

Liquidity Provision and Financial Stability

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Abstract

When financial intermediaries' key characteristic is provision of liquidity through their liabilities, with financial frictions the financial sector in the aggregate is likely to over-accumulate equity, thus decreasing liquidity provision and household welfare. Aggregate household welfare is therefore decreasing in the level of aggregate intermediary equity even though the individual value of intermediaries is increasing in equity, which is why intermediaries over-accumulate equity. Subsidizing intermediary dividends can improve welfare by encouraging earlier payout and decreasing aggregate equity in the financial sector. This policy increases the likelihood that intermediaries provide more liquidity and improves the stability of the economy, even though asset prices fall.

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1 INTRODUCTION

Banks and other financial intermediaries produce short-term, money-like liabilities that provide liquidity services—notably, but not exclusively, deposits, as many non-bank financial intermediaries produce money-like liabilities. In the language of [Dang et al. \(2017\)](#), the output of banks is *debt*, not loans or investments held as assets. The convenience yield on short-term debt allows intermediaries to cheaply finance long-maturity, illiquid assets, allowing them to earn higher net returns and pay dividends to their equity investors (i.e., profit maximization). However, financial frictions create maturity and liquidity mismatch in intermediaries' balance sheets, which makes liquidity transformation a source of financial instability.

Standard macro-finance models (e.g. [Bernanke and Gertler, 1989](#); [Kiyotaki and Moore, 1997](#); [Bernanke, Gertler and Gilchrist, 1999](#); [Brunnermeier and Sannikov, 2014](#)) suppose that intermediaries play crucial roles in investment and capital allocation, and so output is linked to the level of equity in the financial sector. Conventional wisdom argues that larger equity buffers and policies to recapitalize financial intermediaries will improve welfare by reducing financial instability. Policy makers and economists have generally advocated for measures that ensure banks and other intermediaries have larger equity buffers so that they are better prepared to weather adverse shocks.

While the case for capitalization is strong, there are additional issues for policy to consider. While bank failures have real consequences ([Ashcraft, 2005](#)), the extent to which banks boost output more generally is debatable. In a cross-sectional study of U.S. commercial banks, [Egan, Lewellen and Sunderam \(2021\)](#) find that the majority of the variation in banks' market-to-book

ratios stemmed from their liquidity services, not from the productivity of their assets. In other words, banks add value by producing safe-assets via liabilities, not by screening and monitoring loans, suggesting that many standard macro-finance models over-emphasize output gains from intermediation and do not pay enough attention to the liquidity services given by banks. We show that the conventional wisdom for increased bank equity is not always true when intermediation activities do not generate significant gains in real output. If the primary role of banks and other financial intermediaries is liquidity provision, then the financial sector *in the aggregate* over-accumulates equity. Our paper presents a benchmark case in which the economy can indeed have too little debt/leverage, and that equity issuance on its own does not imply more debt/leverage is bad unless coupled with significant productivity gains from intermediation.

We develop a continuous-time stochastic general equilibrium model in which the sole purpose of financial intermediaries (“banks” going forward) is liquidity provision. Households demand deposits because bank debt provides them liquidity value. Banks use deposits to hold a risky asset (a Lucas tree). In the baseline model, banks have no special expertise in managing assets compared to households, but the liquidity value of bank debt provides a financing advantage. The level of output is the same regardless of whether banks or households hold the assets (output is produced by a tree). While banks can issue debt freely, equity issuance is costly, so banks are concerned about solvency. As a result, banks earn time-varying risk premia as compensation for providing households liquidity. When bank equity levels are higher, banks require lower risk premia and invest more in risky assets.

We first consider an analytical version of this model that borrows from [Caballero and Sim-](#)

sek (2019). In this simpler set up, we prove that household welfare can decrease with aggregate bank equity—i.e., in equilibrium households are better off when banks have less equity, not more. Because banks hold equity as a buffer against bankruptcy and can freely pay dividends, banks' value function is increasing in their individual equity level, which is why the economy can over-accumulate equity. Consequently, the marginal *social* value of aggregate bank equity can be negative even when the marginal *private* value of bank equity is positive. In equilibrium, banks are too concerned about their solvency constraints and over-accumulate equity, trading off too much of their liquidity services for greater financial stability.

The intuition for this result is as follows. The aggregate consequences of bank equity depend on the cyclicity of aggregate leverage and asset prices, but banks do not internalize the dynamics in asset prices and the banking sector that their collective decisions induce. If for example bank leverage and asset prices were both increasing in bank equity, then household welfare would also be increasing in aggregate equity: liquidity provision would be greatest when bank equity is high. However, the aggregate provision of liquidity depends on the behavior of asset prices whenever leverage increases after losses (see He, Kelly and Manela, 2017, for empirical evidence of countercyclical leverage). If asset prices do not fall too quickly, then liquidity provision decreases with bank equity. How much asset prices fall when the financial sector suffers losses depends on how well intermediaries can absorb the losses and whether financial assets get sold outside of the financial sector.

At the same time, when banks accumulate equity, all else equal, they borrow less to hold the same portfolio, which decreases liquidity provision. Because financial intermediation does not

generate gains in output, flow utility falls, though potentially with the benefit of improved stability (a dynamic gain). These two effects coupled together are sufficiently powerful that even when bank are well-capitalized, household welfare may be lower than if they had held the risky assets themselves.

Our results extend to a robust environment with risk-averse households in which aggregate risk follows a Brownian motion. This setting features long-run instability and must be solved numerically using global solution methods. Like in the analytical setup, higher bank equity levels can decrease welfare for the same reasons discussed previously, even though equilibrium features endogenous asset price volatility and long-run deviations from the stochastic steady state. Our results continue to hold even when intermediation by banks provide some productivity gains as long as fire sales are not too severe. Indeed, the general equilibrium consequences of incorporating productivity losses from fire sales can strengthen our results, suggesting that depressed liquidity provision, not capital misallocation, is likely to be the greater source of inefficiency when banks are distressed.

Using this numerical model, we also consider the implications for policy. Encouraging dividend payouts *in good times* is welfare-improving for two reasons. First, by discouraging equity accumulation, the policy encourages liquidity provision in good times. Second, and perhaps surprisingly, a policy of discouraging equity accumulation *in good times* decreases endogenous risk in the economy, which decreases the severity of fire sales. Subsidizing dividends in good times need not come at the cost of greater instability.

Related Literature. The paper most closely related to ours is [Gorton and Winton \(2017\)](#). They also point out that there exists a trade off between maintaining safe levels of equity and liquidity provision. Because lending activity produces valuable information, there are social costs when banks default. To reduce the probability of default, banks need to hold more capital. However, higher levels of equity mechanically imply lower levels of debt, which decreases the supply of liquid assets. In general equilibrium, the marginal costs of larger equity buffers outweigh the marginal benefits. Our analysis differs from theirs in two ways. First, we take the liquidity value of debt as given and instead focus on asset pricing dynamics when banks face time-varying risk aversion due to limited equity issuance. Second, we demonstrate that our model's results hold within a numerical model that features long-run financial instability.

[Moreira and Savov \(2017\)](#) studies liquidity provision within a continuous-time environment focusing on the relationship between macroeconomic instability and the transformation of risky assets into liquid securities by shadow banks. In contrast, we take the liquidity value of bank debt as given and study the social trade off between bank capitalization and liquidity provision in the presence of limited equity issuance. The numerical extension we develop borrows from [Phelan \(2016\)](#) but differs in one notable aspect: in [Phelan \(2016\)](#) there exists a trade off between financial stability and economic growth because financial intermediation leads to more efficient investment, while in the baseline model of our paper banks only provide liquidity value. This difference explains why social welfare increases in the level of bank equity in [Phelan \(2016\)](#) but can decrease in our paper.

[Diamond \(1984\)](#) suggests that banks increase investment through their role in screening and monitoring risky and illiquid projects. Banks can also generate economic value without directly affecting real activity. [Diamond and Dybvig \(1983\)](#) and [Gorton and Pennacchi \(1990\)](#) show that households demand bank deposits because deposits provide liquidity.

Finally, our analysis contributes to the growing continuous-time macrofinance literature. [Brunnermeier and Sannikov \(2014\)](#) and [He and Krishnamurthy \(2012, 2013, 2019a\)](#) demonstrate the ability of globally solved nonlinear continuous-time models to explain financial crises and to evaluate policies that target financial stability. [Caballero and Simsek \(2020, 2019\)](#) extend these tools so that they are tractable enough for use in analytical models. Additional work in this field include [Adrian and Boyarchenko \(2012\)](#), [Maggiore \(2017\)](#), [Di Tella \(2017\)](#), and [Van der Gucht \(2019\)](#).

2 ANALYTICAL MODEL WITH TWO STATES

The economy is populated by households and banks, which are owned by households. The only asset is a Lucas tree. Bank deposits earn a liquidity yield, and thus banks have an advantage for financing the tree. The financial friction is costly equity issuance. Our analytical model borrows from [Phelan \(2016\)](#) and [Caballero and Simsek \(2019\)](#). Omitted proofs are in Appendix A.

2.1 *Setup*

Technology, Environment, and Markets Time is continuous and infinite. There are two states in the model, $s \in \{1, 2\}$. Output in the economy is determined by a tree with a stochastic dividend. Since the tree is the only source of output in the economy, we refer to it directly as “the risky asset.”

The dividend (output) at time t in state s , $Y_{t,s}$, evolves according to

$$\frac{\dot{Y}_{t,s}}{Y_{t,s}} = g_s. \quad (1)$$

where $\dot{Y}_{t,s}$ denotes the time derivative of $Y_{t,s}$ and g_s is the state-dependent growth rate. Let λ_s denote the probability of a transition away from state s . The economy starts in state $s = 1$, and a transition to state $s = 2$ occurs according to a Poisson process with intensity λ . Once the economy transitions to state $s = 2$, the economy remains in state $s = 2$. Thus, $\lambda_1 = \lambda$ and $\lambda_2 = 0$.

We now state our first assumption on parameters.

Assumption 1. *The growth rate in state $s = 1$ is greater than in state $s = 2$, i.e. $g_1 > g_2$.*

We model the dynamics of the dividend process in this way to obtain as simple a model as possible. Downside risk needs to exist for the costly equity issuance constraint to affect equilibrium outcomes. However, upside risk is unnecessary to obtain our main results and is isomorphic to the current set up with only downside risk. See Appendix D for details. Alternatively, state $s = 1$ can be interpreted as an initial “boom” state that is followed by a transition to “normal” growth.

We postulate that the price of the tree per dividend $q_{t,s}$ follows

$$\frac{\dot{q}_{t,s}}{q_{t,s}} = \mu_{q,t,s}. \quad (2)$$

Any stochasticity in $q_{t,s}$ arises from the state transition, which causes a jump in the price of $q_{t,s}$. Otherwise, the path of $q_{t,s}$ is deterministic. We will refer to $q_{t,s}$ as the asset price. Since there is no shock to the tree’s productivity except after a state transition, the capital gains process on the tree

per dividend is just

$$\frac{d(q_{t,s}y_{t,s})/dt}{q_{t,s}y_{t,s}} = g_s + \frac{\dot{q}_{t,s}}{q_{t,s}}. \quad (3)$$

Therefore, the return on the tree is

$$dr_{y,t,s} = \frac{1}{q_{t,s}} + g_s + \frac{\dot{q}_{t,s}}{q_{t,s}}. \quad (4)$$

Finally, we assume that markets are incomplete. There is only a market for the tree, which cannot be shorted, and a market for risk-free bonds in zero net supply. If banks could insure against the transition from state $s = 1$ to state $s = 2$, then their portfolio choice would not be affected by financial frictions.

Households There is a continuum of risk-neutral households denoted by $h \in [0, 1]$ with initial wealths w_0^h . Households have a discount rate r , may consume both positive and negative amounts (though in equilibrium their consumption will always be positive), and have “liquidity-in-the-utility-function” with constant marginal utility over bank deposits. Lifetime utility is given by equation (5)

$$V_{t,s} = \mathbb{E}_{t,s} \left[\int_t^\infty e^{-r(\tau-t)} \left(c_{\tau,s}^h + \phi_L \delta_{\tau,s}^h \right) d\tau \right], \quad (5)$$

where r is the discount rate, $c_{t,s}^h$ is flow consumption, $\delta_{t,s}^h$ are bank deposits, and $\phi_L > 0$ is the liquidity preference. It follows that the expected return on any asset that households own (whether the tree or bank equity) must be r , and the expected the return on deposits must be $r - \phi_L$, owing

to the liquidity value.

We model the liquidity value of bank deposits directly in the utility function as a modeling convenience. Deposits have liquidity values for a variety of reasons outside of the model, which we leave out to simplify exposition (see for example [Diamond and Dybvig \(1983\)](#), [Gorton and Pennacchi \(1990\)](#), or [Lagos and Wright \(2005\)](#)). We impose constant marginal utility also to simplify the exposition. In Appendix A, we prove that all our results are robust to a setting in which ϕ_L is an exogenous function of the aggregate state variable. Since the behavior of deposit rates is of first-order empirical importance ([Acharya and Mora, 2015](#); [Drechsler, Savov and Schnabl, 2017](#)), in Section 3, we further consider a more flexible specification with risk aversion and concave utility for deposits that produces pro-cyclical deposit spreads (countercyclical deposit rates). Our main results continue to hold in that environment.

Banks There is a continuum of banks, denoted by $b \in [0, 1]$, with initial book value (“equity”) n_0^b . Banks invest in the risky asset and issue deposits. Banks are owned by households, who choose portfolio holdings, dividend payouts, and equity issuance. The balance sheet of a bank at time t is $y_{t,s}^b q_{t,s} = \delta_{t,s}^b + n_{t,s}^b$, where $y_{t,s}^b$ is holdings of the risky asset and $\delta_{t,s}^b$ is deposits (debt). Because of unmodeled financial frictions, banks are subject to two constraints. First, banks cannot issue equity.¹ Second, the value of banks’ assets minus liabilities $n_{t,s}^b$ cannot become negative (bankruptcy).

Banks’ objective is to maximize the present value of dividends discounted at rate r (households’ time preference) subject to their constraints. Because banks can borrow using debt at an interest rate $r_d = r - \phi_L < r$, banks will never choose a capital structure that is completely equity. And because banks cannot have negative equity, to avoid bankruptcy banks will never choose a capital

structure that is completely debt.

We interpret banks in our model as representing financial intermediaries beyond simply deposit-taking commercial banks but also capturing other bank-like financial intermediaries whose liabilities provide money-like services. Hence, the “banking sector” in our model is meant to capture this portion of the financial sector as a whole. Empirically, [He, Kelly and Manela \(2017\)](#) find that the main buyers of asset sales from investment banks and hedge funds are commercial banks. In our model, we would interpret such trades as trades between banks, not between banks and households.

Portfolio Choice By homogeneity and price-taking, the maximized value of a bank with equity n_t^b can be written as

$$\theta_{t,s} n_{t,s}^b \equiv \max_{\{y_{t,s} \geq 0, d\zeta_{t,s}^b \geq 0\}} \mathbb{E}_{t,s} \left[\int_t^\infty e^{-r(\tau-t)} d\zeta_{\tau,s}^b \right], \quad (6)$$

where $\theta_{t,s}$ is the marginal value of equity, i.e., the proportionality coefficient that summarizes how market conditions affect the value of the bank’s value function per dollar of equity. The marginal value of bank equity $\theta_{t,s}$ can never be less than one because banks can always pay out the full value of equity, guaranteeing a value of at least $n_{t,s}^b$. Banks choose $y_{t,s}$ shares in the tree and issue dividends at rate $d\zeta_{t,s}^b$ subject to the transition equations

$$dn_{t,s}^b = (n_{t,s}^b r_d + q_{t,s} y_{t,s} (dr_{y,t,s} - r_d) dt - d\zeta_{t,s}^b, \quad (7)$$

$$n_{t,s'}^b = n_{t,s}^b + y_{t,s} (q_{t,s'} - q_{t,s}). \quad (8)$$

Equation (7) describes the dynamics of a bank’s equity between state transitions. A bank earns the deposit rate plus the excess returns from the risky asset minus any dividends paid. Equation (8)

describes how a bank's equity changes after a transition from state s to s' . Given holdings $y_{t,s}$ of the tree, the change in a bank's equity is proportional to the change in the price of the risky asset.

Using this representation, we can formulate a bank's portfolio choice problem as a Hamilton-Jacobi-Bellman (HJB) equation, which yields the following proposition.

Proposition 1. *A bank prices shares of the tree according to*

$$dr_{y,t,s} - r_d = -\lambda_s \frac{\theta_{t,s'}}{\theta_{t,s}} \frac{q_{t,s'} - q_{t,s}}{q_{t,s}}, \quad (9)$$

and issues dividends only when $\theta_{t,s} = 1$. Otherwise, the bank pays zero dividends.

Households price shares of the tree according to

$$\frac{1}{q_{t,s}} + g_s + \frac{\dot{q}_{t,s}}{q_{t,s}} - r = 0. \quad (10)$$

Parameter Restrictions To ensure a sensible equilibrium, we make the following assumptions on parameters.

Assumption 2. *The discount rate and growth rate in state $s = 1$ satisfy $r > g_1$.*

Assumption 3. *The liquidity value of deposits in state $s = 1$ satisfies $\phi_L < \min\{r - g_1, \lambda\}$.*

Assumption 4. *The discount rate and probability of transition in state $s = 1$ satisfy $r > \lambda$.*

Assumption 5. *The discount rate, liquidity value of deposits, growth rates of dividends in state*

$s \in \{1, 2\}$, and probability of transition in state $s = 1$ satisfy

$$g_1 > \frac{\phi_L}{\lambda} r + \left(1 - \frac{\phi_L}{\lambda}\right) (g_2 + \phi_L).$$

Assumptions 2, 3, and 4 guarantee that the tree has a positive price in both states. Assumptions 4 and 5 guarantee that we have a nontrivial equilibrium where banks are not always sufficiently capitalized in state $s = 1$. Assumptions 2 and 5 are not incompatible because Assumption 3 implies $\phi_L/\lambda < 1$, hence we may increase r faster than g_1 to achieve Assumption 2 while maintaining the inequality in Assumption 5. Finally, Assumptions 1, 3, and 5 imply $g_1 > g_2 + \phi_L$.

2.2 Equilibrium

A competitive equilibrium is characterized by the market price for the tree, together with portfolio allocations and consumption decisions such that given prices, agents optimize and markets clear. The steps to solve equilibrium are similar to those found in Caballero and Simsek (2020, 2019).

Because banks have a limited ability to issue equity, their decisions will depend on their level of equity, hence equilibrium depends on banks' aggregate level of equity. Define $N_{t,s} = \int n_{t,s}^b db$ as aggregate bank equity. Since the tree dividend grows geometrically and the bank problem is homogeneous, the equilibrium state variable is aggregate bank equity as a fraction of the total value of the tree, or a variant of the "wealth distribution." This variable is $\eta_{t,s} \equiv N_{t,s}/(q_{t,s}Y_{t,s})$.

Proposition 2. *The equilibrium law of motion for $\eta_{t,s}$ will be endogenously given as*

$$\frac{\dot{\eta}_{t,s}}{\eta_{t,s}} = \frac{1}{q_{t,s}} - \lambda_s \frac{\theta_{t,s'} q_{t,s'}}{\theta_{t,s} q_{t,s}} \left(\frac{\eta_{t,s'}}{\eta_{t,s}} - 1 \right) - d\Xi_{t,s}, \quad (11)$$

where $d\Xi_{t,s}$ is an impulse variable creating a reflecting barrier. Furthermore, $d\Xi_{t,s} = d\zeta_{t,s}/N_{t,s}$, where $d\Xi_{t,s} = \int d\zeta_{t,s}^b db$.

By construction, our analytical model can be treated as a two-period model. We characterize equilibrium in both states by solving backwards in time.

State $s = 2$. The economy is deterministic in this state because there are no future state transitions. Since banks can borrow at the cheaper deposit rate r_d and face no risk, they finance themselves entirely with debt and own the entire tree. As a consequence, $\theta_{t,2} = 1$ and $n_{t,2} = 0$. By (9), the per-dividend price of the tree is

$$q_{t,2} = q_2 = \frac{1}{r_d - g_2}. \quad (12)$$

State $s = 1$. The economy may transition into state $s = 2$ with positive probability. As a result, the economy features nontrivial dynamics. We first characterize the economy as a system of three autonomous differential equations, which will serve as the building blocks for later derivations.

Lemma 1. *The marginal value of bank equity and banks' share of aggregate wealth follow*

$$\dot{\theta}_{t,1} = (\lambda + \phi_L - r_d)\theta_{t,1} - 2\lambda + \theta_{t,1}d\Xi_{t,1}, \quad (13)$$

$$\dot{\eta}_{t,1} = \left(\frac{1}{q_{t,1}} + \lambda \frac{q_2}{\theta_{t,1}q_{t,1}} - d\Xi_{t,1} \right) \eta_{t,1}. \quad (14)$$

Suppose banks hold positive quantities of the tree. Then the asset price follows

$$\dot{q}_{t,1} = \left(r_d - g_1 + \frac{\lambda}{\theta_{t,1}} \right) q_{t,1} - 1 - \frac{\lambda}{\theta_{t,1}} q_2. \quad (15)$$

If households also hold positive quantities of the tree, then $q_{t,1}$ is given by

$$q_{t,1} = \frac{\lambda}{(r_d - g_2)(\lambda - \phi_L \theta_{t,1})}. \quad (16)$$

Finally, if households own the entire tree, then $\eta_{t,1} = 0$, and $q_{t,1} = 1/(r - g_1)$.

Lemma 1 allows us to immediately prove two corollaries.

Corollary 1. If $\theta_{t,1} > 1$, then the share of aggregate bank wealth, $\eta_{t,1}$, increases with time.

Corollary 2. Suppose one of the two following conditions is true:

$$(i) \quad r > \lambda + 2\phi_L,$$

$$(ii) \quad \theta_{t,1} < \lambda/\phi_L.$$

Then the marginal value of banks' equity, $\theta_{t,1}$, strictly decreases with time until $\theta_{t,1} = 1$.

These two corollaries provide a rough characterization of equilibrium dynamics in state $s = 1$. Contingent on no state transition occurring, as time passes, banks accumulate equity ($\dot{\eta}_{t,1} > 0$), and the marginal value of their equity accordingly decreases. When we prove that $\theta_{t,1}$ is a function of $\eta_{t,1}$, we show that Corollary 2(ii) is always satisfied across the state space. The key reason why banks will not always pay dividends in state $s = 1$ (or equivalently $\theta_{t,1} \equiv 1$) is that the asset price

decreases after a state transition. When equity is very low, the decline in asset prices can trigger bankruptcy. To avoid this outcome, banks retain their equity until they are sufficiently capitalized.

