Advanced Economic Growth: Lecture 7, Innovation by Incumbents and Cumulative Research

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Introduction

- Key aspect of growth: interplay between innovations and productivity improvements from entry.
- Large part of productivity growth comes from improvements by continuing plants.
- For example: Bartelsman and Doms (2000) and Foster, Haltiwanger and Krizan (2000): entry and exit account for about 25% of average TFP growth, with the remaining accounted for by continuing plants.
- Thus models in which firms continually invest in technology and productivity are important for understanding differences across firms and plants and also across countries.
- Schumpeterian models thus far generate growth only by entry.
    - Models of expanding input or product variety not useful for the study of the set of questions either.
- Discuss models that feature productivity growth by continuing plants (firms).

Innovation by Incumbents and Entrants: Model I

- Continuous time.
- Representative household with the standard CRRA preferences.
- Population constant at L and labor is supplied inelastically.
- Resource constraint at time t as usual:

$$C(t) + X(t) + Z(t) \leq Y(t), \tag{1}$$

- Production function of the unique final good:

$$Y(t) = \frac{1}{1-\beta}\left[\int_0^1 q(\nu,t)^{\beta} x(\nu,t \mid q)^{1-\beta} d\nu\right] L^{\beta}, \tag{2}$$

where:

  - $x(\nu, t \mid q)$=quantity of the machine of type ν of quality $q(\nu, t)$.
  - measure of machines is again normalized to 1.

Innovation by Incumbents and Entrants: Model II

- Exponent β in q: no effects on results on growth, but implies firms with different productivity levels will have different levels of sales (predictions about size distribution of firms).
- Engine of growth: quality improvements, now driven by two types of innovations:
  1. Innovation by incumbents
  2. Creative destruction by entrants.
- "Quality ladder" for each machine type:

$$q(\nu, t) = \lambda^{n(\nu,t)} q(\nu, s) \text{ for all } \nu \text{ and } t,$$

where:

- $\lambda > 1$ and $n(\nu, t)$ is the number of *incremental* innovations on this machine line between $s \leq t$ and t.
- s is the date at which this particular type of technology was first invented, with quality $q(\nu, s)$ at that point.

Innovation by Incumbents and Entrants: Model III

- Incumbent has fully enforced patent on machines that it has developed.
- This patent does not prevent entrants leapfrogging the incumbent's machine quality.
- At time $t = 0$, each machine line starts with some quality $q(\nu, 0) > 0$ owned by an incumbent.
- Incremental innovations only by the incumbent producer, i.e. "tinkering" innovations (consistent with case study evidence, e.g., Freemen, 1982, or Scherer, 1984):
  - If spend $z(\nu, t)\, q(\nu, t)$ of the final good for innovation on quality $q(\nu, t)$, then flow rate of innovation $\phi z(\nu, t)$ for $\phi > 0$
  - More formally: for any interval $\Delta t > 0$, the probability of one incremental innovation is $\phi z(\nu, t)\, \Delta t$ and the probability of more than one is $o(\Delta t)$ (with $o(\Delta t) / \Delta t \to 0$ as $\Delta t \to 0$).
  - Such innovation results in new machine of quality $\lambda q(\nu, t)$.

Innovation by Incumbents and Entrants: Model IV

- Alternatively, new firm (entrant) can innovate over existing machines in machine line ν at time t:
  - If current quality is $q(\nu, t)$, spending one unit of the final good gives flow rate of innovation $\eta(\hat{z}(\nu, t)) / q(\nu, t)$.
  - $\eta(\cdot)$ is a strictly decreasing, continuously differentiable function.
  - $\hat{z}(\nu, t)$ is total of R&D by new entrants towards machine line ν at time t.
  - Innovation leads to new machine of quality $\kappa q(\nu, t)$, where $\kappa > \lambda$.
- Note:
  - Innovation by entrants more "radical" than by incumbents, supported from studies of innovation.
  - Incumbents also have access to the technology for radical innovation, but Arrow replacement effect implies they would never use it (entrants will make zero profits from it, so profits of incumbents would be negative).
  - Strictly decreasing function η, captures "external" diminishing returns (new entrants "fishing out of the same pond").

Innovation by Incumbents and Entrants: Model V

- Each entrant attempting R&D is potentially small, take $\eta\left(\hat{z}\left(\nu, t\right)\right)$ as given.
- Assume that $z\eta\left(z\right)$ is strictly increasing in z: greater aggregate R&D towards a machine line increases the overall probability of discovering a superior machine.
- $\eta\left(z\right)$ satisfies Inada-type assumptions:

$$\lim_{z \to \infty} \eta\left(z\right) = 0 \text{ and } \lim_{z \to 0} \eta\left(z\right) = \infty. \tag{3}$$

- Once a machine of quality $q\left(\nu, t\right)$ has been invented, any quantity can be produced at the marginal cost ψ, $\psi \equiv 1 - \beta$.

Innovation by Incumbents and Entrants: Model VI

- Thus total amount of expenditure on the production of intermediate goods at time t:

$$X(t) = \int_0^1 \psi x(\nu, t)\, d\nu, \tag{4}$$

where $x(\nu, t)$ is the quantity of this machine used in final good production.

- Total expenditure on R&D is sum of R&D by incumbents and entrants ($z(\nu, t)$ and $\hat{z}(\nu, t)$):

$$Z(t) = \int_0^1 [z(\nu, t) + \hat{z}(\nu, t)]\, q(\nu, t)\, d\nu, \tag{5}$$

where $q(\nu, t)$ refers to the highest quality of the machine of type ν at time t (recall: higher-quality machines is proportionately more difficult).

Innovation by Incumbents and Entrants: Model VII

- Allocation. Time paths of $[C(t), X(t), Z(t)]_{t=0}^{\infty}$, $[z(\nu,t), \hat{z}(\nu,t)]_{\nu\in[0,1],t=0}^{\infty}$, $[p^x(\nu,t \mid q), x(\nu,t), V(\nu,t \mid q)]_{\nu\in[0,1],t=0}^{\infty}$, $[r(t), w(t)]_{t=0}^{\infty}$.
- Equilibrium. Allocation in which R&D decisions by entrants maximize their net present discounted value, pricing, quantity and R&D decisions by incumbents maximize their net present discounted value, consumers choose the path of consumption and allocation of spending across machines and R&D optimally, and the labor market clears.

Innovation by Incumbents and Entrants: Model VIII

- Profit-maximization by the final good sector implies the demand for machines of highest-quality:

$$x(\nu, t \mid q) = p^x (\nu, t \mid q)^{-1/\beta} q(\nu, t) L \quad \text{for all } \nu \in [0,1] \text{ and all } t, \tag{6}$$

- Unconstrained monopoly price is usual formula as a constant markup over marginal cost.
- No limit price assumption:

$$\kappa > \left(\frac{1}{1-\beta} \right)^{\frac{1-\beta}{\beta}}, \tag{7}$$

- By implication, incumbents that make further innovations can also charged the unconstrained monopoly price.

Equilibrium I

- Since demand for machines in (6) is iso-elastic and $\psi = 1 - \beta$, profit-maximizing monopoly price:

$$p^x (\nu, t \mid q) = 1. \tag{8}$$

- Combining with (6):

$$x (\nu, t \mid q) = qL. \tag{9}$$

- Flow profits of a firm with monopoly rights on the machine quality q:

$$\pi (\nu, t \mid q) = \beta qL. \tag{10}$$

- Substituting (9) into (2), total output is:

$$Y (t) = \frac{1}{1 - \beta} Q (t) L, \tag{11}$$

with average quality $Q (t) \equiv \int_0^1 q(\nu, t) d\nu$

Equilibrium II

- Aggregate spending on machines:

$$X(t) = (1-\beta)\, Q(t)\, L. \tag{12}$$

- Labor market is competitive, wage rate:

$$w(t) = \frac{\beta}{1-\beta} Q(t). \tag{13}$$

- Need to determine R&D effort levels by incumbents and entrants.
- Net present value of a monopolist with the highest quality of machine q at time t in machine line ν satisfies HJB ($V(\nu, t \mid q) = V(q)$, etc.):

$$\begin{aligned} r(t)\, V(q) - \dot{V}(q) = & \max_{z(\nu,t|q) \geq o} \{\pi(q) - z(q)\, q \\ & + \phi z(q) \left(V(\lambda q) - V(q)\right) - \eta\left(\hat{z}(q)\right) \hat{z}(q) V(q)\}, \end{aligned} \tag{14}$$

Equilibrium III

- Free entry:

$$\begin{aligned} \eta\left(\hat{z}(\nu,t \mid q)\right) V\left(\nu,t \mid \kappa q\right) &\leq q(\nu,t), \text{ and} \\ \eta\left(\hat{z}(\nu,t \mid q)\right) V\left(\nu,t \mid \kappa q\right) &= q(\nu,t) \text{ if } \hat{z}\left(\nu,t \mid q\right) > 0, \end{aligned} \tag{15}$$

- Incumbent's choice of R&D effort implies similar complementary slackness condition:

$$\begin{aligned} \phi\left(V\left(\nu,t \mid \lambda q\right) - V\left(\nu,t \mid \kappa q\right)\right) &\leq q(\nu,t) \text{ and} \\ \phi\left(V\left(\nu,t \mid \lambda q\right) - V\left(\nu,t \mid \kappa q\right)\right) &= q(\nu,t) \text{ if } z\left(\nu,t \mid q\right) > 0. \end{aligned} \tag{16}$$

- Consumer maximization:

$$\frac{\dot{C}(t)}{C(t)} = \frac{1}{\theta}(r(t) - \rho), \tag{17}$$

$$\lim_{t\to\infty}\left[\exp\left(-\int_0^t r(s)\,ds\right)\int_0^1 V\left(\nu,t \mid q\right)d\nu\right] = 0 \tag{18}$$

Equilibrium and Balanced Growth Path

- Equilibrium is thus time paths of
  - $[C(t), X(t), Z(t)]_{t=0}^{\infty}$ that satisfy (1), (5), (12) and (18)
  - $[z(\nu, t), \hat{z}(\nu, t)]_{\nu \in [0,1], t=0}^{\infty}$ that satisfy (15) and (16);
  - $[p^x(\nu, t \mid q), x(\nu, t), V(\nu, t \mid q)]_{\nu \in [0,1], t=0}^{\infty}$ given by (8), (9) and (14);
  - $[w(t), r(t)]_{t=0}^{\infty}$ that satisfy (13) and (17).
- *BGP* (balanced growth path): equilibrium path in which innovation, output and consumption growth a constant rate.
- Note in BGP aggregates grow at the constant rate but there will be firm deaths and births, and the firm size distribution may also change.

