On the sources of the aggregate risk premium: risk aversion, bubbles or regime-switching? *

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Abstract

We develop and estimate a consumption-based asset pricing model that uses historical US financial data and assumes recursive utility, allowing for priced regime-switching risk and intrinsic bubbles. We also estimate several restricted versions, including only a subset of these features. Priced regime-switching risk is essential to the equity risk premium, explaining more than fifty per cent of it. Furthermore, a model that does not consider regime switching would overestimate the public's risk aversion, mistakenly assigning the observed risk premium to high-risk aversion instead of priced regime-switching. We also find that intrinsic bubbles are statistically significant, and even though they are not crucial in explaining the risk premium, they substantially improve the model's fit at the end of the sample.

Keywords: Equity Risk Premium; Macroeconomic Risk; Stochastic Differential Utility; Markov Chain; Intrinsic Bubbles.

Jel codes: G00, G12, E44, C32

1 Introduction

This paper develops and estimates a consumption-based asset pricing model in which the parameters of the dividends process are subject to occasional discrete changes in regime (Bonomo and Garcia, 1996; Driffill and Sola, 1998). We allow for recursive utility (Epstein and Zin, 1989; Duffie and Epstein, 1992b) and intrinsic bubbles (Froot and Obstfeld, 1991; Driffill and Sola, 1998). An essential feature of our model is that the regime-switching risk is priced (Dai and Singleton, 2003; Bhamra, Kuehn, and Strebulaev, 2010; Chen, 2010) and that we estimate the price of the risk. We also estimate the contribution of intrinsic bubbles to the total risk premium.

We start our analysis by showing that dividend growth, the driving variable, is subject to regime changes. We also show that the price-dividend ratio has periods of explosiveness, consistent with bubbles. We then estimate a reduced form model of price dividend ratios, excess returns, and dividend growth à la Froot and Obstfeld (1991). The main results from this initial exercise are that the data is subject to swings that can be characterised as regime shifts, affecting all model equations. Intrinsic bubbles explain a large part of stock prices when regime switches are erroneously not allowed, but their importance diminishes when we allow for regime changes.[1]

We write down a structural model with features that capture those stylised facts. The most general version of the model assumes recursive utility (Lee and Phillips, 2016), regime-switching (Dai and Singleton, 2003; Bhamra, Kuehn, and Strebulaev, 2010; Chen, 2010), and intrinsic bubbles (Froot and Obstfeld, 1991; Driffill and Sola, 1998). This configuration nests several popular specifications obtained by adding different constraints to the model. We derive model-implied values for relevant financial variables such as the price-dividend ratio, the risk-free rate, the dividend growth process, and expected

[1]A bubble is a nonfundamental solution of the pricing equation that determines stock prices. Intrinsic bubbles are nonfundamental solutions that are a function of dividends. Driffill and Sola (1998) have shown that this solution is also regime-dependent when the dividends change with the regime.

stock returns. We then estimate the structural parameters consistent with the different specifications (corresponding to the general and popular models obtained from constrained versions). We show how crucial estimated parameters, such as the risk aversion coefficients, change across restricted model versions. [2]

One of our main results is that pricing regime switching seems to be a critical determinant of the risk premium, explaining more than 50%. To understand why we should price this risk, consider an investor who holds the stock in a good state or regime characterised by low marginal utility. Then, if a regime shift were to occur, the economy would transition to a bad state where consumption is more valuable to the investor at the margin (Bhamra, Kuehn, and Strebulaev, 2010; Chen, 2010; Arnold, Wagner, and Westermann, 2013). Additionally, the stock price reacts to the regime switch. As the economy moves into a bad state, we expect the asset price to jump downward, so there would be an instant capital loss due to the regime change. Therefore, the stock underperforms when there is a change to a high marginal utility state. In our model, this source of risk is priced in equilibrium. We find empirically that this is an essential determinant of the risk premium.

We also find that intrinsic bubbles explain a minor portion of the risk premium when the model allows regime switches but seem to explain a higher proportion of the risk in the single regime model. We interpret this in the spirit of Driffill and Sola (1998): when there are regime shifts, and the researcher (mistakenly) does not account for those changes, the difference between the fundamentals and the data may be (erroneously) accounted for by intrinsic bubbles. Once those changes in fundamentals are accounted for, the role of the bubbles is minor.

A critical result in the paper is that the estimated values for the risk aversion and the elasticity of intertemporal substitution (EIS) are substantially different in the restricted versions of the model (for example, when we restrict the model to exclude regime-switching

[2]We include state-dependent preference shocks to properly compare power and recursive utility with priced regime-switching risk. However, our analysis shows that these are not required for the paper's main results. Thus, our results do not rely on behavioural elements.

or bubbles). For example, the estimated risk aversion parameter is lower when we allow for priced regime-switching risk. The intuition behind this result is that, when considering all the changes in the utility of consumption as marginal (and not allowing for discrete changes), the risk aversion coefficient would mistakenly capture effects associated with discrete changes in the marginal utility of consumption, which take place as a result of a change in the state of the economy to which the agents would like to insure. In a sense, allowing for regime-switching risk alleviates the typical equity premium puzzle.

1.1 Related literature

There is a vast literature on general equilibrium asset pricing, starting with the paper of Lucas (1978) that attempts to model and match the observed risk premium. The work of Mehra and Prescott (1985) highlighted the need to extend the benchmark time-additive constant relative risk aversion construct to rich utility functional forms. Kreps and Porteus (1978) relaxed time-additivity to disentangle risk aversion and intertemporal substitution while Epstein and Zin (1989) developed a constant elasticity of substitution version of Kreps-Porteus preferences. Recursive preferences generate outcomes with an early or late resolution of uncertainty depending on the relative values of risk aversion and intertemporal substitution parameters. The stochastic differential (recursive) utility approach (Epstein and Zin, 1989, 1991; Weil, 1989, 1990; Duffie and Epstein, 1992a,b), helps to resolve the risk-free rate puzzle by producing lower rates, close to the observed ones. However, the equity premium puzzle still needs to be explained by those models. This gives rise to other papers with new features, such as time-varying preference parameters, reference levels, rare disasters, robust control, heterogeneous preferences, and idiosyncratic risks.

Most of these extensions include extra terms in their stochastic discount factors that help resolve asset pricing puzzles. Some authors, such as Bakshi and Chen (1996), added a "status" variable (a reference level) to achieve this objective. The literature has assumed several status variables, such as wealth level or ratio to a social wealth index. Once interacted with consumption, these status variables give rise to composite terms that enter the utility function. Then, once these terms are included, they modify the parameter values associated with the time preference rate, the risk aversion coefficient, and the elasticity of intertemporal substitution (Smith, 2001; Kenc and Dibooğlu, 2007).

Gordon and St-Amour (2000, 2004) add state-dependent risk preferences by allowing for a time-varying risk aversion coefficient. Time-varying risk aversion and "status" variables result in time-varying interest rates and risk premiums. This approach can produce countercyclical or procyclical risk aversion depending on the parameterisation.

Campbell and Cochrane (1999) obtains time-varying risk aversion and risk premium by assuming that external habit is an essential input in the agents' utility. The consumption in excess of the external habit, a mean-reverting stochastic process, enters the utility function and is enough to produce risk aversion and a risk premium consistent with some data properties. The model of Campbell and Cochrane (1999) has several features replicating those in the data. Using a continuous-time variant of Campbell and Cochrane (1999)), Li (2007) predicts that the risk premium is countercyclical.

Melino and Yang (2003) removes the status variable in the utility function in Gordon and St-Amour (2004) and assumes a recursive utility function. The model of Melino and Yang (2003) matches the historical first two moments of the returns on equity and the risk-free rate when allowing the elasticity of intertemporal substitution to vary with the regime and keeping risk preferences constant.

Berrada, Detemple, and Rindisbacher (2018); Ouysse (2023) consider belief-dependent risk aversion. Stochastic fluctuations in beliefs (i.e., risk aversion) generate unconditional moments and dynamics of the market price of risk, the interest rate, and the stock return volatility and correlations with consumption and dividends, which seem apparent in the data. Berrada, Detemple, and Rindisbacher (2018) assumes that the consumption growth follows a Markov switching process with time-varying transition probabilities.

Hence, fluctuations in the probabilities of consumption growth also contribute to the model outcomes. Ouysse (2023) considers a partially observable, stochastic, and exogenous relative risk aversion (RRA) but departs from Berrada, Detemple, and Rindisbacher (2018) as RRA does not follow a Markov switching regime.

On the other hand, papers that include bubbles (Froot and Obstfeld, 1991) and/or regime-switching in fundamentals (Cecchetti, Lam, and Mark, 1990) try to approximate the model solution to the observed prices either by allowing for the existence of bubbles or by making the fundamental solutions state-dependent. Given that the aggregate dividend equals consumption in equilibrium, many researchers (Campbell and Shiller, 1988; Goyal and Welch, 2003) have also adopted the dividend-based model as an alternative to the consumption-based asset pricing modelling.

Froot and Obstfeld (1991) first extended price-dividend models to incorporate "intrinsic bubbles", capturing the divergence between fundamentals and stock prices. These bubbles, a nonfundamental solution to the pricing equation, are a non-linear function of current dividends and thus model the apparent overreaction of stock prices. By accounting for bubbles, such models seem to characterise equity returns better. Driffill and Sola (1998) incorporate regime shifts in the dividend dynamics, which implies that if the intrinsic bubbles are present, those are also a function of the regimes, while Lee and Phillips (2016) considered a structural model that assumes recursive utility.

This paper differs from the existing papers in several ways. First, our model captures regime-specific discount rates and priced regime risk, unlike Driffill and Sola (1998) and Lee and Phillips (2016). Secondly, we use a structural model that allows us to disentangle the contribution to the risk premium of regime changes and bubbles that would otherwise wrongly be attributed to those other sources of risk. Third, given that we use continuous time, we can obtain the price of regime-switching risk directly from the power and stochastic differential utility functions.

2 Preliminary Analysis

This section analyses the presence of structural breaks and explosiveness in the data. As a first pass, we conduct tests for each feature separately. We conclude that the data is consistent with structural breaks and explosiveness. However, as pointed out by Hamilton and Whiteman (1985), standard explosiveness tests can falsely detect it in the presence of structural breaks. Therefore, we use an econometric model that allows for structural breaks and bubbles to understand better whether both features could be present in the data.

2.1 Data

We use quarterly data spanning the 1927Q1-2019Q4 period. We obtained a series of US stock prices, dividends, consumer price index, and nominal 3-month interest rates from the database constructed by Robert Shiller. We take the last value of the available monthly series as the quarterly value for stock prices and the consumer price index. We use data on the nominal 3-month interest rate and the consumer price index for the real risk-free rate and compute ex-post real interest rates.

2.2 Testing for the presence of structural breaks and explosiveness

We start our analysis by confirming that two of the model's main features are present in the data, that is, that the fundamentals are subject to changes in regime and that the price-dividend ratio has periods of explosiveness that may be characterised as a bubble. We first test the single regime hypothesis against the two regimes' alternative in the growth rate of dividends. As is well known, testing the hypothesis that the stochastic process under analysis can be characterised as a linear model against a nonlinear alternative is subject to the usual difficulties that arise from the fact that the transition probabilities are unidentified under the null hypothesis of linearity, thus violating conventional regularity conditions for likelihood-based inference (see for example Davies (1977) and Davies (1987). To overcome these difficulties, we carry out the standardised likelihood ratio procedure proposed by Hansen (1992, 1996). This procedure requires evaluating the likelihood function across a grid of different values for the transition probabilities and each state-dependent parameter.

The standardised likelihood ratio statistics value is 8.271, with a zero p-value under the null hypothesis for all the bandwidths under consideration. The p-values are computed according to the method described by Hansen (1996), using 1,000 random draws from the relevant limiting Gaussian processes and bandwidth parameter M$\in \{0, 1, \ldots, 4\}$). This is strong evidence favouring Markov regime switching, especially since Hansen's test is conservative by construction.

Secondly, we also perform a test of explosiveness in price-dividend ratios. We use the generalised sup-ADF (GSADF) test introduced in Phillips, Shi, and Yu (2015). Due to the possibility of over-rejection in finite samples, we use the bootstrapping procedure proposed in Caravello, Psaradakis, and Sola (2023). The value of the GSADF statistic is 3.8885, which is above the bootstrapped 95% critical value (3.3825). Thus, we can reject the null of no explosiveness.

2.3 Econometric model

Having established that regime switching and explosiveness are present in the data, we now present a model that extends the regime-switching dividend model of Driffill and Sola (1998) by allowing for regime-dependent discount rates. By allowing this extension, we explore whether their results are substantially changed, implying that switching discount rates would better model stock prices.[3] The model estimated here does not impose stark parametric assumptions on preferences. Thus, it can be interpreted as a model that captures the statistical properties of the series and, in some sense, would provide the best fit that a

[3]Allowing the discount rate of the reduced form model to be regime-dependent facilitates the comparison between the empirical results of the reduced form model and those of the consumption-based model presented in the next section.

structural model can attain.

The reduced-form model consists of the following three equations:

$$\frac{P_t}{D_t} = k_{s_t} + a_{s_t} D_t^{\eta-1} + \sigma^a_{s_t} \varepsilon^a_t, \tag{1a}$$

$$\Delta \ln D_t = \mu_{s_t} + \sigma_{s_t} \varepsilon_t, \tag{1b}$$

$$r_t = r^f_{s_t} + \sigma^b_{s_t} \varepsilon^b_t, \tag{1c}$$

Eq. (1a) represents the evolution of the price-dividend ratio, with the term $a_{s_t} D_t^{\eta-1}$ representing an intrinsic bubble.[4] Here, P is the price of a dividend-paying stock, D is the dividend payment, k_{s_t}, and $\sigma^a_{s_t} \varepsilon^a_t$ are the regime-specific deterministic and stochastic components of P_t/D_t, respectively. Eq. (1b) describes the dividend growth process with regime-dependent drift μ_{s_t} and variance σ_{s_t} parameters. Eq. (1c) allows the discount rates to be regime-dependent. In this equation, $r^f_{s_t}$ and $\sigma^b_{s_t} \varepsilon^b_t$ denote the regime-specific risk-free rate and the market price of risk, respectively. All error terms ε_t, ε^a_t, and ε^b_t are independently and identically distributed.[5]

Figure 1 shows actual and fitted prices using the model presented in Eqs. (1a) - (1c). The top panel presents a model estimated under the restriction that $a_{s_t} = 0$ for both states, i.e. allowing for structural breaks but not bubbles. The bottom panel shows the fitted values for an estimated model allowing bubbles. As expected, the unrestricted model fits the data better than the restricted version. [6] Nevertheless, the fit of the model that only considers switching fundamentals appears to be worse than the base model only at the end of the sample. Furthermore, even though the data statistically favours the bubbles

[4] Details of the derivation of the intrinsic bubble are presented in Appendix A.