Using Lemma 1, we can show that $\eta_{t,1}$ is indeed the single state variable required to characterize equilibrium.

Proposition 3. *There exists an equilibrium in which the marginal value of banks' equity and the per-dividend price of the tree in state $s = 1$ can be written as functions of banks' share of aggregate wealth, $\theta_{t,1} = \theta_1(\eta_{t,1})$ and $q_{t,1} = q_1(\eta_{t,1})$. The state space is the interval $[0, \bar{\eta}_1]$, where $\bar{\eta}_1$ is endogenously determined, banks own the entire tree at $\bar{\eta}_1$, and the boundary conditions of this equilibrium are*

(i) When $\eta_1 = 0$,

$$\theta_1(0) = \frac{\lambda}{\phi_L} \left(\frac{g_1 - g_2 - \phi_L}{r - g_2 - \phi_L} \right),$$

$$q_1(0) = \frac{1}{r - g_1}.$$

(ii) When $\eta_1 = \bar{\eta}_1$,

$$\theta_1(\bar{\eta}_1) = 1,$$

$$q_1(\bar{\eta}_1) = \frac{1 + \lambda q_2}{r - \phi_L - g_1 + \lambda}$$

$$\theta'_1(\bar{\eta}_1) = 0,$$

$$q'_1(\bar{\eta}_1) = 0.$$

We sketch the main ideas of the proof here. First, we prove that the asset price tree is a function of θ_1 , $q_{t,1} = q_1(\theta_{t,1})$. We then apply this mapping to show that $\theta_{t,1}$ can be written as a function of $\eta_{t,1}$. Finally, we utilize the boundary conditions to construct a path in time which coincides with the desired equilibrium and thereby pins down $\bar{\eta}_1$.

Corollary 3. *Over the state space $[0, \bar{\eta}_1)$, θ_1 is a decreasing function of η_1 . When households hold a positive fraction of the tree, q_1 is a decreasing function of η_1 .*

Proof. The first claim follows from the fact that $\dot{\theta}_1 < 0$ and $\dot{\eta}_1 > 0$ over the state space $[0, \bar{\eta}_1)$. The second claim follows from the fact that q_1 is an increasing function of θ_1 when households hold a positive fraction of the tree. $\square$

Allocations of Claims to the Tree's Dividends. The allocation of the tree can be characterized in terms of $\eta_{t,s}$ and the asset price. Let $\psi_{t,s} = \int y_{t,s}^b db$ be the fraction of the tree owned by banks. It is immediate from our analysis of state $s = 2$ that $\psi_{t,2} \equiv 1$. The following proposition characterizes leverage in state $s = 1$.

Proposition 4. *The fraction of the tree owned by banks in state $s = 1$ is given by*

$$\psi_{t,1} = \min \left\{ \eta_{t,1} \left(\frac{q_{t,1}}{q_{t,1} - q_2} \right), 1 \right\}. \quad (17)$$

We obtain a fixed point for $\bar{\eta}_1$ as a corollary:

Corollary 4. *The upper bound of the state space $\bar{\eta}_1$ is given by*

$$\bar{\eta}_1 = 1 - \frac{q_2}{q_1(\bar{\eta}_1)}, \quad (18)$$

and it is strictly positive but less than 1.

2.3 Welfare Analysis

Household welfare corresponds to the equilibrium expected present discounted value of flow utility over consumption and liquidity given the current tree dividend $Y_{t,s}$ and bank capitalization $\eta_{t,s}$. Since banks are owned by households (they are not competing agents), banks' payoffs are not included in welfare calculations. When households receive dividends from banks, households reinvest in assets or in deposits to earn the liquidity premium. By market clearing, aggregate consumption is $C_{t,s} = Y_{t,s}$ and aggregate liquidity is $D_{t,s} = (\psi_{t,s} - \eta_{t,s})q_{t,s}Y_{t,s}$. Thus, household welfare is given by

$$V_s(\eta_{t,s}, Y_{t,s}) = \mathbb{E}_t \left[\int_t^\infty e^{-r(\tau-t)} (Y_{\tau,s} + \phi_L(\psi_{\tau,s} - \eta_{\tau,s})q_{\tau,s}Y_{\tau,s}) d\tau \right].$$

Because households are risk neutral and their investments earn expected return r and $r - \phi_L$ for deposits, expected discounted utility is equal to wealth. Household wealth includes the debt and equity invested in banks and any tree they own. Their tree holdings are worth $(1 - \psi_{t,s})q_{t,s}Y_{t,s}$, their debt is worth $\psi_{t,s}q_{t,s}Y_{t,s} - N_{t,s}$, and bank shares are worth $\theta_{t,s}N_{t,s}$. The expected value of dividends from banks is $\theta(\eta_{t,s})N_{t,s} = \theta(\eta_{t,s})q(\eta_{t,s})\eta_{t,s}Y_{t,s}$. The total wealth is thus

$$V_{t,s} = q_{t,s}Y_{t,s} + (\theta_{t,s} - 1)N_{t,s} = (1 + (\theta_{t,s} - 1)\eta_{t,s})q_{t,s}Y_{t,s}.$$

Thus, household welfare per unit of dividend $Y_{t,s}$ is given by

$$J_s(\eta_{t,s}) = (1 + (\theta_{t,s} - 1) \eta_{t,s}) q_{t,s}.$$

In the absence of financial frictions, welfare equals $\frac{1}{r - \phi_L - g_s} = \frac{1}{r_d - g_s}$, which is also the asset price. Note that total wealth exceeds $q_t Y_t$ whenever $\theta_t > 1$. In other words, total wealth in the economy exceeds the market value of capital. This is precisely the consequence of financial frictions: banks' value exceed the market value of their capital, due to intermediation and financial frictions. Without financial frictions, $\theta_t = 1$ always; financial frictions make possible $\theta_t > 1$, which creates a distortion between the market value of capital and the value of bank holdings. Variations in θ_t is precisely why policy can increase welfare with financial frictions. In state $s = 2$, banks effectively do not face any financial frictions, hence

$$J_{t,2} = q_2 = \frac{1}{r - \phi_L - g_2}.$$

While bank equity does not matter for welfare in state $s = 2$, it does matter in state $s = 1$.

Welfare Result 1 Our first result shows that welfare can decrease with respect to η when households hold a positive fraction of the tree.

Lemma 2. Suppose $\psi_{t,1} \in (0, 1)$. If

$$r > \frac{1}{\phi_L}(\lambda^2 + (\lambda - \phi_L)(\lambda - \phi_L - g_2)), \quad (19)$$

then the time derivative at time t of $(\theta_{t,1} - 1)\eta_{t,1}$ is negative.

Proposition 5. *If $\psi_1(\eta_1) < 1$ and (19) is true, then household welfare per dividend $J_1(\eta_1)$ decreases with η_1 .*

Proof. Across the state space $[0, \bar{\eta}_1]$, $\theta_{t,1} < \lambda/\phi_L$, so Corollary 2 implies $\theta_{t,1}$ decreases with time, hence $q_{t,1}$ also has a negative time derivative by (16). By Lemma 2, the terms inside the parentheses have negative time derivatives. Recall that $J_{t,1} = q_{t,1}(1 + (\theta_{t,1} - 1)\eta_{t,1})$. Since $q_{t,1}$ and $(\theta_{t,1} - 1)\eta_{t,1}$ are both decreasing with time, $J_{t,1}$ must also have a negative time derivative. Finally, Corollary 1 implies

$$\frac{dJ_{t,1}}{d\eta_{t,1}} = \frac{\dot{J}_{t,1}}{\dot{\eta}_{t,1}} < 0$$

when $\psi_{t,1} < 1$. □

Lemma 2 is true as long as r is sufficiently large. The only constraint on the size of r is given by Assumption 5, but because the coefficient on r is $\phi_L < \lambda$, we can increase r and g_1 as much as needed to guarantee Assumption 5 and Proposition 5 jointly hold.

Welfare Result 2 Our second result shows that welfare may be lower even if banks have healthy levels of equity.

Proposition 6. *If the parameters satisfy*

$$\phi_L^2 - \phi_L(r - g_2 + \lambda) + \lambda(g_1 - g_2) > 0, \tag{20}$$

then $q_1(\bar{\eta}_1) < q_1(0)$, hence household welfare is higher when banks have no equity than when banks are fully capitalized.

The next proposition states conditions on parameters under which Proposition 6 is true.

Proposition 7. *If g_1 satisfies*

$$g_1 \leq \frac{(r - g_2)^2}{4\lambda} + \frac{1}{2}(r + g_2) + \frac{\lambda}{4}, \quad (21)$$

and if $\phi_L \in (0, \bar{\phi}_L)$, where

$$\bar{\phi}_L = \min \left\{ \frac{r - g_2 + \lambda - \sqrt{(r - g_2 + \lambda)^2 - 4\lambda(g_1 - g_2)}}{2}, \lambda \right\},$$

then (20) holds.

If g_1 is instead greater than the RHS of (21), then for all $\phi_L \in (0, \lambda)$, (20) holds.

Welfare Result 3 Our third result shows that a dividend subsidy can locally improve welfare.

The equilibrium effect of a dividend subsidy is an increase in $\theta_1(\bar{\eta}_1)$ because the subsidy makes it optimal for banks to pay dividends when the marginal value of their equity is above one.

Proposition 8. *If*

$$\frac{(r - \phi_L - g_2)(r - \phi_L - g_1 + 2\lambda)}{(r - \phi_L - g_1 + \lambda)^2} > 1, \quad (22)$$

then a marginal value to household welfare of an increase in θ_1 is positive at $\bar{\eta}_1$.

2.4 Discussion

Both Propositions 5 and 6 imply household welfare is harmed by the existence of banks under certain parameter restrictions. Consider Proposition 5 first.² This result is driven by two factors: (1) the behavior of q_1 , and (2) the behavior of θ_1 . In equilibrium, q_1 decreases with η_1 , which lowers household welfare. To understand why, observe that both households' and banks' asset pricing conditions must bind when households hold a positive fraction of the tree:

$$\begin{aligned}\frac{1}{q_1} + g + \frac{\dot{q}_1}{q_1} - r &= 0, \\ \frac{1}{q_1} + g + \frac{\dot{q}_1}{q_1} - r_d &= \frac{\lambda}{\theta_1} \frac{q_1 - q_2}{q_1}.\end{aligned}$$

The solvency constraint causes banks to account for downside risk, unlike households who only require the return on the risky asset to equal their time preference r , regardless of the dynamics in q_1 . The two equations imply that the liquidity value of debt ϕ_L must equal the risk premium demanded by banks. Because the marginal private value of bank equity θ_1 decreases as banks accumulate equity, the only way for the risk premium to remain constant is for the price of the risky asset to decline with θ_1 as well. In order for q_1 to increase as θ_1 decreases, ϕ_L would need to increase with θ_1 . In other words, due to financial frictions limiting equity issuance, a bank's individual portfolio optimally requires the price of the risky asset to decline as compensation for bearing risk. They fail to internalize that these decisions in the aggregate produce suboptimal asset price dynamics.

The disconnect between the marginal *social* value of bank equity and the marginal *private* value

of bank equity also explains Proposition 5. Across the state space, banks find that the benefits of additional equity ($\theta_1 n_1^b$) outweigh the opportunity cost of issuing dividends (n_1^b). It is therefore privately optimal to accumulate more equity. However, banks do not internalize that their individual decision to accumulate equity has an adverse consequence for aggregate welfare because the private marginal value of their equity may decline too fast. Combined with the required fall in q_1 due to risk premia, banks do not fully take into account the link between their individually optimal decisions and aggregate asset price dynamics. As a result, they fail to see that, in the aggregate, more equity has negative marginal social value.

Put differently, total bank valuations are increasing in the state variable, but household welfare is decreasing in the state variable. If markets were complete, then households and banks could trade equity claims until the valuation and welfare schedules matched. But costly equity issuance means that this market is precisely what is missing, which is why the marginal value of banks and the marginal value to households do not align. A constrained planner, respecting the underlying costly equity issuance friction, could therefore improve welfare by providing incentives for banks to issue dividends (to discourage equity accumulation) in good times. Proposition 8 considers such policies and confirms that the marginal value of a dividend subsidy is positive at $\bar{\eta}_1$.³

Now consider Proposition 6.⁴ When liquidity value is sufficiently low, the price of the risky asset, and thus welfare, can actually be lower with a fully capitalized banking sector than if households had held the entire tree in the first place. The reason is the tension between the role of banks as liquidity providers and their solvency concerns. Any increase in q_1 above what household value the tree ($1/(r - g_1)$) can only be explained by the fact that banks can finance themselves with cheap

debt due to the liquidity services from deposits. Banks are valuable precisely because they issue debt. However, the price $q_1(\bar{\eta}_1)$ does not reach $1/(r_d - g_1)$ since banks cannot go bankrupt after the state transition from $s = 1$ to $s = 2$. When debt has low liquidity value, the downward pressure on the price of the risky asset from financial frictions can be so strong that intermediation by banks fails to increase the price of the risky asset above what households would obtain in a bankless equilibrium.

The behavior of household flow utility also explains why a better capitalized banking sector does not imply higher household welfare in our model. Flow utility is $c_{t,s}^h + \phi_L \delta_{t,s}$, which in equilibrium, after aggregating and applying market-clearing, is

$$Y_{t,s} + \phi_L(\psi_{t,s} - \eta_{t,s})q_{t,s}Y_{t,s} = (1 + \phi_L(\psi_{t,s} - \eta_{t,s})q_{t,s})Y_{t,s}.$$

By (17), flow utility per dividend is

$$1 + \phi_L q_1 \min \left\{ 1 - \eta_1, \eta_1 \left(\frac{q_2}{q_1 - q_2} \right) \right\}.$$

Proposition 9. *Let η_1^ψ be the lower bound such that $\psi_1 = 1$ for all $\eta_1 \in [\eta_1^\psi, \bar{\eta}_1]$.*

- (i) *If q_1 is decreasing at some $\eta_1 \in [\eta_1^\psi, \bar{\eta}_1]$, then flow utility is decreasing.*
- (ii) *There exists an $\varepsilon > 0$ such that on the interval $[\bar{\eta}_1 - \varepsilon, \bar{\eta}_1) \subset [\eta_1^\psi, \bar{\eta}_1]$, household flow utility decreases with η .*

(iii) Whenever q_1 satisfies

$$q_1 < \frac{1 + \frac{\lambda}{\theta_1} q_2}{(1 - \eta_1) \left(r - \phi_L - g_1 + \frac{\lambda}{\theta_1} \right)},$$

flow utility decreases.

This result provides intuition for why household welfare may decrease with η_1 . Since intermediation by banks does not raise the tree's productivity, flow utility is determined by the value of banks' liquidity services. Higher bank capitalization may lower liquidity services because banks do not need to issue as much debt to finance their holdings of the risky asset. Proposition 9(i) is intuitive. If the asset price decreases with η_1 , then banks need less debt to finance their tree holdings, lowering the quantity of liquidity services. Statement (ii) shows that, local to $\bar{\eta}_1$, banks issue less debt, even if q_1 is increasing with η_1 . Statement (iii) shows that banks will borrow less as they accumulate more equity if the asset price is sufficiently low.

On the other hand, if households hold some fraction of the tree, then household flow utility is actually increasing with η_1 , as the following Proposition states.

Proposition 10. *If $\psi_1 < 1$, then household flow utility increases with η_1 .*

This result should be surprising given that the asset price decreases with η_1 when $\psi_1 < 1$, which means that the tree is becoming cheaper and should require less debt to finance. However, the reason is that in this region of the state space banks can increase the fraction of the tree that they own. In equilibrium the fraction of the tree held by banks increases by more than the price of the tree falls, i.e. the elasticity of banks' demand for the tree is larger than 1. But once $\psi_1 = 1$, this

mechanism is no longer present, hence whether the quantity of deposits increases or not depends only on whether q_1 increases at a sufficiently fast rate relative to η_1 .

Robustness to State-Dependent ϕ_L To simplify the exposition, we have assumed that the liquidity value of deposits is constant. However, our welfare results are generally robust when we specify ϕ_L to vary with η . The key condition in Lemma 2 for Proposition 5 is sufficient in the case of state-dependent ϕ_L as long as it holds for the minimum ϕ_L . The implied restriction on ϕ_L is that bank deposits always provide some strictly positive liquidity services to households and does not become vanishingly small. Proposition 8 does not depend on ϕ_L at all. Propositions 6 and 7 continue to hold when ϕ_L varies with η_1 as long as its level at $\bar{\eta}_1$ is neither too small nor too large. For example, if ϕ_L at $\bar{\eta}_1$ coincides with the liquidity value of deposits in $s = 2$, then the results go through exactly as they are stated. Further, as in Proposition 7, a higher g_1 increases the range of parameters for which Proposition 6 holds. The behavior of flow utility becomes more difficult to analytically characterize because it now depends on how quickly and in what direction ϕ_L changes. Proposition 9 always holds if ϕ_L weakly decreases in a neighborhood of $\bar{\eta}_1$ and can possibly hold if it does not increase too quickly. Proposition 10 always holds if ϕ_L weakly increases when $\psi_1 < 1$ and can also hold as long as it does not decrease too quickly.

3 MODEL WITH BROWNIAN UNCERTAINTY

We now extend our model to include robust aggregate risk (Brownian motion) and a richer state space, which combines, with modifications, elements of the models in Brunnermeier and Sannikov

(2014) and Phelan (2016). We also introduce risk aversion (log utility) for households. Omitted proofs and derivations are in Appendix B. In Appendix C we also consider the Brownian model with risk-neutral households, as in the previous section, and show that the behavior of the Brownian models with log and linear utility are quite similar.

3.1 Setup

Technology, Environment, and Markets Time is still continuous and infinite, but there are no longer state transitions, hence we drop the s subscript. Instead the tree's dividend Y_t evolves according to equation (23),

$$\frac{dY_t}{Y_t} = g dt + \sigma dW_t, \quad (23)$$

where W_t is an exogenous Brownian aggregate shock, g is the exogenous growth rate, and σ is the exogenous fundamental volatility.

The risky asset trades in a perfectly competitive market with equilibrium asset price q_t .

We postulate that its law of motion takes the form

$$\frac{dq_t}{q_t} = \mu_{q,t} dt + \sigma_{q,t} dW_t, \quad (24)$$

which will be determined endogenously in equilibrium. The return to owning the risky asset includes the value of the output produced and the capital gains on the value of the tree. By Ito's Lemma, the rate of return is given by

$$dr_{y,t} = \left(\frac{1}{q_t} + g + \mu_{q,t} + \sigma \sigma_{q,t} \right) dt + (\sigma + \sigma_{q,t}) dW_t. \quad (25)$$

The volatility of returns on investments is $\sigma + \sigma_{q,t}$, which includes fundamental risk σ and endogenous price risk $\sigma_{q,t}$.

We denote the risk-free rate by $r_{f,t}$ and the return on deposits by $r_{d,t}$. Since households no longer have linear utility (below), the return on deposits need not be constant as before.

Households and their Portfolio Choice We now suppose that households are risk-averse with concave utility over consumption and deposits. For tractability we assume log utility over consumption. Lifetime utility is now

$$V_t = \mathbb{E}_t \left[\int_t^\infty e^{-r(\tau-t)} \left(\log(c_{h,\tau}) + v \left(\frac{\delta_{h,\tau}}{w_{h,\tau}} \right) \right) d\tau \right], \quad (26)$$

where $v(\delta_{h,t}/w_{h,t})$ captures the convenience yield provided by household deposit holdings, with $v' > 0$ and $v'' \leq 0$. Deposits are scaled by wealth to preserve homogeneity.

Banks The problem for banks is modified in the following two ways. First, because households have log utility, dividends are discounted by households' stochastic discount factor $\xi_t \equiv e^{-rt} c_{h,t}^{-1}$. Second, we relax the assumption that equity issuance is infinitely costly. Now raising one unit of equity costs $1 + \gamma$ units (i.e., γ is the marginal issuance cost). By homogeneity and price-taking, the maximized value of a bank with equity n_t^b can be written as

$$\theta_t n_t^b \equiv \max_{\{y_u \geq 0, d\zeta_u^b\}} \mathbb{E}_t \left[\int_t^\infty \frac{\xi_u}{\xi_t} d\zeta_u^b \right]. \quad (27)$$

The marginal value of equity equals 1 plus the multiplier on the equity-issuance constraint and reflects the aggregate condition of the financial sector. The dynamics of θ_t are determined endogenously in equilibrium with

$$\frac{d\theta_t}{\theta_t} = \mu_{\theta,t} dt + \sigma_{\theta,t} dW_t, \quad (28)$$

with $\sigma_{\theta,t} \leq 0$. See Proposition 14.

3.2 *Equilibrium*

We characterize equilibrium consumption and asset pricing and then the dynamics of the economy.

Equilibrium Consumption and Asset pricing We formally characterize the optimal decisions for households and banks in Propositions 13 and 14 in the Appendix. We summarize the key results here. Because households have log utility, optimal flow consumption is a fraction r of total wealth

$$c_{h,t} = rW_{h,t}. \quad (29)$$

Since output in equilibrium is the tree, aggregate consumption is given by $C_t = Y_t$. Because consumption growth and volatility are respectively g and σ , the risk-free interest rate satisfies

$$r_{f,t} = r + g - \sigma^2. \quad (30)$$

The deposit spread is determined by the marginal utility of deposits

$$r_{f,t} - r_{d,t} = rv'(y_{hd,t}). \quad (31)$$

Finally, optimal household investment implies

$$\mathbb{E}[dr_{y,t}] - r_{f,t} \leq \sigma(\sigma + \sigma_{q,t}), \quad (32)$$

with equality whenever households invest in the tree.

Optimality for banks implies the following. Banks will not pay dividends when $\theta_t \geq 1$; θ_t can never be less than one because banks can always pay out the full value of equity, guaranteeing a value of at least $n_{b,t}$. Banks will raise equity when $\theta_t = 1 + \gamma$; θ_t can never be great than $1 + \gamma$ because banks can always immediately raise equity and pay a dividend. We look for an equilibrium in which banks hold at least some of the tree. Banks' portfolio choice implies

$$\mathbb{E}[dr_{y,t}] - r_{f,t} = \sigma(\sigma + \sigma_{q,t}) + ERP_t - DS_t, \quad (33)$$

where $ERP_t \equiv -\sigma_{\theta,t}(\sigma + \sigma_{q,t})$ is the excess risk premium, and $DS_t \equiv r_{f,t} - r_{d,t}$ is the the deposit spread. Relative to households, banks earn extra returns of at least $DS_t - ERP_t$: they have a funding advantage DS_t but bear the additional risk ERP_t due to financial frictions. Differencing (33) and (32) yields the equilibrium asset pricing condition:

$$DS_t \geq ERP_t, \quad (34)$$

which binds whenever households hold a positive quantity of the tree.