Balanced Growth Path I

- From Euler equation, the requirement that consumption grows at a constant rate in the BGP implies

$$r(t) = r^*$$

- In BGP, must also have $z(\nu, t \mid q) = z(q)$ and $\hat{z}(\nu, t \mid q) = \hat{z}(q)$.
- These imply in BGP $\dot{V}(\nu, t \mid q) = 0$ and $V(\nu, t \mid q) = V(q)$.
- Since profits and costs are both proportional to quality q, $z(q) = z$, $\hat{z}(q) = \hat{z}$ and $V(q) = vq$.
- Look for an "interior" BGP equilibrium (will verify below that exists and is unique).

Balanced Growth Path II

- Incumbents undertake research, thus

$$\phi\left(V\left(\nu,t\mid\lambda q\right)-V\left(\nu,t\mid q\right)\right)=q(\nu,t), \tag{19}$$

- Therefore

$$V\left(q\right)=\frac{q}{\phi\left(\lambda-1\right)}. \tag{20}$$

- The free entry condition then implies $\eta\left(\hat{z}\right)V(\kappa q)=q$ and thus

$$V\left(q\right)=\frac{\beta Lq}{r^{*}+\hat{z}\eta\left(\hat{z}\right)}. \tag{21}$$

- Combining this expression with (19) and (20), we obtain

$$\frac{\phi\left(\lambda-1\right)}{\kappa\eta\left(\hat{z}\right)}=1.$$

Balanced Growth Path III

- Hence the BGP R&D level by entrants $\hat{z}^*$ is defined implicitly by:

$$\hat{z}(q) = \hat{z}^* \equiv \eta^{-1}\left(\frac{\phi(\lambda-1)}{\kappa}\right) \text{ for all } q > 0. \tag{22}$$

- Combining with (21):

$$\begin{aligned} r^* &= \kappa\eta(\hat{z}^*)\beta L - \hat{z}^*\eta(\hat{z}^*) \\ &= \phi(\lambda-1)\beta L - \hat{z}^*\eta(\hat{z}^*). \end{aligned} \tag{23}$$

- From Euler equation, growth rate of consumption and output:

$$g^* = \frac{1}{\theta}\left(\phi(\lambda-1)\beta L - \hat{z}^*\eta(\hat{z}^*) - \rho\right). \tag{24}$$

Balanced Growth Path IV

- (24) determines relationship between $\hat{z}^*$ and g^*. In contrast to standard Shcumpeterian models:

  Remark There is a negative relationship between $\hat{z}^*$ and g^*.

- From (24), g^* is decreasing in $\hat{z}^* \eta \left(\hat{z}^* \right)$ (which is always strictly increasing in $\hat{z}^*$).
- (24) and (22), determine BGP growth rate of the economy, but not how much of productivity growth is driven by creative destruction (entrants) and how much by incumbents.

Balanced Growth Path V

- To determine this, repeat analysis as in standard model:

$$\begin{aligned} Q(t+\Delta t) &= (\lambda \phi z(t) \Delta t) Q(t) + (\kappa \hat{z}(t) \eta(\hat{z}(t)) \Delta t) Q(t) \\ &\quad + ((1 - \phi z(t) \Delta t - \hat{z}(t) \eta(\hat{z}(t)) \Delta t)) Q(t) + o(\Delta t) \end{aligned}$$

- Subtracting $Q(t)$ from both sides, dividing by Δt and taking the limit as $\Delta t \to 0$:

$$g(t) = \frac{\dot{Q}(t)}{Q(t)} = (\lambda - 1) z(t) + (\kappa - 1) \hat{z}(t) \eta(\hat{z}(t)),$$

  which which decomposes growth into the component from incumbent firms (first term) and from new entrants (second term).

- In BGP:

$$g^* = (\lambda - 1) \phi z^* + (\kappa - 1) \hat{z}^* \eta(\hat{z}^*). \tag{25}$$

Balanced Growth Path VI

- Can also verify that this economy does not have any transitional dynamics; if an equilibrium with growth exists, it will involve growth at g^*.
- To ensure equilibrium exists, verify R&D is profitable both for entrants and incumbents.
- The condition that r^* should be greater than ρ is sufficient for there to be positive aggregate growth.
- In addition, r^* should not be so high that transversality condition of the consumers is violated.
- Finally, need to ensure that there is also innovation by incumbents.

Balanced Growth Path VII

- The following condition ensures all three of these requirements:

$$\begin{aligned} & \kappa\eta\left(\hat{z}^{*}\right)\beta L-\left(\theta\left(\kappa-1\right)+1\right)\hat{z}^{*}\eta\left(\hat{z}^{*}\right) \\ > \quad & \rho>\left(1-\theta\right)\left(\kappa\eta\left(\hat{z}^{*}\right)\beta L-\hat{z}^{*}\eta\left(\hat{z}^{*}\right)\right), \end{aligned} \tag{26}$$

  with $\hat{z}^{*}$ given by (22).

- To obtain how much of productivity growth and innovation are driven by incumbents and how much by new entrants, from:

$$\left(\lambda-1\right)\phi z^{*}=\frac{1}{\theta}\left(g^{*}-\rho\right)-\left(\kappa-1\right)\hat{z}^{*}\eta\left(\hat{z}^{*}\right), \tag{27}$$

  with g^{*} given in (24) and $\hat{z}^{*}$ in (22).

Firm-size Dynamics I

- Firm-size dynamics: size of a firm can be measured by its sales:

$$x(\nu, t \mid q) = qL \text{ for all } \nu \text{ and } t.$$

- Suppose that all incumbents do R&D at the rate z^* given by (27) [Why is this is a supposition? What would be the justification?]
- Then, quality of an incumbent firm increases that the flow rate ϕz^*, and it is replaced at the flow rate $\hat{z}^* \eta(\hat{z}^*)$.
- Thus, for Δt sufficiently small:

$$x(\nu, t + \Delta t \mid q) = \begin{cases} \lambda x(\nu, t \mid q) & \text{w. p. } \phi z^* \Delta t + o(\Delta t) \\ 0 & \text{w. p. } \hat{z}^* \eta(\hat{z}^*) \Delta t + o(\Delta t) \\ x(\nu, t \mid q) & \text{w. p. } (1 - \phi z^* - \hat{z}^* \eta(\hat{z}^*)) \Delta t \\ & \quad + o(\Delta t) \end{cases} \tag{28}$$

for all ν and t.

Firm-size Dynamics II

- Thus firms have random growth, and surviving firms expand on average.
- Firms also face a probability of bankruptcy (extinction)
- Let $P(t \mid s, \nu)$ =probability that a particular incumbent firm that started production in machine line ν at time s will be bankrupt by time $t \geq s$:

$$\lim_{t\to\infty} P(t \mid s, \nu) = 1$$

so that each firm will necessarily die eventually.

Summary of Equilibrium

Proposition Consider the above-described economy starting with an initial condition $Q(0) > 0$. Suppose that (3) and (26) are satisfied. Then there exists a unique equilibrium. In this equilibrium growth is always balanced, and technology, $Q(t)$, aggregate output, $Y(t)$, and aggregate consumption, $C(t)$, grow at the rate g^* as in (24) with $\hat{z}^*$ given by (22). Equilibrium growth is driven both by innovation by incumbents and by creative destruction by entrants. Any given firm expands on average as long as it survives, but is eventually replaced by a new entrant with probability one.

Proof of Proposition: Equilibrium I

- Characterization of BGP is given by argument preceding proposition.
- $\hat{z}^*$ is uniquely determined by (22) and given $\hat{z}^*$, (24) gives the unique BGP growth rate.
- To ensure that this is indeed an equilibrium we need to check:
  1. *Positive $\hat{z}^*$:* follows from (3).
  2. *Positive growth:* requires $g^* = (\kappa\eta(\hat{z}^*)\beta L - \hat{z}^*\eta(\hat{z}^*) - \rho)/\theta$. Since $(\theta(\kappa-1)+1) > 1$, the first inequality in (26), is sufficient for this.
  3. *Positive z^*:* from the first inequality (26):

$$\begin{aligned} z^* &= \frac{g^* - (\kappa-1)\hat{z}^*\eta(\hat{z}^*)}{(\lambda-1)\phi} \\ &= \frac{(\phi(\lambda-1)\beta L - \hat{z}^*\eta(\hat{z}^*) - \rho)/\theta - (\kappa-1)\hat{z}^*\eta(\hat{z}^*)}{(\lambda-1)\phi} > 0, \end{aligned}$$

  4. *The transversality condition:* The second inequality in (26) $\Longrightarrow$ $r^* > g^*$, which is necessary and sufficient to ensure this.

Proof of Proposition: Equilibrium II

- Therefore, the BGP is interior and uniquely defined.
- Next show that no transitional dynamics.
- two observations.

[A]Because of the Inada conditions, (3), the free entry condition (15) must hold as equality for all ν, t and q, so that

$$\eta\left(\hat{z}(\nu, t \mid q)\right) V\left(\nu, t \mid \kappa q\right) = q \text{ for all } \nu,\ q \text{ and } t. \tag{29}$$

Since this equation holds for all t and the right-hand side is differentiable in q and t, so must be the left-hand side. Differentiating with respect to t, we obtain

$$\frac{\partial \hat{z}(\nu, t \mid q)/\partial t}{\hat{z}(\nu, t \mid q)} = \frac{1}{\varepsilon_\eta\left(\hat{z}(\nu, t \mid q)\right)} \frac{\dot{V}\left(\nu, t \mid \kappa q\right)}{V\left(\nu, t \mid \kappa q\right)} \text{ for all } q \text{ and } t, \tag{30}$$

where

$$\varepsilon_\eta\left(\hat{z}\right) \equiv -\frac{\eta'\left(\hat{z}\right)\hat{z}}{\eta\left(\hat{z}\right)} > 0.$$

Proof of Proposition: Equilibrium III

[B]The value of a firm with a machine of quality q at time t can be written as

$$V(\nu, t) = \int_t^\infty \exp\left[-\int_t^s \left(r\left(s'\right) + \hat{z}(\nu, s')\eta\left(\hat{z}(\nu, s')\right)\right) ds'\right] \pi(\nu, s)\ ds. \quad (31)$$

This value is always finite since, from observation [A], $\left(r(t) + \hat{z}(\nu, t \mid q)\eta\left(\hat{z}(\nu, t \mid q)\right)\right) > 0$ for all ν, q and t, and because $\pi(\nu, s \mid q) = \beta q L \in (0, \infty)$ and also $V(\nu, t \mid q)$ uniformly bounded away from 0 (unless, $r(t) \to \infty$, which is impossible, since it would violate the transversality condition (18)).

