_i^p(E) \subseteq B_i^p(B_i^p(E)) = F$$

So event F is measurable with respect to i's partition and sometimes, but not always, common p-belief. ([3] $\Rightarrow$ [2]) By definition. ([2] $\Rightarrow$ [1]) Suppose F is sometimes but not always common p-belief. Then (by theorem 1) there exists a non-empty p-evident event E with $E \subseteq B_*^p(F)$. Now $B_i^p(E) = \emptyset$ would contradict $E \neq \emptyset$, since $E \subseteq B_*^p(E) \subseteq B_i^p(E)$; $B_i^p(E) = \Omega$ implies $\Omega = B_i^p(E) \subseteq B_i^p(B_*^p(F)) \subseteq B_i^p(B_i^p(F)) = B_i^p(F)$. Thus $B_i^p(E) = \Omega$, for all $i \in \mathcal{I}$, implies $B_i^p(F) = \Omega$, for all $i \in \mathcal{I}$, contradicting the assumption that F was not always common p-belief.

□

The following example makes clear the importance of the qualifications in each of the three equivalent conditions.

Example 2 $\Omega = \{a,b,c,d\}$; $\mathcal{I} = \{1,2\}$; $\mathcal{P}_1 = (\{a,b\},\{c,d\})$; $\mathcal{P}_2 = (\{a,c\},\{b,d\})$; $\pi_1 = \pi_2 = (\frac{3}{10},\frac{3}{10},\frac{3}{10},\frac{1}{10})$.

Now the $\frac{1}{2}$-evident events are $\emptyset$, $\{a\}$, $\{b\}$, $\{c\}$, $\{a,b\}$, $\{a,c\}$, $\{b,c\}$, $\{a,b,c\}$, $\{b,c,d\}$ and Ω; the contingent $\frac{1}{2}$-evident events are $\{a\}$, $\{b\}$, $\{c\}$, $\{a,b\}$, $\{a,c\}$. Events $\{a,b\}$ and $\{a,c\}$ are elements of 1 and 2's partitions, respectively, and are common $\frac{1}{2}$-belief at some but not all states. But if $\frac{1}{2} < p \leq \frac{3}{4}$, the p-evident events are $\emptyset$, $\{a,b,c\}$ and Ω; none of these events is contingent p-evident. No event in any individual's partition is common p-belief anywhere. For example, $B_1^p(\{a,c\}) = \{c,d\}$, $B_2^p(\{a,c\}) = \{a,c\}$, $B_*^p(\{a,c\}) = \{c\}$, $B_1(B_*^p(E)) = \{c,d\}$, and $B_2^p(B_*^p(E)) = \emptyset$.

3 Common $\frac{1}{2}$-belief

MS's characterization of common p-belief did not require that the individuals' posterior beliefs were derived from the same prior. The following result about common $\frac{1}{2}$-belief does. Say that the common prior assumption [CPA] holds if $\pi_i = \pi$, for some π, for all $i \in \mathcal{I}$. Say that there is non-trivial information if $\mathcal{P}_i \neq \{\Omega\}$, for some $i \in \mathcal{I}$.

Theorem 2 *Suppose [1] there is non-trivial information, [2] there are two individuals and [3] the CPA holds. Then there exists a contingent $\frac{1}{2}$-evident event.*

Before giving the proof, we give some examples to give some intuition why it is true.

Example 3 *Let* $\Omega = \{a, b\}$; $\mathcal{I} = \{1, 2\}$; $\mathcal{P}_1 = (\{a, b\})$; $\mathcal{P}_2 = (\{a\}, \{b\})$; $\pi = (\pi_a, \pi_b)$.

The event $\{a\}$ is contingent $\frac{1}{2}$-evident if $\pi_a \geq \pi_b$. The event $\{b\}$ is contingent $\frac{1}{2}$-evident if $\pi_b \geq \pi_a$.

Example 4 *Let* $\Omega = \{a, b, c, d\}$; $\mathcal{I} = \{1, 2\}$; $\mathcal{P}_1 = (\{a, b\}, \{c, d\})$; $\mathcal{P}_2 = (\{a, c\}, \{b, d\})$; $\pi = (\pi_a, \pi_b, \pi_c, \pi_d)$.

Choose $\omega^* \in \arg\max_{\omega \in \Omega} \pi_\omega$. Then the event $\{\omega^*\}$ is contingent $\frac{1}{2}$-evident.