Equilibrium Dynamics We solve for the global equilibrium dynamics using the methods in Brunnermeier and Sannikov (2014). As before, the state variable η_t is the share of aggregate wealth held by banks. Equilibrium consists of a law of motion for η_t and asset allocations and prices as functions of η . The asset prices are $q(\eta), \theta(\eta)$, and the flow allocation is $\psi(\eta)$; we suppress time subscripts to reduce notation unless necessary for clarity. We derive the evolution of η_t using Ito's Lemma and the equations for returns and budget constraints.

Lemma 3. *The equilibrium law of motion of η_t will be endogenously given as*

$$\frac{d\eta_t}{\eta_t} = \mu_{\eta,t} dt + \sigma_{\eta,t} dW_t + d\Xi_t, \quad (35)$$

where $d\Xi_t$ is an impulse variable creating a regulated diffusion. Furthermore,

$$\begin{aligned} \mu_{\eta,t} &= \frac{1}{q_t} + \frac{(\psi_t - \eta_t)}{\eta_t} ((\sigma_{ch,t} - \sigma_{\theta,t}) - (\sigma + \sigma_{q,t})) (\sigma + \sigma_{q,t}), \\ \sigma_{\eta,t} &= \frac{(\psi_t - \eta_t)}{\eta_t} (\sigma + \sigma_{q,t}), \\ d\Xi_t &= \frac{d\zeta_t}{N_t}, \end{aligned}$$

where $d\zeta_t = \int d\zeta_t^b db$ and $\psi_t = \int y_t^b db$ is the fraction of the tree held by banks.

Because total household wealth satisfies $W_h = N_h + \theta N_b = q(1 + (\theta - 1)\eta)Y$, market clearing

for consumption implies $rq(1 + (\theta - 1)\eta)Y = Y$ and hence

$$q(\eta) = \frac{1}{r(1 + (\theta(\eta) - 1)\eta)}. \quad (36)$$

This means that, while there are two dynamic valuation variables (q and θ), it is sufficient to solve for the dynamics of θ as a function of η . We solve for equilibrium by converting the equilibrium conditions into a system of differential equations (“ODE”) in θ . Given $\theta(\eta), \theta'(\eta)$ We can get equilibrium returns and allocations to get $\theta''(\eta)$. Equation (36) allows us to solve for $q(\eta)$ and $q'(\eta)$. We solve the ODE using appropriate boundary conditions.

Proposition 11 (Equilibrium). *The equilibrium domain of the functions $q(\eta), \theta(\eta)$, and $\psi(\eta)$ is an interval $[\underline{\eta}, \bar{\eta}]$. The function $q(\eta)$ is increasing, $\theta(\eta)$ is decreasing, and the following boundary conditions hold: (i) $\theta(\bar{\eta}) = 1$; (ii) $\theta(\underline{\eta}) = 1 + \gamma$; (iii) $\theta'(\bar{\eta})\bar{\eta} + \theta(\bar{\eta}) = 1$; (iv) $\theta'(\underline{\eta})\underline{\eta} + \theta(\underline{\eta}) = 1$. Over $[\underline{\eta}, \bar{\eta}]$, $\theta_t \geq 1$ and $d\zeta_t = 0$. At $\bar{\eta}$, $d\zeta_t > 0$, and at $\underline{\eta}$, $d\zeta_t < 0$, creating regulated barriers for the process η_t .*

The first two boundary conditions reflect optimal policy for dividends/equity issuance. The last two conditions are smooth pasting conditions implying $q'(\bar{\eta}) = 0$ and $q'(\underline{\eta}) = 0$. Note that combining the conditions at $\bar{\eta}$ implies that $\theta'(\bar{\eta}) = 0$. If the price function and marginal value of bank equity are twice-continuously differentiable, then equations (24) and (28) are functions of η :

$$\frac{dq_t}{q_t} = \mu_q(\eta_t)dt + \sigma_q(\eta_t)dW_t, \quad \frac{d\theta_t}{\theta_t} = \mu_\theta(\eta_t)dt + \sigma_\theta(\eta_t)dW_t, \quad (37)$$

where the drift and variance terms are determined by the derivatives of $q(\eta)$ and $\theta(\eta)$. (For the

remainder of the paper, the dependence on the state-variable η_t is suppressed for notational ease.)

3.3 Welfare Analysis

The model aggregates up to a representative household with life time utility

$$V_t = \mathbb{E}_\tau \left[\int_t^\infty e^{-r(\tau-t)} (\log(C_\tau) + v(D_\tau/W_{h,\tau})) d\tau \right], \quad (38)$$

where aggregate consumption and debt are given by $C_t = Y_t$ and $D_t = \psi_t q_t Y_t - N_{b,t} = (\psi_t - \eta_t) q_t Y_t$, as before. Because households have log utility, we can write their value function as

$$V_t = \frac{\log(W_{h,t})}{r} + H_t,$$

where H_t is a wealth-independent term. From market clearing, household wealth satisfies $W_h = N_h + \theta N_b = q(1 + (\theta - 1)\eta)Y = Y/r$. Hence, the value function can be re-written as:

$$V(\eta, Y) = \frac{\log(Y)}{r} + H(\eta) + \frac{C}{r},$$

where C is a constant. The function $V(\eta, Y)$ satisfies the HJB

$$rV(\eta, Y) = \log(Y) + v(X(\eta)) + V_\eta \eta \mu^\eta + \frac{(\sigma^\eta \eta)^2}{2} V_{\eta\eta} + gY V_Y + \frac{(Y\sigma)^2}{2} V_{YY}, \quad (39)$$

where $X(\eta) \equiv D(\eta)/W_h(\eta) = \frac{\psi-\eta}{1+(\theta-1)\eta}$ is liquidity services as a fraction of total wealth. Plugging in $V(\eta, Y) = \frac{\log(Y)}{r} + H(\eta) + \frac{C}{r}$, this simplifies to

$$rH(\eta) + C = v(X(\eta)) + H'(\eta)\eta\mu^\eta + \frac{(\sigma^\eta\eta)^2}{2}H''(\eta) + \left(g - \frac{\sigma^2}{2}\right), \quad (40)$$

Writing the value function in this way clarifies the source of welfare variations in the economy. Since consumption is exogenously determined by the tree, the discounted expected utility from output is given by $\frac{C}{r} = \frac{g-\sigma^2/2}{r}$. The function $H(\eta)$ is therefore the expected discounted value of the convenience yield $v(X(\eta))$ from deposits.

The behavior of aggregate liquidity is therefore central for households welfare, and we can characterize the welfare function in a particular case. The welfare function H is decreasing in η whenever the aggregate bank valuation $\theta(\eta_t)N_t = \theta(\eta_t)q(\eta_t)\eta_t Y_t$ is increasing in η , and whenever banks always hold the entire tree in equilibrium (i.e., $\psi(\eta) = 1$ for all η). The first condition is natural, that bank values are greatest when capitalization levels are higher. (We have not been able to prove that this must be true in equilibrium, but it is true in every numerical solution we've considered.) The second condition could be a requirement of the environment, i.e., that the tree must be held by intermediaries (a condition that is common in other macro-finance papers), or could reflect high convenience yields.

Proposition 12. *Suppose that in equilibrium bank valuations are increasing in η and banks always hold the entire tree. Then the value function H is decreasing in η with $H(\underline{\eta}) > H(\bar{\eta})$.*

This proposition does *not* require constant marginal utility over deposits but only that utility is increasing in debt. Thus, the convenience yield (spread) on deposits can decrease as banks'

capitalization and *valuation* worsen (deposit rates would increase). The result has an intuitive interpretation. The value function integrates over utility from liquidity services. Whenever utility from liquidity services is decreasing in η , the value function is therefore also decreasing in η since it simply integrates over a decreasing liquidity schedule.

This case serves as a helpful benchmark. Within the current setting, overturning the result can only come from a single mechanism: fire sales, i.e., banks selling the asset to households at depressed prices. For welfare to be increasing in η anywhere requires $\psi < 1$ in equilibrium. In other words, fire sales would be necessary to generate an increasing welfare function. Recall that [He, Kelly and Manela \(2017\)](#) find that the main buyers of asset sales from investment banks and hedge funds are commercial banks. In our model, we would interpret such trades as trades between banks, not between banks and households (i.e., not fire sales).

Outside of the current setting, we might also consider natural modifications to the setup of the model. For example, liquidity from debt could be state-contingent, i.e., $v(x, \eta)$, so that for low η utility from deposits would decline as η decreases. A mechanical decline in the marginal utility for deposits as banks' capitalization and valuation worsen could therefore overturn the results. We numerically consider both of these issues.

3.4 Numerical Solution

The global equilibrium dynamics of the model are comparable to those in [Brunnermeier and San-nikov \(2014\)](#), with the exception that endogenous risk decreases near $\underline{\eta}$ when banks might issue equity to recapitalize. However, like the analytical model, the behavior of household welfare can be quite different from other models in which welfare is highest when banks are well-capitalized.

Calibration To solve the model numerically, we specify the utility for deposits using

$$v(x) = \beta_1 \log(x),$$

so that utility over deposits is concave and strictly increasing. This specification produces a procyclical deposit spread, consistent with the evidence in [Drechsler, Savov and Schnabl \(2017\)](#).

We solve the model numerically with the following parameters: $g = 2\%$, $r = 2\%$, $\sigma = 1.5\%$, $\beta_1 = 0.25$, $\gamma = 2$. The results are not sensitive to g and r . A growth rate of $g = 2\%$ matches consumption growth, and $r = 2\%$ is a common discount rate ([van Binsbergen, Diamond and Grotteria, 2022](#); [He and Krishnamurthy, 2019b](#)).

The parameter β_1 determines the attractiveness of debt finance (the only source of investment advantage for banks) and therefore the frequency of fire sales. We choose an intentionally low value for deposit spreads, implying a conservative funding advantage for banks, to ensure a robust degree of fire sales in the model and biasing our calibration against producing the baseline welfare result. While [Drechsler, Savov and Schnabl \(2017\)](#) report an average deposit spread of 1.26%, we pick a value of $\beta_1 = 0.5/r\% = 0.25$, which implies that the deposit spread never exceeds 0.5%.⁵

The second crucial parameter is volatility. [Hassan, Karels and Peterson \(1994\)](#) find evidence that the volatility of banks' assets is between 0.9% and 2.3%. We choose a middle value of $\sigma = 1.5\%$ to better target leverage levels, which are higher when volatility is lower. In our model leverage is between 5 and 10 about 80% of the time, though it can exceed 30.⁶ Higher volatility causes states with low bank equity to occur with higher frequency.

Finally, approximating the costs associated with equity issuance is a challenging task, most notably because the associated costs are primarily informational. A value of $\gamma = 2$ implies that raising \$1 of equity requires spending \$3, which strikes us as a very large discount. The main results regarding welfare hold as long as γ is not too large, as we show.

Results Figure 1(a) plots household welfare as a function of η , subtracting off $V(\bar{\eta})$, for two economies. The dashed lines show an economy with no fire sales ($\psi = 1$) and the solid lines show an economy with fire sales ($\psi \leq 1$). Most importantly, welfare is decreasing in η in both cases. Figure 1(b) plots bank asset holdings to illustrate when fire sales occur. Figure 1(d) plots the deposit spread in each case, illustrating that the deposit spread is indeed procyclical as found in the data.

[Figure 1 about here.]

While we obtained this welfare result in the analytical model, it should still be surprising for our numerical extension. First, unlike the analytical model, the asset price $q(\eta)$ is increasing everywhere (Figure 1(c)). Second, the private marginal value of bank equity is always greater than 1, which means that increasing bank equity always increases bank value. How do we then explain the decreasing *social* value of equity?

Consider the flow consumption and liquidity services in equilibrium. Since consumption is just the dividend of the tree, consumption (per unit of tree) is constant; bank equity does not affect output. However, liquidity services are greater when debt levels are greater, which occurs when banks use the most leverage. Figure 2(a) plots flow utility in each economy. While fire sales

decrease utility, flow utility is generally decreasing in η , particularly over the most frequent states. Figure 2(b) plots the stationary distribution of the economy. The economy is overall very stable, spending most time near $\bar{\eta}$. Thus the economy spends most time in a range of η where flow utility is decreasing in η . This is precisely why, even though flow utility is not decreasing everywhere in the economy with fire sales, the welfare function is still decreasing.⁷

[Figure 2 about here.]

Total debt equals $(\psi - \eta)qr = L\eta qr$, where $L = (\psi - \eta)/\eta$ is leverage. The asset price q is increasing in η , and yet total debt is generally decreasing (strictly decreasing in the no-fire-sales equilibrium). It is worth decomposing debt provision into two components: leverage and price valuation. For a given level of equity, a higher asset price requires more debt for banks to finance their holdings. Leverage is countercyclical but the asset price is procyclical. Debt provision is countercyclical because bank's adjust their leverage aggressively to absorb the asset, thus cushioning the decrease in the asset price. As discussed regarding Proposition 12, in an economy without fire sales, a sufficient condition for debt provision to be countercyclical is for aggregate bank valuations to be procyclical.

Liquidity services justify the rise in the price of the risky asset as banks accumulate equity because they enable banks to borrow at cheap interest rates. Even though equity accumulation causes banks to value the risky asset at higher prices, it generally also reduces the debt banks issue to finance their purchases, thus lowering liquidity services. Contrast this outcome with Phelan (2016), where intermediation by banks raises the tree's productivity growth so that the behavior of q is determined by productivity as well as liquidity. As a result, the asset price increases quickly

enough that banks provide more, not less, liquidity as they accumulate equity.

3.5 Policy Measures

The main result of our paper is that banks will tend to over-accumulate equity, thus decreasing liquidity provision. A natural policy for a constrained planner that respects the issuance cost is to incentivize banks to de-accumulate equity in good times—in other words, to pay dividends at a lower η . A policy that improves welfare is to subsidize bank dividends at $\bar{\eta}$ so that banks pay dividends when $\theta > 1$ (c.f. Proposition 8). Such a policy decreases asset prices (which would appear to be bad for welfare), but welfare nonetheless improves because the distribution of low outcomes improves. Figure 3(a) plots welfare and Figure 3(b) plots the asset price. Despite the decrease in asset prices, a dividend subsidy increases welfare through two mechanisms.

[Figure 3 about here.]

First, the dividend subsidy incentivizes banks to pay dividends earlier, and thus $\bar{\eta}$ decreases in equilibrium. Because the economy is stable near $\bar{\eta}$ and far from the region with fire sales, a decrease in $\bar{\eta}$ implies that banks use more leverage and issue more debt to finance assets. Thus, the region around the stochastic steady state features more liquidity provision, implying more liquidity on average, and higher welfare.

The second mechanism is that the economy is overall more stable. Because the length of the economy's state space is smaller, the economy features less endogenous risk (Figure 3(c)). Subsidizing dividends decreases endogenous volatility σ_q and decreases the required risk premium. The economy is more stable even though the distribution of outcomes are shifted toward lower η .

This endogenously decreases bank risk aversion so that banks are more willing to hold the risky asset. This is why fire sales are less severe with the dividend subsidy (Figure 3(d)). Thus, this policy has the surprising result of decreasing the severity of crises when they do occur.

It is important to put our result in context. In our model, banks over-accumulate equity *in good times* but the primitive friction is equity issuance costs. In bad times, it is better to get equity *into* the banks rather than to encourage dividend payouts. Accordingly, if regulators could hypothetically decrease the issuance cost γ , then welfare would improve (see Figure 4 below). Additionally, a policy of providing tail risk insurance so that banks are automatically recapitalized when η falls below a threshold also improves welfare. The fundamental financial friction is banks' inability to maintain sufficiently high equity levels. Tail risk insurance addresses this problem by creating a lower bound on the capitalization of banks. We implement insurance following Brunermeier and Sannikov (2014). Like in their model, this insurance increases household welfare. Banks are willing to hold the tree at higher prices, which requires more debt, so liquidity provision increases.⁸ Hence, if policymakers are constrained by the primitive friction of costly equity issuance, incentivizing dividend payouts in good times to discourage over-accumulation of equity is a welfare-improving policy that respect the underlying constraints faced by banks.

In a crisis, policy makers are inclined to restrict dividend payouts. Our results do not overturn this recommendation precisely because the fundamental friction is costly equity issuance, which is binding in bad times. In our model, banks pay dividends only at $\bar{\eta}$. (See Adrian and Duarte (2016) for an example of an optimization problem that leads to banks always paying dividends.) If banks inefficiently pay dividends in bad times, then restricting dividends in bad times would recapitalize

the banks. Restricting dividends in good times would be detrimental to household welfare.

Finally, it is worth noting that in this model there is a limited set of policy options to induce liquidity provision, but in reality there are numerous margins to consider that affect banks' contributions to external liquidity and systemic risk, including incentives for bank managers. As one example, [DeYoung and Huang \(2021\)](#) show empirically that increases in CEO pay-for-performance incentives are associated with economically substantial reductions in both systemic risk and liquidity creation externalities. Increases in CEO pay-for-risk incentives are associated with statistically significant albeit smaller increases in both externalities. We provide a model in which systemic risk is positively associated with liquidity provision and demonstrate a benchmark case in which reducing systemic risk actually decreases welfare.

3.6 *Robustness and Overturning the Result*

The result that welfare is decreasing in η is incredibly robust in the baseline model, holding for a broad range of parameters. We consider two key ways that may overturn our result. We first consider how larger financial frictions (issuance costs) can exacerbate fire sales and evaluate the robustness of our welfare results. We then consider variations in the deposit spread directly. Importantly, [Acharya et al. \(2011\)](#) show that deposit rates increase (spreads decrease) as bank solvency risks increase. Not only that, they show that during the 2007–2009 Financial Crisis, private liquidity provision declined: banks experienced insufficient deposit inflows to match the contraction in credit from other funding sources, and this led to a contraction in credit and liquidity provided by the financial sector. In the context of our model, such a contraction of credit is represented by a decline in ψ , with banks “unable/unwilling” to fully intermediate the financial asset.

Issuance Costs As already noted, the severity of fire sales is a critical determinant of whether welfare is decreasing in η . Welfare can be increasing, or generally increasing over a range, if households frequently hold a large fraction of the tree due to high cost of equity issuance or low liquidity value of debt. When that occurs, liquidity provision is curtailed (households don't finance the tree with liquid debt), and asset prices fall. In these cases, higher equity allows banks to issue more debt so that welfare increases with η . To that end, more severe financial frictions (higher γ) further constrain banks and lead to more severe fire sales. For high γ , welfare may be non-monotonic or even increasing in η .⁹ However, a policy of subsidizing dividends can still be effective at increasing welfare so long as γ is not so large.

[Figure 4 about here.]

Figure 4 plots welfare and bank holdings for $\gamma = 1, 2, 3, 5$. As expected, welfare is higher when issuance costs are lower because financial frictions are mitigated. For low issuance costs, fire sales are less severe (none at all for $\gamma = 1$), but high issuance costs lead to severe fire sales and a welfare function that is therefore increasing in η . Nonetheless, the dividend subsidy in good times can still be welfare improving even when the welfare function is non-decreasing. Figure 5 plots welfare with dividend subsidies at $\bar{\eta}$ for $\gamma = 3$ and 5. When $\gamma = 3$, the welfare function is increasing except near $\bar{\eta}$ (it is decreasing for $\eta > 0.12$) and yet the dividend subsidy increases welfare. This occurs because, as noted, the dividend subsidy decreases endogenous risk, decreasing fire sales, and increasing overall liquidity. When $\gamma = 5$, the welfare function is everywhere increasing and the dividend subsidy is no longer an effective policy, increasing welfare only for the lowest η . It is still true that fire sales become less severe, but because the economy already features severe

fire sales due to severe financial frictions (high issuance costs), decreasing $\bar{\eta}$ shifts the economy toward fire-sale states, mitigating the potential benefits of the policy.

[Figure 5 about here.]

Deposit Spreads The welfare function is decreasing in η because, except in the case of severe fire sales, liquidity services are greatest when η is lowest. We have already shown that this result holds with pro-cyclical deposit spreads, yet there remains good reason to wonder about the robustness of this result. Accordingly, we now consider additional variations in the deposit spread across η . The baseline calibration features pro-cyclical deposit spreads in response to the quantity of debt but not concerns about bank solvency. We now model bank solvency concerns by letting the liquidity preference parameter vary with the state, i.e., $\beta_1(\eta)$.

We consider an ad hoc rule where $\beta_1(\eta)$ falls linearly when $\eta < 0.06$ but is otherwise set to β_1 . This specification captures the idea that close to a crisis deposits respond to solvency concerns, but in good times spreads are constant (information insensitive). We refer to this as exogenous “liquidity adjustment.” Figure 6 plots deposit rates and welfare with this specification. We also vary the overall level of deposit rates/spreads by varying the baseline deposit spread $DS(\bar{\eta})$.¹⁰

[Figure 6 about here.]

This liquidity adjustment can partially or completely overturn the welfare result, depending on the level of deposit spreads. When spreads are relatively high (deposit rates are low, the blue line), deposit rates are the most responsive to the liquidity adjustment but welfare is still decreasing in η . With a larger financing advantage for banks, fire sales are not severe and the decrease in

the deposit spread need not lead to severe fire sales. In this case, the liquidity adjustment turns down the welfare function so that it is non-monotonic but still decreasing for high η . In contrast, when spreads are lower (deposit rates are high, the yellow line), the welfare function is already an increasing function of η . The liquidity adjustment further steepens the welfare function.¹¹

Nonetheless, dividend subsidies can still be welfare improving with the liquidity adjustment for exactly the reasons mentioned above. With the liquidity adjustment considered, dividend subsidies still increase welfare even with the liquidity adjustment and even if deposit spreads are lower than the baseline calibration. In contrast, with more aggressive liquidity adjustment (more rapid and occurring in “distress” instead of “crisis” times), welfare is not generally improved by the subsidy with the liquidity adjustment and can be harmed. See Figure 10 in Appendix B.2.

How exactly the liquidity adjustment is modeled is crucial for determining the welfare consequences of financial frictions and policy in the model. For example, if the liquidity adjustment only occurs for “severe crises” (for very low η), then the dividend subsidy can be welfare improving, as in the baseline model. In contrast, if deposit rates rise sufficiently fast or sufficiently early in the cycle, then welfare will be an increasing function of η and dividend subsidies will harm welfare. Our main results are robust but not global results; they require nuanced application.