- Consider two cases.

Proof of Proposition: Equilibrium IV

Case 1: Suppose that (19), that is, $\phi\left(V\left(\nu, t \mid \lambda q\right) - V\left(\nu, t \mid q\right)\right) = q$, holds for all ν, q and t. Then $V\left(\nu, t \mid q\right)$ is linear in q. Thus

$$V\left(\nu, t \mid q\right) = v\left(t\right) q$$

and (19) can be written as

$$\phi\left(\lambda - 1\right) v\left(t\right) = 1 \text{ for all } t.$$

Therefore, $\dot{v}\left(t\right) = 0$ and $v\left(t\right) = v$ for all t. Moreover, from [A]:

$$\eta\left(\hat{z}(\nu, t \mid \kappa^{-1} q)\right) v\left(t\right) = 1 \text{ for all } t,$$

so that $\hat{z}\left(\nu, t \mid q\right) = \hat{z}\left(t\right)$ for all q and t. Therefore, the value function becomes:

$$r\left(t\right) v = \beta L - \eta\left(\hat{z}\right) v$$

for all t, which implies that $r\left(t\right)$ must be constant, and thus $r\left(t\right) = r^{*}$, $\hat{z}\left(t\right) = \hat{z}^{*}$ and $z\left(t\right) = z^{*}$ for all t.

Proof of Proposition: Equilibrium V

Case 2: Suppose that (19) does not hold for some $\nu \in \mathcal{N} \subset [0,1]$, q and t. Since either $\phi\left(V\left(\nu,t \mid \lambda q\right) - V\left(\nu,t \mid q\right)\right) = q$ or $\phi\left(V\left(\nu,t \mid \lambda q\right) - V\left(\nu,t \mid q\right)\right) < q$ and $z\left(\nu,t \mid q\right) = 0$, and because from observation [A] above, $\eta\left(\hat{z}(\nu,t \mid \kappa^{-1}q)\right) V\left(\nu,t \mid q\right) = \kappa^{-1}q$, the value function (14) in this case can be written as

$$\frac{\dot{V}\left(\nu,t \mid q\right)}{V\left(\nu,t \mid q\right)} = r\left(t\right) + \hat{z}\left(\nu,t \mid q\right)\eta\left(\hat{z}\left(\nu,t \mid q\right)\right) - \kappa\beta L\eta\left(\hat{z}(\nu,t \mid \kappa^{-1}q)\right).$$

Combining this with (30), we obtain a set of differential equations of the form:

$$\frac{\partial\hat{z}(\nu,t \mid \kappa^{-1}q)/\partial t}{\hat{z}(\nu,t \mid \kappa^{-1}q)} = \frac{1}{\varepsilon_{\eta}\left(\hat{z}(\nu,t \mid \kappa^{-1}q)\right)}\left[r\left(t\right) + \hat{z}\left(\nu,t \mid q\right)\eta\left(\hat{z}\left(\nu,t \mid q\right)\right) - \kappa\beta L\eta\left(\hat{z}(\nu,t \mid \kappa^{-1}q)\right)\right],$$

which are all unstable $\Longrightarrow \partial\hat{z}(\nu,t \mid q)/\partial t = 0$ for all $\nu \in [0,1]$ and all t.

Proof of Proposition: Equilibrium VI

- Otherwise, in the limit the free entry condition (15) implies either $V(\nu, t \mid \kappa q) \to \infty$ or $V(\nu, t \mid \kappa q) \to 0$, which are both impossible in view of observation [B]).
- Therefore, $\partial \hat{z}(\nu, t \mid q)/\partial t = 0$ for all ν, q and t, which implies

$$r(t) + \hat{z}(\nu, t \mid \kappa q)\, \eta(\hat{z}(\nu, t \mid \kappa q)) - \beta L \eta(\hat{z}(\nu, t \mid q)) = 0,$$

  for all ν, q and t
- This is only possible if $r(t)$ is constant and thus equal to r^* and $\hat{z}(\nu, t \mid q) = \hat{z}^*$.
- Finally, the result that surviving firms expand on average and that all firms die with probability 1 follows from equation (28).

Some Numbers I

- Choose the following standard numbers:

$$g^* = 0.02,\ \rho = 0.01,\ r^* = 0.05,\text{ and }\theta = 2,$$

  where the last number, the intertemporal elasticity of substitution, is pinned down by the choice of the other three.
- The first three numbers refer to annual rates (implicitly defining $\Delta t = 1$ as one year).
- Normalize $L = 1$.

Some Numbers II

- Much greater uncertainty concerning the remaining parameters. As a benchmark:
  - $\beta = 2/3$ (one third of national income to profits, two thirds to labor).
  - The no limit price requirement implies that $\kappa > 1.7$.
  - Set $\kappa = 3$, so that entry by new firms is sufficiently radical.
  - Innovation by incumbents is relatively minor, choose $\lambda = 1.2$. (implies productivity gains from radical innovation is about ten times that of "tinkering" innovation(i.e., $(\kappa - 1) / (\lambda - 1) = 10$)).
  - Assume functional form for the function $\eta(z)$: $\eta(z) = Bz^{-\alpha}$, $\alpha = 0.5$.
  - Choose ϕ and B to satisfy necessary conditions and ensure g^*.

Some Numbers III

- With these numbers, (22) implies

$$\eta\left(\hat{z}^{*}\right)=0.015 \text{ and } \hat{z}^{*}\eta\left(\hat{z}^{*}\right)=0.003337.$$

- This implies that the contribution of entry to productivity growth is

$$\left(\kappa-1\right)\hat{z}^{*}\eta\left(\hat{z}^{*}\right)=0.0067.$$

- Using (25), the contribution of productivity growth by continuing firms is

$$\begin{aligned}\left(\lambda-1\right)\phi z^{*} &= g^{*}-\left(\kappa-1\right)\hat{z}^{*}\eta\left(\hat{z}^{*}\right)\\ &= 0.02-0.0067\\ &= 0.0133.\end{aligned}$$

- Thus in this parameterization, over two thirds of innovation is driven by incumbents.

Some Numbers IV

- In addition, the value for $\hat{z}^* \eta(\hat{z}^*)$ implies that there is entry of a new product (creative destruction) in each machine line on average once every 7.5 years (recall that $r^* = 0.05$ as the annual interest rate so that $r^* / \hat{z}^* \eta(\hat{z}^*) \approx 7.46$).
- Moreover, $\phi z^* = 0.0667$, so that there are on average 1.2 incremental innovations.
- These numbers are quite typical as we vary the parameter values.
- But, quite different numbers are also possible and quite a bit of uncertainty about what the right parameters should be.

The Effects of Policy on Growth I

- Since Schumpeterian structure, it may be conjectured that entry barriers (or taxes on potential entrance) will have negative effects on economic growth.
- Tax τ_e on R&D expenditure by entrants and a tax τ_i on R&D expenditure by incumbents (can be negative and interpreted as subsidies).
- τ_e can also be interpreted as a more strict patent policy than in the baseline model, where the entrant did not have to pay the incumbent for partially benefiting from its accumulated knowledge.
- Focus on the case in which tax revenues are collected by the government rather than rebated back to the incumbent as patent fees.

The Effects of Policy on Growth II

- Repeating the analysis above, equilibrium conditions:

$$\eta\left(\hat{z}^*\right)) V(\kappa q) = (1 + \tau_e)\, q \text{ or } V(q) = \frac{q\,(1 + \tau_e)}{\kappa\eta\left(\hat{z}^*\right)}. \tag{32}$$

- The equation that determines the optimal R&D decisions of incumbents, (19), is also modified:

$$\phi\left(V(\lambda q) - V(q)\right) = (1 + \tau_i)\, q. \tag{33}$$

- Combining (32) with (33):

$$\phi\left(\frac{(\lambda - 1)(1 + \tau_e)}{\kappa\eta\left(\hat{z}^*\right)(1 + \tau_i)}\right) = 1.$$

- Consequently, the BGP R&D level by entrants $\hat{z}^*$, when their R&D is taxed at the rate τ_e, is given by

$$\hat{z}^* \equiv \eta^{-1}\left(\frac{\phi(\lambda - 1)(1 + \tau_e)}{\kappa(1 + \tau_i)}\right). \tag{34}$$

The Effects of Policy on Growth III

- Equation (21) still applies, so that the the BGP interest rate is

$$r^* = (1+\tau_i)^{-1}\phi(\lambda-1)\beta L - \hat{z}^*\eta(\hat{z}^*), \tag{35}$$

- BGP growth rate is

$$g^* = \frac{1}{\theta}\left((1+\tau_i)^{-1}\phi(\lambda-1) - \hat{z}^*\eta(\hat{z}^*) - \rho\right). \tag{36}$$

- From (36), g^* does not directly depend on τ_e, thus

$$\frac{dg^*}{d\tau_e} = \frac{\partial g^*}{\partial \hat{z}^*}\frac{\partial \hat{z}^*}{\partial \tau_e} > 0.$$

- Opposite of standard Schumpeterian results. Intuition?
- Moreover, as expected

$$\frac{dg^*}{d\tau_i} < 0.$$

Proposition The growth rate of the economy is decreasing in the tax rate on incumbents and increasing in the tax rate (entry barriers) on entrants.

The Effects of Policy on Growth VII

- Surprising result: in Schumpeterian models, making entry more difficult, either with entry barriers or by taxing R&D by entrants, has negative effects on economic growth.
- Despite the Schumpeterian nature of the current model, blocking entry and protecting the incumbents increases equilibrium growth (and welfare)
- *Intuition:*
  - Engine of growth is still quality improvements, but these improvements are undertaken both by incumbents and entrants.
  - Entry barriers, by protecting incumbents, increase their value and greater value by incumbents encourages more R&D investments and faster productivity growth.
  - Taxing entrants makes incumbents more profitable and this encourages further innovation by the incumbents.