The theorem was stated for finite state space, but the definitions extend to countably infinite state spaces. In the following example, the property of the theorem continues to hold.

Example 5 *Let* $\Omega = \aleph$; $\mathcal{I} = \{1, 2\}$; $\mathcal{P}_1 = (\{0\}, \{1, 2\}, \{3, 4\}..)$; $\mathcal{P}_2 = (\{0, 1\}, \{2, 3\}, ..)$; $\pi = (\pi_0, \pi_1, ...)$.

For any $\omega \in \Omega$, if $\pi_\omega \geq \pi_{\omega+1}$, then the event $\{0, 1, .., \omega\}$ is contingent $\frac{1}{2}$-evident; if $\pi_\omega \leq \pi_{\omega+1}$, then the event $\{\omega + 1, \omega + 2, ..\}$ is contingent $\frac{1}{2}$-evident.

In particular, consider the co-ordinated attack problem (Halpern (1986)). Suppose a commander-in-chief sends a messenger to instruct general 1 to attack. General 1 sends the messenger on to general 2. There is some probability ε that the messenger will be lost on any particular journey. So general 2 sends the messenger back to general 1 to confirm receipt of the message. General 1 sends a confirmation in return, and so on, ad infinitum. This situation can be represented by the state space above, where, in state ω, the messenger is captured just after

delivering the ωth message, $\mathcal{P}_i$ represents the information of general i, and $\pi_\omega = \varepsilon(1 - \varepsilon)^\omega$. In this example, an event is contingent $\frac{1}{2}$-evident exactly if it is of the form $\{0, 1, .., \omega\}$, for some $\omega \in \Omega$.

The proof of the theorem will make use of the following lemma. Write $\sim E$ for the complement in Ω of the event E.

Lemma 1 *If $C_i \in \mathcal{P}_i$, $\sim C_i$ is not contingent $\frac{1}{2}$-evident, and the CPA holds, then there exists $j \in \mathcal{I} \backslash \{i\}$ and $C_j \in \mathcal{P}_j$ such that $\pi[\sim C_i \cap C_j] < \pi[C_i \cap C_j]$.*

Proof. (of lemma 1) For any event $C_i \in \mathcal{P}_i$, $B_i^{\frac{1}{2}}[\sim C_i] = \sim C_i$. If $C_i = \Omega$, then the property of the lemma is trivially true. If $C_i \neq \Omega$, then $\sim C_i$ is not $\frac{1}{2}$-evident. Thus there exists $\omega \in \sim C_i$ such that $\omega \notin B_j^{\frac{1}{2}}(\sim C_i)$, for some $j \neq i$. Let $C_j = P_j(\omega)$; then

$$\pi[\sim C_i \cap C_j] < \frac{1}{2}\pi[C_j] = \frac{1}{2}\pi(\sim C_i \cap C_j) + \frac{1}{2}\pi(C_i \cap C_j)$$

Thus $\pi[\sim C_i \cap C_j] < \pi[C_i \cap C_j]$. □

Proof. (of theorem 2) Suppose the premises of the theorem hold, but there does not exist a $\frac{1}{2}$-evident event. By lemma 1, we can construct a sequence of events $\{C_1^k, C_2^k\}_{k=1,\infty}$ such that, for each $k = 1, \infty$,

$$\pi[\sim C_1^k \cap C_2^k] < \pi[C_1^k \cap C_2^k] \tag{1}$$

$$\pi[\sim C_2^k \cap C_1^{k+1}] < \pi[C_2^k \cap C_1^{k+1}] \tag{2}$$

Now suppose that, for some k, $C_1^k = C_1^{k+1} = C_1$. Then we have

$$\pi[\sim C_1 \cap C_2^k] < \pi[C_1 \cap C_2^k] \tag{3}$$

$$\pi[\sim C_2^k \cap C_1] < \pi[C_2^k \cap C_1] \tag{4}$$

Thus event $C_1 \cap C_2^k$ is $\frac{1}{2}$-evident. By assumption of non-trivial information, either C_1 or C_2^k is not equal to Ω, so $C_1 \cap C_2^k$ is contingent $\frac{1}{2}$-evident. Thus, for all $k = 1, \infty$, $C_1^k \neq C_1^{k+1}$, so $C_1^{k+1} \subseteq \sim C_1^k$, and so

$$\pi[C_1^{k+1} \cap C_2^k] \leq \pi[\sim C_1^k \cap C_2^k] \tag{5}$$

By a symmetric argument, for all $k = 1, \infty$, $C_2^k \neq C_2^{k+1}$, so $C_2^{k+1} \subseteq \sim C_2^k$, and so