[5] These assumptions are consistent with those of the theoretical model presented in Appendix A. We also explored a version that allowed for contemporaneously correlated errors, but we found the correlation coefficients to be not significantly different from zero.

[6] The maximised likelihood function for the unrestricted model is 362.293, whereas, for the restricted version, it is 338.935. A standard likelihood ratio test rejects the null of the restricted model in favour of the alternative model with bubbles for any conventional significance level. Notice that in the model that allows for bubbles, the coefficient on the bubble term is also state-dependent, so combining both regime switches and bubbles is validated by the data.

hypothesis, Figure 1 shows that their explanatory power seems similar when considering these two alternative explanations for the evolution of stock prices.

In summary, when estimating a model that only allows for regime changes, presented in the top panel, the fundamentals explain most of the movements in stock prices. Therefore, although statistically intrinsic bubbles seem present in the data, their economic importance must be clarified.

Unfortunately, since the base model is essentially reduced form, we cannot interpret the above results in terms of the deep parameters of an optimising model (such as the risk aversion or intertemporal elasticity of substitution) and have little guidance on how the structural breaks affect investors. For example, it could be the case that a sizeable value of risk aversion is needed to explain a low value for the fundamental component in the base model (lower panel of Figure 1). Furthermore, considering different specifications, it is unclear how the implied deep parameters would change under each estimated model.

To fill the gap, we develop a structural version of the model to answer the questions raised above in terms of the deep parameters of the economy. A structural model enables us to extend the analysis in several directions. For example, we can inquire how the estimated structural parameters change when we modify or restrict the model specifications. We can show how different models (which we can think of as other types of lenses we use to look at the same data) alter our conclusions about the underlying deep parameters. Also, a structural model allows the decomposition of the total risk premium in terms of its different sources. Thus, we can assess how much of the risk premium is due to regime-switching risk, bubbles, and other factors. Finally, a structural model enables the interpretation of the different stylised facts in the data employing general equilibrium arguments.

3 A Structural Model of Stock Prices

We present a consumption-based stock pricing model that nests several cases of interest. The model is an extended, structural version of Driffill and Sola (1998) in continuous time that incorporates a recursive utility function, regime-specific factor risk, and priced regime-switching risk. We use the model for three purposes: (i) to show how the estimates of the deep parameters change under different restricted versions of the model; (ii) to assess how much of the risk premium is explained by the diffusion component, the regime-switching component and bubbles; and (iii) to relate stylised facts found in the data to a general equilibrium model.

3.1 Primitives and Equilibrium

3.1.1 Exogenous States

Consider a stock that generates a random net cash flow (or dividend) stream of D_t per unit of time. A diffusion process with the state (regime) dependent drift and volatility parameters drives dividend flows. A Brownian motion $\{W_t\}$ generates continuous shocks in the economy. In contrast, a Markov chain s_t randomly determines discrete regime changes defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Time is continuous. More specifically, dividends evolve according to the following:

$$\frac{dD}{D} = \mu_s dt + \sigma_s dW, \tag{2}$$

A Poisson process N governs jumps in the state s_t that follows a two-state Markov chain that alternates between states 0 and 1. Two hazard rates, h_0, and h_1, parametrise the process. These rates form the rate matrix of the chain s_t as follows: $H = \begin{bmatrix} -h_0 & h_1 \\ h_0 & -h_1 \end{bmatrix}$. The probability that a transition occurs from state i to state j in a small time interval $(t, t+dt)$ equals $h_i dt$. Similarly, $1 - h_i dt$ is the probability that the process remains in state i. Finally, the regime s_t has the following dynamics:

$$ds = Hsdt + dN, \tag{3}$$

where dN are shocks to ds.

3.1.2 Market Clearing

The economy has two assets: the stock and an instantaneous risk-free bond. We normalise the number of outstanding shares to 1. We assume that the risk-free bond is in zero net supply and that there is no other source of income. Therefore, in equilibrium, we have that $c_t = D_t$ for all t.

3.1.3 Preferences

Consider a representative investor who has to decide how much to consume or save and the composition of the portfolio of assets he decides to hold. The lifetime indirect utility function takes the form

$$J_t = \mathbf{E}_t \int_t^{\infty} f(c_u, J_u) du, \tag{4}$$

where f is a normalised aggregator of consumption and the expected values of future lifetime indirect utility in each period, defined as follows:

$$f(c, J(s), s) = \frac{\delta}{(1-\psi)} \frac{z(s)c^{1-\psi} - [(1-\gamma)J(s)]^{\frac{1-\psi}{1-\gamma}}}{[(1-\gamma)J(s)]^{\frac{1-\psi}{1-\gamma}-1}}, \tag{5}$$

where δ is the time preference parameter, γ is the risk aversion coefficient, $1/\psi$ is the elasticity of intertemporal substitution, and $z(s)$ is a preference shock that only takes two values, is indexed to the regime and reflects the fact that the individual values the marginal utility of consumption differently under a good and a bad regime. This expression collapses to the power utility function when $\psi = \gamma$.[7] The stochastic discount factor takes the following form:

$$M(s_t) = e^{\int_0^t f_J(D_\tau, J(D_\tau, s_\tau))d\tau} f_c(c_t, J(D_t, s_t)). \tag{6}$$

Applying Ito's lemma yields the following differential equation (see Appendix B for its derivation):

$$\frac{dM}{M} = -r_s^f dt - \lambda_s dW - (\Gamma_s - 1)dN, \tag{7}$$

with

$$\Gamma_i = \frac{z_j^{1-\psi}}{z_i^{1-\psi}} \frac{\phi_j^{\psi-\gamma}}{\phi_i^{\psi-\gamma}}, \tag{7a}$$

$$r_i^f = -\delta \frac{1-\gamma}{1-\psi} \left[\left(\frac{\psi-\gamma}{1-\gamma} \right) z_i^{1-\psi} \phi_i^{\psi-1} - 1 \right] + \gamma\mu_i - \tfrac{1}{2}\gamma(1+\gamma)\sigma_i^2 - h_i \left[\Gamma_i - 1\right], \tag{7b}$$

$$\lambda_i = \gamma\sigma_i^2, \tag{7c}$$

where λ_s and $\Gamma_s - 1$ are the market prices of risk due to Brownian (dW) and Poisson (dN) shocks in states i, respectively, and ϕ_i is the regime-specific consumption-wealth ratios. Appendix B derives variables including r_s^f, Γ_s, and ϕ_i. Note that the values of Γ_i satisfy the condition $\Gamma_0\Gamma_1 = 1$. Also, notice that for the power utility function, when $\psi = \gamma$, the price of the switching regime risk, Γ_i, only differs from one if we assume regime-specific preference shocks. The model yields formulas for the state-dependent risk-free rate and the prices of the diffusion and switching risks.

Given that in equilibrium $c = D$ and the law of motion of dividends, J is a function of D, and s, the optimality condition requires that $J(D, s)$ must satisfy the Hamil-

[7] Consumption does not jump when there is a regime change, but what changes is consumption (dividend) growth. Since the recursive utility function captures the discounted value of all future consumption (that, after the change, will grow at a different rate), it will, in turn, affect marginal utility at date t. On the other hand, for a power utility function, a change in the consumption growth rate does not affect the marginal utility of consumption at time t. Then, unless we assume a preference shock, the instant marginal utility would not change when there is a regime shift, and thus, the price of switching risk would be zero. We show in detail in Appendix C the role of this preference shock when assuming a power utility function. It is easy to show that Γ is constant when we assume the existence of this type of regime-specific preference shock.

ton–Jacobi–Bellman (HJB) equation:

$$\sup_{D} \mathcal{D}J_i(D,s) + f(D, J_i, s)dt = 0, \qquad i = 0,1 \tag{8}$$

This can be solved by postulating the following solution:

$$J(D,s) = \frac{[\phi_s D]^{1-\gamma}}{1-\gamma}, \tag{9}$$

where ϕ_0 and ϕ_1 are consumption-wealth ratios in regime 0 and 1, respectively, and are to be determined by solving the HJB equation (see Appendix B).

3.2 Model implications for the observable variables.

This section summarises the formulas the model yields for several important observable variables, such as the price-dividend ratio, the excess return, and the risk-free rate.

Price-dividend ratio. The equilibrium value of the price-dividend ratio is given by:

$$\frac{P_s}{D} = k_s + \sum_{\nu=1}^{2} a_{s,\nu} D^{\eta_\nu - 1}, \tag{10}$$

where

$$k_i = \frac{r_j^e - \mu_j + h_i \Gamma_i}{(r_i^e - \mu_i + h_i \Gamma_i)(r_j^e - \mu_j + h_j \Gamma_j) - h_i h_j}, \tag{11}$$

where r_i^e are expected stock returns conditional on state i, Γ_i is the price of the risk of switching regimes, and η_ν are characteristic roots of the homogeneous component of the pricing equation. The exact expressions for η_ν, r_i^e, and Γ_i can be found in Appendix B. On the other hand, the bubble coefficients $a_{s,\nu}$ are arbitrary by definition. [8] The critical take-away is that the model predicts state-dependent expressions for the price-dividend ratios

[8] The bubble terms are the solution to a homogeneous differential equation and do not have conditions to pin down the relevant coefficients. This, in turn, implies that those coefficients can be estimated as free parameters and will be different from zero only if the bubble exists.

and the intrinsic bubbles. Furthermore, the formulas for k_s and η_i depend on preference parameters (δ, ψ, γ) as well as on the parameters driving the exogenous processes.

Risk-free rate. The risk-free rate is a state-dependent constant. It also depends on the parameters for the process of dividend growth, preference parameters, and the price of the switching risk. The general formula is rather involved, so we relegate it to Appendix B. We will study some simpler restricted cases later on.

Expected stock returns. The state-dependent expected returns are given by (see Appendix E):

$$\mathbf{E}_t\left[\frac{dP_i + D\,dt}{P_i}\right] = R(D, s = i)dt, \tag{12}$$

where

$$R(D, s=i) = r_i^f + \left(\frac{k_i D + \sum_{m=1}^{2} \eta_m a_{i,m} D^{\eta_m}}{k_i D + \sum_{m=1}^{2} a_{i,m} D^{\eta_m}}\right)\gamma\sigma_i^2 + h_i(\Gamma_i - 1)\frac{(k_i - k_j)D + \sum_{m=1}^{2}(a_{i,m} - a_{j,m})D^{\eta_m}}{k_i D + \sum_{m=1}^{2} a_{i,m} D^{\eta_m}} \tag{13}$$

and $a_{m,i}$ is the coefficient for the term with η_m in state i.

Note that the expected excess return depends on the level of dividends if and only if the bubble terms are non-zero. In section 3.4, we analyse the different terms that affect excess returns.

Dividend growth. The growth of dividends is a fundamental driving variable assumed exogenous and state-dependent, as shown in Eq. (2).

3.3 Restricted versions nested in the general model

This section shows that the general model nests many popular specifications used in the literature. We evaluate how imposing different restrictions on the base model highlights the relevant cross-equation relations and the theoretical importance of each model ingredient. Furthermore, we indicate how these ingredients affect the predicted risk-free rate, the price-dividend ratio, and the risk premium.

3.3.1 Power utility

The utility function becomes the power utility when $\gamma = \psi$. This is valid under any of the combinations we explore below. Thus, we will have two possible specifications for each case. We present one with power utility (i.e., imposing $\psi = \gamma$) and another with recursive utility, where ψ and γ may differ.

3.3.2 Recursive utility, single regime, and no bubbles

First, we consider the case where we do not allow regime shifts or bubbles. Under this scenario, the relationship between observable variables and the model is given by:

$$\frac{dD}{D} = \mu dt + \sigma dW, \tag{14}$$

$$\frac{P}{D} = \frac{1}{r^f + \gamma\sigma^2 - \mu}, \tag{15}$$

$$r^f = \delta + \psi\mu - \frac{1}{2}\gamma(1+\psi)\sigma^2, \tag{16}$$

$$\mathbf{E}_t\left[\frac{dP + D\,dt}{P}\right] - r^f dt = \gamma\sigma^2 dt. \tag{17}$$

These are the baseline consumption-based asset pricing model equations with recursive utility. The price-dividend ratio obeys the Gordon formula using the appropriate risk-adjusted discount rate. The risk-free rate, r^f, depends on the time discount factor, δ, the growth rate, μ, the intertemporal elasticity of substitution, ψ, and a precautionary savings term, $\frac{1}{2}\gamma(1+\psi)\sigma^2$. Lastly, $\gamma\sigma^2$ represents the expected excess return. Higher risks, σ^2, or risk aversion, γ, drive up the expected excess return required for the agent to hold the stock in equilibrium.

3.3.3 Recursive utility, single regime, and intrinsic bubbles

When we turn off regime-switching but allow for intrinsic bubbles, the relevant equations are:

$$\frac{dD}{D} = \mu dt + \sigma dW, \tag{14'}$$

$$\frac{P}{D} = \frac{1}{r^f + \gamma\sigma^2 - \mu} + a_1 D^{\eta_1 - 1} + a_2 D^{\eta_2 - 1}, \tag{15'}$$

$$r^f = \delta + \psi\mu - \frac{1}{2}\gamma(1+\psi)\sigma^2, \tag{16'}$$

$$\mathbf{E}_t\left[\frac{dP + Ddt}{P}\right] - r^f dt = \frac{kD + \eta_1 a_1 D^{\eta_1} + \eta_2 a_2 D^{\eta_2}}{kD + a_1 D^{\eta_1} + a_2 D^{\eta_2}}\gamma\sigma^2 dt. \tag{17'}$$

When we allow intrinsic bubbles, the price-dividend ratio is time-varying, and the relation between prices and dividends becomes non-linear. Non-linearity also appears in the expected excess return equation. When the relationship is linear (i.e., no bubble), the instantaneous variance of $\frac{dP}{P}$ is given by σ^2. On the other hand, when there are intrinsic bubbles, dividend changes also affect prices through the induced changes in the value of the bubble term aD^η, a non-linear function of dividends. Therefore, the instantaneous covariance between consumption growth and stock returns is no longer constant: shocks to dividend growth (and hence consumption) affect prices differently depending on the level of dividends.