Equilibrium Firm Size Distribution I

- In equilibrium, there is entry, exit and stochastic growth of firms. $\Longrightarrow$ endogenous of firm size distribution.
- Firm growth consistent with *Gibrat's Law*
  - Gibrat's Law states that firm growth is independent of firm size.
  - Good description of actual firm size dynamics in the data (e.g., Sutton, 1997).
  - Though not always for new firms.
- What about firm size distributions?
- Axtell (2001) US firm size distribution very well approximated by the Pareto distribution with an exponent of one.
- Recall that the *Pareto distribution* is $\Pr[\tilde{x} \leq y] = 1 - \Gamma y^{-\chi}$ for $\Gamma > 0$ and $y \geq \Gamma$.

Equilibrium Firm Size Distribution II

- In the current economy, the size of average firm measured by sales, $x(t)$, grows.
- To look at the firm size distribution, we need to normalize firm sizes by the average size of firm, given by $X(t)$.
- Let the *normalized firm size* be

$$\tilde{x}(t) \equiv \frac{x(t)}{X(t)}.$$

Equilibrium Firm Size: Pareto Distribution I

- Since in equilibrium $\dot{X}(t)/X(t) = g^* > 0$, law of motion for normalized size of leading firm in each industry:

$$\tilde{x}(t+\Delta t) = \begin{cases} \frac{\lambda}{1+g^*\Delta t}\tilde{x}(t) & \text{w. p. } \phi z^* \Delta t \\ \frac{\kappa}{1+g^*\Delta t}\tilde{x}(t) & \text{w. p. } \hat{z}^* \eta(\hat{z}^*)\Delta t + o(\Delta t) \\ \frac{1}{1+g^*\Delta t}\tilde{x}(t) & \text{otherwise} \end{cases}$$

  (still under the assumption that all incumbents undertake R&D z^*).

- Notice that this expression does not refer to the growth rate of a single firm, but to the leading firm in a representative industry, and in particular, when there is entry, this leads to an increase in size rather than extinction.

Proposition If a stationary distribution of (normalized) firm sizes exists, then it is a Pareto distribution with exponent equal to 1, i.e., $\Pr[\tilde{x} \le y] = 1 - \Gamma/y$ with $\Gamma > 0$.

Proof of Proposition: Pareto Distribution of Firm Sizes I

- Suppose that a stationary distribution exists.
- Consider an arbitrary time interval of $\Delta t > 0$ and write

$$\begin{aligned}
\Pr\left[\tilde{x}\left(t+\Delta t\right) \leq y\right] &= \mathbb{E}\left[\mathbf{1}_{\{\tilde{x}(t+\Delta t)\leq y\}}\right] \\
&= \mathbb{E}\left[\mathbf{1}_{\{\tilde{x}(t)\leq y/(1+g^x(t+\Delta t))\}}\right] \\
&= \mathbb{E}\left[\mathbb{E}\left[\mathbf{1}_{\{\tilde{x}(t)\leq y/(1+g^x(t+\Delta t))\}} \mid g^x\left(t+\Delta t\right)\right]\right],
\end{aligned}$$

where $\mathbf{1}_{\{\mathcal{P}\}}$ is the indicator function, so the first equation holds by definition. The second equation also holds by definition once $g^x\left(t+\Delta t\right)$ is designated as the the (stochastic) growth rate of x between t and $t+\Delta t$. Finally, the third equation follows from the Law of Iterated Expectations.

Proof of Proposition: Pareto Distribution of Firm Sizes II

- Next, denoting $\mathbb{G}_t(y) \equiv 1 - \Pr[\tilde{x}(t) \leq y]$:

$$\begin{aligned} \Pr[\tilde{x}(t+\Delta t) \leq y] &= 1 - \mathbb{G}_{t+\Delta t}(y) \\ &= \mathbb{E}\left[1 - \mathbb{G}_t\left(\frac{y}{1 + g^x(t+\Delta t)}\right)\right]. \end{aligned}$$

- Therefore, we obtain the functional equation

$$\begin{aligned} \mathbb{G}_{t+\Delta t}(y) &= \mathbb{E}\left[\mathbb{G}_t\left(\frac{y}{1+g^x(t+\Delta t)}\right)\right] \qquad (3 \\ &= \phi z^* \Delta t \mathbb{G}_t\left(\frac{(1+g^*\Delta t)\,y}{\lambda}\right) \\ &\quad + \hat{z}^*\eta(\hat{z}^*)\,\Delta t \mathbb{G}_t\left(\frac{(1+g^*\Delta t)\,y}{\kappa}\right) \\ &\quad + (1 - \phi z^*\Delta t - \hat{z}^*\eta(\hat{z}^*)\,\Delta t)\,\mathbb{G}_t((1+g^*\Delta t)\,y) + o(\Delta \end{aligned}$$

A stationary equilibrium will correspond to a function $\mathbb{G}(y)$ such that $\mathbb{G}_{t+\Delta t}(y) = \mathbb{G}_t(y) = \mathbb{G}(y)$ for all t and Δt and (37) holds.

Proof of Proposition: Pareto Distribution of Firm Sizes III

- Let us conjecture that $\mathbb{G}(y) = \Gamma y^{-\chi}$ with $\Gamma > 0$. Then

$$
\begin{aligned}
\Gamma y^{-\chi} &= \phi z^{*} \Delta t \Gamma \left(\frac{(1 + g^{*} \Delta t) y}{\lambda} \right)^{-\chi} \\
&\quad + \hat{z}^{*} \eta(\hat{z}^{*}) \Delta t \Gamma \left(\frac{(1 + g^{*} \Delta t) y}{\kappa} \right)^{-\chi} \\
&\quad + (1 - \phi z^{*} \Delta t - \hat{z}^{*} \eta(\hat{z}^{*}) \Delta t) \Gamma ((1 + g^{*} \Delta t) y)^{-\chi} + o(\Delta t).
\end{aligned}
$$

or

$$
\begin{aligned}
& \phi z^{*} \Delta t \left(\frac{(1 + g^{*} \Delta t)}{\lambda} \right)^{-\chi} + \hat{z}^{*} \eta(\hat{z}^{*}) \Delta t \left(\frac{(1 + g^{*} \Delta t)}{\kappa} \right)^{-\chi} \qquad (38) \\
+ \quad & (1 - \phi z^{*} \Delta t - \hat{z}^{*} \eta(\hat{z}^{*}) \Delta t) (1 + g^{*} \Delta t)^{-\chi} + o(\Delta t) \Gamma^{-1} y^{-\chi} = 1.
\end{aligned}
$$

Proof of Proposition: Pareto Distribution of Firm Sizes IV

- Now subtracting 1 from both sides, dividing by Δt, and taking the limit as $\Delta t \to 0$, we obtain

$$\lim_{\Delta t \to 0} \left\{ \phi z^* \left(\frac{(1 + g^* \Delta t)}{\lambda} \right)^{-\chi} + \hat{z}^* \eta(\hat{z}^*) \left(\frac{(1 + g^* \Delta t)}{\kappa} \right)^{-\chi} + \frac{(1 - \phi z^* \Delta t - \hat{z}^* \eta(\hat{z}^*) \Delta t)(1 + g^* \Delta t)^{-\chi} - 1}{\Delta t} + \frac{o(\Delta t)}{\Delta t} \right\} = 0.$$

- Therefore the exponent χ must satisfy

$$\phi(\lambda^\chi - 1) z^* + (\kappa^\chi - 1) \hat{z}^* \eta(\hat{z}^*) - \chi g^* = 0. \tag{39}$$

- It can be easily verified that (39) has two solutions $\chi = 0$ and $\chi^* = 1$, since, by definition, $g^* = \phi(\lambda - 1) z^* + (\kappa - 1) \hat{z}^* \eta(\hat{z}^*)$.

Proof of Proposition: Pareto Distribution of Firm Sizes V

- To see that there are no other solutions, consider the derivative of this function, which is given by

$$g'(\chi) = \phi z^* \lambda^\chi \ln \lambda + \eta(\hat{z}^*) \hat{z}^* \kappa^\chi \ln \kappa - g^*.$$

  Since $\ln a < a - 1$ for any $a > 1$, $g'(0) < 0$. Moreover, $g''(\chi) > 0$, so that the right-hand side of (39) is convex and as $\chi \to \infty$, it limits to infinity. Thus there is a unique nonzero solution, which as we saw above, is $\chi^* = 1$.

- Finally, note that $\chi = 0$ cannot be a solution, since it would imply $\mathbb{G}(y) = \Gamma$ and thus $\mathbb{G}(y) = 0$, which would imply that all firms have normalized size equal to zero, and violate the hypothesis that a stationary firm-size distribution exists.
- It can also be verified that no other function than $\mathbb{G}(y) = \Gamma y^{-\chi}$ with $\Gamma > 0$ can satisfy this functional equation, completing the proof of the proposition.

Equilibrium Firm Size: Pareto Distribution II

- Unfortunately, previous proposition stated under the hypothesis that the station redistribution exists. But:

Proposition In the economy studied here, a stationary firm-size distribution does *not* exist.

- **Proof:**
  - A stationary distribution must take the form $\Pr[\tilde{x} \leq y] = 1 - \Gamma / y$ with $\Gamma > 0$ and Γ should be the minimum normalized firm size.
  - However, the law of motion of firm sizes shows that $\tilde{x}(t)$ can tend to zero. Therefore, Γ must be equal to 0, which implies that there does not exist a stationary firm-size distribution.
- Intuitively, given the random growth process (Gibrat's Law), the distribution of firm sizes will continuously expand.
- The "limiting distribution" will involve all firms being arbitrarily small relative to the average $X(t)$ and a vanishingly small fraction of firms becomes arbitrarily large.

Equilibrium Firm Size: Pareto Distribution III

- Can we have a stationary firm size distribution?
- Let us now relax the "supposition" that all incumbents do R&D at the rate z^*. Instead, $z^*(\tilde{q})$, where

$$\tilde{q}(\nu, t) \equiv \frac{q(\nu, t)}{Q(t)}.$$

- We need

$$\int_0^1 z^*(\tilde{q}(\nu, t))\, d\nu = z^* \text{ for all } t.$$

- In particular, suppose that there exists ε such that

$$z^*(\tilde{q}) = \begin{cases} \bar{z}^* & \text{for all } \tilde{q} > \varepsilon \\ \infty & \text{if } \tilde{q} = \varepsilon. \end{cases}$$

- This implies that $\tilde{q}$ will never fall below ε. Slight deviation from Gibrat's Law.