$$\pi[C_2^{k+1} \cap C_1^{k+1}] \leq \pi[\sim C_2^k \cap C_1^{k+1}] \tag{6}$$

Now combining, in turn, inequalities (6), (2), (5), and (1) in turn, we have, for all $k = 1, \infty$,

$$\pi[C_1^{k+1} \cap C_2^{k+1}] < \pi[C_1^k \cap C_2^k] \tag{7}$$

By the finiteness of the state space, there exist $k_1 \neq k_2$ such that $C_1^{k_1} = C_1^{k_2}$ and $C_2^{k_1} = C_2^{k_2}$. This contradicts inequality (7) above □

Example 3 shows that the bound of $\frac{1}{2}$ in the theorem is tight: if $\pi_a = \pi_b$, then there is no contingent $\frac{1}{2}$-evident event for any $p > \frac{1}{2}$. The theorem is not true if any of the three conditions are dropped. If there is no non-trivial information, then $B_i^p(E) \in \{\emptyset, \Omega\}$, for all events E and individuals $i \in \mathcal{I}$, so there clearly cannot exist a contingent p-evident event. The next two sections demonstrate that the conclusion of the theorem may not hold if either of the other two assumptions are dropped.

4 The Common Prior Assumption is necessary

Proposition 1 *For any $p > 0$, there exists a two player information system with non-trivial information and no contingent p-evident events.*

The following example proves the proposition by construction. Suppose that the economy may be in a good state (G) or a bad state (B), and the President may be Republican (R) or Democrat (D). Two astronauts - one Republican (r) and one Democrat (d) - return to earth after a long and incommunicado mission. Before arrival, each attached a $\frac{1}{2}$ probability to each Presidential candidate, but - loyal party member that each is - each attached probability $\varepsilon < \frac{1}{2}$ to the economy doing well contingent on the other party winning power and probability $1 - \varepsilon$ to the economy doing well contingent on his own party winning power. On arrival, the Republican astronaut is told the party of the current President (but not the state of the economy); the Democrat astronaut is told the state of the economy (but not the party of the current President). This situation can be represented formally as follows.

Example 6 *Let* $\Omega = \{RG, RB, DG, DB\}$; $\mathcal{I} = \{r, d\}$; $\mathcal{P}_r = (\{RG, RB\}, \{DG, DB\})$; $\mathcal{P}_d = (\{RG, DG\}, \{RB, DB\})$; $\pi_r = (\frac{1-\varepsilon}{2}, \frac{\varepsilon}{2}, \frac{\varepsilon}{2}, \frac{1-\varepsilon}{2})$; $\pi_d = (\frac{\varepsilon}{2}, \frac{1-\varepsilon}{2}, \frac{1-\varepsilon}{2}, \frac{\varepsilon}{2})$; *where* $0 < \varepsilon < \frac{1}{2}$.

Observe that, for every event $E \in 2^\Omega \backslash \{\emptyset, \Omega\}$, there exists $i \in \mathcal{I}$ and $\omega \in E$ such that $\pi_i[E|P_i(\omega)] = \varepsilon$. For example, for the singleton event, 'Democrat victory, good economy' ($\{DG\}$), we have $\pi_r[\{DG\}|P_r(DG)] = \pi_r[\{DG\}|\{DG, DB\}] = \varepsilon$. For the two state event, 'Republican victory' ($\{RB, RG\}$), we have $\pi_d[\{RB, RG\}|P_r(RG)] = \pi_d[\{RB, RG\}|\{RG, DG\}] = \varepsilon$.

Thus, if we set $\varepsilon = 0$, there does not exist a contingent p-evident event for any $p > 0$. For any given $p > 0$, we can (by setting $0 < \varepsilon < p$), find an information system where individuals' priors have common support but there is no contingent p-evident event.