3.3.4 Recursive utility, regime-switching, and no bubbles.

When we allow for regime changes but exclude intrinsic bubbles, which is attained by setting $\mathbf{a}_{0,i} = \mathbf{a}_{1,i} = 0$, the observable equations take the form: [9]

$$\frac{dD}{D} = \mu_s dt + \sigma_s dW, \tag{14''}$$

$$\frac{P_s}{D} = k_s, \tag{15''}$$

$$r_s^f = -\delta \frac{1-\gamma}{1-\psi}\left[\left(\frac{\psi-\gamma}{1-\gamma}\right) z_s^{1-\psi} \phi_s^{\psi-1} - 1\right] + \gamma\mu_s - \tfrac{1}{2}\gamma(1+\gamma)\sigma_s^2 - h_s\left[\Gamma_s - 1\right], \tag{16''}$$

$$\mathbf{E}_t\left[\frac{dP_i + D\,dt}{P_i}\right] - r_i^f dt = \left(\gamma\sigma_i^2 - h_i(\Gamma_i - 1)\frac{(k_j - k_i)}{k_i}\right) dt, \tag{17''}$$

where the expressions for the k_s are given in Eq. (B) in Appendix B. First, note that the dividend process is subject to regime shifts, making the risk-free rate, price dividend ratio, and stock expected return state-dependent. The parameters μ and σ are also state-dependent in these three equations. Furthermore, since regime-switching risk is priced, new state-dependent terms appear in the formulas. [10] In what follows, we explain in detail how each of the equations is affected by regime switching.

As we mentioned, dividend growth is subject to exogenous regime shifts that change the value of its drift, μ, and volatility, σ. The impact of regime-switching on dividend growth is direct, as dividends are the exogenous variable driving everything else.

Regime changes modify the interest rate equation in several ways. First, the term that corresponds to the time discounting (which used to be just δ) is now given by $-\delta\frac{1-\gamma}{1-\psi}\left[\left(\frac{\psi-\gamma}{1-\gamma}\right)z_s^{1-\psi}\phi_s^{\psi-1} - 1\right]$. This expression captures the interaction between regime-switching and recursive utility. Secondly, the terms relating to the growth rate of the

[9] We repeat the risk-free rate equation for expositional simplicity.

[10] Notice that several decompositions in the literature show contributions to the risk premium that extend the covariation between the excess returns and consumption (dividend) growth. For example, Ouysse (2023) presents a decomposition using a power utility and no bubbles. This decomposition shows the contribution of curvature risk, which comes from the fact that γ is assumed to be time-varying in that paper.

economy $\gamma\mu_s$ and precautionary savings ($\frac{1}{2}\gamma(1+\gamma)\sigma_s^2$) are now regime-dependent. [11] [12] Finally, the term $h_i(\Gamma_i - 1)$, reflects that we price regime-switching. Recall that the risk-free rate per unit of time is given by $-E_t\left[\frac{dM}{M}\right]$. When regime shifts are allowed, the stochastic discount factor (M) value jumps whenever a regime switch occurs. [13] Jumps in M, which would take place with a positive probability, would, in turn, affect the expected growth rate of the stochastic discount factor ($E_t\left[\frac{dM}{M}\right]$) and hence the risk-free rate.

The justification for pricing regime-switching is the following. First off, it can be shown that Γ_i is the ratio of the state-specific stochastic discount factors, $\Gamma_i = \frac{M_j}{M_i}$ (see the Appendix B for details). Then, whenever there is a regime change and $\Gamma_i > 1$, the agent values a consumption unit more in the new (bad) state. [14] Assume that the economy is in a state i, with Γ_i. Consider an investor holding an instantaneous risk-free bond that pays 1 unit of the consumption good. The investor knows the economy may transition to a bad state with some probability in the next instant. Note that a consumption unit is more valuable to her in a bad state. This possibility makes the risk-free bond more valuable. The reason is that if a regime change were to occur in the next instant, the investor would receive a unit of consumption in a bad state of the world, and she would value that unit more.[15] Thus, under this scenario, the mere existence of priced regime switches makes the bond more valuable, and consequently, its rate of return (r_i^f) is lower in equilibrium. Whenever $\Gamma_i > 1$, the term $-h_i(\Gamma_i - 1)$ is negative, so the risk-free rate is lower than the expression obtained if those regimes were not priced.[16] The same intuition applies if $\Gamma_i < 1$: as you

[11]Note that ψ does not directly affect the precautionary savings term. Interestingly, the relationship between ψ and the precautionary savings expression comes through the value function, Φ_i. This relationship was also present in the single regime model. In a single regime model, a closed-form solution for Φ, substituted back into the main equation, yields the well-known single regime expression where ψ affects both "economic growth" and "precautionary savings."

[12]Notice that for the power utility function, that is when $\gamma = \psi$, then $-\delta\frac{1-\gamma}{1-\psi}\left[\left(\frac{\psi-\gamma}{1-\gamma}\right)z_s^{1-\psi}\phi_s^{\psi-1} - 1\right] = \delta$.

[13]In our model, there are two sources for this jump: the exogenous preference shock and the endogenous change in lifetime utility, which affects the stochastic discount factor under recursive preferences.

[14]Because the value of the stochastic discount factor is M_i, and if $\Gamma > 1$, then $M_j > M_i$ So the agent's valuation for one unit of goods is higher in state j (i.e., the other state) than in the current state.

[15]Consumption risk is still present when regimes are not priced. Jumps add a source of variation in the stochastic discount factor, which appears as the additional terms we are discussing.

[16]If regimes were not priced, the risk-free rate would be given by $r_s^f = -\delta\frac{1-\gamma}{1-\psi}\left[\left(\frac{\psi-\gamma}{1-\gamma}\right)z_s^{1-\psi}\phi_s^{\psi-1} - 1\right] +$

transition to a state where a unit of consumption is less valuable, then the risk-free bond is less appealing, so its rate of return must be higher in equilibrium for the investor to hold it. Continuing the analysis for other observables, note that the price-dividend ratio and the expected excess return are state-dependent constants. Once we remove bubbles, the relation between stock prices and dividends is linear and conditional on staying in the same state. The price of regime-switching affects the price-dividend ratios via the $h_i Gamma_i$ terms in eq. (B).

Finally, the expected return of the risky asset now has a term $\left(-h_i(\Gamma_i - 1)\frac{(k_j - k_i)}{k_i}\right)$, which reflects the priced regime-switching risk. Therefore, the above term depends on the rate of arrival of regime switches h_i, the proportional capital gain/loss that a regime switch would induce due to the jump in prices, $\frac{(k_j - k_i)}{k_i}$, and the price of the risk term, $(\Gamma_i - 1)$.[17] Following the same reasoning, consider an investor holding the stock. Assume once again that, in the current state of the economy, $\Gamma_i > 1$. Then, if a regime shift were to occur, the economy would transition to a bad state where the investor's marginal utility of consumption is higher. Additionally, we must consider how the asset's price reacts to the regime switch. As this is a switch to a bad state, intuitively, we would expect the asset's price to jump by a negative amount $(k_j - k_i)/k_i < 0$, so there would be an instantaneous capital loss due to the regime switch. [18]

Therefore, priced regime-switching would make the risky asset less valuable: it has a negative pay-off when there is a transition to a bad state, in which consumption is more valuable. Thus, the expected return required by the investor to hold that asset in equilibrium is higher. [19] Note that the effect of pricing regime-switching on stock returns is the exact opposite of that of the risk-free bond. Regarding the latter, the safe asset guarantees a unit of consumption if a switch to a bad state occurs. Thus, pricing

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$\gamma\mu_s - \frac{1}{2}\gamma(1+\gamma)\sigma_s^2$. See Appendix B for details.

[17]Recall that $\Gamma_1\Gamma_2 = 1$. Therefore, the magnitude of the term is higher the closer to zero any of the Γ_i terms is: $\Gamma_i - 1$ gets larger in absolute value, and $\Gamma_j - 1 = \dfrac{1}{\Gamma_i} - 1$ gets larger as well.

[18]This will turn out to be true in the estimated model.

[19]The rate at which these changes happen is also an essential factor in the formula.

regime-switching increases the price of the risk-free bond. On the contrary, the risky asset has a negative pay-off if a regime shift occurs: the holder suffers a capital loss precisely when the economy transitions to a state where consumption is more valuable for the agent. Therefore, this characteristic makes the risky asset less attractive, which makes its return higher in equilibrium.

3.3.5 The general model with switching and bubbles

It is important to note that all the above mechanisms are also valid for the general case. However, it is worth noting that the capital gains or losses due to regime switches would now be affected by the state-dependent bubble coefficients. That is, in the presence of bubbles, capital gains/losses due to regime shifts are given by $\frac{(k_i-k_j)D+(a_{i,1}-a_{j,1})D^{\eta_1}+(a_{i,2}-a_{j,2})D^{\eta_2}}{k_i D+a_{i,1}D^{\eta_1}+a_{i,2}D^{\eta_2}}$ instead of $\frac{k_j-k_i}{k_i}$. Thus, the term now depends on the difference of bubble coefficients, $a_{i,m}-a_{j,m}$ for $m=1,2$. It creates an additional term in the excess return formula due to the interaction of both switching and bubbles. If a regime change were to occur, prices would jump because k_i changes to k_j and because the bubble size (captured by the $a_{i,m}$ coefficients) also changes.

3.4 Risk Premium Decomposition

In this subsection, we present a decomposition of the excess return to identify each previously discussed feature. To do so, we re-write the equation for the excess returns in the general model as:

$$\mathbf{E}_t\left[\frac{dP_i + D\,dt}{P_i}\right] - r_i^f dt = \left(\underbrace{\gamma\sigma_i^2}_{\textbf{(A)}} + \underbrace{\gamma\sigma_i^2 \frac{(\eta_1 - 1)a_{i,1}D^{\eta_1} + (\eta_2 - 1)a_{i,2}D^{\eta_2}}{k_i D + a_{i,1}D^{\eta_1} + a_{i,2}D^{\eta_2}}}_{\textbf{(B)}} + \right.$$
$$\underbrace{h_i(\Gamma_i - 1)\frac{(k_i - k_j)}{k_i + a_{i,1}D^{\eta_1 - 1} + a_{i,2}D^{\eta_2 - 1}}}_{\textbf{(C)}} +$$
$$\left. \underbrace{h_i(\Gamma_i - 1)\frac{(a_{i,1} - a_{j,1})D^{\eta_1} + (a_{i,2} - a_{j,2})D^{\eta_2}}{k_i D + a_{i,1}D^{\eta_1} + a_{i,2}D^{\eta_2}}}_{\textbf{(D)}} \right) dt \tag{18}$$

Our proposed decomposition has four terms: Term **A** corresponds to the risk premium predicted by a single regime model without bubbles. It has the usual interpretation: higher risk aversion (γ) or higher risk (σ_i^2) would increase the expected excess return. The term **B** corresponds to the part explained solely by bubbles. As we showed, in a single regime model that allows for bubbles, the sum $\mathbf{A} + \mathbf{B}$ determines the risk premium. As discussed above, the term **B** shows up because bubbles add a source of price variability. Thus, excess returns include a new component as prices now are a non-linear function in dividends. The term **C** corresponds to the part of the risk premium explained solely by switching. As we showed, the sum by $\mathbf{A} + \mathbf{C}$ determines the risk premium in a model with two regimes but without bubbles. The term **D** appears because of the interaction between switching regimes and bubbles. It captures the fact that when there is a regime switch, the bubble coefficients change, which generates an additional reason why prices jump.

4 Estimation

We estimate the general model and three different restricted versions, namely: model 1 features a single-regime without allowing for bubbles; model 2 is also single-regime but allows for bubbles; model 3 allows for regime-switching but excludes bubbles, while model 4 allows for both regime-switching and bubbles. We use the same data as in Section 2.1.

4.1 Econometric Model

The reduced form implied by the structural model is augmented with measurement errors to avoid a stochastic singularity. The equations we use are:

$$\frac{P_t}{D_t} = k(s_t) + a_{s_{t,1}} {D_t}^{\eta_1 - 1} + \sigma^a(s_t)\varepsilon_t^a, \tag{14'''}$$

$$\Delta \log(D_t) = (\mu(s_t) - \frac{\sigma^2(s_t)}{2}) + \sigma(s_t)\xi_t, \tag{15'''}$$

$$r_t^F = r^f(s_t) + \sigma^c(s_t)\varepsilon_t^c, \tag{16'''}$$

$$r_t^S = R(D_t, s_t) + \sigma^b(s_t)\varepsilon_t^b, \tag{17'''}$$

where $\frac{P_t}{D_t}$ is the observed price dividend ratio, $\Delta \log(D_t)$ is quarterly dividend growth, r_t^S is the quarterly observed cum-dividends stock returns, and r_t^F is the ex-post real rate. It is also worth noting that the model is augmented with three measurement errors. These are given by $(\varepsilon_t^a, \varepsilon_t^b, \varepsilon_t^c)$. All of them are assumed to be standard normal, independent and identically distributed. Their standard deviations are given by $(\sigma_{s_t}^a, \sigma_{s_t}^b, \sigma_{s_t}^c,)$ and are state-dependent. Finally, ξ_t is the shock to dividend growth. Once again, we stress that the expressions for $k(s_t), R(D_t, s_t)$, and $r(s_t)$ depend on which specific restricted version we consider.[20]

[20] Note that we only include one of the bubble terms, whereas the theory implies two distinct possible roots: positive and negative. We follow (Froot and Obstfeld, 1991; Driffill and Sola, 1998) by eliminating the coefficient associated with the negative η_2 root. The stock price should go to zero (and not infinity) as dividends go to zero. Therefore, we only include the term corresponding to the positive root, η_1. Therefore,

We estimate the model by maximum likelihood. Appendix F spells out the remaining details.

4.2 Results

We estimate the four alternative versions of the structural model described above for both the recursive and the power utility functions. Below, we discuss several takeaways from our estimation.

4.2.1 Parameter estimates

Risk aversion. Table 1 shows the results for the single regime parameterisations. In contrast, Table 2 shows the results of the switching versions of the model. One salient feature of the inferred structural parameters is that the risk aversion coefficient, γ, is much higher for the single regime models, with estimated values between 3.8 and 5.7, than for the models that allow regime switching, with values between 1.07 and 2.1. We can understand this result regarding the sources of risk premium. If the model does not allow regime switching, the estimated risk aversion has to be high to match the relatively elevated levels of risk premium we observe in the data. Once we allow for priced regime-switching risk, a new source of risk partially explains the observed risk premium and a lower value of γ is needed to match the observed data. A similar effect is apparent when comparing Models 1 (no bubbles) and 2 (allow for bubbles). Table 1 shows that bubbles in Model 2 contribute to the risk premium and that the inferred level of γ is relatively lower. However, the difference between estimated values of γ for Models 1 and 2 is smaller than that between models with or without regime switching.