Equilibrium Firm Size: Pareto Distribution IV

- If the measure of firms at ε is finite, then $\bar{z}^* = z^*$.
- This R&D behavior for incumbents implies that $q(t) \geq \varepsilon Q(t)$ and $x(t) \geq \varepsilon X(t)$, thus

$$\tilde{x}(t) \geq \varepsilon.$$

- Now we can establish the existence of a stationary Pareto distribution with an exponent equal to one.

Proposition Consider the modification described above with $\varepsilon > 0$. Then there exists a unique stationary firm size-distribution given by the Pareto distribution $\Pr[\tilde{x}(t) \leq y] = 1 - \varepsilon / y$.

Proof of Proposition: Existence of Pareto Distribution of Firm Sizes I

- For $\varepsilon > 0$,

$$\Pr\left[\tilde{x}\left(t\right) \leq y\right] = 1 - \varepsilon / y,$$

- Moreover, $\Pr\left[\tilde{x}\left(t\right) = \varepsilon\right] = 0$, so $\bar{z}^{*} = z^{*}$.
- Same argument as above: $\Pr\left[\tilde{x}\left(t\right) \leq y\right] = 1 - \varepsilon / y$ solves the functional equation (37) above.
- Proper probability distribution for any $\varepsilon > 0$.

Equilibrium Firm Size: Pareto Distribution V

- Therefore, this simple model provides a good approximation to the firm size distribution in practice.
- This is despite the fact that it was not designed to study firm size distributions in the first place.
- A consequence of Gibrat's Law except for firms very much behind the average (ε could be arbitrarily small).
- In practice, richer firm dynamics than in the current model.
- Interaction between innovation dynamics and firm size behavior, area for future research.

Step-by-Step Innovations I

- Arrow's replacement effect in standard Schumpeterian models: new entrants undertake innovation on any machine and do not need any developed "know-how".
- But in practice quality improvements may have a major cumulative aspect and incumbents may undertake R&D
- More realistic: only a few firms engaging in innovation and competition in a particular product or machine line (Aghion, Harris, Howitt and Vickers, 2001)
- Cumulative research or "*step-by-step innovation*".

Step-by-Step Innovations II

- Will also endogenize the equilibrium market structure and have a richer analysis of the effects of competition and intellectual property rights policy.
- Recall baseline Schumpeterian models and models of expanding varieties: weaker patent protection and greater competition reduce economic growth.
- But evidence suggests positive (or at least non-monotonic relationship) between competition and growth (e.g., Nickell, Aghion et al.).
- Schumpeterian models with endogenous market structure: effects are more complex, and greater competition (and weaker intellectual property rights protection) sometimes increase growth.

Preferences and Technology I

- Continuous time economy with unique final good.
- Continuum of measure 1 of individuals, each with 1 unit of labor supplied inelastically.
- Representative household preferences:

$$\int_0^{\infty} \exp(-\rho t) \log C(t)\, dt, \tag{40}$$

- Closed economy and final good used only for consumption (i.e., no investment or spending on machines),

$$C(t) = Y(t).$$

- Euler equation from (40):

$$g(t) \equiv \frac{\dot{C}(t)}{C(t)} = \frac{\dot{Y}(t)}{Y(t)} = r(t) - \rho, \tag{41}$$

Preferences and Technology II

- Y produced using a continuum 1 of intermediate goods (Cobb-Douglas production function):

$$\ln Y(t) = \int_0^1 \ln y(\nu, t)\, d\nu, \tag{42}$$

- $y(\nu, t)$ is the output of νth intermediate at time t.
- Take price of the final good (or the ideal price index of the intermediates) as the numeraire.
- $p^y(\nu, t)$=price of intermediate ν at time t.
- Free entry into the final good sector.
- Thus, final good producers' demand for intermediates:

$$y(\nu, t) = \frac{Y(t)}{p^y(\nu, t)}, \quad \text{for all } \nu \in [0, 1]. \tag{43}$$

Preferences and Technology III

- Each intermediate $\nu \in [0, 1]$ comes in two different varieties, each produced by one of two infinitely-lived firms.
- These two varieties are perfect substitutes and compete a la Bertrand.
- No other firm is able to produce in this industry.
- Firm $i = 1$ or 2 in industry ν has technology:

$$y(\nu, t) = q_i(\nu, t)\, l_i(\nu, t) \tag{44}$$

  where $l_i(\nu, t)$ is employment level and $q_i(\nu, t)$ level of technology at time t.
- The only difference between firms is their technology, determined endogenously.
- Each consumer holds balanced portfolio of the shares of all firms. Thus, the objective function of each firm is to maximize expected profits.

Preferences and Technology IV

- (44) implies the marginal cost of producing intermediate ν for firm i at time t is

$$MC_i(\nu, t) = \frac{w(t)}{q_i(\nu, t)} \tag{45}$$

  where $w(t)$ is the wage rate in the economy at time t.

- Denote the *technological leader* by i and the *follower* by $-i$, so:

$$q_i(\nu, t) \geq q_{-i}(\nu, t).$$

- Bertrand competition implies all intermediates will be supplied by leader at the limit price :

$$p_i^y(\nu, t) = \frac{w(t)}{q_{-i}(\nu, t)}. \tag{46}$$

Preferences and Technology V

- (43) implies demand for intermediates:

$$y(\nu, t) = \frac{q_{-i}(\nu, t)}{w(t)} Y(t). \tag{47}$$

- R&D by the leader or the follower stochastically leads to innovation.
- When leader innovates, its technology improves by factor $\lambda > 1$.
- The follower can undertake R&D to catch up with the frontier technology.
- This innovation is for follower's variant and results from its own R&D. Hence no infringement of patent of the leader and no payments to the leader.
- R&D for both have the same costs and the same probability of success.

Preferences and Technology VI

- Each firm (in every industry) has access to the R&D technology (innovation possibilities frontier):

$$z_i(\nu, t) = \Phi(h_i(\nu, t)), \tag{48}$$

  where:

  - $z_i(\nu, t)$=flow rate of innovation at time t
  - $h_i(\nu, t)$=workers hired by firm i in industry ν to work in the R&D process at t.
  - Φ is twice continuously differentiable and satisfies $\Phi'(\cdot) > 0$, $\Phi''(\cdot) < 0$, $\Phi'(0) < \infty$
  - there exists $\bar{h} \in (0, \infty)$ such that $\Phi'(h) = 0$ for all $h \geq \bar{h}$.

- Note since $\Phi'(0) < \infty$ there is no Inada condition when $h_i(\nu, t) = 0$.
- Last assumption ensures there is an upper bound on the flow rate of innovation (to simplify proofs).

Preferences and Technology VII

- Cost for R&D is $w(t)\, G(z_i(\nu, t))$ where

$$G(z_i(j,t)) \equiv \Phi^{-1}(z_i(j,t)), \tag{49}$$

- Assumptions on Φ imply G is twice continuously differentiable, $G'(\cdot) > 0$, $G''(\cdot) > 0$, $G'(0) > 0$ and $\lim_{z \to \bar{z}} G'(z) = \infty$, where $\bar{z} \equiv \Phi(\bar{h})$ is the maximal flow rate of innovation.
- Suppose leader i in industry ν at time t has a technology level of

$$q_i(\nu, t) = \lambda^{n_i(\nu,t)}, \tag{50}$$

- Follower $-i$'s technology at time t is

$$q_{-i}(\nu, t) = \lambda^{n_{-i}(\nu,t)}, \tag{51}$$

where $n_i(\nu, t) \geq n_{-i}(\nu, t)$.

Preferences and Technology VIII

- Denote technology gap in industry ν at time t by

$$n(\nu, t) \equiv n_i(\nu, t) - n_{-i}(\nu, t)$$

- If leader undertakes innovation within a time interval of Δt, technology gap rises to $n(\nu, t + \Delta t) = n(\nu, t) + 1$ (probability of two or more innovations is $o(\Delta t)$).
- If follower undertakes innovation during the interval Δt, then $n(\nu, t + \Delta t) = 0$.
- Intellectual property rights (IPR): patent held by the technological leader expires at the exponential rate $\kappa < \infty$, in which case, the follower can close the technology gap.

Preferences and Technology IX

- Thus, the law of motion of the technology gap in industry ν is:

$$n(\nu, t+\Delta t) = \begin{cases} n(\nu,t)+1 & \text{w. prob.} \quad z_i(\nu,t)\Delta t + o(\Delta t) \\ 0 & \text{w. prob.} \quad (z_{-i}(\nu,t)+\kappa)\Delta t + o(\Delta t) \\ n(\nu,t) & \text{otherwise} \end{cases} \tag{52}$$

  where:

  - $o(\Delta t)$ =second-order terms (e.g. the probabilities of more than one innovations within Δt).
  - $z_i(\nu,t)$ and $z_{-i}(\nu,t)$=flow rates of innovation by the leader and the follower,
  - κ =flow rate at which follower is allowed to copy the technology of the leader.

Preferences and Technology X

- In the first line, the flow rate of innovation is $2z(\nu, t)$, since the two firms are neck-and-neck and undertake the same amount of research effort $z(\nu, t)$.
- Instantaneous "operating" profits (i.e.,profits exclusive of R&D expenditures and license fees) for leader i in industry ν at time t are:

$$\begin{aligned} \Pi_i(\nu, t) &= \left[p_i^y(\nu, t) - MC_i(\nu, t)\right] y_i(\nu, t) \\ &= \left(\frac{w(t)}{q_{-i}(\nu, t)} - \frac{w(t)}{q_i(\nu, t)}\right) \frac{Y(t)}{p_i^y(\nu, t)} \\ &= \left(1 - \lambda^{-n(\nu, t)}\right) Y(t) \end{aligned} \quad (53)$$

- (53) implies that there will be zero profits in an industry that is *neck-and-neck*, i.e., when $n(j, t) = 0$.
- Followers make zero profits, since they have no sales.