5 A bound on the number of players is necessary to guarantee the existence of contingent p-evident events

Proposition 2 *For any $p > 0$, there exists an information system with non-trivial information, a common prior but no contingent p-evident event. In particular, for any positive integer n, we can construct such an n person information system with no contingent p-evident event, for any $p > \frac{1}{n}$.*

The proposition is proved by construction by the following example. Suppose that a commander-in-chief has to co-ordinate the actions of n generals, who are situated on a line of n hills. Label them (from west to east) 1 through n. The commander-in-chief sends a messenger to convey an attack order to the generals. The messenger delivers the message either from west to east (from 1 to n), or from east to west (in reverse order from n to 1). There is always some chance that the messenger will be captured. Thus the possible sets of generals receiving the attack order are: $\emptyset$, $\{1\}$, $\{n\}$, $\{1, 2\}$, $\{n-1, n\}$,.. , $\{1, 2, .., n-1\}$, $\{2, 3, .., n\}$, $\{1, 2, .., n\}$. Suppose that the capture rates are such that each of these $2n$ possibilities is equally likely. This example can be represented as follows.

Example 7 *Let* $\Omega = \{1, 2.., 2n\}$; $\mathcal{I} = \{1, 2, .., n\}$; $\mathcal{P}_1 = (\{1, 2, .., n\}, \{n+1, n+2, .., 2n\})$; $\mathcal{P}_2 = (\{2, 3, .., n+1\}, \{n+2, .., 2n, 1\})$... $\mathcal{P}_k = (\{k, k+1, .., n+k-1\}, \{n+k, .., 2n, 1, .., k-1\})$... $\mathcal{P}_n = (\{n, n+1, .., 2n-1\}, \{2n, 1, .., n-1\})$; $\pi = (\frac{1}{2n}, \frac{1}{2n}, ..., \frac{1}{2n})$.

The formal example corresponds to the above description where we interpret the first information set of each general k to represent the event "k receives the attack order". Thus each state $\omega \in \{1, .., n\}$ represents a situation where generals 1 through ω receive the attack order, while each state $\omega \in \{n+1, .., 2n-1\}$ represents a situation where generals $\omega - n + 1$ through n receive the attack order. State $2n$ represents a situation where nobody receives the attack order.

Choose p such that $\frac{1}{n} < p \leq \frac{2}{n}$. Thus $\omega \in B_i^p(C)$ exactly if $P_i(\omega) \cap C$ has at least two elements (for all events C and $i \in \mathcal{I}$). Let $m = \frac{n}{2}$,

if n is even, $m = \frac{n+1}{2}$, if n is odd. Let $E = \{m, m+1, .., n+m-1\}$. Then $B^p_m(E) = E = \{m, m+1, .., n+m-1\}$; $B^p_{m-1}(E) = \{m-1, m, .., n+m-2\}$; $B^p_{m+1}(E) = \{m+1, m+2, .., n+m\}$; $B^p_k(E) = \Omega$, for all $k \notin \{m-1, m, m+1\}$. So $B^p_*(E) = \{m+1, m+2, .., n+m-2\}$. Now $B^p_m(B^p_*(E)) = \{m, m+1, .., n+m-1\}$; $B^p_{m-1}(B^p_*(E)) = \{m-1, m, .., n+m-2\}$; $B^p_{m+1}(B^p_*(E)) = \{m+1, m+2, .., n+m\}$; $B^p_{m-2}(B^p_*(E)) = \{m-2, m, .., n+m-3\}$; $B^p_{m+2}(B^p_*(E)) = \{m+2, m+3, .., n+m+1\}$; $B^p_k(B^p_*(E)) = \Omega$, for all $k \notin \{m-2, m-1, m, m+1, m+2\}$. Thus $[B^p_*]^2(E) = \{m+2, m+3, .., n+m-3\}$. By induction, $[B^p_*]^l(E) = \{m+l, m+l+1, .., n+m-l-1\}$, if $l \leq \frac{n-1}{2}$ and $[B^p_*]^l(E) = \emptyset$, if $l > \frac{n-1}{2}$. But now by the cyclical symmetry of the information structure of this example, $[B^p_*]^l(E) = \emptyset$, for all $l > \frac{n-1}{2}$ and all non-trivial events E measurable with respect to some individual's partition. By corollary 1, there does not exist a contingent p-evident event.

6 Economic Relevance

The following definition and theorem are slight (many person) generalizations of results in Morris, Rob and Shin (1993). Consider a game of incomplete information defined on the information system. Each individual $i \in \mathcal{I} = \{1, .., n\}$ has a set of possible actions A_i. Let $A = A_1 \times .. \times A_n$; individual i's payoffs are then represented by a function $g_i : A \times \Omega \to \Re$. Assume that each individual i knows his own payoff (so that $g_i(a; \omega') = g_i(a; \omega)$ for all $a \in A$, if $\omega' \in P^i(\omega)$), but not the payoffs of others. Write a_{-i} for the set of $(n-1)$-tuple of actions excluding i's. For a given such $(n-1)$-tuple of strategies, a^*_{-i}, we will be interested in the set of $(n-1)$-tuples where at least one player plays according to a^*_{-i}, i.e. $h_i(a^*_{-i}) \equiv \{a \in A | a_j = a^*_j, \text{ for some } j \neq i\}$.