We also assess whether using the power utility specification is supported by the data. We perform a likelihood ratio test of the null $\gamma = \psi$ against the alternative that allows for different values of these parameters in all models. We fail to reject the null hypothesis for

we set $a_{s_t,2} = 0$ for both states.

conventional significance levels in all cases.

Interpretation of regimes. Focusing on models that allow regime switching (Table 2), the results show that the mean of dividends growth is lower and the variance bigger in regime 0 than in regime 1. In this model, prices reflect an extra compensation for risk, which arises because of the possibility of switching to a different state of nature. In particular, state 0 has large volatility and lower expected dividend growth. Note also that for models 3 and 4, estimates of Γ_1 are greater than 1. It means that when the economy is in state 1, a regime shift to state 0 (the "bad" one) will increase the stochastic discount factor. [21] This is consistent with the standard intuition: bad states are associated with a high marginal utility of consumption. [22] Also, as the estimated $k_0 < k_1$, there is an instant capital loss when there is a change from regime 1 to regime 0. Thus, priced regime-switching makes the stock less valuable since it has a negative pay-off when there is a change to a high marginal utility state (i.e., the bad one). As explained below, priced regime-switching risk will account for an essential part of the observed excess returns.

The role of bubbles. The right panel of Table 1 shows the estimation results of the single regime models allowing bubbles. We notice that, under the maintained assumption of no structural breaks, the data supports the existence of an intrinsic bubble, both in the power and recursive utility case. Even though the point estimates for the bubble coefficients are insignificant, the likelihood ratios statistics are 2(1646.237 - 1614.489) = 63.496 and 2(1646.210 - 1614.302) = 63.816, for the recursive and power specifications, respectively; therefore, the bubbles are jointly significant for both cases. Also, we find that the estimated variance of the price-dividend equation is considerably smaller in model 2 than in model 1, showing that the contribution of the bubble to the fit of this equation is considerable.

[21] Since $\Gamma_1 = \dfrac{M_0(D)}{M_1(D)}$, then $\Gamma_1 > 1$ implies $M_0(D) > M_1(D)$

[22] It is worth noting that we estimate Γ_0 as a free parameter, so the data seems to favour a parametrisation in which the bad state in terms of the dividend process (low μ and high σ^2) is also the bad state in terms of the stochastic discount factor.

The right panel of Table 2 shows parameter estimates for the model that allows for regime switching and bubbles. The importance of bubbles seems lower: the estimated variance of the price-dividend equation in Model 3 is similar to that of Model 4 (for both regimes). However, we statistically reject the null of no bubbles when allowing regime switches.[23] As we show below, adding bubbles only seems to improve the model's fit at the end of the sample and explains a minor portion of the risk premium. Thus, although statistically significant, its quantitative (economic) contribution is minor once we account for regime switching.

The role of preference shocks. As we argued in section 3.1.3, the reason for having preference shocks is to allow for the price of the regime-switching risk when using a power utility function. However, this is not needed when we assume recursive utility. Thus, within the context of a recursive utility function and a regime-switching process for fundamentals, we can formally test the importance of having preference shocks in the utility function using a likelihood ratio test. In that context, the unrestricted likelihood corresponds to Models 3 and 4, while the restricted model (under the null) corresponds to a model with no preference shocks, that is, when $\frac{z_1}{z_0} = 1$. The likelihood ratio statistics are $2(1958.332 - 1957.108) = 2.448$ and $2(1974.424 - 1973.346) = 2.156$ for the cases with (Model 3) and without bubbles (Model 4), respectively. Thus, we cannot reject the null at the 10 per cent value.[24] This shows that adding preference shocks is of no quantitative importance for the results presented in the paper when using recursive utility functions.

4.2.2 Model-implied observable moments and interpretation of stylised facts

Table 3 shows key predicted moments of the data, that is, the predicted risk-free rate, the expected excess returns, and the fundamental component of the price-dividend ratio, k.

[23]The likelihood ratio statistics are 2(1974.424 - 1958.333) = 32.182 for the recursive and 2(1973.54 - 1957.88) = 31.32 for the power utility function. These statistics are smaller than in the model with no regime-switching.

[24]The results for observables and the risk premium decomposition are essentially the same as those of the baseline specification; therefore, we omit them.

When comparing models 1 and 2, we find that the average risk-free rate explained by the model is larger when we allow bubbles. We also find similar values across models for the risk premium and the price-dividend ratio. When considering Model 3, we find that the risk-free rate is roughly the same in both states and is comparable in magnitude to the values found for Models 1 and 2. Regarding Model 3, the average risk premium is higher in the bad state (state 0) and lower in the good state—the estimated risk premium in Models 1 and 2 lies between the state-specific values for Models 3. Similarly, k is smaller in bad states and larger in good states, and the single regime models' estimates lie between these state-dependent estimates. Finally, the results for Model 4 are qualitatively similar to those of Model 3. However, once we allow for bubbles, k becomes smaller. As bubbles account for a (small) part of the observed price-dividend ratio, the inferred fundamental component for the price-dividend ratio becomes smaller.

Figure 3 shows the data for price-dividend ratios, the values predicted by models 3 and 4, and the fitted regime probabilities. As we can see, our model estimates that the economy has been in a good state (i.e., high growth and low variance) since the mid-1990s. We can also see that the fitted price-dividend ratio for that period is higher than in the previous periods. Thus, models 3 and 4 support a narrative that there was an apparent permanent switch in the price-dividend ratio, particularly from the 1990s: a regime shift in the process for dividend growth. In good states, as average dividend growth is high and variance is low, the price-dividend ratios are higher than those of the bad state. The public expects higher cash flows associated with higher expected dividend growth and lower discount rates.[25]

[25]Our findings complement those of Monache, Petrella, and Venditti (2021), who in particular find that the fall change in the expected return is the main driver of the rise in the price-dividend ratios throughout the sample.

4.2.3 Risk premium decomposition

Table 4 shows a decomposition of the risk premium for each considered model. For Model 1, the expected excess return is solely explained by $\gamma\sigma^2$, and, as a result, the point estimate of γ is higher than those obtained from the other models. Model 2 has an additional source of risk (the bubble term) to account for the observed level of risk premium. Nevertheless, it is worth noting that the term $\gamma\sigma^2$ still explains most of the excess return. Thus, although important, intrinsic bubbles do not seem to be the main driver of the risk premium in model 2.

Furthermore, in model 3, when we allow for priced regime switching, the switching component explains most of the risk premium. Our results indicate that this is an essential source of the risk premium. This result is important because many models that allow switching do not price this risk. Note that the inferred regimes are typically associated with changes in the mean and variance of the dividends growth process and do not necessarily coincide with booms and recessions.[26] The identified regimes are similar to those in Bonomo and Garcia (1996) and Driffill and Sola (1998): a high drift, low variance state (the good state, which corresponds to our state 1), and low drift, high variance state (the bad state, which is given by state 0 in our estimation). Lastly, in model 4, we account for both regime-switching and bubbles and find that the total risk premium predicted by the model is now higher than in the other models. In Model 4, the regime-switching component explains much of the excess return, while the pure bubble component (Term **B**) is negligible.

[26] Our estimated expected excess return (for all specifications in Table 2) is countercyclical. Specifically, it has a significant (at all conventional levels) negative correlation with detrended output measured as suggested in Hamilton (2018).

5 Conclusions

Using a structural model that features regime switching and bubbles, this paper finds: i) evidence of regime shifts in the parameters of the process that drives dividend growth, which help to explain the higher price dividend ratios and lower excess returns present in the data, particularly from the 1990s onwards; ii) estimates of the price of the regime-switching risk for each state, which measure the importance of a change of regime in the marginal utility of the individuals; iii) that the data do not reject the use of the power utility function; iv) that intrinsic bubbles are statistically significant, can improve the fit at the end of the sample but do not seem essential to explain the expected excess returns; v) that pricing regime-switching risk is crucial to explain the observed excess return; vi) estimates of the risk aversion coefficient that are lower than those obtained in models without breaks or bubbles. This helps to resolve the equity premium puzzle.

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6 Tables and Figures

6.1 Tables

	No Regime-Switching			
	Model 1		Model 2	
Parameter	Recursive	Power	Recursive	Power
δ (p.a)	0.001	0.001	0.001	0.001
ψ	4.029	4.818	3.674	3.863
	(1.198)	(0.591)	(0.913)	(0.333)
γ	5.729	4.818	4.188	3.863
	(1.724)	(0.591)	(1.654)	(0.333)
μ	0.006	0.005	0.005	0.005
	(0.002)	(0.000)	(0.002)	(0.000)
σ	0.033	0.0333	0.035	0.035
	(0.001)	(0.001)	(0.001)	(0.001)
a			0.018	0.018
			(0.018)	(0.018)
σ^a	64.687	64.687	58.257	58.243
	(2.772)	(2.762)	(2.427)	(2.395)
σ^b	0.097	0.097	0.097	0.097
	(0.002)	(0.002)	(0.002)	(0.002)
σ^c	0.008	0.008	0.009	0.009
	(0.000)	(0.000)	(0.000)	(0.000)
log-likelihood	1614.49	1614.30	1646.24	1646.21

Table 1: Estimated Parameters for each Specification.
Standard error in parenthesis. All parameters are expressed in quarterly values except for δ. Parameter δ was obtained by maximizing the likelihood using a grid, and it is reported in annualised terms.

	Regime-Switching			
	Model 3		Model 4	
Parameter	Recursive	Power	Recursive	Power
$\delta(p.a)$	0.0010	0.0330	0.0010	0.0120
ψ	5.2495	1.0754	4.7646	2.1058
	(5.408)	(0.266)	(4.396)	(0.183)
γ	1.3650	1.0754	1.7055	2.1058
	(0.287)	(0.266)	(0.197)	(0.183)
p	0.9968	0.9971	0.9986	0.9993
	(0.002)	(0.002)	(0.001)	(0.000)
q	0.9930	0.9920	0.9937	0.9848
	(0.005)	(0.002)	(0.004)	(0.002)
μ_0	-0.004	-0.005	-0.002	-0.001
	(0.002)	(0.002)	(0.001)	(0.001)
μ_1	0.0083	0.0083	0.0084	0.0083
	(0.001)	(0.001)	(0.001)	(0.001)
σ_0	0.0452	0.0452	0.0485	0.0481
	(0.001)	(0.001)	(0.001)	(0.001)
σ_1	0.0172	0.0172	0.0165	0.0165
	(0.001)	(0.000)	(0.001)	(0.001)
σ_0^a	19.3744	19.3868	18.7863	18.7880
	(0.728)	(0.730)	(0.979)	(1.012)
σ_1^a	59.1425	59.1489	57.4500	57.3194
	(4.507)	(4.421)	(4.082)	(3.742)
σ_0^b	0.1221	0.1221	0.1221	0.1225
	(0.004)	(0.004)	(0.004)	(0.004)
σ_1^b	0.0671	0.0670	0.0670	0.0669
	(0.002)	(0.002)	(0.003)	(0.002)
σ_0^c	0.0105	0.0105	0.0105	0.0105
	(0.001)	(0.001)	(0.000)	(0.000)
σ_1^c	0.0057	0.0057	0.0057	0.0057
	(0.000)	(0.000)	(0.000)	(0.000)
$a_{0,1}$			0.0141	0.0251
			(0.025)	(0.043)
$a_{1,1}$			0.0036	0.0077
			(0.008)	(0.017)
Γ_0	0.2557	0.2235	0.0976	0.0518
	(0.151)	(0.131)	(0.072)	(0.039)
Γ_1	3.9097	4.4726	10.2365	19.2957
	(2.304)	(2.624)	(7.579)	(14.699)
log-likelihood	1958.33	1957.88	1974.42	1973.54

Table 2: Estimated Parameters for each specification.
Standard error in parenthesis. All parameters are expressed in quarterly values except for δ. Parameter δ was obtained by maximizing the likelihood using a grid, and it is reported in annualised terms.

	No Regime-Switching				Regime-Switching			
	Model 1		Model 2		Model 3		Model 4	
	Recursive	Power	Recursive	Power	Recursive	Power	Recursive	Power
r^f_0	0.0274	0.0274	0.0336	0.0274	0.0274	0.0275	0.0299	0.0296
r^f_1					0.0286	0.0286	0.0286	0.0286
$\mathbf{E}_t(r^S - r^f \mid s_t = 0)$	0.0252	0.0213	0.0253	0.0235	0.0342	0.0361	0.0422	0.0742
$\mathbf{E}_t(r^S - r^f \mid s_t = 1)$					0.0208	0.0218	0.0218	0.0232
k_0	32.9336	32.9078	31.0485	32.9335	20.7915	20.8060	19.8251	19.7112
k_1					43.8082	43.7939	41.0512	40.5265

Table 3: Observable Averages Predicted by each of the Model Specifications. All variables are reported in annualized terms.

Model	Preferences	Term A	Term B	Term C	Term D	Total
Model 1	Recursive	0.0252	-	-	-	0.0252
	Power	0.0213	-	-	-	0.0213
Model 2	Recursive	0.0205	0.0048	-	-	0.0253
	Power	0.0190	0.0045	-	-	0.0235
Model 3	Recursive	0.0062	-	0.0210	-	0.0272
	Power	0.0048	-	0.0237	-	0.0285
Model 4	Recursive	0.0087	0.0015	0.0234	-0.0044	0.0292
	Power	0.0103	0.0018	0.0399	-0.0048	0.0472
Data						0.039691

Table 4: Risk premium Decomposition.
Terms are computed using the decomposition presented in Eq. (18). All figures are reported in annualized terms. A cell with "-" means that the source of risk does not exist for the case.