Preferences and Technology XI

- Cobb-Douglas in (42) leads to simple form of the profits (53), that depend only on technology gap of industry and aggregate output.
- Hence technology gap in each industry the only industry-specific payoff-relevant state variable.
- The objective function of each firm is to maximize the net present discounted value of net profits (operating profits minus R&D expenditures and plus or minus patent fees).
- Each firm takes $[r(t)]_{t=0}^{\infty}$, $[Y(t)]_{t=0}^{\infty}$, $[w(t)]_{t=0}^{\infty}$, the R&D decisions of all other firms, and policies as given.
- Note even though technology and output in each sector are stochastic, total output, $Y(t)$, given by (42) is nonstochastic.

Equilibrium I

- $\boldsymbol{\mu}(t) \equiv \{\mu_n(t)\}_{n=0}^{\infty}$ =distribution of industries over different technology gaps, with $\sum_{n=0}^{\infty} \mu_n(t) = 1$.
  - e.g., $\mu_0(t)$ =fraction of industries in which the firms are neck-and-neck at time t.
- Focus on Markov Perfect Equilibria (MPE), where strategies are only functions of the payoff-relevant state variables.
- Rules out implicit collusive agreements between the follower and the leader and eliminates dependence on industry ν.
- Thus refer to R&D decisions by z_n for the technological leader that is n steps ahead and by z_{-n} for a follower that is n steps behind.
- List of decisions by the leader and follower with technology gap n at time t :
  - $\xi_n(t) \equiv \left\langle z_n(t), p_i^y(\nu, t), y_i(\nu, t) \right\rangle$ and
  - $\xi_{-n}(t) \equiv z_{-n}(t)$.

Equilibrium II

- $\boldsymbol{\xi}$=whole sequence of decisions at every state, $\boldsymbol{\xi}(t) \equiv \{\xi_n(t)\}_{n=-\infty}^{\infty}$.
- Allocation. Time paths of:
  - decisions for a leader that is $n = 0, 1, ..., \infty$ steps ahead, $[\xi_n(t)]_{t=0}^{\infty}$,
  - decisions for a follower that is $n = 1, ..., \infty$ steps behind, $[\xi_{-n}(t)]_{t=0}^{\infty}$,
  - wages and interest rates $[w(t), r(t)]_{t=0}^{\infty}$, and
  - industry distributions over technology gaps $[\boldsymbol{\mu}(t)]_{t=0}^{\infty}$.
- Markov Perfect Equilibrium. Time paths $[\boldsymbol{\xi}^*(t), w^*(t), r^*(t), Y^*(t)]_{t=0}^{\infty}$ such that
  1. $[p_i^{y*}(\nu, t)]_{t=0}^{\infty}$ and $[y_i^*(\nu, t)]_{t=0}^{\infty}$ implied by $[\boldsymbol{\xi}^*(t)]_{t=0}^{\infty}$ satisfy (46) and (47);
  2. $[\mathbf{z}^*(t)]_{t=0}^{\infty}$ are best responses to themselves, i.e., $[\mathbf{z}^*(t)]_{t=0}^{\infty}$ maximizes the expected profits of firms taking aggregate $[Y^*(t)]_{t=0}^{\infty}$, $[w^*(t), r^*(t)]_{t=0}^{\infty}$, and $[\mathbf{z}^*(t)]_{t=0}^{\infty}$ as given;
  3. $[Y^*(t)]_{t=0}^{\infty}$ is given by (42); and
  4. labor and capital markets clear at all times given the factor prices $[w^*(t), r^*(t)]_{t=0}^{\infty}$.

Equilibrium III

- Since only the technological leader produces, labor demand in industry ν with technology gap $n(\nu, t) = n$ is:

$$l_n(t) = \frac{\lambda^{-n} Y(t)}{w(t)} \text{ for } n \geq 0. \tag{54}$$

- In addition, there is demand for labor from R&D of both followers and leaders in all industries.
- Using (48) and the definition of G, industry demands for R&D labor are:

$$h_n(t) = \begin{cases} G(z_n(t)) + G(z_{-n}(t)) & \text{if } n \geq 1 \\ 2G(z_0(t)) & \text{if } n = 0 \end{cases}, \tag{55}$$

where $z_{-n}(t)$ refers to the R&D effort of a follower that is n steps behind.

Equilibrium IV

- Note in industry with neck-and-neck competition ($n = 0$), there is twice the demand coming from the two "symmetric" firms.
- Labor market clearing:

$$1 \geq \sum_{n=0}^{\infty} \mu_n(t) \left[\frac{1}{\omega(t) \lambda^n} + G(z_n(t)) + G(z_{-n}(t)) \right], \tag{56}$$

and $\omega(t) \geq 0$, with complementary slackness, where

$$\omega(t) \equiv \frac{w(t)}{Y(t)} \tag{57}$$

is the labor share at time t.

- (56) uses the fact that total supply is equal to 1.
- If demand falls short of 1, $w(t)$ and thus $\omega(t)$ have to be equal to zero (but this will never be the case in equilibrium).

Equilibrium V

- Demand consists of demand for:
  - production (terms with ω in denominator),
  - R&D workers from the neck-and-neck industries $(2G(z_0(t))$ when $n = 0)$ and
  - R&D workers from leaders and followers in other industries $(G(z_n(t)) + G(z_{-n}(t))$ when $n > 0)$.
- Index of aggregate quality (not the average, but reflects the Cobb-Douglas aggregator):

$$\ln Q(t) \equiv \int_0^1 \ln q(\nu, t)\, d\nu. \tag{58}$$

- Given this, equilibrium wage is:

$$w(t) = Q(t)\, \lambda^{-\sum_{n=0}^{\infty} n\mu_n(t)}. \tag{59}$$

Steady-State Equilibrium I

- Steady-state (Markov Perfect) equilibria: distribution of industries $\boldsymbol{\mu}(t) \equiv \{\mu_n(t)\}_{n=0}^{\infty}$ is stationary, $\omega(t)$ defined in (57) and g^* is constant over time.
- If the economy is in steady state at $t = 0$, then

$$Y(t) = Y_0 e^{g^* t}$$

$$w(t) = w_0 \exp(g^* t)$$

- And, thus

$$\omega(t) = \omega^* \text{ for all } t \geq 0$$

- Assume parameters such that g^* is positive but not large enough to violate the transversality conditions.
- Hence the net present values of each firm at all t is finite and the maximization problem of a leader that is $n > 0$ steps ahead can be written recursively.

Steady-State Equilibrium II

- The value function for a firm that is n steps ahead (or $-n$ steps behind) is:

$$\begin{aligned} r(t)V_n(t) - \dot{V}_n(t) &= \max_{z_n(t)}\{[\Pi_n(t) - w^*(t)G(z_n(t))] \\ &\quad + z_n(t)[V_{n+1}(t) - V_n(t)] \\ &\quad + (z^*_{-n}(t) + \kappa)[V_0(t) - V_n(t)]\}. \end{aligned} \tag{60}$$

- In steady state, $V_n(t)$ will also grow at constant g^* for all $n \in \mathbb{Z}_+$.
- Define normalized values as

$$v_n(t) \equiv \frac{V_n(t)}{Y(t)} \tag{61}$$

for all n.

- These will be independent of time in steady state, i.e., $v_n(t) = v_n$.

Steady-State Equilibrium III

- Using (61) and the fact that from (41), $r(t) = g(t) + \rho$, (60) can be written as:

$$\rho v_n = \max_{z_n} \{ \left(1 - \lambda^{-n}\right) - \omega^* G(z_n) + z_n [v_{n+1} - v_n], + [z_{-n}^* + \kappa] [v_0 - v_n] \} \text{ for all } n \geq 1, \quad (62)$$

where z_{-n}^* is the equilibrium value of R&D by a follower n steps behind, and ω^* is the steady-state labor share (z_n is now explicitly chosen to maximize v_n).

- Value for neck-and-neck firms:

$$\rho v_0 = \max_{z_0} \{ -\omega^* G(z_0) + z_0 [v_1 - v_0] + z_0^* [v_{-1} - v_0] \}, \quad (63)$$

- Values for followers:

$$\rho v_{-n} = \max_{z_{-n}} \{ -\omega^* G(z_{-n}) + [z_{-n} + \kappa] [v_0 - v_{-n}] \}.$$

Steady-State Equilibrium IV

- Value functions and decision for followers should not depend on steps behind (a single innovation is sufficient to catch-up). Thus,

$$\rho v_{-1} = \max_{z_{-1}} \left\{ -\omega^* G\left(z_{-1}\right) + \left[z_{-1} + \kappa\right]\left[v_0 - v_{-1}\right] \right\}, \tag{64}$$

  where v_{-1} is the value of any follower (irrespective of how many steps behind it is).

- The maximization problems yield the profit-maximizing R&D decisions:

$$\begin{aligned} z_n^* &= \max\left\{ G'^{-1}\left(\frac{[v_{n+1} - v_n]}{\omega^*} \right), 0 \right\} && (65) \\ z_{-n}^* &= \max\left\{ G'^{-1}\left(\frac{[v_0 - v_{-n}]}{\omega^*} \right), 0 \right\} && (66) \\ z_0^* &= \max\left\{ G'^{-1}\left(\frac{[v_1 - v_0]}{\omega^*} \right), 0 \right\}, && (67) \end{aligned}$$

Steady-State Equilibrium V

- Since G is twice continuously differentiable and strictly concave, G'^{-1} is continuously differentiable and strictly increasing.
- Hence innovation rates, z_n^*'s, are increasing in the incremental value $v_{n+1} - v_n$ of moving to the next step and decreasing in the cost of R&D, as measured by ω^*.
- Note also that since $G'(0) > 0$, R&D levels can be equal to zero.
- The response of z_n^*, to $v_{n+1} - v_n$, is the key economic force. Relaxing IPR has two effects:
  - *Disincentive effect:* a policy that reduces the patent protection of leaders $n+1$ steps ahead (increasing κ) makes being $n+1$ steps ahead less profitable, thus reduce $v_{n+1} - v_n$ and z_n^*.
  - *Composition effect*: typically, $\{v_{n+1} - v_n\}_{n=0}^{\infty}$ is a decreasing sequence, so z_{n-1}^* is higher than z_n^* for $n \geq 1$. Weaker patent protection (shorter patent lengths) shift more industries into the neck-and-neck state and potentially increase the equilibrium level of R&D .