As a final note on this quantitative section, it is worth noting that the deposit spreads in all these scenarios are intentionally lower than the empirical estimates. Since welfare is more likely to be decreasing in η when deposit spreads are high, and since the effect of the liquidity adjustment is muted when deposit spreads are high, we have intentionally biased our results toward non-decreasing welfare functions with a prominent role for the liquidity adjustment. Choosing a higher

β_1 to target spreads closer to 1% or even higher would result in an economy with no fire sales and in which the liquidity adjustment had much less significant welfare implications. We consider this calibration in Appendix B.2.

4 CONCLUSION

We considered a model in which financial intermediaries have a funding advantage but not necessarily a production advantage compared to households and other investors. In the presence of financial frictions (costly equity issuance) we showed that in the aggregate banks are likely to over-accumulate equity to avoid bankruptcy. As a result, the financial sector is more capitalized than is socially optimal. A policy of subsidizing dividends *in good times* to encourage leverage *in good times* improves household welfare. Such a policy increases liquidity provision and decreases overall systemic risk.

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NOTES

1. In Section 3, we relax this assumption and allow banks to issue equity at a finite cost.
2. The parameters required for this Proposition to hold are not unreasonable. Choosing $r = 4\%$, $\phi_L = 1\%$, $g_2 = 1\%$ and $\lambda < 2\%$ satisfy the inequality. With these parameters, we also only require $g_1 > 2.33\%$ to satisfy Assumption 5. If λ is interpreted as the probability of a rare disaster, then a value less than 2% does not seem unreasonable *a priori*. Note that the Proposition would still hold for even lower r because in the proof, we simplify the required hypothesis by finding conditions which guarantee $(\theta_1 - 1)\eta_1$ decreases as well as q_1 . As long as q_1 decreases faster than $(\theta_1 - 1)\eta_1$ increases, r can be even lower relative to the other parameter values.
3. The same set of parameters satisfying Proposition 5 ($r = 4\%$, $\phi_L = 1\%$, $g_2 = 1\%$, $\lambda < 2\%$, and $g_1 > 2.33\%$) also satisfy (22).
4. Because we have not assumed any lower bounds on ϕ_L except that it is positive, we can always find a sufficiently low

ϕ_L so that λ falls within the interval established by Proposition 7. Inspection of (19) and 7 suggests that Propositions 5 and 6 may not both hold at the same time. Parameters would have to guarantee that (19) holds; g_1 is greater than the RHS of (21); and Assumptions 2 and 5 are not violated.

5. Additionally, the return on average assets for all U.S. banks is typically on the order of 1% (FRED USROA). If we impose that the tree requires intermediation (banks always hold the tree) then equilibrium is not very sensitive to this parameter (however, the overall *level* of the asset price, and thus welfare, does depend heavily on this variable). When that is not the case, the value of β_1 determines the frequency of fire sales in equilibrium because it determines the deposit spread. We consider a calibration with higher deposit spreads in Appendix B.2.
6. He and Krishnamurthy (2012) document average leverage for commercial banks and broker-dealers of 8.3 and 25 respectively, and Acharya et al. (2011) show that when leverage is measured as total assets over common equity, leverage rose over 2000–2007 from 15 to 22 for commercial banks and from 17 to 35 for investment banks.
7. See Figure 7 in Appendix B.2 for the evolutions of η and q .
8. To be clear, such a hypothetical policy does not respect the primitive friction of costly equity issuance. Such a policy should be understood as a theoretical exercise illustrating how relieving the friction would affect welfare and equilibrium dynamics.
9. Appendix B.2.4 examines the equilibrium consequences of productivity gains from financial intermediation. If issuance costs are not too severe (as in our baseline calibration), then productivity gains from intermediation counterintuitively stabilize the banking sector and reduces the probability of fire sales, which strengthens our result that welfare decreases in η .
10. As a technical note, we now specify deposit utility as $\beta_1 \log(x + 1)$ so that deposit utility is strictly positive. This change doesn't meaningfully change the behavior of spreads, but it's necessary because if utility over deposits is negative, then a lower β_1 implies *higher* utility, so the liquidity adjustment would mechanically improve welfare,

which is the opposite of our objective. The change in the specification requires a recalibration to β_1 since now the deposit spread is $\frac{r\beta_1 q^* r}{(\psi - \eta)q^* r + 1}$ instead of $\frac{r\beta_1}{(\psi - \eta)}$.

11. The responsiveness of the spread is important for the determining the equilibrium consequences of the liquidity adjustment. If the utility over deposits was instead linear, then variations in debt would not lead to variations in deposit rates. We consider this case in Appendix [B.2](#).

APPENDIX

A PROOFS AND DERIVATIONS FOR THE ANALYTICAL TWO-STATE MODEL

While the main body of the paper assumed constant marginal utility from debt, to demonstrate the robustness of our results, we allow the marginal utility from debt to now depend on the aggregate capitalization of the banking sector $\eta_{t,s}$, i.e. utility from debt is now $\phi_{L,s}(\eta_{t,1})\delta_{t,s}^h$. Because the liquidity value of deposits in $s = 2$ must be constant (equilibrium is essentially static), we economize on notation by denoting $\phi_L(\eta_{t,1}) \equiv \phi_{L,1}(\eta_{t,1})$ and $\phi_{L,2}(\eta_{t,2}) \equiv \phi_{L,2}$. For our purposes, it is sufficient to assume $\phi_L(\eta_{t,1})$ is Lipschitz continuous. Further, we modify Assumptions [3](#) and [5](#). We now require that ϕ_L has an upper bound $\bar{\phi}_L < \min\{r - g_1, \lambda\}$ and is bounded away from zero. We also impose $\phi_{L,2} < \min\{r - g_2, \lambda\}$. and assume that

$$g_1 > \frac{\phi_L(\eta_1)}{\lambda}r + \left(1 - \frac{\phi_L(\eta_1)}{\lambda}\right)(g_2 - \phi_{L,2}) \quad \text{for all } \eta_1.$$

A.1 Results Characterizing Equilibrium

Proof of Proposition 1. In this proof, we suppress the b superscript. It is convenient to replace the control $y_{t,s}$ with $x_{t,s} = q_{t,s}y_{t,s}/n_{t,s}^b$, which is the share of a bank's equity invested in the risky asset.

The Hamilton-Jacobi-Bellman (HJB) equation is

$$\begin{aligned} r\theta_{t,s}n_{t,s} &= \max_{x_{t,s}, d\zeta_{t,s}^b \geq 0} d\zeta_{t,s} + \mathbb{E}[d(\theta_{t,s}n_{t,s})] \\ &= \max_{x_{t,s}, d\zeta_{t,s} \geq 0} d\zeta_{t,s} + \theta_{t,s}(n_{t,s}(r_{d,t,s} + x_{t,s}(dr_{y,t,s} - r_{d,t,s})) - d\zeta_{t,s}) + \frac{\partial(\theta_{t,s}n_{t,s})}{\partial t} \\ &\quad + \lambda_s \left(\theta_{t,s'}n_{t,s} \left(1 + x_{t,s} \frac{q_{t,s'} - q_{t,s}}{q_{t,s}} \right) - \theta_{t,s}n_{t,s} \right). \end{aligned}$$

The HJB has first-order conditions (FOCs)

$$\begin{aligned} (x_{t,s}) : \quad & 0 = \theta_{t,s}n_{t,s}(dr_{y,t,s} - r_{d,t,s}) + \lambda_s \theta_{t,s'}n_{t,s} \frac{q_{t,s'} - q_{t,s}}{q_{t,s}} \\ (d\zeta_{t,s}) : \quad & 0 \geq 1 - \theta_{t,s} \text{ subject to the additional constraint } d\zeta_{t,s} \geq 0 \end{aligned}$$

Re-arranging the first FOC yields

$$dr_{y,t,s} - r_{d,t,s} = -\lambda_s \frac{\theta_{t,s'}}{\theta_{t,s}} \frac{q_{t,s'} - q_{t,s}}{q_{t,s}},$$

which is equation (9). The second FOC establishes a bang-bang control such that, whenever $\theta_{t,s} = 1$, $d\zeta_{t,s} > 0$, and whenever $\theta_{t,s} > 1$, $d\zeta_{t,s} = 0$ due to assumption of limited equity issuance. This observation proves the remainder of Proposition 1.

The household asset pricing condition is immediate from their risk neutrality and lack of consumption constraints. □

Proof of Proposition 2. From banks' asset pricing condition (9),

$$g_s + \frac{\dot{q}_{t,s}}{q_{t,s}} = -\frac{1}{q_{t,s}} + r_{d,t,s} - \lambda_s \frac{\theta_{t,s'}}{\theta_{t,s}} \frac{q_{t,s'} - q_{t,s}}{q_{t,s}},$$

hence aggregate wealth evolves according to

$$\frac{\partial(q_{t,s}Y_{t,s})/\partial t}{q_{t,s}Y_{t,s}} = -\frac{1}{q_{t,s}} + r_{d,t,s} - \lambda_s \frac{\theta_{t,s'}}{\theta_{t,s}} \frac{q_{t,s'} - q_{t,s}}{q_{t,s}}.$$

It follows that

$$\frac{\partial(1/(q_{t,s}Y_{t,s}))/\partial t}{1/(q_{t,s}Y_{t,s})} = \frac{1}{q_{t,s}} - r_{d,t,s} + \lambda_s \frac{\theta_{t,s'}}{\theta_{t,s}} \frac{q_{t,s'} - q_{t,s}}{q_{t,s}}.$$

Finally, because we can aggregate banks into a representative bank, the law of motion for banks' aggregate equity is the same as the law of motion for an individual bank's equity, hence

$$\begin{aligned} \frac{\dot{\eta}_{t,s}}{\eta_{t,s}} &= r_{d,t,s} - x_{t,s} \lambda_s \frac{\theta_{t,s'}}{\theta_{t,s}} \frac{q_{t,s'} - q_{t,s}}{q_{t,s}} - \frac{1}{n_{t,s}} d\zeta_{t,s} \mathbb{1}_{\theta_{t,s}=1} + \frac{1}{q_{t,s}} - r_{d,t,s} + \lambda_s \frac{\theta_{t,s'}}{\theta_{t,s}} \frac{q_{t,s'} - q_{t,s}}{q_{t,s}} \\ &= \frac{1}{q_{t,s}} - (x_{t,s} - 1) \lambda_s \frac{\theta_{t,s'}}{\theta_{t,s}} \left(\frac{q_{t,s'}}{q_{t,s}} - 1 \right) - \frac{1}{n_{t,s}} d\zeta_{t,s} \mathbb{1}_{\theta_{t,s}=1}. \end{aligned}$$

Using the constraint for the change in wealth after a transition (8), we have

$$\begin{aligned}
\left(1 + x_{t,s} \frac{q_{t,s'} - q_{t,s}}{q_{t,s}}\right) &= \frac{n_{t,s'}}{n_{t,s}} = \frac{n_{t,s'} / (q_{t,s'} k_{t,s})}{n_{t,s} / (q_{t,s} k_{t,s})} \frac{q_{t,s'}}{q_{t,s}} \\
x_{t,s} \frac{q_{t,s'} - q_{t,s}}{q_{t,s}} &= \frac{\eta_{t,s'}}{\eta_{t,s}} \frac{q_{t,s'}}{q_{t,s}} - 1 \\
x_{t,s} \left(\frac{q_{t,s'}}{q_{t,s}} - 1\right) &= \frac{\eta_{t,s'}}{\eta_{t,s}} \frac{q_{t,s'}}{q_{t,s}} - 1 + \frac{q_{t,s'}}{q_{t,s}} - \frac{q_{t,s'}}{q_{t,s}} \\
(x_{t,s} - 1) \left(\frac{q_{t,s'}}{q_{t,s}} - 1\right) &= \frac{q_{t,s'}}{q_{t,s}} \left(\frac{\eta_{t,s'}}{\eta_{t,s}} - 1\right)
\end{aligned}$$

Substituting this back into the law of motion for $\eta_{t,s}$ and setting $d\Xi_{t,s} \equiv d\zeta_{t,s} \mathbb{1}_{\theta_{t,s}=1} / n_{t,s}$ yields

$$\frac{\dot{\eta}_{t,s}}{\eta_{t,s}} = \frac{1}{q_{t,s}} - \lambda_s \frac{\theta_{t,s'} q_{t,s'}}{\theta_{t,s} q_{t,s}} \left(\frac{\eta_{t,s'}}{\eta_{t,s}} - 1\right) - \Xi_t.$$

Note that this law of motion parallels those found in [Caballero and Simsek \(2019\)](#). In their framework, the impulse variable term on the RHS would be replaced by r , and log utility would imply an equilibrium price of $q_{t,s} = 1/r$ (absent belief differences). In addition, the transition in the marginal values of a claim on the tree's dividends for banks parallels the transition in the share of wealth held by pessimists. □

Proof of Lemma 1. Suppose banks hold positive quantities of the tree. Given the per-dividend price of the tree and the marginal value of bank equity in state $s = 2$, the evolution of $\eta_{t,1}$ is given

by

$$\begin{aligned}\frac{\dot{\eta}_{t,1}}{\eta_{t,1}} &= \frac{1}{q_{t,1}} - \lambda \frac{q_2}{\theta_{t,1} q_{t,1}} \left(\frac{0}{\eta_{t,1}} - 1 \right) - d\Xi_{t,1} \\ &= \frac{1}{q_{t,1}} + \lambda \frac{q_2}{\theta_{t,1} q_{t,1}} - d\Xi_{t,1}.\end{aligned}$$

This yields (14).

Since banks hold positive quantities of the tree, (9) binds. Re-arranging this equation obtains (15). If households also hold the tree, then (10) implies the system of equations

$$\begin{aligned}\frac{1}{q_{t,1}} + g_1 + \frac{\dot{q}_{t,1}}{q_{t,1}} - r &= 0, \\ \frac{1}{q_{t,1}} + g_1 + \frac{\dot{q}_{t,1}}{q_{t,1}} - r + \phi_L(\eta_{t,1}) &= -\lambda \frac{1}{\theta_{t,1}} \left(\frac{q_2}{q_{t,1}} - 1 \right).\end{aligned}$$

Differencing the two yields

$$\phi_L(\eta_{t,1}) = -\lambda \frac{1}{\theta_{t,1}} \left(\frac{q_2}{q_{t,1}} - 1 \right),$$

which implies

$$\begin{aligned}\phi_L(\eta_{t,1}) q_{t,1} &= -\frac{\lambda}{\theta_{t,1}} (q_2 - q_{t,1}) \\ q_{t,1} \left(\phi_L(\eta_{t,1}) - \frac{\lambda}{\theta_{t,1}} \right) &= -\frac{\lambda}{\theta_{t,1} (r_{d,2} - g_2)} \\ q_{t,1} \left(\frac{\phi_L(\eta_{t,1}) \theta_{t,1} - \lambda}{\theta_{t,1}} \right) &= -\frac{\lambda}{\theta_{t,1} (r_{d,2} - g_2)}\end{aligned}$$

Dividing through yields (16).

To derive (13), first plug the FOCs into the HJB and utilize the fact that $\theta_{t,2} = 1$.

$$r\theta_{t,1}n_{t,1} = \theta_{t,1}n_{t,1}r_{d,1} + \frac{\partial(\theta_{t,1}n_{t,1})}{\partial t} + \lambda(1 - \theta_{t,1})n_{t,1}.$$

Dividing through by $\theta_{t,1}n_{t,1}$ and plugging in for the deposit rate, we obtain

$$r = r - \phi_L(\eta_{t,1}) + \frac{1}{\theta_{t,1}n_{t,1}} \frac{\partial(\theta_{t,1}n_{t,1})}{\partial t} + \lambda \left(\frac{1}{\theta_{t,1}} - 1 \right)$$

Simplifying further yields

$$\phi_L(\eta_{t,1}) = \frac{(\partial(\theta_{t,1}n_{t,1}))/(\partial t)}{\theta_{t,1}n_{t,1}} + \lambda \left(\frac{1}{\theta_{t,1}} - 1 \right). \quad (41)$$

Since $n_{t,1} = \eta_{t,1}q_{t,1}Y_{t,1}$,

$$\begin{aligned} \frac{\dot{N}_{t,1}}{n_{t,1}} &= \frac{\dot{\eta}_{t,1}}{\eta_{t,1}} + \frac{\dot{q}_{t,1}}{q_{t,1}} + \frac{\dot{Y}_{t,1}}{Y_{t,1}}, \\ \frac{\partial(\theta_{t,1}n_{t,1})}{\partial t} &= \theta_{t,1}n_{t,1} \left(\frac{\dot{\theta}_{t,1}}{\theta_{t,1}} + \frac{\dot{N}_{t,1}}{n_{t,1}} \right). \end{aligned}$$

After aggregating 41, it follows that

$$\begin{aligned} \phi_L(\eta_{t,1}) &= \frac{\dot{\theta}_{t,1}}{\theta_{t,1}} + \frac{\dot{\eta}_{t,1}}{\eta_{t,1}} + \frac{\dot{q}_{t,1}}{q_{t,1}} + \frac{\dot{Y}_{t,1}}{Y_{t,1}} + \lambda \left(\frac{1}{\theta_{t,1}} - 1 \right) \\ &= \frac{\dot{\theta}_{t,1}}{\theta_{t,1}} + \frac{\dot{\eta}_{t,1}}{\eta_{t,1}} + \frac{\dot{q}_{t,1}}{q_{t,1}} + g_1 + \lambda \left(\frac{1}{\theta_{t,1}} - 1 \right). \end{aligned}$$

Re-arranging yields an ODE for $\theta_{t,1}$:

$$\dot{\theta}_{t,1} = \lambda(\theta_{t,1} - 1) + \theta_{t,1} \left(\phi_L(\eta_{t,1}) - \frac{\dot{\eta}_{t,1}}{\eta_{t,1}} - \frac{\dot{q}_{t,1}}{q_{t,1}} - g_1 \right)$$

Note that (15) holds even when households own claims on the tree's dividends since we only used banks' asset pricing condition to derive it. Assuming $\theta_{t,1} > 1$, substitute (15) as well as (14) the previous ODE to obtain

$$\begin{aligned} \dot{\theta}_{t,1} &= \lambda(\theta_{t,1} - 1) + \theta_{t,1} \left(\phi_L(\eta_{t,1}) - \frac{1}{q_{t,1}} - \lambda \frac{q_2}{\theta_{t,1} q_{t,1}} - \frac{\dot{q}_{t,1}}{q_{t,1}} - g_1 \right) + \theta_{t,1} d\Xi_{t,1} \\ &= \lambda(\theta_{t,1} - 1) - \lambda \frac{q_2}{q_{t,1}} + \theta_{t,1} \left(\phi_L(\eta_{t,1}) - \frac{1}{q_{t,1}} - \frac{\dot{q}_{t,1}}{q_{t,1}} - g_1 \right) + \theta_{t,1} d\Xi_{t,1} \\ &= \lambda(\theta_{t,1} - 1) - \lambda \frac{q_2}{q_{t,1}} + \theta_{t,1} \left(\phi_L(\eta_{t,1}) - \frac{1}{q_{t,1}} - g_1 - (r_{d,1} - g_1) + \frac{1}{q_{t,1}} + \lambda \frac{1}{\theta_{t,1}} \frac{q_2 - q_{t,1}}{q_{t,1}} \right) + \theta_{t,1} d\Xi_{t,1} \\ &= \lambda(\theta_{t,1} - 1) + \lambda \left(\frac{q_2 - q_{t,1}}{q_{t,1}} - \frac{q_2}{q_{t,1}} \right) + \theta_{t,1} (\phi_L(\eta_{t,1}) - r_{d,1}) + \theta_{t,1} d\Xi_{t,1} \\ &= \lambda(\theta_{t,1} - 1) - \lambda + \theta_{t,1} (\phi_L(\eta_{t,1}) - r_{d,1}) + \theta_{t,1} d\Xi_{t,1} \end{aligned}$$

Collecting similar terms yields (13).

The final claim follows from the economy's stationarity when $\eta_{t,1} = 0$ due to limited equity issuance and households' asset pricing condition (10). $\square$

Proof of Corollary 1. We have $\theta_{t,1} > 1$ if and only if $d\Xi_{t,1} = 0$. When $d\Xi_{t,1}$, the RHS of $\dot{\eta}_{t,1}/\eta_{t,1}$ in (14) is positive since the asset price and $\theta_{t,1}$ will be positive in equilibrium. $\square$

Proof of Corollary 2. When $\theta_{t,1} > 1$, $d\Xi_{t,1} = 0$. If $\phi_L(\eta_{t,1}) \leq \bar{\phi}_L$ for all $\eta_{t,1}$ and $r \geq \lambda + 2\bar{\phi}_L$, it is immediate that $\dot{\theta}_{t,1} < 0$ for any value of $\theta_{t,1}$ (see (13)).

Alternatively, if $\theta_{t,1} < \lambda/\phi_L(\eta_{t,1})$, then

$$\begin{aligned} (\lambda + \phi_L(\eta_{t,1}) - r_{t,1}^L)\theta_1 - 2\lambda &< (\lambda + 2\phi_L(\eta_{t,1}) - r)\frac{\lambda}{\phi_L(\eta_{t,1})} - 2\lambda = \lambda \left(\frac{\lambda}{\phi_L(\eta_{t,1})} + 2 - \frac{r}{\phi_L(\eta_{t,1})} - 2 \right) \\ \Rightarrow (\lambda + \phi_L(\eta_{t,1}) - r_{t,1}^L)\theta_1 - 2\lambda &< \frac{\lambda}{\phi_L(\eta_{t,1})}(\lambda - r) < 0 \end{aligned}$$

where the final inequality follows from Assumption 4 ($r > \lambda$).

The remaining claim that θ_1 stops decreasing when it equals 1 follows from the deterministic evolution of equilibrium variables and banks' ability to issue dividends. $\square$

Proof of Proposition 3. It is convenient to derive the boundary conditions first because they are important for proving that $\theta_{t,1}$ and $q_{t,1}$ are functions of $\eta_{t,1}$. Afterward, we prove that, given the appropriate initial condition, $\theta_{t,1} = 1$ in finite time. This length of time will also be important for the proof. Once we have justified these two claims, we show that (i) the system of differential equations in Lemma 1 imply a unique solution; (ii) the asset price and the marginal value of bank equity may be written as functions of η ; and (iii) the existence of the endogenous endpoint $\bar{\eta}_1$.

Boundary Conditions. Assume for the derivation of the boundary conditions that $q_{t,1} = q_1(\eta_{t,1})$ and $\theta_{t,1} = \theta_1(\eta_{t,1})$.