6.2 Figures

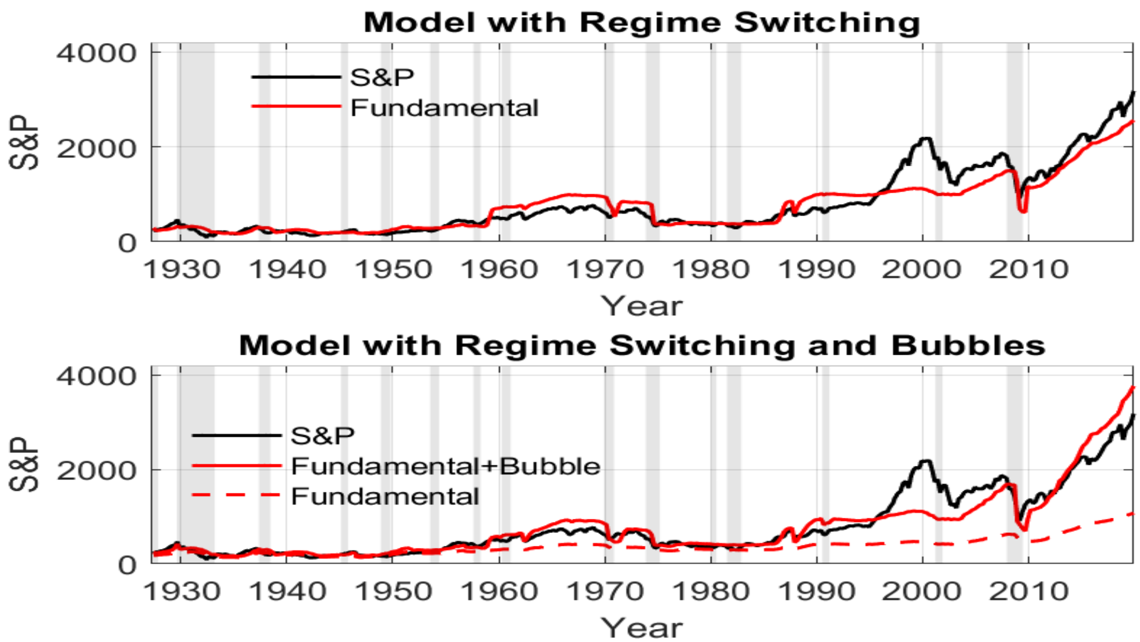

Figure 1: Actual and fitted prices using the model given by Eqs. (1a) - (1c). The top panel shows the model's fitted values that allow for regime-switching but not for bubbles. The bottom panel shows the fitted values for the model that allow for both regime switching and bubbles.

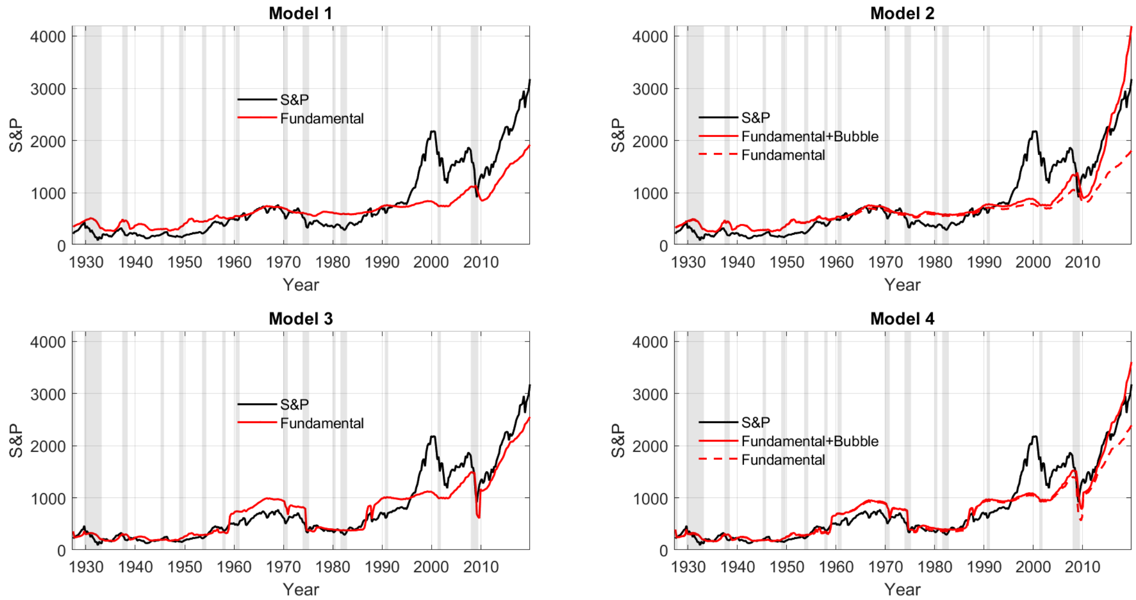

Figure 2: Stock prices and fitted values by each model. In all cases, recursive utility is used.

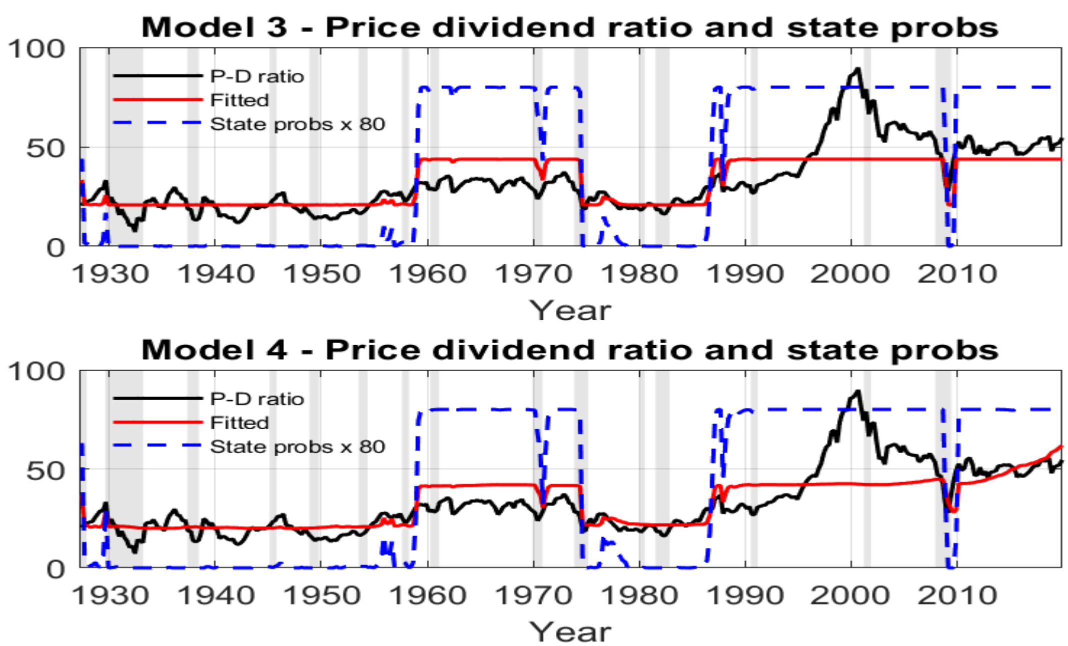

Figure 3: Actual and fitted price dividend ratios and transition probabilities for models 3 and 4.

Appendices

Appendix A Derivations for the Preliminary Analysis

We present the model used in Section 2. Time is discrete. The pricing equation is given by:

$$P_t = \mathbf{E}_t\Big\{e^{-r(s_t)}[P_{t+1}(s_{t+1}) + D_t]\Big\}, \tag{A.1}$$

(the price P_t might be seen as the start-of-period price, with the dividend D_t paid at the end of the period). The discount rate is regime-specific. The mean and standard deviation of dividend growth is state-dependent:

$$\Delta \ln D_t = \mu_{s_t} + \sigma_{s_t}\varepsilon_t, \tag{A.2}$$

where $\mu_{s_t} = (\mu_0(1-s_t) + \mu_1 s_t)$ and $\sigma_{s_t} = (\sigma_0(1-s_t) + \sigma_1 s_t)$. The fundamental solution to (A.1) is linear in dividends, with a regime-specific coefficient: $P_t^{pv} = k_{s_t} D_t$. These coefficients can be determined using (A.1) and (A.2) as:

$$\begin{aligned} k_0 &= ((qe^{-r_0} + (1-q)e^{-r_1}) + qk_0b_0 + (1-q)k_1b_1), \\ k_1 &= ((pe^{-r_1} + (1-p)e^{-r_0}) + pk_1b_1 + (1-p)k_0b_0), \end{aligned} \tag{A.3}$$

where $b_0 = e^{(\mu_0 - r_0 + \frac{1}{2}\sigma_0^2)}$ and $b_1 = e^{(\mu_1 - r_1 + \frac{1}{2}\sigma_1^2)}$. These two equations determine (k_0, k_1) as a function of all the model parameters.

The bubble component B_t has to solve the following difference equation:

$$B_t = \mathbf{E}_t\Big\{e^{-r_{s_t}} B_{t+1}(s_{t+1})|I_t\Big\}. \tag{A.4}$$

The intrinsic bubble has the form $B_t = a_i D_t^{\lambda}$. Using this expression, (A.2) and (A.4), we get the following relations:

$$a_0 D_t^\lambda = a_0 q D_t^\lambda e^{(\lambda\mu_0 - r_0 + \frac{1}{2}\lambda^2\sigma_0^2)} + (1-q) a_1 D_t^\lambda e^{(\lambda\mu_1 - r_1 + \frac{1}{2}\lambda^2\sigma_1^2)},$$
$$a_1 D_t^\lambda = a_0 (1-p) D_t^\lambda e^{(\lambda\mu_0 - r_0 + \frac{1}{2}\lambda^2\sigma_0^2)} + p a_1 D_t^\lambda e^{(\lambda\mu_1 - r_1 + \frac{1}{2}\lambda^2\sigma_1^2)}.$$

From these two, we can obtain the following expressions for the ratio $\frac{a_1}{a_0}$

$$\frac{a_1}{a_0} = \frac{1 - q e^{(\lambda\mu_0 - r_0 + \frac{1}{2}\lambda^2\sigma_0^2)}}{(1-q) e^{(\lambda\mu_1 - r_1 + \frac{1}{2}\lambda^2\sigma_1^2)}} \tag{A.5}$$

$$\frac{a_1}{a_0} = \frac{(1-p) e^{(\lambda\mu_0 - r_0 + \frac{1}{2}\lambda^2\sigma_0^2)}}{1 - p e^{(\lambda\mu_1 - r_1 + \frac{1}{2}\lambda^2\sigma_1^2)}} \tag{A.6}$$

We can use Eq. (A.5) and Eq. (A.6)) to get a solution for λ and $\frac{a_1}{a_0}$. One of the bubble coefficients is left as a free parameter to be estimated. The theoretical model that is estimated is given by the following three equations:

$$\frac{P_t}{D_t} = k_{s_t} + c_{s_t} D_t^{\lambda-1} + \theta_{s_t} \nu_t, \tag{A.7a}$$
$$\Delta \ln D_t = \mu_{s_t} + \sigma^d_{s_t} u_t, \tag{A.7b}$$
$$r_t = r_{s_t} + \sigma^r_{s_t} \xi_t. \tag{A.7c}$$

These are Eq. (1a)-(1c) in the main text.

Appendix B Recursive Utility with State-Dependent Preferences

This Appendix derives the solution of the model assuming a recursive utility function and that the instantaneous return (aggregator) function f is subject to a preference shock. The randomness in the model is not only governed by a diffusion process with constant drift and volatility parameters but also has an additional source of uncertainty through which the dynamics of the state variables shift between different regimes at random times. In other words, the economy has two types of shocks: continuous shocks generated by a Brownian motion and infrequent shocks generated by a two-state Markov chain $\{s_t\}$. We suppose consumption growth alternates between two regimes: 'μ_i - σ_i' and μ_j - σ_j', such that s_t is a two-state Markov chain alternating between states 1 and 2. Suppose the rate matrix of the chain s_t is $H = (h_{ij})$; $1 \leqslant i, j \leqslant 2$, where H is a $Q-$matrix and $h_{ij} > 0$ for $i \neq j$, $h_{1j} + h_{2j} = 0$ for $j = 1, 2$ so $h_{11} < 0$ and $h_{22} < 0$:

$$H = \begin{bmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{bmatrix} = \begin{bmatrix} -h_1 & h_2 \\ h_1 & -h_2 \end{bmatrix}. \tag{B.1}$$

The regime s_t has the following dynamics

$$ds = Hs dt + dN. \tag{B.2}$$

The probability that a transition occurs from state i to state j in a small time interval $(t, t + dt)$ is equal to $h_i dt$. Similarly, $1 - h_i dt$ is the probability that the process remains in state i, and dN are shocks to ds.

In this case, the corresponding value function J and aggregator f are given by

$$
\begin{aligned}
J(c_t, s_t) &= E_t \int_t^{\infty} f(c_\tau, J_\tau) d\tau, && \text{(B.3)} \\
f(c, J(s), s) &= \frac{\delta}{(1-\psi)} \frac{z(s)^{1-\psi} c^{1-\psi} - [(1-\gamma)J(s)]^{\frac{1-\psi}{1-\gamma}}}{[(1-\gamma)J(s)]^{\frac{1-\psi}{1-\gamma}-1}}, && \text{(B.4)}
\end{aligned}
$$

where $z(s)$ is the preference shock. In equilibrium, the representative household consumes as much as the dividends that it receives and faces the following dividend process:

$$
\frac{dD}{D} = \mu_{Ds} dt + \sigma_{Ds}) dW.
$$

The value function in terms of measurable dividends is

$$
J(D_t, s_t) = E_t \left[\int_t^{\infty} f(D_\tau, J_\tau) d\tau \right].
$$

The stochastic discount factor in equilibrium is defined as:

$$
M_t = Y(D_t, s_t)\, f_c(D_t, J(D_t, s_t)), \tag{B.5}
$$

where $Y(D_t, s_t) = e^{\int_0^t f_J(D_\tau, J(D_\tau, s_\tau)) d\tau}$.