Steady-State Equilibrium VI

- Given equilibrium R&D decisions, $\boldsymbol{\mu}^*$ has to satisfy accounting relations:

$$\left(z_{n+1}^* + z_{-1}^* + \kappa\right)\mu_{n+1}^* = z_n^* \mu_n^* \text{ for } n \geq 1, \tag{68}$$

$$\left(z_1^* + z_{-1}^* + \kappa\right)\mu_1^* = 2z_0^* \mu_0^*, \tag{69}$$

$$2z_0^* \mu_0^* = z_{-1}^* + \kappa. \tag{70}$$

- In (68), exit from state $n+1$ (leader going one more step ahead or follower catching-up the leader) is equated to entry into this state (leader from the state n making one more innovation).
- In (69), accounting for state 1 taking into account that entry comes from innovation by either of the two firms that are competing neck-and-neck.
- In (70) for state 0, entry comes from innovation by a follower in any industry with $n \geq 1$.

Steady-State Equilibrium VII

- Labor market clearing condition in steady state:

$$1 \geq \sum_{n=0}^{\infty} \mu_n^* \left[\frac{1}{\omega^* \lambda^n} + G\left(z_n^*\right) + G\left(z_{-n}^*\right) \right] \text{ and } \omega^* \geq 0, \tag{71}$$

with complementary slackness.

Proposition The steady-state growth rate is given by

$$g^* = \ln \lambda \left[2\mu_0^* z_0^* + \sum_{n=1}^{\infty} \mu_n^* z_n^* \right]. \tag{72}$$

Proof of Steady-State Equilibrium Growth I

- (57) and (59) imply

$$Y(t) = \frac{w(t)}{\omega(t)} = \frac{Q(t)\,\lambda^{-\sum_{n=0}^{\infty} n\mu_n^*(t)}}{\omega(t)}.$$

- Since $\omega(t) = \omega^*$ and $\{\mu_n^*\}_{n=0}^{\infty}$ are constant in steady state, $Y(t)$ grows at the same rate as $Q(t)$. Thus,

$$g^* = \lim_{\Delta t \to 0} \frac{\ln Q(t + \Delta t) - \ln Q(t)}{\Delta t}.$$

- During an interval Δt, in the fraction μ_n^* of the industries with technology gap $n \geq 1$ the leaders innovate at a rate $z_n^* \Delta t + o(\Delta t)$ and in the fraction μ_0^* of the industries with technology gap of $n = 0$, both firms innovate.
- Thus total innovation rate is $2z_0^* \Delta t + o(\Delta t)$.

Proof of Steady-State Equilibrium Growth II

- Since each innovation increases productivity by a factor λ, we obtain the preceding equation. Combining these observations:

$$\ln Q(t+\Delta t) = \ln Q(t) + \ln \lambda \left[2\mu_0^* z_0^* \Delta t + \sum_{n=1}^{\infty} \mu_n^* z_n^* \Delta t + o(\Delta t) \right].$$

- Subtracting $\ln Q(t)$, dividing by Δt and taking the limit as $\Delta t \to 0$ gives (72).

Steady-State Equilibrium VIII

- Steady-state growth comes from two sources:
  1. R&D decisions of leaders or of firms in neck-and-neck industries.
  2. The distribution of industries across different technology gaps, $\boldsymbol{\mu}^* \equiv \{\mu_n^*\}_{n=0}^{\infty}$.
- Latter channel reflects composition effect: the relationship between competition (or intellectual property rights protection) and growth is more complex, because such policies will change the equilibrium market structure.

Definition A steady-state equilibrium is given by $\langle \boldsymbol{\mu}^*, \ \mathbf{v}, \ \ \mathbf{z}^*, \ \omega^*, \ g^* \rangle$ such that the distribution of industries $\boldsymbol{\mu}^*$ satisfy (68), (69) and (70), the values $\mathbf{v} \equiv \{v_n\}_{n=-\infty}^{\infty}$ satisfy (62), (63) and (64), the R&D decisions $\mathbf{z}^*$ are given by, (65), (66) and (67), the steady-state labor share ω^* satisfies (71) and the steady-state growth rate g^* is given by (72).

Bounded and Increasing Value Sequence

Proposition In a steady state equilibrium, we have $v_{-1} \leq v_0$ and $\{v_n\}_{n=0}^{\infty}$ forms a bounded and strictly increasing sequence converging to some positive value v_∞.

Proof of Proposition: Bounded and Increasing Value Sequence I

- Let $\{z_n\}_{n=-1}^{\infty}$ be the R&D decisions of a firm and $\{v_n\}_{n=-1}^{\infty}$ be the sequence of values, taking $\{z_n^*\}_{n=-1}^{\infty}$, $\{\mu_n^*\}_{n=-1}^{\infty}$, ω^* and g^*, as given.
- By choosing $z_n = 0$ for all $n \geq -1$, the firm guarantees $v_n \geq 0$ for all $n \geq -1$.
- Moreover, since $\pi_n \leq 1$ for all $n \geq -1$, we have $v_n \leq 1/\rho$ for all $n \geq -1$, establishing that $\{v_n\}_{n=-1}^{\infty}$ is a bounded sequence, with $v_n \in [0, 1/\rho]$ for all $n \geq -1$.
- *Proof of* $v_1 > v_0$:
    - Suppose $v_1 \leq v_0$.
    - Then (67) implies $z_0^* = 0$, and by symmetry in equilibrium, (63) implies $v_0 = v_1 = 0$.
    - As a result, from (66) we obtain $z_{-1}^* = 0$.
    - (62) implies when $z_{-1}^* = 0$, $v_1 \geq \left(1 - \lambda^{-1}\right) / \left(\rho + \kappa\right) > 0$, yielding a contradiction.

Proof of Proposition: Bounded and Increasing Value Sequence II

- *Proof of $v_{-1} \leq v_0$:*
  - Suppose $v_{-1} > v_0$.
  - (66) implies $z^*_{-1} = 0$, which leads to $v_{-1} = \kappa v_0 / (\rho + \kappa)$.
  - This contradicts $v_{-1} > v_0$ since $\kappa / (\rho + \kappa) < 1$ (given that $\kappa < \infty$).
- *Proof of $v_n < v_{n+1}$:*
  - Suppose $v_n \geq v_{n+1}$.
  - (65) implies $z^*_n = 0$, and (62) becomes

$$\rho v_n = \left(1 - \lambda^{-n}\right) + z^*_{-1} [v_0 - v_n] + \kappa [v_0 - v_n]. \quad (73)$$

  - Also from (62), the value for state $n + 1$ satisfies

$$\rho v_{n+1} \geq \left(1 - \lambda^{-n-1}\right) + z^*_{-1} [v_0 - v_{n+1}] + \kappa [v_0 - v_{n+1}]. \quad (74)$$

Proof of Proposition: Bounded and Increasing Value Sequence III

- Combining:

$$\left(1-\lambda^{-n}\right)+z_{-1}^{*}\left[v_{0}-v_{n}\right]+\kappa\left[v_{0}-v_{n}\right] \geq$$
$$1-\lambda^{-n-1}+z_{-1}^{*}\left[v_{0}-v_{n+1}\right]+\kappa\left[v_{0}-v_{n+1}\right].$$

- Since $\lambda^{-n-1}<\lambda^{-n}$, $v_n<v_{n+1}$, contradicting $v_n \geq v_{n+1}$, and establishing $v_n<v_{n+1}$.

- Thus $\{v_n\}_{n=-1}^{\infty}$ is nondecreasing and $\{v_n\}_{n=0}^{\infty}$ is (strictly) increasing.
- Since a nondecreasing sequence in a compact set must converge, $\{v_n\}_{n=-1}^{\infty}$ converges to its limit point, v_∞, which must be strictly positive, since $\{v_n\}_{n=0}^{\infty}$ is strictly increasing and has a nonnegative initial value.

Steady-State Equilibrium IX

- Potential difficulty: need to determine R&D levels and values for an infinite number of firms, since the technology gap can take any value.
- But we can restrict attention to a finite sequence of values:

Proposition There exists $n^* \geq 1$ such that $z_n^* = 0$ for all $n \geq n^*$.

Steady-State Equilibrium X

- Next show $\mathbf{z}^* \equiv \{z_n^*\}_{n=0}^{\infty}$ is a decreasing sequence: technological leaders that are further ahead undertake less R&D.
- Intuitively, benefits of further R&D investments are decreasing in the technology gap, since greater values of the gap translate into smaller increases in the equilibrium markup (recall (53)).
- Composition effect matters because leaders that are sufficiently ahead of their competitors undertake little R&D.
- Thus, all else equal, closing the technology gap will increase R&D spending and equilibrium growth.
- But note this may not always increase welfare, especially if there is a strong business stealing effect.

Proposition In any steady-state equilibrium, we have $z_{n+1}^* \leq z_n^*$ for all $n \geq 1$ and moreover, $z_{n+1}^* < z_n^*$ if $z_n^* > 0$. Furthermore, $z_0^* > z_1^*$ and $z_0^* \geq z_{-1}^*$.

Proof of Proposition: Decreasing R&D Investments Sequence I

- From equation (65),

$$\delta_{n+1} \equiv v_{n+1} - v_n < v_n - v_{n-1} \equiv \delta_n \tag{75}$$

is sufficient to establish that $z^*_{n+1} \leq z^*_n$.

- Let us write:

$$\bar{\rho} v_n = \max_{z_n} \left\{ \left(1 - \lambda^{-n}\right) - \omega^* G\left(z_n\right) + z_n \left[v_{n+1} - v_n\right] + \left(z^*_{-1} + \kappa\right) v_0 \right\}, \tag{76}$$

where $\bar{\rho} \equiv \rho + z^*_{-1} + \kappa$.