When $\eta_1 = 0$, households must own the entire tree, so

$$q_1(0) = \frac{1}{r - g_1}.$$

Note that Assumption 5 implies $q_1(0) > q_2$, so even when households hold the entire tree in $s = 1$,

their valuation will exceed their valuation in $s = 2$, when banks can fund their entire portfolio with debt. By continuity, we require $q_1(\eta_1) \rightarrow q_1(0)$ as $\eta_1 \rightarrow 0$. This behavior follows from the observation that as η_1 approaches zero, banks cannot own the entire tree. Suppose they did except at $\eta_1 = 0$. By continuity of q_1 as a function of η_1 , there cannot be a jump in the asset price to satisfy $q_1(0) = 1/(r - g_1)$. Since $q_1(0) > q_2$, continuity also implies, given some $\varepsilon \in (0, q_2)$, there exists a $\delta > 0$ such that for all $\eta_1 \in [0, \delta)$, $q_1(\eta_1) > q_2 + \varepsilon$. Let $q_{\min,1}^\varepsilon = \min_{\eta_1 \in [0, \delta)} q_1(\eta_1)$ and let $q_{\max,1}^\varepsilon$ be similarly defined. Because $q_{\min,1}^\varepsilon > q_2 + \varepsilon$ and q_1 is the price per dividend, a state transition when $\eta_1 \in [0, \delta)$ produces a total loss of at least εY_1 . Further, the value of bank equity is at most $\eta_1 q_{\max,1}^\varepsilon Y_1$, as we assumed banks hold the entire tree. The banks' remaining equity after a state transition is then bounded above as follows:

$$\underbrace{\eta_1 q_1(\eta_1) Y_1}_{\text{bank equity}} - \underbrace{(q_1(\eta_1) - q_2) Y_1}_{\text{loss after state transition}} \leq \underbrace{\eta_1 q_{\max,1}^\varepsilon Y_1}_{\text{maximal equity}} - \underbrace{\varepsilon Y_1}_{\text{minimal loss}}.$$

As $\eta_1 \rightarrow 0$, the RHS must cross zero, hence the bank would be insolvent after a state transition, a contradiction.

We derive the boundary condition for θ_1 at $\eta = 0$ using the fact that banks do not own the entire tree near $\eta = 0$. Since banks do not own the entire tree, the asset price is given by (16):

$$q_1(\eta) = \frac{\lambda}{\lambda - \phi_L(\eta) \theta_1(\eta)} q_2.$$

Continuity requires

$$\frac{1}{r - g_1} = \frac{\lambda}{\lambda - \phi_L(0)\theta_1(0)} q_2 = \frac{\lambda}{\lambda - \phi_L(0)\theta_1(0)} \frac{1}{r - \phi_{L,2} - g_2}.$$

This expression re-arranges to

$$\theta_1(0) = \frac{\lambda}{\phi_L(0)} \left(1 - \frac{r - g_1}{r - g_2 - \phi_{L,2}} \right) = \frac{\lambda}{\phi_L(0)} \left(\frac{g_1 - g_2 - \phi_{L,2}}{r - g_2 - \phi_{L,2}} \right).$$

For a nontrivial equilibrium, we need $\theta_1(0) \in (1, \lambda/\phi_L(0))$. The lower bound is necessary because banks issue dividends when $\theta_{t,1} = 0$, and the upper bound is necessary to guarantee a positive finite asset price (see the denominator of (16)). Assumption 2 immediately implies $\theta_1(0) < \lambda/\phi_L(0)$, and Assumption 5 implies $\theta_1(0) > 1$:

$$\begin{aligned} g_1 &> \frac{\phi_L(0)}{\lambda} r + \left(1 - \frac{\phi_L(0)}{\lambda} \right) (g_2 + \phi_{L,2}) \\ g_1 &> \frac{\phi_L(0)}{\lambda} (r - g_2 - \phi_{L,2}) + g_2 + \phi_{L,2} \\ \frac{\lambda}{\phi_L(0)} (g_1 - g_2 - \phi_{L,2}) &> r - g_2 - \phi_{L,2} \\ \frac{\lambda}{\phi_L(0)} \frac{g_1 - g_2 - \phi_{L,2}}{r - g_2 - \phi_{L,2}} &> 1 \\ \theta_1(0) &> 1. \end{aligned}$$

Now consider $\bar{\eta}_1$. The existence of $\bar{\eta}_1$ follows from banks' ability to issue dividends and their choice, in equilibrium, to issue dividends only when $\theta_1 = 1$. Since dynamics are deterministic and η always increase with time, once banks start issuing dividends, they continue to issue dividends.

As a result, η will stay at $\bar{\eta}_1$, hence $\dot{\theta}_1$ and $\dot{q}_1$ must both be zero. Since banks issue dividends at $\bar{\eta}_1$, $\theta_1(\bar{\eta}_1) = 1$. To calculate $q(\bar{\eta}_1)$, substitute $\dot{q}_1(\bar{\eta}_1) = 0$ into (15).

The remaining boundary conditions are the first derivatives of θ_1 and q_1 at $\bar{\eta}_1$. Plug $\theta_1(\bar{\eta}_1) = 1$ into (13) and (41) to obtain

$$\begin{aligned}\dot{\theta}_1 &= 2\phi_L(\eta_1) - r - \lambda + d\Xi_1, \\ \phi_L(\eta_1) - g_1 &= \dot{\theta}_1 + \frac{\dot{\eta}_1}{\eta_1} + \frac{\dot{q}_1}{q_1}\end{aligned}$$

By Assumption 5, $g_1 > \phi_L(\eta)$ for all η , hence we cannot have $\dot{\theta}_1 = \dot{q}_1 = \dot{\eta}_1 = 0$, or the second equation would not hold. We will choose the following conditions at $\eta_1 = \bar{\eta}_1$: $\dot{\theta}_1 = \dot{q}_1 = 0$ and $\dot{\eta}_1 = (\phi_L(\bar{\eta}_1) - g_1)\bar{\eta}_1$. These choices clearly satisfy the second equation, and they are crucial for our proof of equilibrium's existence below. These choices also have economic meaning. Setting $\dot{\eta}_1 = (\phi_L(\bar{\eta}_1) - g_1)\bar{\eta}_1 < 0$ creates a reflecting barrier at $\bar{\eta}_1$ since $\dot{\eta}_1 > 0$ for $\eta_1 < \bar{\eta}_1$. This barrier corresponds to banks paying out dividends when their share of aggregate wealth is sufficiently high that they are no longer concerned about their solvency constraint. Because the model is deterministic contingent on no state transition, once $\eta_1 = \bar{\eta}_1$, the per-dividend price of the tree and the marginal value of bank equity should not change with time.

As further justification, consider two others cases, which are consistent with (41): (1) $\dot{q}_1 = \dot{\eta}_1 = 0$ and $\dot{\theta}_1 \neq 0$, or (2) $\dot{\theta}_1 = \dot{\eta}_1 = 0$ and $\dot{q}_1 \neq 0$.

In the first case, the per-dividend price of the tree satisfies

$$q_1 = \frac{1 + \lambda q_2}{r - \phi_L(\bar{\eta}_1) - g_1 + \lambda}$$

by (15). It follows that

$$\frac{\dot{\eta}_1}{\eta_1} = r - \phi_L(\bar{\eta}_1) - g_1 + \lambda - d\Xi_1 \Rightarrow d\Xi_1 = r - \phi_L(\bar{\eta}_1) - g_1 + \lambda.$$

Plug this into the ODE for θ_1 .

$$\dot{\theta}_1 = -\lambda + 2\phi_L(\bar{\eta}_1) - r + r + \lambda - \phi_L(\bar{\eta}_1) - g_1 = \phi_L(\bar{\eta}_1) - g_1 < 0.$$

This result implies that as time progresses, θ_1 decreases below 1, which violates banks' FOC for dividend issuance.

In the second case, imposing $\dot{\theta}_1 = 0$ implies

$$d\Xi_1 = r + \lambda - 2\phi_L(\bar{\eta}_1).$$

Imposing $\dot{\eta} = 0$ requires

$$0 = \frac{1 + \lambda q_2}{q_1} - r - \lambda + 2\phi_L(\bar{\eta}_1) \Rightarrow q_1 = \frac{1 + \lambda q_2}{r + \lambda - 2\phi_L(\bar{\eta}_1)}.$$

Then (15) implies

$$\begin{aligned}
\dot{q}_1 &= (r - \phi_L(\bar{\eta}_1) - g_1)q_1 - 1 - \lambda(q_2 - q_1) \\
&= \frac{r + \lambda - \phi_L(\bar{\eta}_1) - g_1}{r + \lambda - 2\phi_L(\bar{\eta}_1)}(1 + \lambda q_2) - (1 + \lambda q_2) \\
&= \left(\frac{r + \lambda - \phi_L(\bar{\eta}_1) - g_1}{r + \lambda - 2\phi_L(\bar{\eta}_1)} - 1 \right) (1 + \lambda q_2) \\
&= \frac{-\phi_L(\bar{\eta}_1) - g_1 + 2\phi_L(\bar{\eta}_1)}{r + \lambda - 2\phi_L(\bar{\eta}_1)}(1 + \lambda q_2) \\
&= \frac{1 + \lambda q_2}{r + \lambda - 2\phi_L(\bar{\eta}_1)}(\phi_L(\bar{\eta}_1) - g_1)
\end{aligned}$$

Observe that the fraction on the RHS is just q_1 , so

$$\frac{\dot{q}_1}{q_1} = \phi_L(\bar{\eta}_1) - g_1 < 0.$$

However, if $\dot{q}_1 < 0$ when $\theta_1 = 1$ and $\eta_1 = \bar{\eta}_1$, then q_1 declines (in time), causing $\dot{\eta}_1 > 0$ (see (14)).

To maintain the steady state, the impulse variable $d\Xi_1$ has to increase so that $\dot{\eta}_1 = 0$. However, a higher impulse causes $\dot{\theta}_1 > 0$, pushing θ_1 above 1. From banks' FOCs, as soon as $\theta_1 > 1$, $d\Xi_1 = 0$, causing $\dot{\eta}_1 > 0$ again. Therefore, a point in the phase space (q_1, θ_1, η_1) satisfying $\dot{q}_1 \neq 0$ and $\dot{\theta}_1 = \dot{\eta}_1 = 0$ will not be stable.

We conclude the derivations of the boundary conditions by applying the chain rule to acquire

$$\begin{aligned}
\dot{\theta}_1 &= \frac{d\theta_1}{d\eta_1} \dot{\eta}_1 \\
\dot{q}_1 &= \frac{dq_1}{d\eta_1} \dot{\eta}_1.
\end{aligned}$$

Since $\dot{\theta}_1 = \dot{q}_1 = 0$ and $\dot{\eta}_1 < 0$, we must have

$$\frac{d\theta_1}{d\eta_1} = \frac{dq_1}{d\eta_1} = 0.$$

Note that this provides additional reason why we cannot have $\dot{\eta}_1 = 0$ and either $\dot{\theta}_1 \neq 0$ or $\dot{q}_1 \neq 0$.

Otherwise, we would have to have an exploding first derivative with respect to η_1 in order for θ_1 and q_1 to be functions of η_1 .

Banks hold the entire capital stock at $\bar{\eta}_1$ Proceed by contradiction and suppose they did not.

Using $\theta_1 = 1$ and the asset price (16) when households hold some claims, households' asset pricing condition (10) implies

$$\begin{aligned} \frac{\dot{q}_1}{q_1} &= r - \frac{1}{q_1} - g_1 = r - \frac{\lambda - \phi_L(\bar{\eta}_1)}{\lambda} (r - \phi_L(\bar{\eta}_1) - g_2) - g_1 \\ &= \frac{\phi_L(\bar{\eta}_1)}{\lambda} r + \frac{\lambda - \phi_L(\bar{\eta}_1)}{\lambda} (\phi_L(\bar{\eta}_1) + g_2) - g_1 < 0 \end{aligned}$$

by Assumption 5. For $\dot{\eta}_1$, we have

$$\begin{aligned} \frac{\dot{\eta}_1}{\bar{\eta}_1} &= \frac{1 + \lambda q_2}{\lambda q_2} (\lambda - \phi_L(\bar{\eta}_1) - r - \lambda + 2\phi_L(\bar{\eta}_1)) \\ &= \frac{\lambda - \phi_L(\bar{\eta}_1)}{\lambda (r - \phi_L(\bar{\eta}_1))^{-1}} + \lambda - \phi_L(\bar{\eta}_1) - r - \lambda + 2\phi_L(\bar{\eta}_1) \\ &= -\frac{\phi_L(\bar{\eta}_1)}{\lambda} (r - \phi_L(\bar{\eta}_1)) + r - \phi_L(\bar{\eta}_1) - r + \phi_L(\bar{\eta}_1) \\ &= -\frac{\phi_L(\bar{\eta}_1)}{\lambda} (r - \phi_L(\bar{\eta}_1)) - \frac{\lambda - \phi_L(\bar{\eta}_1)}{\lambda} g_2 < 0. \end{aligned}$$

Although $\dot{\eta}_1 < 0$, the asset price q_1 will continue to decrease. If $\dot{\theta}_1 = 0$ due to $d\Xi_1$ guaranteeing θ_1 remains at 1, eventually q_1 will become sufficiently low that $\dot{\eta}_1 = 0$, resulting in the second case of impossible boundary conditions discussed above ($\dot{q}_1 \neq 0$ and $\dot{\eta}_1 = \dot{\theta}_1 = 0$). This is a contradiction. Thus, banks hold the entire capital stock at $\bar{\eta}_1$. Note that this conclusion is related to the required boundary condition $\dot{q}_1(\bar{\eta}_1) = 0$.

$\theta_{t,1}$ reaches 1 in finite time By Assumption 4,

$$\lambda - r < \lambda + 2\phi_L(\eta_{t,1}) - r < 3\lambda - r.$$

so the ODE (13) satisfies the inequalities

$$(\lambda - r)\theta_{t,1} - 2\lambda < \dot{\theta}_{t,1} = (\lambda + 2\phi_L(\eta_{t,1}) - r)\theta_{t,1} - 2\lambda < (3\lambda - r)\theta_{t,1} - 2\lambda. \quad (42)$$

Define the “alternative” differential equations

$$\dot{\theta}_{t,1}^l = (\lambda - r)\theta_{t,1}^l - 2\lambda, \quad (43)$$

$$\dot{\theta}_{t,1}^u = (3\lambda - r)\theta_{t,1}^u - 2\lambda, \quad (44)$$

with initial conditions $\theta_{0,1}^l = \theta_{0,1}^u = \theta_{0,1}$. Then we may integrate both sides with respect to time to obtain

$$\theta_{t,1}^l < \theta_{t,1} < \theta_{t,1}^u. \quad (45)$$

Since ϕ_L is Lipschitz continuous and the function $f(x) = x$ is bounded over a finite interval, the product $\phi_L(\eta_1) \cdot \theta_{t,1}$ will also be Lipschitz. Therefore, the solutions to (43), (44), and (13) will exist and be unique over the relevant range. The quantities $\theta_{t,1}^l$ and $\theta_{t,1}^u$ are essentially alternative trajectories for $\theta_{t,1}$. The “slowest” $\theta_{t,1}$ can change is captured by the upper limit $\theta_{t,1}^l$, and the “fastest” $\theta_{t,1}$ can change is captured by the upper limit $\theta_{t,1}^u$.

To solve for $\theta_{t,1}^l$, observe that

$$\begin{aligned}\dot{\theta}_{t,1}^l - (\lambda - r)\theta_{t,1}^l &= -2\lambda \\ e^{-(\lambda-r)t} \dot{\theta}_{t,1}^l - e^{-(\lambda-r)t} (\lambda - r)\theta_{t,1}^l &= -2\lambda e^{-(\lambda-r)t} \\ \frac{d}{dt} \left(e^{-(\lambda-r)t} \theta_{t,1}^l \right) &= -2\lambda e^{-(\lambda-r)t}\end{aligned}$$

Integrating yields

$$\begin{aligned}e^{-(\lambda-r)t} \theta_{t,1}^l \Big|_0^t &= \frac{2\lambda}{\lambda - r} e^{-(\lambda-r)t} \Big|_0^t \\ e^{-(\lambda-r)t} \theta_{t,1}^l - \theta_{0,1} &= \frac{2\lambda}{\lambda - r} e^{-(\lambda-r)t} - \frac{2\lambda}{\lambda - r}.\end{aligned}$$

Re-arranging yields

$$\theta_{t,1}^l = \begin{cases} -e^{(\lambda-r)t} \left(\frac{2\lambda}{\lambda-r} - \theta_{0,1} \right) + \frac{2\lambda}{\lambda-r}, & \text{if } \theta_{t,1}^l > 1 \\ 1 & \text{otherwise.} \end{cases} \quad (46)$$

Similar steps yield

$$\theta_{t,1}'' = \begin{cases} -e^{(3\lambda-r)t} \left(\frac{2\lambda}{3\lambda-r} - \theta_{0,1} \right) + \frac{2\lambda}{3\lambda-r}, & \text{if } \theta_{t,1}^l > 1 \text{ and } r \neq 3\lambda, \\ \theta_{0,1} e^{-2\lambda t} & \text{if } \theta_{t,1}^l > 1 \text{ and } r = 3\lambda, \\ 1 & \text{otherwise.} \end{cases} \quad (47)$$

Define

$$\bar{\theta}_1 = \frac{\lambda}{\phi_{\bar{\theta}_1, L}} \left(\frac{g_1 - \phi_{L,2} - g_2}{r - \phi_{L,2} - g_2} \right) < \frac{\lambda}{\phi_{\bar{\theta}_1, L}}, \quad (48)$$

$$\phi_{\bar{\theta}_1, L} = \phi_L(0). \quad (49)$$

The inequality follows from Assumption 2. Note that $\bar{\theta}_1$ is our desired boundary condition for $\theta_1(0)$, but since we have not proven yet that θ_1 is a function of η_1 , we use $\bar{\theta}_1$ to keep our notation clear. If we set $\theta_{0,1} = \bar{\theta}_1$, then Corollary 2 implies that $\theta_{t,1}$ decreases with time. Equations (46) and (47) then imply the existence of $t^{l,*}, t^{u,*} \in (0, \infty)$ such that

$$1 = -e^{(\lambda-r)t^{l,*}} \left(\frac{2\lambda}{\lambda-r} - \bar{\theta}_1 \right) + \frac{2\lambda}{\lambda-r}, \quad (50)$$

$$1 = \begin{cases} -e^{(3\lambda-r)t^{u,*}} \left(\frac{2\lambda}{3\lambda-r} - \bar{\theta}_1 \right) + \frac{2\lambda}{3\lambda-r} & \text{if } r \neq 3\lambda \\ \bar{\theta}_1 e^{-2\lambda t^{u,*}} & \text{if } r = 3\lambda. \end{cases} \quad (51)$$

By (45), $t^{l,*} < t^{u,*}$. By continuity of $\theta_{t,1}$ with respect to time, the strict monotonicity of $\theta_{t,1}$, and the intermediate value theorem, there exists a unique $t^* \in (t^{l,*}, t^{u,*})$ such that $\theta_{t^*,1} = 1$.

Existence and uniqueness of solutions to equilibrium ODEs Define the functions

$$f_{\eta}(x, y, z; x^*) = \begin{cases} \left(\frac{1}{z} + \frac{\lambda}{y} \frac{q_2}{z}\right)x & \text{if } x < x^* \\ -(\phi_L(x) - g_1)x & \text{if } x \geq x^*. \end{cases} \quad (52)$$

$$f_{\theta}(x, y, z) = \begin{cases} (\lambda + 2\phi_L(x) - r)y - 2\lambda & \text{if } x > 0, y > 1, \\ 0 & \text{if } x = 0 \text{ or } y \leq 1. \end{cases} \quad (53)$$

$$f_q(x, y, z) = \left(r - \phi_L(x) - g_1 + \frac{\lambda}{y}\right)z - 1 - \frac{\lambda}{y}q_2. \quad (54)$$

The last argument x^* for f_{η} denotes the dependence of the function on some value x^* , which determines the location of the discontinuity in f_{η} . The conditions for f_{θ} depend on η because if $\eta = 0$ and banks have zero equity, then the marginal value of bank equity for a hypothetical bank will remain constant forever since banks will never accumulate equity. Then we may represent the equilibrium system of differential equations as

$$\begin{pmatrix} \dot{\eta}_{t,1} \\ \dot{\theta}_{t,1} \\ \dot{q}_{t,1} \end{pmatrix} = \begin{pmatrix} f_{\eta}(\eta_{t,1}, \theta_{t,1}, q_{t,1}; \bar{\eta}_1) \\ f_{\theta}(\eta_{t,1}, \theta_{t,1}, q_{t,1}) \\ f_q(\eta_{t,1}, \theta_{t,1}, q_{t,1}) \end{pmatrix} = \mathbf{f}(\eta_{t,1}, \theta_{t,1}, q_{t,1}; \bar{\eta}_1). \quad (55)$$

We analyze the vector field described by $\mathbf{f}$ on a compact set. Since $\eta_{t,1}$ is a wealth share, we may restrict $\eta_{t,1} \in [0, 1]$. We restrict $\theta_{t,1} \in [1, \bar{\theta}_1]$ because this interval is the economically relevant range. We can bound $q_{t,1}$ below by q_2 because we shall show below that it is a lower bound for any q_1 along the locus $\dot{q}_{t,1} = 0$. There are many finite upper bounds on $q_{t,1}$ that suffice for our analysis.

To prove one such bound must exist, notice that f_q does not imply finite-time explosion in $q_{t,1}$ when $(\eta_{t,1}, \theta_{t,1}) \in [0, 1] \times [1, \bar{\theta}_1]$. Therefore, if we only consider initial conditions $(\eta_{0,1}, \theta_{0,1}, q_{0,1})$ in an open ball around $(0, \bar{\theta}_1, 1/(r - g_1))$ with some sufficiently small radius, then there must exist some $\bar{Q} < \infty$ such that $q_{t,1} < \bar{Q}$ for all $t \in [0, t^*]$. Define $\mathbf{D} = [0, 1] \times [1, \bar{\theta}_1] \times [q_2, \bar{Q}]$. To simplify the analysis, we focus on the restriction $(f_{\eta,c}, f_{\theta,c}, f_{q,c}) \equiv \mathbf{f}_c \equiv \mathbf{f}|_{\mathbf{D}}$. Since ϕ_L is Lipschitz continuous, $\mathbf{f}_c$ satisfies the necessary Lipschitz condition to guarantee that solutions to $\mathbf{f}$ exist and are unique over $\mathbf{D}$.