Hereafter we will drop the subscript t. The process for $M(s, D)$ can be written as:

$$
dM(s, D) = f_c(c, J(D, s)) dY(s) + Y(s)\, df_c(c, J(D, s)) + \langle Y(s) f_c(c, J(D, s))\,,\, ds \rangle. \tag{B.6}
$$

We conjecture the following solution for J,

$$
J(D, s) = \frac{[\phi(s) D]^{1-\gamma}}{1-\gamma}.
$$

Note that in equilibrium $c = D$ and $dc = dD$. Dividing Eq. (B.6) by Eq. (B.5) gives

$$\begin{aligned} \frac{dM_i}{M_i} &= \frac{dY_i}{Y_i} + \frac{df_{c_i}}{f_{c_i}} + h_i\left[\frac{Y_j f_{c_j}}{Y_i f_{c_i}} - 1\right]dt - \left[\frac{Y_j f_{c_j}}{Y_i f_{c_i}} - 1\right]dN_i, \\ &= \frac{dY_i}{Y_i} + \frac{df_{c_i}}{f_{c_i}} + h_i\left[\frac{z_j^{1-\psi}}{z_i^{1-\psi}}\frac{\phi_j^{\psi-\gamma}}{\phi_i^{\psi-\gamma}} - 1\right]dt - \left[\frac{z_j^{1-\psi}}{z_i^{1-\psi}}\frac{\phi_j^{\psi-\gamma}}{\phi_i^{\psi-\gamma}} - 1\right]dN_i, \end{aligned} \tag{B.7}$$

First, in order to compute $\frac{dY}{Y}$, note that:

$$f_{V_i} = \frac{\delta}{(1-\psi)}\left[\frac{z_i^{1-\psi}D^{1-\psi}}{[(1-\gamma)J_i]^{\frac{1-\psi}{1-\gamma}}}(\psi-\gamma) + (\gamma-1)\right] = \frac{\delta}{1-\psi}\Big[\frac{z_i^{1-\psi}(\psi-\gamma)}{\phi_i^{1-\psi}} + (\gamma-1)\Big]$$

Which does not depend on D. Furthermore f_{V_i} can depend on the state. But as $Y_t = e^{(\int_0^t f_J(s_\tau)d\tau)}$, a regime shift would not make Y_t jump. Therefore,

$$\begin{aligned} \frac{dY}{Y} &= f_J dt \\ &= \frac{\delta}{1-\psi}\Big[\frac{z_i^{1-\psi}(\psi-\gamma)}{\phi_i^{1-\psi}} + (\gamma-1)\Big]dt \end{aligned}$$

On the other hand, to obtain $\frac{df_c}{f_c}$ we use that c=D in equilibrium and obtain differentials for D. Thus, using the conjecture for J, we get:

$$f_c(D, J(D,s)) = \frac{\delta\, z_i^{1-\psi}D^{-\psi}}{[(1-\gamma)J_i]^{\frac{1-\psi}{1-\gamma}-1}} = \frac{\delta\, z_i^{1-\psi}D^{-\gamma}}{\phi_i^{\gamma-\psi}} = \delta z_i^{1-\psi}\phi_i^{\psi-\gamma}D^{-\gamma}$$

Thus,

$$\begin{aligned} df_c &= \frac{\partial f_c(D, J(D,s))}{\partial D}dD + \frac{1}{2}\frac{\partial^2 f_c(D, J(D,s))}{\partial D^2}(dD)^2 \\ &= (-\gamma\mu_{s_t} + \frac{1}{2}\gamma(1+\gamma)\sigma_{s_t}^2)f_c(D, J(D,s))dt - \gamma\sigma_{s_t}f_c(D, J(D,s)dW, \end{aligned}$$

and

$$\frac{df_c}{f_c} = (-\gamma\mu_{s_t} + \frac{1}{2}\gamma(1+\gamma)\sigma_{s_t}^2)dt - \gamma\sigma_{s_t}dW.$$

Therefore, the stochastic discount factor satisfies the following.

$$\begin{aligned}\frac{dM}{M} &= \Big[\frac{\delta}{1-\psi}\Big[\frac{z_i^{1-\psi}(\psi-\gamma)}{\phi_i^{1-\psi}} + (\gamma-1)\Big] + (-\gamma\mu_{s_t} + \frac{1}{2}\gamma(1+\gamma)\sigma_{s_t}^2)\Big]dt - \gamma\sigma_{s_t}dW \\ &+ h_i\left[\frac{z_j^{1-\psi}}{z_i^{1-\psi}}\frac{\phi_j^{\psi-\gamma}}{\phi_i^{\psi-\gamma}} - 1\right]dt - \left[\frac{z_j^{1-\psi}}{z_i^{1-\psi}}\frac{\phi_j^{\psi-\gamma}}{\phi_i^{\psi-\gamma}} - 1\right]dN_i,\end{aligned}$$

Note that $\Gamma_i - 1 = \frac{z_j^{1-\psi}}{z_i^{1-\psi}}\frac{\phi_j^{\psi-\gamma}}{\phi_i^{\psi-\gamma}} - 1$ is the price of regime switching. It is easy to show that $\Gamma_0\Gamma_1 = 1$. Having armed with the expression dM/M, one can determine an expression for the risk-free interest rate using the definition $r^f(s_t)dt = -E\Big(\frac{dM(s)}{M(s)}|F_t, s\Big)$:

$$r_i^f = -\delta\frac{1-\gamma}{1-\psi}\Big[\Big(\frac{\psi-\gamma}{1-\gamma}\Big)z_i^{1-\psi}\phi_i^{\psi-1} - 1\Big] + \gamma\mu_i - \tfrac{1}{2}\gamma(1+\gamma)\sigma_i^2 - h_i\left[\frac{z_j^{1-\psi}}{z_i^{1-\psi}}\frac{\phi_j^{\psi-\gamma}}{\phi_i^{\psi-\gamma}} - 1\right]. \quad \text{(B.8)}$$

We solve ϕ_0 and ϕ_1 by the use of the Hamilton-Jacobi-Bellman equation,

$$\sup_D \left[J_{D_i}\mathbf{E}_t\Big\{dD_i + h_i(J_j - J_i)dt + \frac{1}{2}J_{DD_i}[dD_i]^2\Big\}\right] + f(D, J_i)dt = 0. \quad \text{(B.9)}$$

Plugging the relevant derivatives into the above equation yields

$$\frac{\delta}{1-\psi}[z_i^{1-\psi}\phi_i^{\psi-\gamma} - \phi_i^{1-\gamma}] + h_i[\frac{\phi_j^{1-\gamma} - \phi_i^{1-\gamma}}{1-\gamma}] + \phi_i^{1-\gamma}[\mu_i - \frac{1}{2}\sigma_i^2\gamma] = 0.$$

By simplifying it, we obtain a neater equation:

$$\delta\frac{1-\gamma}{1-\psi}z_i^{1-\psi}\phi_i^{\psi-\gamma} + [(1-\gamma)\mu_i - \gamma(1-\gamma)\sigma_i^2 - \delta\frac{1-\gamma}{1-\psi}]\phi_i^{1-\gamma} + h_i[\phi_j^{1-\gamma} - \phi_i^{1-\gamma}] = 0. \quad \text{(B.10)}$$

We have two equations (one for each state), so we must solve a 2-by-2 system to find ϕ_0, ϕ_1. This system mirrors the result in Chen (2010) for two states. It's easy to see that the solution to the system is homogeneous of degree 1 in z_0, z_1 [27]. We exploit that property for the calibration of Γ_i.

Solving for prices

Once we find the closed-form value function solutions, we can describe their asset pricing implications. To obtain an expression for the price of the stock as a function of the states and dividends, $P(D_t, s_t)$, we utilize the fact that those prices satisfy the standard asset pricing equation:

$$0 = \frac{D}{P}dt + E_t\left[\frac{dM}{M} + \frac{dP}{P} + \frac{dM}{M}\frac{dP}{P}\right]. \tag{B.11}$$

We substitute the law of motion of $\frac{dM}{M}$ from Eq. (7) in the main text (after plugging ϕ_i in Eq. (B.10)) into the asset pricing equation in (B.11), and then apply Ito's lemma (denoting $P(D, s)$ with P_s, we will use both notations interchangeably) to find that $P(D, s)$ satisfies (this is shown in Appendix D):

$$\begin{aligned} r_0^f P_0 &= D + [\mu_0 - \gamma\sigma_0^2]P_{0D}D + \frac{1}{2}\sigma_0^2 P_{0DD}D^2 + h_0\Gamma_0[P_1 - P_0], \\ r_1^f P_1 &= D + [\mu_1 - \gamma\sigma_1^2]P_{1D}D + \frac{1}{2}\sigma_1^2 P_{1DD}D^2 + h_1\Gamma_1[P_0 - P_1], \end{aligned} \tag{B.12}$$

We guess that the general solution has the following functional form:

$$P(D, s) = k_s D + \sum_{\nu=1}^{4} a_{s,\nu} D^{\eta_\nu}, \tag{B.13}$$

where $a_{s,\nu}$ for $\nu = 1, 2, 3, 4$ are free parameters. The first term is the stock's fundamental value, whereas the non-linear terms correspond to the intrinsic bubble. In the sequel, we derive expressions for k_s and λ_i as functions of the deep parameters of the model.

[27] Take $z^* = (z_0^*, z_1^*)$ and the corresponding solution $\phi^* = (\phi_0^*, \phi_1^*)$. Now, scale z by a factor of λ. Using Eq (B.10), it is straightforward to verify the guess that $\lambda\phi^*$ is a solution to the system.

Fundamental solution

For expositional clarity, let us focus first on the fundamental term of Eq. (B.12). With the help of the first term of the postulated solution in (B.13), we obtain expressions for k_s. We have that k_0, k_1 satisfy:

$$\begin{aligned} r_0^f k_0 &= 1 + [\mu_0 - \gamma\sigma_0^2]k_0 + h_0\Gamma_0[k_1 - k_0], \\ r_1^f k_1 &= 1 + [\mu_1 - \gamma\sigma_1^2]k_1 + h_1\Gamma_1[k_0 - k_1], \end{aligned} \tag{B.14}$$

or in matrix form:

$$\begin{pmatrix} r_0^f - [\mu_0 - \gamma\sigma_0^2] + h_0\Gamma_0 & -h_0\Gamma_0 \\ -h_1\Gamma_1 & r_1^f - [\mu_1 - \gamma\sigma_1^2] + h_1\Gamma_1 \end{pmatrix} \begin{pmatrix} k_0 \\ k_1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

We can solve this linear system inverting the matrix to obtain (k_0, k_1) [28] as

$$k_i = \frac{r_j^e - \mu_j + h_i\Gamma_i}{(r_i^e - \mu_i + h_i\Gamma_i)(r_j^e - \mu_j + h_j\Gamma_j) - h_i h_j}$$

for $i = 0, 1$ and $j \neq i$; where $r_i^e = r_i^f + \gamma\sigma_i^2$

Non-Fundamental Solutions

We now turn to the non-fundamental solutions (i.e., the intrinsic bubble terms).[29] Note that the homogeneous part of the system above corresponds to the terms derived from $E[d(M_t P_t)] = 0$. Any solution to this equation can be seen as the price of an asset that never pays any cash flow. That is why we can think of it as bubbles: the only source of their price is that they are expected to have a higher price in the future. Therefore, the

[28] We need $\left(r_0^f - [\mu_0 - \gamma\sigma_0^2] + h_0\Gamma_0\right)\left(r_1^f - [\mu_1 - \gamma\sigma_1^2] + h_1\Gamma_1\right) - h_0h_1 > 0$ to obtain positive values of k_0 and k_1.

[29] Theoretical results about the possible existence of bubbles in sequential equilibria have been explored in the literature (see Kocherlakota (1992), Kocherlakota (2008)).

extra terms in $P(D,s)$ must solve the homogeneous system:

$$\begin{aligned}[]
[\mu_0-\gamma\sigma_0^2]P_{0D}D+\frac{1}{2}\sigma_0^2P_{0DD}D^2+h_0\Gamma_0[P_1-P_0] &= r_0^fP_0,\\
[\mu_1-\gamma\sigma_1^2]P_{1D}D+\frac{1}{2}\sigma_1^2P_{1DD}D^2+h_1\Gamma_1[P_0-P_1] &= r_1^fP_1.
\end{aligned} \tag{B.15}$$

As mentioned above, we conjecture the solutions to be $P^B(D,s)=\mathbf{a}(\mathbf{s})D^\eta$. Using this guess yields:

$$\begin{aligned}
\left([\mu_0-\gamma\sigma_0^2]\eta+\frac{1}{2}\sigma_1^2\eta(\eta-1)-h_0\Gamma_0-r_0^f\right)a_0 &= h_0\Gamma_0a_1,\\
\left([\mu_1-\gamma\sigma_1^2]\eta+\frac{1}{2}\sigma_2^2\eta(\eta-1)-h_1\Gamma_1-r_1^f\right)a_1 &= h_1\Gamma_1a_0,
\end{aligned} \tag{B.16}$$

By multiplying both equations, we can eliminate the bubble coefficients to obtain:

$$G_0(\eta)G_1(\eta)=h_0h_1, \tag{B.17}$$

where $G_i(\eta)$ is given by

$$G_i(\eta)=(\mu_i-\gamma\sigma_i^2)\eta+\frac{1}{2}\sigma_i^2\eta(\eta-1)-h_i\Gamma_i-r_i^f \qquad \text{for } i=0,1.$$

Equation (B.17) has four distinct roots, with $\eta_1>\eta_2>0$ and $\eta_4<\eta_3<0$. Thus, the general solution to the homogeneous system is

$$\begin{aligned}
P^B(D,s=0) &= \textstyle\sum_{\nu=1}^4 a_{0,\nu}{D_t}^{\eta_\nu},\\
P^B(D,s=1) &= \textstyle\sum_{\nu=1}^4 a_{1,\nu}{D_t}^{\eta_\nu}.
\end{aligned}$$

As it is standard practice, we impose the boundary condition that $\lim_{D\to0^+}P(D,s)=0$. So $a_{i,3}=a_{i,4}=0$ for $i=0,1$. Thus, from now on, we will only write the two terms corresponding to the positive roots of eq. (B.17).

Appendix C Role of the preference parameter when pricing regime-switching risk assuming a power utility function

Several authors, such as Bhamra, Kuehn, and Strebulaev (2010), Chen (2010) and Dai and Singleton (2003), express the discounting factor change as:

$$\frac{dM(s)}{M(s)} = -{r^f}_s dt - \lambda_s dW - (\Gamma_s - 1)dN \tag{C.1}$$

Dai and Singleton 2003 have a reduced-form model that treats Γ_i as a constant. In a structural model, changes in Γ_i arise as a discrete change in the utility of consumption. This change is a consequence of the change in the state, which affects the consumption growth rate. The issue is that consumption does not jump when the regime changes. When assuming Epstein and Zin's utility function, the utility function jumps at time t because it incorporates all the future expected changes, while when using a power utility function, the change would only affect the regime-specific utility directly when assuming a preference shock. Below, we show this algebraically.

Assume, as before, that the states follow a two-regime Markov process

$$ds = Hs dt + dN. \tag{C.2}$$

Consider now the following pricing kernel $M(s) = e^{-\delta t}u'[c(s)]$. Then, the process for $M(s)$ can be written as

$$dM(s) = -\delta e^{-\delta t}u'[c(s)]dt + e^{-\delta t}u''[c(s)]dc(s) + \frac{1}{2}e^{-\delta t}u'''[c(s)]dc^2(s) + \langle e^{-\delta t}u'[c(s)], ds\rangle,$$

where the discrete impact term, $\langle e^{-\delta t}u'[c(s)], ds\rangle$, resulting from a regime shift is

$$\langle e^{-\delta t}u'[c(s)], ds\rangle = \langle e^{-\delta t}u'[c(s)], Hsdt\rangle + \langle e^{-\delta t}u'[c(s)], dN\rangle.$$

Here $\langle .,.\rangle$ is the inner product operator used in the following way: If x, y are (column) vectors in R^N, we write $\langle x, y\rangle = x'y$ for their scalar (inner) product. Furthermore, the components of the discrete impact term can be, in turn, expressed as:

$$\begin{aligned}
\langle e^{-\delta t}u_i'[c(s)], Hsdt\rangle &= (h_1[e^{-\delta t}\left(u_2'[c]-u_1'[c]\right)]dt,\ h_2[e^{-\delta t}\left(u_1'[c]-u_2'[c]\right)]dt)' \\
&\quad \text{and} \\
\langle e^{-\delta t}u_i'[c(s)], dN\rangle &= [e^{-\delta t}\left(u_1'[c]-u_2'[c]\right)]dN_1.
\end{aligned}$$

The second term follows from the fact that $dN = (dN_1, dN_2)$ and $dN_1 = -dN_2$.