Definition 3 *Action profile $a_* \in A$ is* p-dominant *at state ω if, for each $i \in \mathcal{I}$ and each $\lambda \in \Delta(A_{-i})$ such that* $\sum_{a_{-i} \in h_i(a^*_{-i})} \lambda(a_{-i}) \geq p$,

$$\sum_{a_{-i} \in A_{-i}} \lambda(a_{-i}) g_i(a^*_i, a_{-i}; \omega) > \sum_{a_{-i} \in A_{-i}} \lambda(a_{-i}) g_i(a_i, a_{-i}; \omega), \text{ for all } a_i \in A_i$$

Thus action a^*_i is the unique best response of individual i to any conjecture about the other individuals' actions which puts probability at least p on some other individual playing a^*_j.

Example 8 *For any information system, consider payoffs as follows. Each individual has two possible actions, $A_i = \{C, R\}$ ("charge" or "retreat"). Payoffs are $g_i(a; \omega) = 0$, if $a_i = R$; $g_i(a; \omega) = -1$, if $a_i = C$ and $a_j = R$, for any $j \in \mathcal{I}$; $g_i(a; \omega) = x_i(\omega)$, if $a_j = C$, for all $j \in \mathcal{I}$.*

Then "all retreat" is a p-dominant action profile at ω if and only if $x_i(\omega) < \frac{p}{1-p}$, for all $i \in \mathcal{I}$.

Now we have:

Theorem 3 *Suppose that [1] action profile a^* is p-dominant at every state [2] action a^*_i is a dominant action for some individual at some state; and [3] there does not exist a contingent $(1-p)$-evident event. Then the unique strategy profile surviving iterative deletion of dominated strategies has each a^* played everywhere.*

Proof. A pure strategy for individual i is a mapping $s_i : \Omega \to A_i$. Let U_i be set of such strategies which survive iterated deletion of dominated strategies. Let $N_i = \{\omega \in \Omega | s_i(\omega) \neq a^*_i, \text{ for some } s_i \in U_i\}$. Define $N_{-i} \equiv \cap_{j \neq i} N_j$ and $N^* \equiv \cap_{j \in \mathcal{I}} N_j$. By [1] we have, for all $i \in \mathcal{I}$, $\pi_i[\sim N_{-i} | P_i(\omega)] \geq p \Longrightarrow \omega \notin N_i$. Thus $\omega \in N_i \Longrightarrow \pi_i[\sim N_{-i} | P_i(\omega)] < p \Longrightarrow \pi_i[N_{-i} | P_i(\omega)] > 1-p$. Since $\omega \in N_i \Longrightarrow P_i(\omega) \subseteq N_i$, we have $N^* \subseteq N_i \subseteq B^{1-p}_i(N^*)$. Thus N^* is $(1-p)$-evident. By [2], $N_j \neq \Omega$, for some $j \in \mathcal{I}$. Since $B^{1-p}_j(N^*) \subseteq N_j$, we have $B^{1-p}_j(N^*) \neq \Omega$. Now by [3], $B^{1-p}_j(N^*) = \emptyset$. But then $N_j = \emptyset$, so $N_i = \emptyset$, for all $i \in \mathcal{I}$.

The following example illustrates the power of theorem 3 as the number of individuals becomes large.

Example 9 *Consider the incomplete information game with the information system of example 7 and the payoffs of example 8. Suppose that $x_i(\omega) < 0$, for some $\omega \in \Omega$ and $i \in \mathcal{I}$; and $x_i(\omega) < n-1$, for all $\omega \in \Omega$ and $i \in \mathcal{I}$. Then the unique strategy profile surviving iterative deletion of dominated strategies has each general always retreating.*

7 Conclusion

This note examined finite state spaces. The algebraic intuition for theorem 2 seems to extend to countable state spaces. Example 7 suggested that it might possible to extend theorem 2 to show that every n person information system with non-trivial information and a common prior has some contingent $\frac{1}{n}$-evident events. However, the particular proof of theorem 2 does not seem to generalize in any obvious way to either many individuals or countable state spaces.

References

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