Proof of Proposition: Decreasing R&D Investments Sequence II

- Since z^*_{n+1}, z^*_n and z^*_{n-1} are maximizers of the value functions v_{n+1}, v_n and v_{n-1}, (76) implies:

$$\begin{aligned}
\bar{\rho} v_{n+1} &= 1 - \lambda^{-n-1} - \omega^* G\left(z^*_{n+1}\right) + z^*_{n+1}\left[v_{n+2} - v_{n+1}\right] \\
&\quad + \left(z^*_{-1} + \kappa\right) v_0, \\
\bar{\rho} v_n &\geq 1 - \lambda^{-n} - \omega^* G\left(z^*_{n+1}\right) + z^*_{n+1}\left[v_{n+1} - v_n\right] \\
&\quad + \left(z^*_{-1} + \kappa\right) v_0, , \\
\bar{\rho} v_n &\geq 1 - \lambda^{-n} - \omega^* G\left(z^*_{n-1}\right) + z^*_{n-1}\left[v_{n+1} - v_n\right] \\
&\quad + \left(z^*_{-1} + \kappa\right) v_0, , \\
\bar{\rho} v_{n-1} &= 1 - \lambda^{-n+1} - \omega^* G\left(z^*_{n-1}\right) + z^*_{n-1}\left[v_n - v_{n-1}\right] \\
&\quad + \left(z^*_{-1} + \kappa\right) v_0 .
\end{aligned} \tag{77}$$

Proof of Proposition: Decreasing R&D Investments Sequence III

Now taking differences with $\bar{\rho} v_n$ and using the definition of δ_n's, we obtain

$$\begin{aligned}
\bar{\rho}\delta_{n+1} &\leq \lambda^{-n}\left(1-\lambda^{-1}\right) + z^*_{n+1}\left(\delta_{n+2}-\delta_{n+1}\right) \\
\bar{\rho}\delta_{n} &\geq \lambda^{-n+1}\left(1-\lambda^{-1}\right) + z^*_{n-1}\left(\delta_{n+1}-\delta_{n}\right).
\end{aligned}$$

Therefore,

$$\left(\bar{\rho}+z^*_{n-1}\right)\left(\delta_{n+1}-\delta_n\right) \leq -k_n + z^*_{n+1}\left(\delta_{n+2}-\delta_{n+1}\right),$$

where $k_n \equiv (\lambda-1)^2 \lambda^{-n-1} > 0$.

- To obtain a contradiction, suppose $\delta_{n+1}-\delta_n \geq 0$.
- From the previous equation, this implies $\delta_{n+2}-\delta_{n+1} > 0$ since k_n is strictly positive.

Proof of Proposition: Decreasing R&D Investments Sequence IV

- Repeating this argument successively, if $\delta_{n'+1} - \delta_{n'} \geq 0$, then $\delta_{n+1} - \delta_n > 0$ for all $n \geq n'$.
- However, we know $\{v_n\}_{n=0}^{\infty}$ is strictly increasing and converges to a constant v_∞. This implies that $\delta_n \downarrow 0$, which contradicts the hypothesis that $\delta_{n+1} - \delta_n \geq 0$ for all $n \geq n' \geq 0$, and establishes that $z_{n+1}^* \leq z_n^*$.
- To see that the inequality is strict when $z_n^* > 0$, note that we have already established (75), i.e., $\delta_{n+1} - \delta_n < 0$, thus if equation (65) has a positive solution, then $z_{n+1}^* < z_n^*$.
- *Proof of* $z_0^* \geq z_{-1}^*$:
  - (63) can be written as

$$\rho v_0 = -\omega^* G\left(z_0^*\right) + z_0^*\left[v_{-1} + v_1 - 2v_0\right]. \tag{78}$$

  - We have $v_0 \geq 0$ from previous Proposition.

Proof of Proposition: Decreasing R&D Investments Sequence V

- Suppose $v_0 > 0$. Then (78) implies $z_0^* > 0$ and

$$\begin{aligned} v_{-1} + v_1 - 2v_0 &> 0 \qquad (79) \\ v_1 - v_0 &> v_0 - v_{-1}. \end{aligned}$$

- This inequality combined with (67) and (66) yields $z_0^* > z_{-1}^*$.
- Suppose next $v_0 = 0$.
- Inequality (79) now holds as a weak inequality and implies that $z_0^* \geq z_{-1}^*$.
- Moreover, since $G(\cdot)$ is strictly convex and z_0^* is given by (67), (78) then implies $z_0^* = 0$ and thus $z_{-1}^* = 0$.

Steady-State Equilibrium XI

- Thus the highest amount of R&D is undertaken in neck-and-neck industries.
- It is generally not possible to find a close form for growth because of the endogenous market structure.
- But a steady state equilibrium exists

Proposition A steady-state equilibrium $\langle \boldsymbol{\mu}^*, \ \mathbf{v}, \ a^*_{-1}, \ \mathbf{x}^*, \ \omega^*, \ g^* \rangle$ exists. Moreover, in any steady-state equilibrium $\omega^* < 1$, $g^* > 0$. For any steady-state R&D decisions, the steady-state distribution of industries $\boldsymbol{\mu}^*$ is uniquely determined.

Steady-State Equilibrium XII

- In addition, one can consider various different types of patent policies.
  - Licensing: followers can use information of leaders
  - State dependent patenting: patent protection depends on technology gap.

Proposition Consider the state-dependent IPR policy $\langle \boldsymbol{\eta}, \boldsymbol{\zeta} \rangle$ and suppose that $G'^{-1}\left(\left(1-\lambda^{-1}\right) / \left(\rho + \eta_1\right)\right) > 0$. Then a steady-state equilibrium $\left\langle \boldsymbol{\mu}^*, \mathbf{v}, a^*_{-1}, \mathbf{x}^*, \omega^*, g^* \right\rangle$ exists. Moreover, in any steady-state equilibrium $\omega^* < 1$. In addition, if either $\eta_1 > 0$ or $x^*_{-1} > 0$, then $g^* > 0$.

- See Acemoglu and Akcigit (2006) for the proofs on both results.

Steady-State Equilibrium XI

- Important features:
  - Equilibrium markups are endogenous and evolve over time as a function of competition between firms producing in same product line.
  - When a firm is sufficiently ahead of its rival it undertakes less R&D.
- Hence contrary to the baseline Schumpeterian model and to all expanding varieties models, greater competition may lead to higher growth rates.
- Greater competition generated by closing the gap between the followers and leaders induces the leaders to undertake more R&D in order to escape the competition from the followers.
- In practice, the effect of industrial organization more complex because of dynamic interactions (see below)

Numerical Computations

- Analyze the structure of equilibrium and optimal (growth-maximizing or welfare-maximizing) policies numerically.
- Empirical work

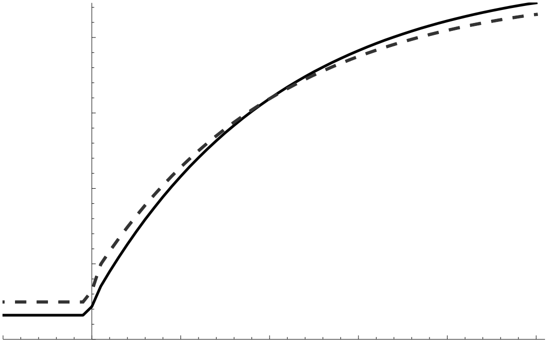
$$\text{Innovation}\,(t) = B_0 \exp(\kappa t)\,(\text{R\&D inputs})^{\gamma}, \tag{80}$$

  with γ between 0.1 and 0.6.
- Let us take

$$x = Bh^{\gamma} \tag{81}$$

  and $\gamma = 0.35$ to start with (and then check robustness).
- Growth rate $g^* = 0.02$ and baseline $\lambda = 1.05$ (major innovation on average in three years)
- Choose B to match the growth rate.

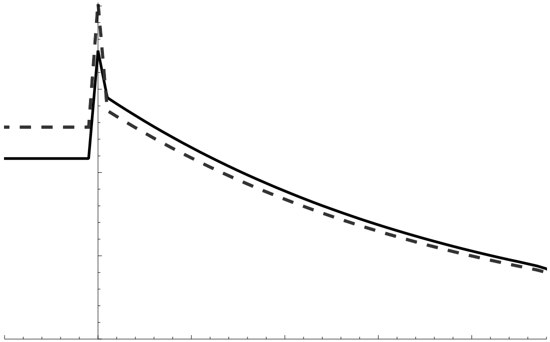
Example of Value Functions



Value Functions

Solid line no licensing, dotted line with licensing.

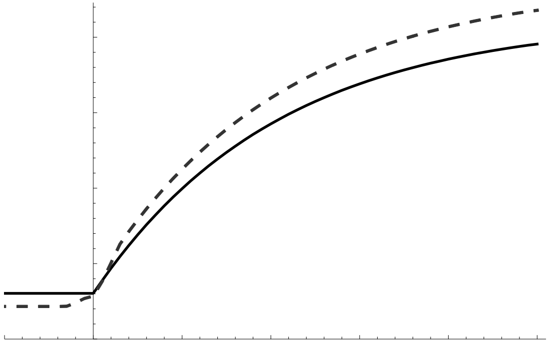
Example of R&D Distributions



R&D Distributions

Solid line no licensing, dotted line with licensing.

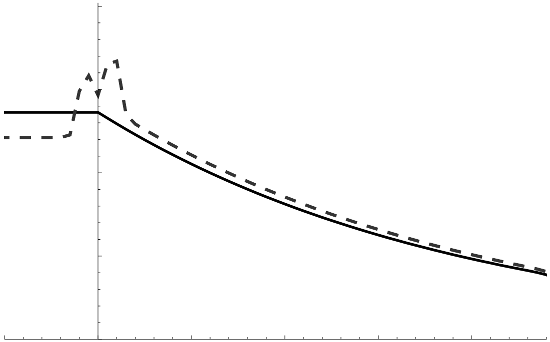
Example of Value Functions



Value Functions

Solid line no state-dependent IPR, dotted line with state-dependent IPR.

Example of R&D Distributions



R&D Distribution

Solid line no state-dependent IPR, dotted line with state-dependent IPR.

What Is Happening?

- *Trickle-Down of Incentives*: by providing IPR protection to firms that are significantly ahead of others, you also provide incentives to firms that are less further ahead
  - these firms invest more in R&D to reach levels where they have greater protection.
- Given this effect, it is also optimal to weaken patent protection for firms that are only a few steps ahead of their rivals
  - strengthen the trickle-down effect
- Licensing also boosts growth, because it avoids duplicative R&D.

Conclusions

- Schumpeterian models enable to make greater contact with the industrial organization of innovation: process of creative destruction implies market structure evolves endogenously.
- But in the baseline model, all R&D by entrants and markups are constant.
- These features can be relaxed to obtain more realistic and richer models of innovation behavior.
- Important issues:
    - the contribution of incumbents and entrants to productivity growth.
    - the effects of competition and patent protection on innovation (which are potentially quite different than in the baseline models).
- These models might provide a useful framework for the analysis of industrial policies.