To arrive at the desired result, it is convenient to note that, conditional on $s = i$, $i = \{1, 2\}$ and denoting $dM_i = dM(s = i)$, we can write:

$$\begin{aligned}
dM_1 &= -\delta e^{-\delta t}u_1'[c]dt + e^{-\delta t}u_1''[c]dc_1 + \frac{1}{2}e^{-\delta t}u_1'''[c]dc_1^2 \\
&\quad + h_1[e^{-\delta t}\left(u_2'[c]-u_1'[c]\right)]dt + [e^{-\delta t}\left(u_1'[c]-u_2'[c]\right)]dN_1,
\end{aligned}$$

$$\begin{aligned}
dM_2 &= -\delta e^{-\delta t}u_2'[c]dt + e^{-\delta t}u_2''[c]dc_2 + \frac{1}{2}e^{-\delta t}u_2'''[c]dc_2^2 \\
&\quad + h_2[e^{-\delta t}\left(u_1'[c]-u_2'[c]\right)]dt + [e^{-\delta t}\left(u_2'[c]-u_1'[c]\right)]dN_2.
\end{aligned}$$

Rearranging terms, we obtain the following expressions:

$$\begin{aligned}
\frac{dM_1}{M_1} &= -\delta dt + \frac{cu_1''[c]}{u_1'[c]}\frac{dc_1}{c_1} + \frac{1}{2}\frac{c^2u_1'''[c]}{u_1'[c]}\left(\frac{dc_1}{c_1}\right)^2 \\
&\quad + h_1[\frac{(u'[c_2]-u'[c_1])}{u'[c_1]}]dt + [\frac{(u'[c_1]-u'[c_2])}{u'[c_1]}]dN_1,
\end{aligned}$$

$$\frac{dM_2}{M_2} = -\delta dt + \frac{c_2 u_2''[c]}{u_2'[c]}\frac{dc_2}{c_2} + \frac{1}{2}\frac{c^2 u_2'''[c]}{u_2'[c]}\left(\frac{dc_2}{c_2}\right)^2 + h_2[\frac{(u_1'[c] - u_2'[c])}{u_2'[c]}]dt + [\frac{(u_2'[c] - u_1'[c])}{u_2'[c]}]dN_2.$$

To relate the results of this appendix to Eq. (7) in the text, it is convenient to assume a functional form, which includes a preference shock, for the utility function:

$$u_i[c(s)] = \frac{z_i c^{1-\gamma}}{1-\gamma},\ i = 1, 2, \tag{C.3}$$

and a stochastic process for consumption,

$$\frac{dc_i}{c_i} = \mu_{c_i} dt + \sigma_{c_i} dW \tag{C.4}$$

We can then write the explicit forms for $\frac{dM_1}{M_1}$ and $\frac{dM_2}{M_2}$ as

$$\frac{dM_1}{M_1} = \left(-\delta - \gamma\mu_{c_1} + \frac{1}{2}\gamma(\gamma+1)\sigma_{c_1}^2 + h_1[\frac{(u_2'[c] - u_1'[c])}{u_1'[c]}]\right) dt \tag{C.5}$$

$$-\gamma\sigma_{c_1} dW_t + [\frac{(u_1'[c] - u_2'[c])}{u_1'[c]}]dN_1 \tag{C.6}$$

$$\frac{dM_2}{M_2} = \left(-\delta - \gamma\mu_{c_2} + \frac{1}{2}\gamma(\gamma+1)\sigma_{c_2}^2 + h_2[\frac{(u_1'[c] - u_2'[c])}{u_2'[c]}]\right) dt \tag{C.7}$$

$$-\gamma\sigma_{c_2} dW_t + [\frac{(u_2'[c] - u_1'[c])}{u_2'[c]}]dN_2 \tag{C.8}$$

since $\frac{cu''[c]}{u'[c]} = -\gamma$ and $\frac{c^2 u'''[c]}{u'[c]} = \gamma(\gamma+1)$.

Given that $r_s^f dt = -E(\frac{dM(s)}{M(s)}|F_t, s)$, conditional on $s = 1$ and $s = 2$ the above expressions for $\frac{dM_1}{M_1}$ and $\frac{dM_2}{M_2}$ imply that:

$$r_1^f = \left(\delta + \gamma\mu_{c_1} - \frac{1}{2}\gamma(\gamma+1)\sigma_{c_1}^2 - h_1[\frac{(u_2'[c] - u_1'[c])}{u_1'[c]}]\right) \tag{C.9}$$

$$r_2^f = \left(\delta + \gamma\mu_{c_2} - \frac{1}{2}\gamma(\gamma+1)\sigma_{c_2}^2 - h_2[\frac{(u_1'[c] - u_2'[c])}{u_1'[c]}]\right) \tag{C.10}$$

Therefore, using the expressions in C.5 and C.7, we can see the assumption on the stochastic discount process in the text Eq. (7)

$$\frac{dM(s)}{M(s)} = -r_s^f dt - \lambda_{s_t} dW_t - (\Gamma_s - 1)[dN_t],$$

is equivalent, given the utility specification in C.3, to the assumption that, conditional on $s = 1$, the risk-free rate r_1^f satisfies equation C.9 and, conditional on $s = 2$, the risk-free rate r_2^f satisfies C.10.

What is more, a comparison of the expressions in equations C.5 and C.7 with the expression in Eq. (7) allows us to conclude that the price of the diffusion risk is $\lambda_i = \gamma\sigma_{c_i}$ and that the market prices of regime-switching risk are $\Gamma_1 - 1 = \frac{(u_2'[c]-u_1'[c])}{u_1'[c]} = \frac{z_2-z_1}{z_1}$ and $\Gamma_2 - 1 = \frac{(u_1'[c]-u_2'[c])}{u_2'[c]} = \frac{z_1-z_2}{z_2}$, respectively.

Now notice that 1) These terms are constant and 2) that for the power utility function, eliminating the preference shock implies that there is no price of switching risk since, as we said above, consumption does not jump (and what jumps is the rate of growth of consumption).

Appendix D Details for Regime-Dependent Stock Price Evaluation

Stock Price valuation.

Following Cochrane (2005) we write the regime-switching version of the no-arbitrage stock price valuation equation as follows:

$$0 = M_s D dt + \mathbf{E}[d(M_s P(D, s))]. \tag{D.1}$$

Applying Ito's lemma breaks the last term $d(M(s)P(D, s))$:

$$d(M(s)P(D, s)) = P(D, s)dM(s) + M(s)dP(D, s) + dM(s)dP(D, s). \tag{D.2}$$

Following Bhamra, Kuehn, and Strebulaev (2010), Chen (2010) and Dai and Singleton (2003) we write $dM(s)$ as:

$$dM(s) = -r_s^f M(s)dt - \lambda_s M(s)dW - M(s)(\Gamma_s - 1)dN \tag{D.3}$$

where r_s^f is the risk-free rate of return, λ_s is the (regime-dependent) market price of continuous risk (diffusion risk), Γ_s is the market price of a shift from regime $s = j$ to regime i $(i \neq j; i, j = 0, 1)$, and dW is the increment of a standard Wiener process.

By substituting Eq. (D.3) for dM in Eq. (D.1) we derive the following expression:

$$r^f P(D, s)dt = Ddt + E[dP(D, s)] + E\left[\frac{dM(s)dP(D, s)}{M(s)}\right]. \tag{D.4}$$

To obtain the final result presented in the text, we need to determine expressions for $E[dP(D, s)]$ and $E[\frac{dM(s)dP(D,s)}{M(s)}]$.

(i) Derivation of $E(dP(D, s))$.

Let $P_i = P(D, s(t) = i)$ for $i = 0, 1$ and $P = (P_0, P_1)$ be a 2×1 row vector consisting of elements P_0, P_1. Let $\langle ., . \rangle$ be the inner product operator used in the following way: If x, y are (column) vectors in $\mathbb{R}^N$, we write $\langle x, y \rangle = x'y$ for their scalar (inner) product. When x is a matrix, and y is a (column) vector, $\langle x, y \rangle = diag(xy)$ denotes the diagonal matrix with vector xy on its diagonal.

Then applying Ito's lemma to cases with regime shifts along the lines of Elliott, Aggoun, and Moore (1995) we can express the change in the stock price as $dP = dP(D, s)$ as in Shen and Elliott (2015):

$$\begin{aligned} dP &= \langle dP, s\rangle + \langle P, ds\rangle \\ &= P_D(D,s)dc + \tfrac{1}{2}P_{DD}(D,s)(dD)^2 + \langle P, Hs dt\rangle + \langle P, dN\rangle, \\ &= \underbrace{\left(\mu_s P_D(D,s)D + \tfrac{1}{2}\sigma_s^2 P_{DD}(D,s)D^2\right)dt + \sigma_s P_D(D,s)D dW_t}_{\text{due to the diffusion}} \\ &\quad + \underbrace{\langle P, Hs dt\rangle + \langle P, dN\rangle}_{\text{due to the discrete shifts}} \end{aligned} \tag{D.5}$$

where the subscripts D and DD denote, respectively, the first and second partial derivatives of variable P with respect to D.

Notice that we can express $\langle P, Hsdt\rangle = (h_0[P_1 - P_0]dt, h_1[P_0 - P_1]dt)'$ and, using the fact that $dN = (dN_0, dN_1)$ and that $dN_0 = -dN_1$, we can write $\langle P, dN\rangle = [P_0 - P_1]dN_0$. Applying this property gives the following two equations for $s = 0$ or $s = 1$:

$$\begin{aligned} dP_0 &= \left(\mu_0 P_{0D} D + \tfrac{1}{2}\sigma_0^2 P_{0DD} D^2 + h_0[P_1 - P_0]\right)dt + \sigma_0 P_{0D} D dW + [P_0 - P_1]dN_0 \\ dP_1 &= \left(\mu_1 P_{1D} D + \tfrac{1}{2}\sigma_1^2 P_{1DD} D^2 + h_1[P_0 - P_1]\right)dt + \sigma_1 P_{1D} D dW + [P_1 - P_0]dN_1 \end{aligned} \tag{D.6}$$

By taking expectations, we obtain

$$\begin{aligned} E(dP_0) &= \left(\mu_0 P_{0D} D + \tfrac{1}{2}\sigma_0^2 P_{0cc} D^2 + h_0[P_1 - P_0]\right)dt, \\ E(dP_1) &= \left(\mu_1 P_{1D} D + \tfrac{1}{2}\sigma_1^2 P_{1DD} D^2 + h_1[P_0 - P_1]\right)dt. \end{aligned}$$

(ii) Derivation of $E(\frac{dM(s)dP(D,s)}{M(s)})$.

To arrive at the final solution, we derive, for each regime an expression of the product of the product of $dM(s)dP(D, s)$.

$$\begin{aligned} \frac{dM_0}{M_0}dP_0 &= -\lambda_0\sigma_0 P_{0D}D\,dt - (\Gamma_0 - 1)[P_0 - P_1]dN_0^2 \\ \frac{dM_1}{M_1}dP_1 &= -\lambda_1\sigma_1 P_{1D}D\,dt - (\Gamma_1 - 1)[P_1 - P_0]dN_1^2 \end{aligned} \tag{D.7}$$

where $(dN)^2$=$(dN_0^2, dN_1^2)'$=$(h_0 dt, h_1 dt)'$. [30] Substituting this result in Eq. (D.7) yields:

$$\begin{aligned} E(\tfrac{dM_0 dP_0}{M_0}) &= -\lambda_0\sigma_0 P_{0D}D\,dt - (\Gamma_0 - 1)h_0[P_0 - P_1]dt, \\ E(\tfrac{dM_1 dP_1}{M_1}) &= -\lambda_1\sigma_1 P_{1D}D\,dt - (\Gamma_1 - 1)h_1[P_1 - P_0]dt. \end{aligned} \tag{D.8}$$

Plugging Eq. (D.8) in Eq. (D.4), viz.,

$$r_s^f P(D,s)dt = D\,dt + E\,dP(D,s) + E\frac{dM(s)dP(D,s)}{M(s)}. \tag{D.9}$$

gives us

$$\begin{aligned} r_0^f P_0 &= D + (\mu_0 - \lambda_0\sigma_0)P_{0D}D + \tfrac{1}{2}\sigma_0^2 P_{0DD}D^2 + h_0(\Gamma_0)[P_1 - P_0] \\ r_1^f P_1 &= D + (\mu_1 - \lambda_1\sigma_1)P_{1D}D + \tfrac{1}{2}\sigma_2^2 P_{1DD}D^2 + h_1(\Gamma_1)[P_0 - P_1] \end{aligned} \tag{D.10}$$

Appendix E Risk Premium and Switching Regimes

E.1 Switching Regimes

In this appendix, we show how to derive the excess holding returns when regime changes occur. These derivations hold when using a recursive or power utility function. [31] The only difference would be the exact expressions of r_i^f, Γ_i, λ_i, and other parameters

[30] An expression for $(dN)^2$ can be obtained using the results presented in Lemma 1.3 in Appendix B of Elliott, Aggoun, and Moore (1995):

$$(dN)^2 = diag(Hs)dt - diag(s)H'dt - H\,diag(s)dt.$$

where $diag(x)$ denotes the diagonal matrix with vector x on its diagonal. This expression simplifies to $(dN)^2 = diag((h_0 dt, h_1 dt)')$. Notice that the authors use a different notation from that of this paper, and states are defined as vectors, $s_0 = (1,0)'$ and $s_1 = (0,1)'$, which makes $(dN)^2$ a matrix and not a vector as in our paper, and where the relevant elements of our vector are the diagonal elements of this matrix.

[31] This is because the power utility is a particular case of the recursive utility function.

as functions of the deep parameters of the model: these will change depending on the specification of the utility function. For example, in the power utility case, $r_i^f = \delta + \gamma\mu_{ci} - \frac{1}{2}\gamma(\gamma+1)(\sigma_{ci})^2 - h_i(\Gamma_i - 1)$, whereas in the recursive utility case, r_i^f is given by Eq. (B.8).