Steady states Observe that

$$\begin{aligned} f_q(0, 1/(r - g_1), \bar{\theta}_1) &= \left(r - \phi_{\bar{\theta}_1, L} - g_1 + \frac{\lambda}{\bar{\theta}_1} \right) \frac{1}{r - g_1} - 1 - \frac{\lambda}{\bar{\theta}_1} \frac{1}{r - \phi_{L,2} - g_2} \\ &= \frac{r - \phi_{\bar{\theta}_1, L} - g_1}{r - g_1} - 1 + \frac{\lambda}{\bar{\theta}_1} \frac{r - \phi_{L,2} - g_2 - (r - g_1)}{(r - g_1)(r - \phi_{L,2} - g_2)} \\ &= \frac{r - \phi_{\bar{\theta}_1, L} - g_1 - (r - g_1)}{r - g_1} + \frac{\lambda}{\bar{\theta}_1} \frac{g_1 - \phi_{L,2} - g_2}{(r - g_1)(r - \phi_{L,2} - g_2)} \\ &= \frac{-\phi_{\bar{\theta}_1, L}}{r - g_1} + \frac{\lambda}{\bar{\theta}_1} \frac{\phi_{\bar{\theta}_1, L}}{\lambda} \frac{1}{\bar{\theta}_1} \frac{1}{r - g_1} = 0. \end{aligned}$$

This observation as well as inspection of f_η and f_θ establish the existence of two steady states at $(0, 1/(r - g_1), \bar{\theta}_1)$ and $(\bar{\eta}_1, \bar{q}_1, 1)$, where $\bar{\eta}_1$ and $\bar{q}_1$ are yet to be determined.

Saddle path stability for q The locus of points for which $f_q(\cdot) = 0$ satisfies the fixed point

$$q^\ell = \frac{\theta + \lambda q_2}{(r - \phi_L(\eta) - g_1)\theta + \lambda}, \quad (56)$$

which defines the function $q^\ell(\eta)$. Note that $\bar{q}_1$ lies on the locus q^ℓ when $\eta = \bar{\eta}$ and $\theta = 1$.

We claim that if a trajectory is at a point $(\hat{\eta}, \hat{\theta}, \hat{q})$, then $\hat{q} < q^\ell(\hat{\eta}, \hat{\theta})$ implies $f_{q,c}(\hat{\eta}, \hat{\theta}, \hat{q}) < 0$, and $\hat{q} > q^\ell(\hat{\eta}, \hat{\theta})$ implies $f_{q,c}(\hat{\eta}, \hat{\theta}, \hat{q}) > 0$. In other words, $\hat{q} < q^\ell(\hat{\eta}, \hat{\theta})$ if and only if $f_{q,c}(\hat{\eta}, \hat{\theta}, \hat{q}) < 0$.

0. Observe that

$$\begin{aligned} f_{q,c}(\hat{\eta}, \hat{\theta}, \hat{q}) &= \left(r - \phi_L(\hat{\eta}) - g_1 + \frac{\lambda}{\hat{\theta}} \right) \hat{q} - 1 - \frac{\lambda}{\hat{\theta}} q_2 \\ &= \left(r - g_1 + \frac{\lambda}{\hat{\theta}} \right) \hat{q} - 1 - \frac{\lambda}{\hat{\theta}} q_2 - \hat{q} \phi_L(\hat{\eta}). \end{aligned}$$

To establish the sign of $f_{q,c}(\hat{\eta}, \hat{\theta}, \hat{q})$, notice that

$$f_{q,c}(\hat{\eta}, \hat{\theta}, q^\ell) - f_{q,c}(\hat{\eta}, \hat{\theta}, \hat{q}) = \left(r - g_1 + \frac{\lambda}{\hat{\theta}} - \phi_L(\hat{\eta}) \right) (q^\ell - \hat{q}).$$

Because $f_{q,c}(\hat{\eta}, \hat{\theta}, q^\ell) = 0$, it suffices to show that $\hat{q} < q^\ell$ if and only if $f_{q,c}(\hat{\eta}, \hat{\theta}, q^\ell) - f_{q,c}(\hat{\eta}, \hat{\theta}, \hat{q}) > 0$.

This statement is true because $\hat{\theta} \in [1, \bar{\theta}_1]$ and ϕ_L is bounded above by $r - g_1$ by Assumption 3, hence the coefficient on $q^\ell - \hat{q}$ is positive.

Finally, recall that $f_{\eta,c}(\cdot) > 0$ for $\eta < \bar{\eta}_1$ and $f_{\theta,c}(\cdot) < 0$ by Corollaries 1 and 2. As argued in the subsequent section, these corollaries imply θ can always be represented as a function of η . Since $\hat{q} < q^\ell$ implies that $\hat{q}$ will decrease while $\hat{q} > q^\ell$ implies that $\hat{q}$ will increase, we may conclude that there exists a saddle-path stable function $q(\eta)$ satisfying $\lim_{t \rightarrow \infty} q(\eta_t) = \bar{q}_1$ starting from any initial $\eta_t \in (0, \bar{\eta}_1]$.

Identifying the path for θ and remaining boundary conditions By Corollaries 1 and 2, θ and η must evolve strictly monotonically in time. For any set of paths satisfying the system of differential equations in Lemma 1, θ can always be represented as a function of η . Given this observation, we may identify the endogenous endpoint $\bar{\eta}_1$ by integrating the system of differential equations in Lemma 1 forward in time with the initial conditions $\theta_{0,1} = \bar{\theta}_1$, $q_{0,1} = 1/(r - g_1)$, and $\eta_{0,1} = \varepsilon > 0$ until $\theta_{t^*,1} = 1$, where t^* is the associated stopping time. Let $\eta_{t^*,1}^\varepsilon$ denote the value of $\eta_{t^*,1}$ when $t = t^*$ and $\eta_{0,1} = \varepsilon$. Because the vector flows induced by $\mathbf{f}_c$ do not cross, $\bar{\eta}_1$ is the unique value in $(0, 1]$ such that if $\varepsilon \downarrow 0$, then $\eta_{t^*,1}^\varepsilon \rightarrow \bar{\eta}_1$. Further, the value which $q_{t^*,1}$ obtains in the limit as $\varepsilon \downarrow 0$ is precisely $\bar{q}_1$.

Verifying that q_2 is a lower bound for $\mathbf{D}$ We now verify that q_2 is a reasonable lower bound for q . Observe that

$$\begin{aligned} f_q(\eta, \theta, q_2) &= \left(r - \phi_L(\eta, \theta, q_2) - g_1 + \frac{\lambda}{\theta} \right) q_2 - 1 - \frac{\lambda}{\theta} q_2 \\ &= \frac{r - \phi_L(\eta, \theta, (r - \phi_{L,2} - g_2)^{-1}) - g_1}{r - \phi_{L,2} - g_2} - 1 < 0. \end{aligned}$$

The fraction is strictly less than 1 because Assumption 5 implies $g_1 > \phi_{L,2} + g_2$, and ϕ_L is a strictly positive function.

Since (η, θ) are arbitrary, $q_2 < q^\ell$ for all (η, θ) . Our previous analysis established a number of facts about the vector field for q . First, q^ℓ weakly decreases with η and strictly increases with θ . Second, the boundaries for the equilibrium path of q are on q^ℓ . Third, $\dot{q} < 0$ if and only if $q < q^\ell$. These three observations imply that q_2 must be a lower bound for the equilibrium path of q as well.

With this last claim, we conclude the proof of equilibrium existence. $\square$

Proof of Proposition 4. After the transition from state 1 to state 2, banks will pay out all their net worth because they can finance themselves entirely with debt. This result implies that the transition constraint (8) becomes

$$0 = N_{t,1} \left(1 + x_{t,1} \frac{q_2 - q_{t,1}}{q_{t,1}} \right).$$

Re-arranging and substituting in for $x_{t,1}$ imply

$$\frac{\psi_{t,1}}{\eta_{t,1}} = \frac{q_{t,1}}{q_{t,1} - q_2}.$$

Multiplying through by $\eta_{t,1}$ and noting $\psi_{t,1} \in [0, 1]$ give the result. $\square$

Proof of Corollary 4. Plug the boundary condition $q_1(\bar{\eta}_1)$ into the equation for $\psi_{t,1}$ (17) and re-arrange to obtain (18). By Assumption 5, $q_2 < q_1(\bar{\eta}_1)$ hence $\bar{\eta}_1 \in (0, 1)$. $\square$

A.2 Welfare

Proof of Lemma 2. The key condition (19) now becomes

$$r > \frac{1}{\phi_L(\eta_{t,1})} (\lambda^2 + (\lambda - \phi_L(\eta_{t,1}))(\lambda - \phi_{L,2} - g_2)). \quad (57)$$

If $\underline{\phi}_L = \inf \phi_L(\eta_{t,1})$, then a sufficient condition for (57) is

$$r > \frac{1}{\underline{\phi}_L} \left(\lambda^2 + (\lambda - \underline{\phi}_L)(\lambda - \phi_{L,2} - g_2) \right). \quad (58)$$

Differentiate $(\theta_{t,1} - 1)\eta_{t,1}$ with respect to time.

$$\begin{aligned} \frac{d}{dt}((\theta_{t,1} - 1)\eta_{t,1}) &= \frac{1}{\theta_{t,1}\eta_{t,1}} \left(\lambda + 2\phi_L(\eta_{t,1}) - r + \frac{1}{q_{t,1}} + \frac{\lambda}{\theta_{t,1}} \left(\frac{q_2}{q_{t,1}} - 2 \right) \right) - \eta_{t,1} \left(\frac{1}{q_{t,1}} + \frac{\lambda}{\theta_{t,1}} \frac{q_2}{q_{t,1}} \right) \\ &= \left(\frac{1}{q_{t,1}} + \frac{\lambda}{\theta_{t,1}} \frac{q_2}{q_{t,1}} \right) \left(\frac{1 - \theta_{t,1}\eta_{t,1}^2}{\theta_{t,1}\eta_{t,1}} \right) + \frac{1}{\theta_{t,1}\eta_{t,1}} \left(\lambda + 2\phi_L(\eta_{t,1}) - r - 2\frac{\lambda}{\theta_{t,1}} \right) \end{aligned}$$

Multiply both sides by $\theta_{t,1}\eta_{t,1}$.

$$\theta_{t,1}\eta_{t,1} \left(\frac{d}{dt}((\theta_{t,1} - 1)\eta_{t,1}) \right) = \frac{1}{q_{t,1}} \left(1 + \frac{\lambda}{\theta_{t,1}} q_2 \right) (1 - \theta_{t,1}\eta_{t,1}^2) + \lambda + 2\phi_L(\eta_{t,1}) - r - 2\frac{\lambda}{\theta_{t,1}}$$

Substitute q_2 .

$$\begin{aligned} \theta_{t,1}\eta_{t,1} \left(\frac{d}{dt}((\theta_{t,1} - 1)\eta_{t,1}) \right) &= \frac{1}{q_{t,1}} \left(1 + \frac{\lambda}{\theta_{t,1}(r - \phi_{L,2} - g_2)} \right) (1 - \theta_{t,1}\eta_{t,1}^2) \\ &\quad + \lambda + 2\phi_L - r - 2\frac{\lambda}{\theta_{t,1}}. \end{aligned}$$

Since $\psi_{t,1} < 1$ by assumption, Corollary 2 holds. By Proposition 3, $\psi_{t,1} = 1$ when $\eta_{t,1} = \bar{\eta}_1$, so we must have $\eta_{t,1} < \bar{\eta}_1$, hence $\theta_{t,1} > 1$. Finally, Assumption 3 implies $\phi_L(\eta_{t,1}) < \lambda$. Apply these

observations.

$$\begin{aligned}
\theta_{t,1}\eta_{t,1}\left(\frac{d}{dt}((\theta_{t,1}-1)\eta_{t,1})\right) &= \frac{1}{q_{t,1}}\left(1+\frac{\lambda}{\theta_{t,1}(r-\phi_{L,2}-g_2)}\right)(1-\theta_{t,1}\eta_{t,1}^2) \\
&\quad +\lambda+2\phi_L(\eta_{t,1})-r-2\frac{\lambda}{\theta_{t,1}} \\
&< \frac{1}{q_{t,1}}\left(1+\frac{\lambda}{\theta_{t,1}(r-\phi_{L,2}-g_2)}\right)(1-\theta_{t,1}\eta_{t,1}^2) \\
&\quad +\lambda+2\phi_L(\eta_{t,1})-r-2\phi_L(\eta_{t,1}) \\
&= \frac{1}{q_{t,1}}\left(1+\frac{\lambda}{\theta_{t,1}(r-\phi_{L,2}-g_2)}\right)(1-\theta_{t,1}\eta_{t,1}^2)+\lambda-r \\
&< \frac{1}{q_{t,1}}\left(1+\frac{\lambda}{r-\phi_{L,2}-g_2}\right)+\lambda-r.
\end{aligned}$$

Substitute $q_{t,1}$ with (16) and again apply $\theta_{t,1} > 1$.

$$\begin{aligned}
\theta_{t,1}\eta_{t,1}\left(\frac{d}{dt}((\theta_{t,1}-1)\eta_{t,1})\right) &< \frac{(r-\phi_{L,2}-g_2)(\lambda-\phi_L(\eta_{t,1})\theta_{t,1})}{\lambda}\left(\frac{r-\phi_{L,2}-g_2+\lambda}{r-\phi_{L,2}-g_2}\right)+\lambda-r \\
&= \frac{\lambda-\phi_L(\eta_{t,1})\theta_{t,1}}{\lambda}(r-\phi_{L,2}-g_2+\lambda)+\lambda-r \\
&< \frac{\lambda-\phi_L(\eta_{t,1})}{\lambda}(r-\phi_{L,2}-g_2+\lambda)+\lambda-r
\end{aligned}$$

Take the RHS of the inequality, multiply by λ , and re-arrange.

$$\begin{aligned}
& \lambda \left(\frac{\lambda - \phi_L(\eta_{t,1})}{\lambda} (r - \phi_{L,2} - g_2 + \lambda) + \lambda - r \right) \\
&= (\lambda - \phi_L(\eta_{t,1}))(r - \phi_{L,2} - g_2 + \lambda) + \lambda^2 - \lambda r \\
&= (\lambda - \phi_L(\eta_{t,1}) - \lambda)r + \lambda(\lambda - \phi_L(\eta_{t,1}) + \lambda) - (\lambda - \phi_L(\eta_{t,1}))(\phi_{L,2} + g_2) \\
&= -\phi_L(\eta_{t,1})r + \lambda^2 + (\lambda - \phi_L)(\lambda - \phi_{L,2} - g_2).
\end{aligned}$$

By (19), this expression is negative. □

Proof of Proposition 6. With variable ϕ_L , (20) becomes

$$\phi_L(\bar{\eta}_1)\phi_{L,2} - \phi_{L,2}(r - g_2 + \lambda) + \lambda(g_1 - g_2) > 0, \quad (59)$$

and we also require $\phi_L(\bar{\eta}_1) \leq \phi_{L,2}$. A sufficient condition for the latter inequality is $\bar{\phi}_L \leq \phi_{L,2}$.

By the hypothesis,

$$\begin{aligned}
0 &< \phi_L(\bar{\eta}_1)\phi_{L,2} - \phi_{L,2}(r - g_2 + \lambda) + \lambda(g_1 - g_2) \\
&= \phi_L(\bar{\eta}_1)\phi_{L,2} - \phi_{L,2}(r - g_2 + \lambda) + (r - g_1)(r - g_2) - (r - g_1)(r - g_2) + \lambda(r - g_2 - r + g_1) \\
&= \phi_L(\bar{\eta}_1)\phi_{L,2} - \phi_{L,2}(r - g_2 + \lambda) + (r - g_1 + \lambda)(r - g_2) - (r - g_2 + \lambda)(r - g_1) \\
&= \phi_L(\bar{\eta}_1)\phi_{L,2} - \phi_{L,2}(r - g_2) + \phi_{L,2}(r - g_1) - \phi_{L,2}(r - g_1 + \lambda) \\
&\quad + (r - g_1 + \lambda)(r - g_2) - (r - g_2 + \lambda)(r - g_1).
\end{aligned}$$

Re-arranging terms yields

$$\begin{aligned} & (r - g_1 + \lambda)(r - g_2) - \phi_{L,2}(r - g_2) - \phi_{L,2}(r - g_1 + \lambda) + \phi_L(\bar{\eta}_1)\phi_{L,2} \\ & > (r - g_2 + \lambda)(r - g_1) - \phi_{L,2}(r - g_1). \end{aligned}$$

Since $\phi_L(\bar{\eta}_1) \leq \phi_{L,2}$, we may replace $\phi_{L,2}(r - g_2)$ on the LHS of the inequality with $\phi_L(\bar{\eta}_1)(r - g_2)$:

$$\begin{aligned} & (r - g_1 + \lambda)(r - g_2) - \phi_L(\bar{\eta}_1)(r - g_2) - \phi_{L,2}(r - g_1 + \lambda) + \phi_L(\bar{\eta}_1)\phi_{L,2} \\ & > (r - g_2 + \lambda)(r - g_1) - \phi_{L,2}(r - g_1). \end{aligned}$$

Divide both sides by $r - g_1$ and combine terms.

$$\frac{(r - \phi_L(\bar{\eta}_1) - g_1 + \lambda)(r - \phi_{L,2} - g_2)}{r - g_1} > r - \phi_{L,2} - g_2 + \lambda.$$

Now divide both sides by $r - \phi_{L,2} - g_2$.

$$\frac{r - \phi_L(\bar{\eta}_1) - g_1 + \lambda}{r - g_1} > 1 + \frac{\lambda}{r - \phi_{L,2} - g_2} = 1 + \lambda q_2.$$

Then divide by $r - \phi_L(\bar{\eta}_1) - g_1 + \lambda$.

$$\frac{1}{r - g_1} > \frac{1 + \lambda q_2}{r - \phi_L(\bar{\eta}_1) - g_1 + \lambda}.$$

The LHS is $q_1(0)$, and the RHS is $q_1(\bar{\eta}_1)$. □

Proof of Proposition 7. First, consider the case where $\phi_L(\bar{\eta}_1) = \phi_{L,2}$. Then (59) coincides with (20), which is an inequality on a polynomial. The roots of that polynomial

$$x^2 - x(r - g_2 + \lambda) + \lambda(g_1 - g_2) = 0$$

are given by

$$x_{\pm} = \frac{r - g_2 + \lambda \pm \sqrt{(r - g_2 + \lambda)^2 - 4\lambda(g_1 - g_2)}}{2}.$$

We first show that (21) implies real roots:

$$\begin{aligned} g_1 &< \frac{(r - g_2)^2}{4\lambda} + \frac{1}{2}(r + g_2) + \frac{\lambda}{4} \\ 4\lambda g_1 &< (r - g_2)^2 + 2(r + g_2)\lambda + \lambda^2 \\ 4\lambda g_1 &< (r - g_2)^2 + 2(r - g_2)\lambda + 4\lambda g_2 + \lambda^2 \\ 4\lambda(g_1 - g_2) &< (r - g_2)^2 + 2(r - g_2)\lambda + \lambda^2 = (r - g_2 + \lambda)^2. \end{aligned}$$

Since $g_1 > g_2$, $\lambda(g_1 - g_2) > 0$. By Assumptions 1, 2, and 3, $r - g_2 + \lambda > 0$. When (21) is true, $\phi_+ \geq \phi_- > 0$. However, observe that $\phi_+ \geq r - g_2 + \lambda > \lambda$. By Assumption 4, $\phi_L < \lambda$, hence ϕ_L can never be larger than ϕ_+ and therefore lies in the interval $(0, \phi_-)$. Observe that $\bar{\phi}_L$ is precisely ϕ_- . If the inequality points the other direction, then the roots are complex, hence $\phi_L \in (0, \lambda)$.

Now consider the case where $\phi_L(\bar{\eta}_1) < \phi_{L,2}$. We claim that the proposition follows as long as $\phi_L(\bar{\eta}_1)$ is sufficiently large. Thus, it would be sufficient to impose a larger lower bound on $\underline{\phi}_L$.

Equation (59) re-arranges to

$$\phi_L(\bar{\eta}_1) > r - g_2 + \lambda - \frac{\lambda}{\phi_{L,2}}(g_1 - g_2) = r - g_2 - \frac{\lambda}{\phi_{L,2}}(g_1 - g_2 - \phi_{L,2}).$$

By Assumption 5, the RHS is strictly less than $r - g_2$, hence for sufficiently big g_1 , the permissible lower bound on $\phi_L(\bar{\eta}_1)$ is zero. Otherwise, the liquidity value of deposits needs to be large enough.

□

Proof of Proposition 8. When ϕ_L varies with η_1 , the condition (22) becomes

$$1 < \frac{(r - \phi_{L,2} - g_2)(r - \phi_L(\bar{\eta}_1) - g_1 + 2\lambda)}{(r - \phi_L(\bar{\eta}_1) - g_1 + \lambda)^2}$$

By Corollary 4, household welfare at $\bar{\eta}_1$ can be written as

$$J_1(\bar{\eta}_1) = \left(1 + (\theta - 1) \left(1 - \frac{q_2}{q_1(\bar{\eta}_1)}\right)\right) q_1(\bar{\eta}_1) = q_1(\bar{\eta}_1) + (\theta - 1)(q_1(\bar{\eta}_1) - q_2).$$

First, we calculate the derivative of the capital price at $\bar{\eta}_1$ with respect to θ_1 . Note that the level of q_1 when we do not impose $\theta_1(\bar{\eta}_1) = 1$ is

$$q_1(\theta) = \frac{1 + \frac{\lambda}{\theta} q_2}{r - \phi_L(\bar{\eta}_1) - g_1 + \frac{\lambda}{\theta}}.$$

Differentiate with respect to θ .

$$\begin{aligned}
\frac{\partial}{\partial \theta} q_1(\theta) &= (-\lambda \theta^{-2} q_2) (r - \phi_L(\bar{\eta}_1) - g_1 + \frac{\lambda}{\theta}) - \left(1 + \frac{\lambda}{\theta} q_2\right) \left(r - \phi_L(\bar{\eta}_1) - g_1 + \frac{\lambda}{\theta}\right)^{-2} (-\lambda \theta^{-2}) \\
&= -\frac{\lambda \theta^{-2}}{r - \phi_L(\bar{\eta}_1) - g_1 + \frac{\lambda}{\theta}} \left[q_2 - \frac{1 + \frac{\lambda}{\theta} q_2}{r - \phi_L(\bar{\eta}_1) - g_1 + \frac{\lambda}{\theta}} \right] \\
&= -\lambda \theta^{-2} \frac{(r - \phi_L(\bar{\eta}_1) - g_1 + \frac{\lambda}{\theta}) q_2 - 1 - \frac{\lambda}{\theta} q_2}{(r - \phi_L(\bar{\eta}_1) - g_1 + \frac{\lambda}{\theta})^2} \\
&= -\lambda \theta^{-2} \frac{\frac{r - \phi_L(\bar{\eta}_1) - g_1}{r - \phi_{L,2} - g_2} - 1}{(r - \phi_L(\bar{\eta}_1) - g_1 + \frac{\lambda}{\theta})^2}.
\end{aligned}$$

Evaluate the derivative at 1 to acquire

$$\frac{\partial}{\partial \theta} q_1(1) = -\lambda \frac{\frac{r - \phi_L(\bar{\eta}_1) - g_1}{r - \phi_{L,2} - g_2} - 1}{r - \phi_L(\bar{\eta}_1) - g_1 + \lambda)^2}.$$

This derivative is strictly negative because $g_1 > g_2 + \phi_{L,2}$.