We start by noting that allowing for regime shifts in the diffusion and risk premium parameters gives the following pricing equation:

$$\frac{dP(D,s)}{P(D,s)} + \frac{D\,dt}{P(D,s)} = -\frac{dM(s)}{M(s)} - \frac{dM(s)}{M(s)}\frac{dP(D,s)}{P(D,s)}. \tag{E.1}$$

Then, to derive an expression for the excess return we need to derive the results for the right-hand side terms of the above equation. These terms were shown to be:

$$\begin{aligned}
\frac{dM_0}{M_0} &= -r^f dt - \lambda_0 dW - (\Gamma_0 - 1)dN_0,\\
\frac{dM_1}{M_1} &= -r^f dt - \lambda_1 dW - (\Gamma_1 - 1)dN_1\\
\frac{dP_0}{P_0} &= \left(\mu_1 \frac{P_{0D}D}{P_0} + \frac{1}{2}\sigma_0^2 \frac{P_{0DD}D^2}{P_0} + h_0 \frac{P_1 - P_0}{P_0}\right)dt + \sigma_0 \frac{P_{0D}D}{P_0}dW + \frac{P_0 - P_1}{P_0}dN_0\\
\frac{dP_1}{P_1} &= \left(\mu_2 \frac{P_{1D}D}{P_1} + \frac{1}{2}\sigma_1^2 \frac{P_{1DD}D^2}{P_1} + h_1 \frac{P_0 - P_1}{P_1}\right)dt + \sigma_1 \frac{P_{1D}D}{P_1}dW + \frac{P_1 - P_0}{P_1}dN_1
\end{aligned}$$

By multiplying the growth in the stochastic discount factor by the growth in prices, we obtain the following:

$$\begin{aligned}
\frac{dM_0}{M_0}\frac{dP_0}{P_0} &= \left(-\lambda_0 dW - (\Gamma_0 - 1)dN_0\right)\left(\sigma_0 \frac{P_{0D}D}{P_0}dW + \frac{P_0 - P_1}{P_0}dN_0\right)\\
\frac{dM_1}{M_1}\frac{dP_1}{P_1} &= \left(-\lambda_1 dW - (\Gamma_1 - 1)dN_1\right)\left(\sigma_1 \frac{P_{1D}D}{P_1}dW + \frac{P_1 - P_0}{P_1}dN_1\right)
\end{aligned}$$

Substituting in Eq. (E.1) we obtain excess return equations. Notice that these equations are valid regardless of the existence of bubbles:

$$\begin{aligned}
\frac{dP_0(D)+Ddt}{P_0(D)}-r^f dt &= \lambda_0 dW+(\Gamma_0-1)dN_0+\Big(\lambda_0 dW+(\Gamma_0-1)dN_0\Big)\Big(\sigma_0\frac{P_{0D}D}{P_0}dW+\frac{P_0-P_1}{P_0}dN_0\Big)\\
\frac{dP_1(D)+Ddt}{P_1(D)}-r^f dt &= \lambda_1 dW+(\Gamma_1-1)dN_1+\Big(\lambda_1 dW+(\Gamma_1-1)dN_1\Big)\Big(\sigma_1\frac{P_{1D}D}{P_1}dW+\frac{P_1-P_0}{P_1}dN_1\Big),
\end{aligned}$$

which simplifies to the following expression:

$$\begin{aligned}
\frac{dP_0+Ddt}{P_0}-r^f dt &= \gamma\sigma_0^2\frac{P_{0D}D}{P_0}dt+(\Gamma_0-1)h_0\frac{P_0-P_1}{P_0}dt+\gamma\sigma_0 dW+(\Gamma_0-1)dN_0\\
\frac{dP_1+Ddt}{P_1}-r^f dt &= \gamma\sigma_1^2\frac{P_{1D}D}{P_1}dt+(\Gamma_1-1)h_1\frac{P_1-P_0}{P_1}dt+\gamma\sigma_1 dW+(\Gamma_1-1)dN_1
\end{aligned}$$

From these expressions, we can, substituting conveniently, derive the risk premium with or without bubbles simply using the appropriate price. For example, if stock prices only reflect fundamentals, then the state-dependent price equations and derivatives are: $P_0 = k_0D$, $P_{0D} = k_0$, $P_1 = k_1D$, $P_{1D} = k_1$. Note that $dN_0^2 = h_0dt$ and $dN_1^2 = h_1dt$ and $\lambda_i = \gamma\sigma_i$. This gives rise to the following expressions:

$$\begin{aligned}
\frac{P_{0D}D}{P_0} &= 1\\
\frac{P_{1D}D}{P_1} &= 1\\
\frac{P_0-P_1}{P_0} &= \frac{(k_0-k_1)}{k_0}\\
\frac{P_1-P_0}{P_1} &= \frac{(k_1-k_0)}{k_1}
\end{aligned}$$

Finally, we arrive at the final expressions for the excess holding returns:

$$\begin{aligned}\frac{dP_0 + D dt}{P_0} - r^f dt &= \left(\gamma\sigma_0^2 + h_0(\Gamma_0 - 1)\frac{k_0 - k_1}{k_0}\right)dt + \gamma\sigma_0 dW + (\Gamma_0 - 1)dN_0 \\ \frac{dP_1 + D dt}{P_1} - r^f dt &= \left(\gamma\sigma_1^2 + h_1(\Gamma_1 - 1)\frac{k_1 - k_0}{k_1}\right)dt + \gamma\sigma_1 dW + (\Gamma_1 - 1)dN_1\end{aligned}$$

Needless to say, if the economy is not subject to abrupt changes in regime, the above expressions collapse to:

$$\frac{dP + D dt}{P} - r^f dt = \gamma\sigma^2 dt + \gamma\sigma dW$$

To compute the excess holding returns when there are changes in regimes and intrinsic bubbles, we need to substitute in Eq. (B.13) the relevant equations that determine the evolution of stock prices, that is: $P_0 = k_0 D + a_{0,1}D^{\eta_1} + a_{0,2}D^{\eta_2}$, $P_1 = k_1 D + a_{1,1}D^{\eta_1} + a_{1,2}D^{\eta_2}$, $P_{0D} = k_0 + \eta_1 a_{0,1}D^{\eta_1 - 1} + \eta_2 a_{0,2}D^{\eta_2 - 1}$ and $P_{1D} = k_1 + \eta_1 a_{1,1}D^{\eta_1 - 1} + \eta_2 a_{1,2}D^{\eta_2 - 1}$. Using these expressions for P_0, P_1, P_{0D}, P_{1D} trivially gives the equations for $\frac{P_{0D}D}{P_0}$, $\frac{P_{1D}D}{P_1}$, $\frac{P_0 - P_1}{P_0}$ and $\frac{P_1 - P_0}{P_1}$:

$$\begin{aligned}\frac{P_{0D}D}{P_0} &= \frac{k_0 D + \eta_1 a_{0,1}D^{\eta_1} + \eta_2 a_{0,2}D^{\eta_2}}{k_0 D + a_{0,1}D^{\eta_1} + a_{0,2}D^{\eta_2}}, \\ \frac{P_{1D}D}{P_1} &= \frac{k_1 D + \eta_1 a_{1,1}D^{\eta_1} + \eta_2 a_{1,2}D^{\eta_2}}{k_1 D + a_{1,1}D^{\eta_1} + a_{1,2}D^{\eta_2}}, \\ \frac{P_0 - P_1}{P_0} &= \frac{(k_0 - k_1)D + (a_{0,1} - a_{1,1})D^{\eta_1} + (a_{0,2} - a_{1,2})D^{\eta_2}}{k_0 D + a_{0,1}D^{\eta_1} + a_{0,2}D^{\eta_2}}, \\ \frac{P_1 - P_0}{P_1} &= \frac{(k_1 - k_0)D + (a_{1,1} - a_{0,1})D^{\eta_1} + (a_{1,2} - a_{0,2})D^{\eta_2}}{k_1 D + a_{1,1}D^{\eta_1} + a_{1,2}D^{\eta_2}}.\end{aligned}$$

Substituting the above equations back into Eq. (13) yields the following excess holding returns:

$$
\begin{aligned}
\frac{dP_0 + D dt}{P_0} - r^f dt &= \Big(\frac{k_0 D + \eta_1 a_{0,1} D^{\eta_1} + \eta_2 a_{0,2} D^{\eta_2}}{k_0 D + a_{0,1} D^{\eta_1} + a_{0,2} D^{\eta_2}}\Big)\gamma\sigma_1^2 dt \\
&\quad + h_0(\Gamma_0 - 1)\frac{(k_0 - k_1)D + (a_{0,1} - a_{1,1})D^{\eta_1} + (a_{0,2} - a_{1,2})D^{\eta_2}}{k_0 D + a_{0,1} D^{\eta_1} + a_{0,2} D^{\eta_2}} dt \\
&\quad + \gamma\sigma_0 dW + (\Gamma_0 - 1) dN_0, \\
\frac{dP_1 + D dt}{P_1} - r^f dt &= \Big(\frac{k_1 D + \eta_1 a_{1,1} D^{\eta_1} + \eta_2 a_{1,2} D^{\eta_2}}{k_1 D + a_{1,1} D^{\eta_1} + a_{1,2} D^{\eta_2}}\Big)\gamma\sigma_1^2 dt \\
&\quad + h_1(\Gamma_1 - 1)\frac{(k_1 - k_0)D + (a_{1,1} - a_{0,1})D^{\eta_1} + (a_{1,2} - a_{0,2})D^{\eta_2}}{k_1 D + a_{1,1} D^{\eta_1} + a_{1,2} D^{\eta_2}} dt \\
&\quad + \gamma\sigma_1 dW + (\Gamma_1 - 1) dN_1.
\end{aligned}
$$

When there are no changes in the regime, these expressions collapse the single regime expression:

$$
\frac{dP + D dt}{P} - r^f dt = \Big(\frac{kD + \eta_1 a_1 D^{\eta_1}}{kD + a_1 D^{\eta_1}}\Big)\gamma\sigma^2 dt + \gamma\sigma dW.
$$

Appendix F Construction of the Likelihood for the General Model and estimation procedure

We estimate the regime-switching model using procedures that are identical to those described in Hamilton (1989, 1994), except that in this case, the price-dividend ratio, the dividend equation, the real interest rate, and the stock returns depend on the state. Also, note that k_0 and k_1 satisfy the system described in Equation (B). The solution of the system for ϕ_0 and ϕ_1, which is consistent with the theory, can only be solved numerically. The program calls a subroutine that solves the relevant equations numerically, so each line search is assured of satisfying the conditions imposed by the model. We write the density of the data y_t conditional on the state s and the history of the system as

$$
\begin{aligned}
P(y_t|s_t, y_{t-1}, ..., y_1) \quad = \quad & \frac{1}{(2\pi)^{.5}\sigma(s_t)} \exp\left(-(\sigma_s^2)^{-1}\left(\Delta\log(D_t) - \left[\mu(s_t) - \frac{\sigma^2(s_t)}{2}\right]\right)^2\right) \\
& \times \frac{1}{(2\pi)^{.5}\sigma_a(s_t)} \exp\left(-(\sigma_a(s_t)^2)^{-1}\left(\frac{P_t}{D_t} - \left[k_{s_t} + c_{s_t} D^{\lambda-1}\right]\right)^2\right) \\
& \times \frac{1}{(2\pi)^{.5}\sigma_b(s_t)} \exp\left(-(\sigma_b(s_t)^2)^{-1}\left(r_t^S - R(D_t, s_t)\right)^2\right) \\
& \times \frac{1}{(2\pi)^{.5}\sigma_c(s_t)} \exp\left(-(\sigma_c(s_t)^2)^{-1}\left(r_t^F - r^f(s_t)\right)^2\right),
\end{aligned}
$$

where y_t is a 5×1 vector containing $\Delta\log(D_t)$, the rate of growth of dividends; $\frac{P_t}{D_t}$, the observed price dividend ratio; D, real dividends; r_t^F , the ex-ante real interest rate; and r_t^S, the observed cum-dividends stock returns. The likelihood is maximized with respect to $(\delta,\ \phi,\ \gamma,\ p,\ q,\ \mu_0,\ \mu_1,\ \sigma_0,\ \sigma_1,\ \sigma_{a,0},\ \sigma_{a,1},\ \mathbf{a}_{1,0},\ \mathbf{a}_{1,1},\ \mathbf{a}_{2,0},\ \mathbf{a}_{2,1},\ \sigma_{b,0},\ \sigma_{b,1},\ \sigma_{c,0},\ \sigma_{c,1},\ z_0,\ z_1)$

Note that the system to be solved is homogeneous of degree one in (z_0, z_1). Thus, we normalize $z_0 = 1$ and leave z_1 as a free parameter to be estimated. Then, using that $\Gamma_i = \frac{z_j^{1-\psi}}{z_i^{1-\psi}} \frac{\phi_j^{\psi-\gamma}}{\phi_i^{\psi-\gamma}}$ (and that we set one z_j equal one), we can reparameterize the model in terms of Γ_i, since there is a one-to-one mapping between the value of Γ_i and the value of z_i given all other parameters.

The filter performs the following calculations in each line search of the numerical optimization algorithm (given parameter values).

1. Compute h_0, h_1 using that $h_0 = \dfrac{q}{1-q}$ and $h_1 = \dfrac{p}{1-p}$.

2. Obtain the values of ϕ_0 and ϕ_1 which solve

$$
\frac{\delta}{1-\psi}\left[z_i^{1-\psi}\phi_i^{\psi-\gamma} - \phi_i^{1-\gamma}\right] + h_i\left[\frac{\phi_j^{1-\gamma} - \phi_i^{1-\gamma}}{1-\gamma}\right] + \phi_i^{1-\gamma}\left[\mu_i - \tfrac{1}{2}\sigma_i^2\gamma\right] = 0, \quad i = 0, 1, j \neq i
$$

numerically. That gives (ϕ_0, ϕ_1) as a function of the remaining parameters. Use those values to compute Γ_i using Eq. (7a).

3. Using the values for h_i and Γ_i obtained in previous steps alongside the remaining parameters, we compute the four roots of Eq. (B.17). Keep only the two positive roots.

4. Using Eq. (B), obtain (k_0, k_1). Use Eq. (7b) to compute $r^f(s_t)$ and Eq. (13)) to obtain the function $R(D, s)$.

Due to numerical issues, we obtain the maximum in a multi-step procedure: i) Define a dense grid on δ. We set, in annualized terms, $\delta \in [0.001, 0.05]$ with 0.001 increments; ii) for each fixed value of δ on the grid, maximize over all other parameters and obtain the maximum; iii) estimate of δ corresponds to the value that attains the maximum within the grid, and the estimates of other parameters correspond to maximizers for this given value of δ.