Now differentiate welfare with respect to θ_1 .

$$\begin{aligned}
&\frac{\partial}{\partial \theta_1} q_1 \Big|_{\theta_1=1} + q_1 - q_2 \\
&= -\lambda \frac{\frac{r - \phi_L(\bar{\eta}_1) - g_1}{r - \phi_{L,2} - g_2} - 1}{r - \phi_L(\bar{\eta}_1) - g_1 + \lambda)^2} + \frac{1 + \lambda q_2}{r - \phi(\bar{\eta}_1) - g_1 + \lambda} - \frac{1}{r - \phi_{L,2} - g_2} \\
&= -\lambda q_2 \frac{r - \phi_L(\bar{\eta}_1) - g_1 - \frac{1}{q_2}}{r - \phi_L(\bar{\eta}_1) - g_1 + \lambda)^2} + \frac{1 + \lambda q_2}{r - \phi(\bar{\eta}_1) - g_1 + \lambda} - \frac{1}{r - \phi_{L,2} - g_2} \\
&= -\frac{\lambda q_2}{r - \phi_L(\bar{\eta}_1) - g_1 + \lambda} + \frac{\lambda}{(r - \phi_L(\bar{\eta}_1) - g_1 + \lambda)^2} + \frac{1 + \lambda q_2}{r - \phi_L(\bar{\eta}_1) - g_1 + \lambda} - \frac{1}{r - \phi_{L,2} - g_2} \\
&= \frac{1}{r - \phi_L(\bar{\eta}_1) - g_1 + \lambda} \left(1 + \frac{\lambda}{r - \phi_L(\bar{\eta}_1) - g_1 + \lambda} \right) - \frac{1}{r - \phi_{L,2} - g_2}.
\end{aligned}$$

Multiply through by $r - \phi_{L,2} - g_2$ and re-arrange. Condition (22) implies that this derivative is indeed positive. $\square$

Proof of Proposition 9. For the proposition to hold, we need to impose that either $\phi_L(\eta_1)$ weakly decreases with η_1 or that ϕ_L increases at a sufficiently slow rate.

Since $\psi_1 = 1$, the flow utility per dividend is

$$1 + \phi_L(\eta_1)q_1(1 - \eta_1).$$

Claim (i) immediately follows from the hypothesis if $\phi_L(\eta_1)$ is weakly decreasing with η_1 . The claim should also hold as long as ϕ_L does not increase too quickly with η_1 .

Assume now that $\phi_L(\eta_1)$ is differentiable and differentiate the flow utility per dividend with respect to η_1 .

$$\frac{d}{d\eta_1} \phi_L(\eta_1)q_1(1 - \eta_1) = -(\phi_L(\eta_1)q_1) + (1 - \eta_1)(\phi_L(\eta_1)q'_1 + \phi'_L(\eta_1)q_1).$$

This derivative is negative if and only if

$$1 > (1 - \eta_1) \frac{\phi_L(\eta_1)q'_1 + \phi'_L(\eta_1)q_1}{\phi_L(\eta_1)q_1} = \frac{(1 - \eta_1)}{\eta_1} \frac{\dot{q}_1}{q_1} + (1 - \eta_1) \frac{\phi'_L(\eta_1)}{\phi_L(\eta_1)}$$

Assume $\theta_1 > 1$. Plug in (14) and (15).

$$\frac{1 - \eta_1}{\dot{\eta}_1} = \frac{1 - \eta_1}{\eta_1} \frac{1}{(1 + \lambda q_2 / \theta_1) / q_1} = \frac{1 - \eta_1}{\eta_1} \frac{q_1}{1 + \frac{\lambda}{\theta_1} q_2}$$

$$\frac{(1 - \eta_1)}{\dot{\eta}_1} \frac{\dot{q}_1}{q_1} + (1 - \eta_1) \frac{\phi'_L(\eta_1)}{\phi_L(\eta_1)} = \frac{1 - \eta_1}{\eta_1} \frac{1}{1 + \frac{\lambda}{\theta_1} q_2} \left(\left(r_{d,1} - g_1 + \frac{\lambda}{\theta_1} \right) q_1 - 1 - \frac{\lambda}{\theta_1} q_2 \right) + (1 - \eta_1) \frac{\phi'_L(\eta_1)}{\phi_L(\eta_1)}.$$

This quantity is less than 1 if and only if

$$\frac{\eta_1}{1 - \eta_1} > \frac{\left(r_{d,1} - g_1 + \frac{\lambda}{\theta_1} \right) q_1}{1 + \frac{\lambda}{\theta_1} q_2} - 1 + \eta_1 \frac{\phi'_L(\eta_1)}{\phi_L(\eta_1)} \Rightarrow \frac{1}{1 - \eta_1} - \eta_1 \frac{\phi'_L(\eta_1)}{\phi_L(\eta_1)} > \frac{\left(r - \phi_L(\eta_1) - g_1 + \frac{\lambda}{\theta_1} \right) q_1}{1 + \frac{\lambda}{\theta_1} q_2}$$

Re-arranging yields

$$q_1 < \left(\frac{1}{1 - \eta_1} - \eta_1 \frac{\phi'_L(\eta_1)}{\phi_L(\eta_1)} \right) \frac{1 + \frac{\lambda}{\theta_1} q_2}{\left(r - \phi_L - g_1 + \frac{\lambda}{\theta_1} \right)}.$$

This matches the condition in (iii) when $\phi'_L(\eta_1) = 0$, and if the condition in (iii) holds, then the above condition is also satisfied when $\phi'_L(\eta_1) \leq 0$.

For claim (ii), recall that

$$q_1(\bar{\eta}_1) = \frac{1 + \lambda q_2}{r - \phi_L(\bar{\eta}_1) - g_1 + \lambda}.$$

If $\theta_1 = 1$, then

$$\frac{1 + \frac{\lambda}{\theta_1} q_2}{(1 - \eta_1) \left(r - \phi_L(\bar{\eta}_1) - g_1 + \frac{\lambda}{\theta_1} \right)} = \frac{1 + \lambda q_2}{(1 - \eta_1) (r - \phi_L(\bar{\eta}_1) - g_1 + \lambda)} > \frac{1 + \lambda q_2}{r - \phi_L(\bar{\eta}_1) - g_1 + \lambda} = q_1(\bar{\eta}_1).$$

The condition from claim (iii) is therefore satisfied at $\eta_1 = \bar{\eta}_1$ whenever $\phi'_L(\eta_1) \leq 0$. If $\phi'_L(\eta_1) > 0$, then as long as $\phi'_L(\eta_1)$ and $\bar{\eta}_1$ are sufficiently small, the key condition from claim (iii) should still hold. Since θ_1 , q_1 , and ϕ_L are continuous functions of η_1 , there exists an $\varepsilon > 0$ such that for all $\eta_1 \in [\bar{\eta}_1 - \varepsilon, \bar{\eta}_1)$,

$$q_1 - \frac{1 + \frac{\lambda}{\theta_1} q_2}{(1 - \eta_1) \left(r - \phi_L(\bar{\eta}_1) - g_1 + \frac{\lambda}{\theta_1} \right)} < 0,$$

which proves the second claim. □

Proof of Proposition 10. For the proposition to hold, we need to impose that either $\phi_L(\eta_1)$ weakly increases with η_1 or that ϕ_L decreases at a sufficiently slow rate.

When $\psi_1 < 1$, the flow utility per dividend becomes

$$1 + \phi_L(\eta_1) q_1 \eta_1 \frac{q_2}{q_1 - q_2} = 1 + \phi_L(\eta_1) q_2 \eta_1 \frac{q_1}{q_1 - q_2}.$$

By Corollary 3, q_1 is decreasing. For functions of the form $f(x) = x/(x - a)$ for positive a and $x > a$, decreasing x increases f . When $\phi_L(\eta_1)$ weakly increases, then utility is increasing in η_1 .

In the case that $\phi_L(\eta_1)$ strictly decreases, we derive a lower bound for how quickly it can

decrease. Assume ϕ_L is differentiable and differentiate

$$\begin{aligned}
& \phi'_L(\eta_1)q_2\eta_1\frac{q_1}{q_1-q_2} + \phi_L(\eta_1)q_2\left(\frac{q_1}{q_1-q_2} + \eta_1(q'_1(q_1-q_2)^{-1} - q_1(q_1-q_2)^{-2}q'_1)\right) \\
&= \phi'_L(\eta_1)q_2\eta_1\frac{q_1}{q_1-q_2} + \phi_L(\eta_1)\frac{q_2}{q_1-q_2}(q_1 + \eta_1q'_1(1 - q_1(q_1-q_2)^{-1})) \\
&= \phi'_L(\eta_1)q_2\eta_1\frac{q_1}{q_1-q_2} + \phi_L(\eta_1)q_2\frac{q_1}{q_1-q_2}\left(1 - \eta_1\frac{q'_1}{q_1}\frac{q_2}{q_1-q_2}\right).
\end{aligned}$$

Recall that

$$\begin{aligned}
\eta_1\frac{q'_1}{q_1} &= \frac{\eta_1}{\dot{\eta}_1}\frac{\dot{q}_1}{q_1} = \left(\frac{q_1}{1 + \lambda\frac{q_2}{\theta_1}}\right)\left(\frac{r - \phi_L(\eta_1) - g_1 + \frac{\lambda}{\theta_1}}{q_1} - \frac{1}{q_1}\left(1 + \frac{\lambda}{\theta_1}q_2\right)\right) \\
&= \frac{r - \phi_L(\eta_1) - g_1 + \frac{\lambda}{\theta_1}}{1 + \frac{\lambda}{\theta_1}q_2} - 1.
\end{aligned}$$

It follows that the derivative of flow utility per dividend is

$$\begin{aligned}
& \phi'_L(\eta_1)q_2\eta_1\frac{q_1}{q_1-q_2} + \phi_L(\eta_1)q_2\frac{q_1}{q_1-q_2}\left(1 - \left(\frac{r - \phi_L(\eta_1) - g_1 + \frac{\lambda}{\theta_1}}{1 + \frac{\lambda}{\theta_1}q_2} - 1\right)\frac{q_2}{q_1-q_2}\right) \\
&= q_2\frac{q_1}{q_1-q_2}\left(\phi'_L(\eta_1)\eta_1 + \phi_L(\eta_1)\left(1 - \left(\frac{r - \phi_L(\eta_1) - g_1 + \frac{\lambda}{\theta_1}}{1 + \frac{\lambda}{\theta_1}q_2} - 1\right)\frac{q_2}{q_1-q_2}\right)\right).
\end{aligned}$$

The sign of this derivative is positive if and only if

$$\phi'_L(\eta_1) > -\frac{\phi_L(\eta_1)}{\eta_1}\left(1 - \left(\frac{r - \phi_L(\eta_1) - g_1 + \frac{\lambda}{\theta_1}}{1 + \frac{\lambda}{\theta_1}q_2} - 1\right)\frac{q_2}{q_1-q_2}\right).$$

Since $\psi_1 < 1$, we know what q_1 is in closed form and obtain that

$$\begin{aligned}
q_1 - q_2 &= \frac{\lambda}{(r - \phi_{L,2} - g_2)(\lambda - \phi_L(\eta_1)\theta_1)} - \frac{1}{r - \phi_{L,2} - g_2} \\
&= \frac{1}{r - \phi_{L,2} - g_2} \left(\frac{\lambda}{\lambda - \phi_L(\eta_1)\theta_1} - 1 \right) \\
&= q_2 \frac{\phi_L(\eta_1)\theta_1}{\lambda - \phi_L(\eta_1)\theta_1} \\
\frac{q_2}{q_1 - q_2} &= \frac{\lambda - \phi_L(\eta_1)\theta_1}{\phi_L(\eta_1)\theta_1}.
\end{aligned}$$

The condition then simplifies to

$$\phi'_L(\eta_1) > -\frac{\phi_L(\eta_1)}{\eta_1} \left(1 - \left(\frac{r - \phi_L(\eta_1) - g_1 + \frac{\lambda}{\theta_1}}{1 + \frac{\lambda}{\theta_1} q_2} - 1 \right) \frac{\lambda - \phi_L(\eta_1)\theta_1}{\phi_L(\eta_1)\theta_1} \right).$$

Thus, as long as $\phi_L(\eta_1)$ does not decrease too quickly, flow utility continues to increase with η_1 . □

B ADDITIONAL RESULTS FOR THE LOG-UTILITY BROWNIAN MODEL

B.1 Proofs and Additional Propositions

B.1.1 Household Optimality

Proposition 13. *Let $\sigma_{ch,t}$ denote the volatility of a household's consumption. Then households choose $y_{h,t}$ and $c_{h,t}$ as well as price deposits such that (i) $\mathbb{E}[dr_{y,t}] - r_{f,t} \leq \sigma_{ch,t}(\sigma + \sigma_{q,t})$, and the*

inequality binds if and only if $y_{h,t} > 0$; (ii) $c_{h,t} = r(n_{h,t} + U_t)$; (iii) $\sigma_{ch,t} = \omega_t \sigma_{nh,t} + (1 - \omega_t) \sigma_{U,t}$; (iv) $dr_{y,t} = r_{f,t} - rv'(1 - y_{h,t})$.

Proof of Proposition 13, households' optimality decisions. Each household owns a bank, so their wealth $w_{h,0}$ equals the sum of their “non-bank net worth” $n_{h,0}$ and the value of the shares in bank equity. We characterize the household problem as follows. Conjecture that households have a twice-differentiable value function

$$V_t = \frac{\log(w_{h,t})}{r} + h_t = \frac{\log(n_{h,t} + \theta_t n_{b,t})}{r} + h_t,$$

where h_t is independent of the level of household wealth. By Ito's lemma, households' HJB is

$$\begin{aligned} \log(w_{h,t}) + rh_t &= \max_{y_{h,t} \geq 0, y_{hd,t}, c_{h,t}, l_t} \log(c_{h,t}) + v(y_{hd,t}) \\ &\quad + \frac{1}{r(n_{h,t} + \theta_t n_{b,t})} \mu_{wh,t} w_{h,t} - \frac{1}{2r(n_{h,t} + \theta_t n_{b,t})^2} (\sigma_{wh,t} w_{h,t})^2 \\ &= \max_{y_{h,t} \geq 0, y_{hd,t}, c_{h,t}, l_t} \log(c_{h,t}) + v(y_{hd,t}) + \frac{1}{r} \mu_{wh,t} - \frac{1}{2r} \sigma_{wh,t}^2. \end{aligned}$$

Let $\omega_t \equiv n_{h,t}/w_{h,t}$. Then $w_{h,t}$ follows the process

$$\begin{aligned} \frac{dw_{h,t}}{w_{h,t}} &= \{ \omega_t \mu_{nh,t} + (1 - \omega_t)(\mu_{\theta,t} + \mu_{nb,t} + \sigma_{\theta,t} \sigma_{nb,t}) \} dt \\ &\quad + \{ \omega_t \sigma_{nh,t} + (1 - \omega_t)(\sigma_{\theta,t} + \sigma_{nb,t}) \} dW_t. \end{aligned} \tag{60}$$

Further, because $\sigma_{nh,t} = y_{h,t}(\sigma + \sigma_{q,t})$,

$$\frac{\partial \sigma_{wh,t}}{\partial y_{h,t}} = \omega_t \frac{\partial \sigma_{n,t}}{\partial y_{h,t}} = \omega_t(\sigma + \sigma_{q,t}).$$

Finally, households' law of motion for their non-bank net worth can also be written as

$$\frac{dn_{h,t}}{n_{h,t}} = r_{f,t} + y_{hd,t}(r_{d,t} - r_{f,t}) + y_{h,t}(dr_{y,t} - r_{f,t}) - \frac{c_{h,t}}{n_{h,t}} dt.$$

where $y_{hd,t} = 1 - y_{h,t}$ is the portfolio weight on deposits. Given these observations and (60), the first-order conditions are

$$\begin{aligned} (y_{h,t}) : \quad 0 &= \frac{\omega_t}{r} (\mathbb{E}[dr_{y,t}] - r_{f,t}) - \frac{1}{r} \sigma_{wh,t} \frac{\partial \sigma_{wh,t}}{\partial y_{h,t}} \\ &= \omega_t (\mathbb{E}[dr_{y,t}] - r_{f,t} - \sigma_{wh,t}(\sigma + \sigma_{q,t})) \\ \Rightarrow \mathbb{E}[dr_{y,t}] &= r_{f,t} + \sigma_{wh,t}(\sigma + \sigma_{q,t}) \\ (y_{hd,t}) : \quad 0 &= v'(y_{hd,t}) \omega_t + \frac{\omega_t}{r} (r_{d,t} - r_{f,t}) \\ \Rightarrow r_{d,t} &= r_{f,t} - rv'(y_{hd,t}) \\ (c_{h,t}) : \quad 0 &= \frac{1}{c_{h,t}} - \frac{1}{r} \omega_t \frac{1}{n_{h,t}} = \frac{1}{c_{h,t}} - \frac{1}{rw_{h,t}} \\ \Rightarrow c_{h,t} &= rw_{h,t} \end{aligned}$$

The remaining equations follow letting $\sigma_{ch,t} = \sigma$ and $\mu_{ch,t} = g$. □

B.1.2 Bank Optimality

Proposition 14. Assume θ_t and $c_{h,t}$ follow the finite diffusions

$$\begin{aligned}\frac{d\theta_t}{\theta_t} &= \mu_{\theta,t} dt + \sigma_{\theta,t} dW_t, \\ \frac{dc_{h,t}}{c_{h,t}} &= \mu_{ch,t} dt + \sigma_{ch,t} dW_t,\end{aligned}$$

with $\sigma_{\theta,t} \leq 0$. Then $\theta_t n_t$ represents the maximal future expected payoff that a bank with book value n_t can attain, and $\{y_{b,t}, d\zeta_t\}$ is optimal if and only if (i) $\theta_t \geq 1 \forall t$, (ii) $d\zeta_t > 0$ only when $\theta_t = 1$, $d\zeta_t < 0$ only when $\theta_t = 1 + \gamma$, and $d\zeta_t = 0$ otherwise, (iii) $\mu_{\theta,t} = r_{f,t} - r_{d,t} + \sigma_{ch,t} \sigma_{\theta,t}$, (iv) $\mathbb{E}[dr_{y,t}] - r_{d,t} \leq (\sigma_{ch,t} - \sigma_{\theta,t})(\sigma + \sigma_{q,t})$, with strict equality when $y_{b,t} > 0$, (v) The transversality condition $\mathbb{E}[\xi_t \theta_t n_t] \rightarrow 0$ holds under $\{y_{b,t}, d\zeta_t\}$.

Proof of Proposition 14, banks' optimality decisions. Homogeneity and price-taking imply that banks' value function takes the form $U_t = \theta_t n_{b,t}$, where θ_t is the marginal value of banks' equity. Further, households' discount factor is $\xi_t = e^{-rt} c_{h,t}^{-1}$ because households have log utility. Therefore, the HJB can be written as

$$r\theta_t n_{b,t} c_{h,t}^{-1} = \max_{y_{b,t} \geq 0, d\zeta_t, l_t} d\zeta_t c_{h,t}^{-1} + \mathbb{E}[d(\theta_t n_{b,t} c_{h,t}^{-1})], \quad (61)$$

subject to the budget and net-worth constraints. Conjecture that θ_t and $c_{h,t}$ both follow diffusions.

By Ito's product rule,

$$\frac{d(\theta_t n_{b,t})}{\theta_t n_{b,t}} = (\mu_{\theta,t} + \mu_{nb,t} + \sigma_{\theta,t} \sigma_{nb,t}) dt + (\sigma_{\theta,t} + \sigma_{nb,t}) dW_t.$$

By Ito's product and quotient rules,

$$\begin{aligned} & (\theta_t n_{b,t} c_{h,t}^{-1}) \mathbb{E} \left[\frac{d(\theta_t n_{b,t} c_{h,t}^{-1})}{\theta_t n_{b,t} c_{h,t}^{-1}} \right] \\ &= \mu_{\theta,t} + \mu_{nb,t} + \sigma_{\theta,t} \sigma_{nb,t} + \sigma_{ch,t}^2 - \mu_{ch,t} - \sigma_{ch,t} (\sigma_{\theta,t} + \sigma_{nb,t}). \end{aligned}$$

After dropping the differential dt , equation (61) becomes

$$\begin{aligned} r \theta_t n_{b,t} c_{h,t}^{-1} = & \max_{y_{b,t} \geq 0, d\zeta_{b,t} \geq 0} c_{h,t}^{-1} d\zeta_{b,t} + \theta_t n_{b,t} c_{h,t}^{-1} (\mu_{\theta,t} + \mu_{nb,t} - \mu_{ch,t} + \sigma_{ch,t}^2 \\ & + \sigma_{\theta,t} \sigma_{nb,t} - \sigma_{ch,t} \sigma_{nb,t} - \sigma_{ch,t} \sigma_{\theta,t}). \end{aligned}$$

Dividing through by $\theta_t n_{b,t} c_{h,t}^{-1}$, substituting wealth terms, and plugging $y_{bd,t} = 1 - y_{b,t}$ obtains

$$\begin{aligned} r = & \max \frac{d\zeta_{b,t}}{\theta_t n_{b,t}} + \mu_{\theta,t} + r_{d,t} + y_{b,t} (\mathbb{E}[dr_{y,t}] - r_{d,t}) - \frac{d\zeta_{b,t}}{n_{b,t}} \\ & - \mu_{ch,t} + \sigma_{ch,t}^2 + y_{b,t} \sigma_{\theta,t} (\sigma + \sigma_{q,t}) - y_{b,t} \sigma_{ch,t} (\sigma + \sigma_{q,t}) - \sigma_{ch,t} \sigma_{\theta,t} \end{aligned}$$

Unlike the household problem, we may plug this constraint directly into the problem because banks do not have money-in-the-utility demand or costs from raising bank deposits which would introduce spreads into the bank problem.

By the stochastic maximum principle (see Brunnermeier and Sannikov (2016)), the risk-free interest rate satisfies

$$r_{f,t} = r + \mu_{ch,t} - \sigma_{ch,t}^2.$$

Add $\mu_{ch,t} - \sigma_{ch,t}^2$ to both sides of the HJB and re-arrange.

$$\begin{aligned} r_{f,t} = \max \frac{d\zeta_{b,t}}{n_{b,t}} \left(\frac{1}{\theta_t} - 1 \right) + \mu_{\theta,t} - \sigma_{ch,t} \sigma_{\theta,t} + r_{f,t} + r_{d,t} - r_{f,t} \\ + y_{b,t} (\mathbb{E}[dr_{y,t}] - r_{d,t} + (\sigma_{\theta,t} - \sigma_{ch,t})(\sigma + \sigma_{q,t})) \\ + r_{f,t} - r_{d,t} = \max \frac{d\zeta_{b,t}}{n_{b,t}} \left(\frac{1}{\theta_t} - 1 \right) + \mu_{\theta,t} - \sigma_{ch,t} \sigma_{\theta,t} \\ + y_{b,t} (\mathbb{E}[dr_{y,t}] - r_{d,t} + (\sigma_{\theta,t} - \sigma_{ch,t})(\sigma + \sigma_{q,t})). \end{aligned}$$

The first-order conditions imply

$$d\zeta_{b,t} > 0 \text{ if } \theta_t \leq 1; \quad d\zeta_{b,t} = 0 \text{ if } \theta_t \in (1, 1 + \gamma); \quad d\zeta_{b,t} < 0 \text{ if } \theta_t \geq 1 + \gamma;$$

$$\mathbb{E}[dr_{y,t}] - r_{d,t} = (\sigma_{ch,t} - \sigma_{\theta,t})(\sigma + \sigma_{q,t}).$$

The FOC on asset holdings can be re-written as

$$\mathbb{E}[dr_{y,t}] - r_{f,t} = (\sigma_{ch,t} - \sigma_{\theta,t})(\sigma + \sigma_{q,t}) - (r_{f,t} - r_{d,t}).$$

Plugging these first-order conditions into the HJB and re-arranging yields

$$\mu_{\theta,t} = r_{f,t} - r_{d,t} + \sigma_{ch,t}\sigma_{\theta,t}.$$

The equations in the text follow letting $\sigma_{ch,t} = \sigma$ and $\mu_{ch,t} = g$. □

B.1.3 Characterizing Equilibrium

Proof of Lemma 3, law of motion for η_t . After substituting banks' asset pricing condition and using $x_{b,t} = \psi_t/\eta_t$, the process for banks' aggregate equity $N_{b,t}$ is

$$\begin{aligned} \frac{dN_{b,t}}{N_{b,t}} &= \left(r_{d,t} + \frac{\psi_t}{\eta_t} (\sigma_{ch,t} - \sigma_{\theta,t})(\sigma + \sigma_{q,t}) \right) dt - \frac{d\zeta_{b,t}}{N_{b,t}} + \frac{\psi_t}{\eta_t} (\sigma + \sigma_{q,t}) dW_t \\ &= \left(r_{d,t} + \frac{\psi_t}{\eta_t} (\sigma_{ch,t} - \sigma_{\theta,t})(\sigma + \sigma_{q,t}) \right) dt - \frac{d\zeta_{b,t}}{N_{b,t}} + \frac{\psi_t}{\eta_t} (\sigma + \sigma_{q,t}) dW_t. \end{aligned}$$

The law of motion for the value of the tree is

$$\frac{d(q_t Y_t)}{q_t Y_t} = (\mu_{q,t} + g + \sigma \sigma_{q,t}) dt + (\sigma + \sigma_{q,t}) dW_t.$$

Because banks' asset pricing condition always holds in equilibrium, we may write

$$\begin{aligned} \mu_{q,t} &= r_{f,t} - DS_t + (\sigma_{ch,t} - \sigma_{\theta,t})(\sigma + \sigma_{q,t}) - \frac{1}{q_t} - g - \sigma \sigma_{q,t}, \\ &= r - DS_t - \sigma_{\theta,t}(\sigma + \sigma_{q,t}) - \frac{1}{q_t}. \end{aligned}$$

Substituting into the law of motion for the value of capital

$$\frac{d(q_t Y_t)}{q_t Y_t} = \left(r - DS_t - \sigma_{\theta,t}(\sigma + \sigma_{q,t}) - \frac{1}{q_t} + g + \sigma \sigma_{q,t} \right) dt + (\sigma + \sigma_{q,t}) dW_t.$$

By Ito's quotient rule,

$$\frac{d(1/(q_t Y_t))}{1/(q_t Y_t)} = \left((\sigma + \sigma_{q,t})^2 - \left(r - DS_t - \sigma_{\theta,t}(\sigma + \sigma_{q,t}) - \frac{1}{q_t} + g + \sigma \sigma_{q,t} \right) \right) dt - (\sigma + \sigma_{q,t}) dW_t.$$

Using Ito's product rule,

$$\begin{aligned} \frac{d\eta_t}{\eta_t} &= \left(r_{d,t} + \frac{\psi_t}{\eta_t} (\sigma_{ch,t} - \sigma_{\theta,t})(\sigma + \sigma_{q,t}) \right) dt - \frac{d\zeta_{b,t}}{N_{b,t}} \\ &\quad + \left((\sigma + \sigma_{q,t})^2 - \left(r - DS_t - \sigma_{\theta,t}(\sigma + \sigma_{q,t}) - \frac{1}{q_t} + g + \sigma \sigma_{q,t} \right) \right) dt - (\sigma + \sigma_{q,t}) dt \\ &\quad - \frac{\psi_t}{\eta_t} (\sigma + \sigma_{q,t})^2 dt + \left(\frac{\psi_t}{\eta_t} - 1 \right) (\sigma + \sigma_{q,t}) dW_t. \end{aligned}$$

The drift simplifies to

$$\mu_{\eta,t} = \left(\frac{\psi_t}{\eta_t} - 1 \right) ((\sigma_{ch,t} - \sigma_{\theta,t}) - (\sigma + \sigma_{q,t}))(\sigma + \sigma_{q,t}) + \frac{1}{q_t} - \frac{d\zeta_{b,t}}{N_{b,t}}.$$

Define $d\Xi_t \equiv d\zeta_{b,t}/N_{b,t}$ as a control creating an upper reflecting barrier. The drift and volatility of $d\eta_t/\eta_t$ now match (35).

□

Proof of Proposition 11. The proposition has the same proof as the analogous statements in [Phe-](#)

lan (2016); Chen and Phelan (2021). We derive the smooth pasting condition here. Suppose there exists a market for claims on the aggregate capital stock. Arbitrage pricing for the risky asset requires that

$$q_t Y_t = \mathbb{E}_t \left[\int_t^\infty \frac{\xi_\tau}{\xi_t} \text{Div}_\tau d\tau \right] = C_{h,t} \mathbb{E}_t \left[\int_t^\infty e^{-r(s-t)} \frac{\text{Div}_s}{C_{h,\tau}} d\tau \right]$$

where Div_t are dividends from capital. Define $G_t \equiv q_t Y_t / C_{h,t}$. Then the previous equation simplifies to

$$G_t = \mathbb{E}_t \left[\int_t^\infty e^{-r(\tau-t)} \frac{\text{Div}_s}{C_{h,\tau}} d\tau \right],$$

which implies the equation

$$rG_t = \frac{\text{Div}_t}{C_{h,t}} + \mathbb{E}[dG_t].$$

Drop time subscripts to reduce clutter and plug in for household consumption.

$$G \equiv \frac{qY}{C_h} = \frac{qY}{rq(1 + (\theta - 1)\eta)K} = \frac{1}{r(1 + (\theta - 1)\eta)}.$$

If there is a reflecting barrier at $\eta = \eta^R$, then smooth pasting requires

$$0 = \frac{\partial}{\partial \eta} G(\eta^R) = -G \frac{\theta'(\eta^R)\eta^R + \theta(\eta^R) - 1}{1 + (\theta(\eta^R) - 1)\eta^R}.$$

Note that this condition is equivalent to requiring $q'(\eta^R) = 0$, which is the standard smooth pasting condition for an asset price. The derivative of G with respect to η captures the marginal benefit of increasing η at the barrier, and it should equal the marginal cost of the barrier. However, because banks pay the cost of reflection, someone holding a claim to the aggregate capital stock faces zero marginal cost. Therefore, the smooth-pasting condition for q_t at a barrier η^R is

$$\theta'(\eta^R)\eta^R + \theta(\eta^R) = 1. \quad (62)$$

□

B.1.4 Decreasing Household Value Function

Proof of Proposition 12. We first begin with a lemma.

Lemma 4. Consider the following ODE

$$rV(\eta) = u(\eta) + \mu_\eta(\eta)V'(\eta) + \frac{\sigma_\eta^2(\eta)}{2}V''(\eta).$$

over the range be $\eta \in [\underline{\eta}, \bar{\eta}]$, subject to two boundary conditions $V'(\bar{\eta}) = 0$ and $V'(\underline{\eta}) = 0$. Suppose that $r > 0$, μ_η and σ_η^2 are both strictly positive functions of η . The function $u(\eta)$ is strictly decreasing in η . Then V is decreasing over $[\underline{\eta}, \bar{\eta}]$ with $V(\bar{\eta}) > V(\underline{\eta})$.

Proof. We can rearrange the HJB ODE to:

$$V'' = \frac{2}{\sigma_\eta^2} (rV - u - \mu_\eta V')$$

First, we can say $V(\underline{\eta}) < \frac{u(\underline{\eta})}{r}$, because the RHS is the value of receiving $u(\underline{\eta})$ forever, which is the highest possible value. But since η moves around, the discounted value has to be lower. Hence, $V''(\underline{\eta}) < 0$, which means that $V' < 0$ local to 0.

Second, $V(\bar{\eta}) > \frac{u(\bar{\eta})}{r}$, by similar logic. If the process stayed at 1 forever, the RHS is the payoff you'd get, which is the lowest possible value. But since η moves around, you can only get higher than that. Hence, $V''(\bar{\eta}) > 0$. Thus, the second derivative must flip signs at some point.

Let η_1 be the first time that $V'' = 0$. Note that V is decreasing over $[0, \eta_1]$ and $V'(\eta_1) < 0$. Then we have

$$rV(\eta_1) = u(\eta_1) + \mu_{\eta} V'(\eta_1) < u(\eta_1),$$

which means that $rV < u$ over $[\underline{\eta}, \eta_1]$.

Let η_2 be the highest η such that $V'' = 0$. Note that V is decreasing over $[\eta_2, \bar{\eta}]$ and $V'(\eta_2) < 0$. Again we have

$$rV(\eta_2) = u(\eta_2) + \mu_{\eta} V'(\eta_2) < u(\eta_2),$$

which again means that $rV(\eta_2) < u(\eta_2)$.

If V is increasing anywhere over $[\eta_1, \eta_2]$, there there must be a local minimum and a local maximum, given the derivatives at η_1 and η_2 . For a local minimum, there must be an $\eta_3 \in [\eta_1, \eta_2]$ such that $V'(\eta_3) = 0$ and $V''(\eta_3) > 0$. From the HJB, this would require that $rV(\eta_3) > u(\eta_3)$, thus any local minimum occurs above u . For a local maximum, there must be an $\eta_4 \in [\eta_1, \eta_2]$ such that $V'(\eta_4) = 0$ and $V''(\eta_4) < 0$. This could only occur if $rV(\eta_3) < u(\eta_3)$. Thus any local maximum occurs below u .

However, this is impossible. Since $rV(\eta_1) < u(\eta_1)$ and $V'(\eta_1) < 0$, any local minimum would

require the function rV crossing u and then a local maximum would have to occur there. But a local maximum must occur below u . By the same logic, Since $rV(\eta_2) < u(\eta_2)$ and $V'(\eta_2) < 0$, and since any local maximum must occur below u , that would require a local minimum to occur below u , which is impossible. Hence, it is not possible for $V' > 0$ over $[\eta_1, \eta_2]$ and hence there are no local maxima over $[\underline{\eta}, \bar{\eta}]$. $\square$

Applying the lemma is straightforward. First, because $\underline{\eta}$ and $\bar{\eta}$ are regulated barriers, the smooth pasting conditions require $H'(\underline{\eta}) = 0$ and $H'(\bar{\eta}) = 0$. Second, utility from liquidity is decreasing in η . Aggregate liquidity as a fraction of total wealth is

$$X(\eta) \equiv \frac{D(\eta)}{W_h(\eta)} = \frac{1 - \eta}{1 + (\theta - 1)\eta} = (1 - \eta)qr$$

where we impose that $\psi = 1$. This term is the fraction of household wealth not held in bank equity. $X(\eta)$ is decreasing if $1 - X(\eta)$ is increasing.

$$1 - X(\eta) = \frac{\theta\eta}{1 + (\theta - 1)\eta} = \theta\eta qr,$$

which is r times the bank valuation. By assumption this is increasing, and therefore X is decreasing. Since the convenience yield v is increasing in liquidity services, $v(X(\eta))$ is decreasing in η . $\square$

B.2 Additional Figures

First, Figure 7 plots the drift and volatility terms for the state variable η and the asset price q . The system is generally stable, as evidenced by the positive μ_η that is higher at low η , reflecting

higher risk premia and greater leverage. Greater leverage also increases systemic risk, though, as evidenced by the shape of σ_η (higher for low η). Endogenous volatility is reflected in the positive σ_q , which is hump-shaped in η .

[Figure 7 about here.]

B.2.1 *Liquidity Adjustment with Linear Utility for Deposits*

Figure 8 illustrates the same liquidity adjustment with linear deposit utility. As is clear, the deposit rate is now entirely exogenous, and welfare is a strictly increasing function in η regardless of the overall level of deposit spreads. The results here are similar to those in the main text.

[Figure 8 about here.]

Figure 9 considers an even more aggressive adjustment, with spreads adjusting $\eta < 0.1$ and at twice the rate. Welfare is strictly increasing for each case with the liquidity adjustment; the aggressive liquidity adjustment is definitively overturning the welfare result.

[Figure 9 about here.]

B.2.2 *Dividend Subsidies with Liquidity Adjustment*

Dividend subsidies can still be welfare improving even with the liquidity adjustment. Figure 10 plots welfare with dividend subsidies comparing several cases, varying the overall level of deposits and the degree of liquidity adjustments. Figure 10(a) and (b) consider the baseline liquidity adjustment but with high and low spreads, so that the welfare function is increasing with the liquidity

adjustment. Even in this case with the liquidity adjustment, dividend subsidies increase welfare, though more so for higher spreads. The bottom panels (c) and (d) consider the more aggressive liquidity adjustment. In these cases the dividend subsidy improves welfare only for low η and otherwise decreases welfare.

[Figure 10 about here.]

B.2.3 *High Deposit Spreads*

We now calibrate the deposit spreads to better match the values in the empirical literature. [Drechsler, Savov and Schnabl \(2017\)](#) report an average deposit spread of 1.26%, and the return on average assets for all U.S. banks is typically on the order of 1% (FRED USROA). We therefore calibrate $\beta_1 = 0.5292$ so that the deposit spread at $\bar{\eta}$ is 1.26%. Figure 11 plots equilibrium variables in this case. Importantly, the economy endogenously features no fire sales (banks' funding advantage is sufficiently large).

[Figure 11 about here.]

Figure 12 plots welfare and bank holdings varying issuance costs. In contrast to the baseline case, there are no fire sales for $\gamma < 5$, and only for $\gamma = 5$ is the welfare function non-decreasing. In the baseline calibration, welfare was non-decreasing for $\gamma = 3$.

[Figure 12 about here.]

Finally, we consider how the high-spread model performs with the liquidity adjustment and dividend subsidies in Figure 13. In both of these economies there are no fire sales ($\psi = 1$), which

means that the liquidity adjustment affects welfare mechanically by decreasing convenience utility. With the baseline liquidity adjustment, the prior results hold. With the aggressive liquidity adjustment, the welfare function is increasing in η (because of the mechanical response of convenience utility) and therefore dividend subsidies hurt welfare.

[Figure 13 about here.]

B.2.4 *Variations in Household Growth Rate*

We extend the model to include productivity losses when households hold the tree in the form of lower productivity growth. With log utility, solving the model tractably requires constant output (so that q and q' are analytical functions of θ and η), and so growth losses are a tractable way of incorporating productivity differences across agents. In particular, the productivity of household shares grow at the rate $\underline{g} \leq g$. A few equilibrium equations are modified by the lost growth when $\psi < 1$, but the most important change is the asset price condition, which now requires

$$g - \underline{g} + DS \geq ERP,$$

where banks now have an advantage owing to the deposit spread and the higher growth rate. If $\underline{g}$ is sufficiently small, then equilibrium features no fire sales ($\psi = 1$). Figure 14 plots the equilibrium asset price, bank holdings, normalized welfare, and drift of η for $\underline{g} = 0.02, 0.0195, 0.018$.

[Figure 14 about here.]

A lower household growth rate has two important implications for equilibrium. First, a lower

$\underline{g}$ leads to a lower asset price. But second and more importantly, a lower $\underline{g}$ leads to fewer fire sales (higher ψ) and therefore a more stable economy (note that μ_η is generally higher for low $\underline{g}$). Importantly, as a result, the slope of the welfare function becomes *more* negative for low $\underline{g}$. If $\underline{g} < 0.018$ then the economy features no fire sales and equilibrium is identical to the equilibrium with $\underline{g} = 0.018$, which also features no fire sales.

The reason for these results is that banks endogenously earn higher returns when $\underline{g}$ is lower. All else equal, fire sales become less probable in equilibrium. This outcome is an example of the *intermediation paradox* pointed out by Phelan (2016). Although productivity losses create adverse feedback between fire sales and output in bad times, the economy is counterintuitively more stable due to general equilibrium effects.

Due to these dynamics, welfare becomes more and more a decreasing function of η the lower $\underline{g}$ is. If banks can stave off fire sales through equity issuance, then high levels of equity become less valuable. The trade off between liquidity provision and equity shifts further in favor of liquidity provision as a result. Fire sales eliminate the trade off between liquidity provision and equity. During fire sales, banks deleverage because they are net worth constrained. Increasing the banking sector's equity level in bad times loosens financial frictions, so banks become willing to borrow more and increase liquidity provision.

These results suggest that an economy may be more vulnerable to financial instability when banks primarily provide liquidity. Large productivity losses are stabilizing because they raise the expected returns earned by banks due to general equilibrium effects. Higher returns help banks recapitalize faster, which promotes long-run stability. Without this endogenous “hedge” when

bank equity is low, banks have to deleverage through fire sales sooner, making financial crises more likely. This conclusion also implies that the main source of inefficiency from a distressed banking sector is not capital misallocation but depressed liquidity provision.

C MODEL WITH RISK-NEUTRAL AGENTS AND BROWNIAN MOTION

C.1 Setup and Equilibrium

With linear utility over consumption and deposits, it remains the case that $r_f = r$ and $r_d = r - \phi_L$. The risk-neutral case requires a few modifications compared to the log utility case. First, banks' optimality conditions are slightly modified. Second, the drift of η is modified.

Proposition 15. *Consider a finite process*

$$\frac{d\theta_t}{\theta_t} = \mu_{\theta,t} dt + \sigma_{\theta,t} dW_t, \quad (63)$$

with $\sigma_{\theta,t} \leq 0$. Then $\theta_t n_t$ represents the maximal future expected payoff that a bank with book value n_t can attain, and $\{y_t, d\zeta_t\}$ is optimal if and only if

1. $\theta_t \geq 1$ for all t , and $d\zeta_t > 0$ only when $\theta_t = 1$,
2. $\theta_t \leq 1 + \gamma$ for all t , and $d\zeta_t < 0$ only when $\theta_t = 1 + \gamma$
3. $\mu_{\theta,t} = \phi_L$,
4. $\mathbb{E}[dr_{y,t}] - r_d \leq -\sigma_{\theta,t}(\sigma + \sigma_{q,t})$ with strict equality when $y_t > 0$,

5. The transversality condition $\mathbb{E}[e^{-rt} \theta_t n_t] \rightarrow 0$ holds under $\{y_t, d\zeta_t\}$.

Hence, $-\sigma_{\theta,t}(\sigma + \sigma_{q,t})$ represents the bank's required risk-premium (or level of risk aversion). Banks will not pay dividends when $\theta_t \geq 1$, and, banks will issue equity when $\theta_t = 1 + \gamma$ (θ_t can never exceed $1 + \gamma$ since banks can always raise equity at a cost γ). The proposition has the same proof as the analogous statement in [Phelan \(2016\)](#) and in this paper.

Lemma 5. *The equilibrium law of motion of η_t will be endogenously given with*

$$\begin{aligned}\mu_{\eta,t} &= -\frac{(\psi_t - \eta_t)}{\eta_t}(\sigma + \sigma_{q,t})(\sigma_{\theta,t} + \sigma + \sigma_{q,t}) + \frac{1}{q_t}, \\ \sigma_{\eta,t} &= \frac{(\psi_t - \eta_t)}{\eta_t}(\sigma + \sigma_{q,t}), \\ d\Xi_t &= \frac{d\zeta_t}{N_t},\end{aligned}$$

where $d\zeta_t = \int d\zeta_t^b db$ and $\psi_t = \int y_t^b db$ is the fraction of the tree held by banks.

Proof of Lemma 5. We derive the law of motion for η_t in the more general case with productivity differences, growth differences, and an occasionally binding exogenous leverage constraint L . When banks hold y_t shares of the tree, dividends are produced at rate ay_t , and when households hold y_t shares of the tree, dividends are produced at rate $\underline{a}y_t$, with $\underline{a} \leq a$. Additionally, the productivity of the tree grows at rate g when held by banks and $\underline{g}$ when held by households,

First, we have

$$\begin{aligned}\frac{d(1/(q_t Y_t))}{1/(q_t Y_t)} &= ((\sigma + \sigma_{q,t})^2 - g\psi_t - (1 - \psi_t)\underline{g} - \mu_{q,t} - \sigma\sigma_{q,t})dt - (\sigma + \sigma_{q,t})dW_t \\ &= ((\sigma + \sigma_{q,t})^2 + (1 - \psi_t)(g - \underline{g}) - g - \mu_{q,t} - \sigma\sigma_{q,t})dt - (\sigma + \sigma_{q,t})dW_t.\end{aligned}$$

If leverage constraints do not bind, then

$$\frac{dN_t}{N_t} = \left(r - \phi_L - \frac{\psi_t}{\eta_t} \sigma_{\theta,t} (\sigma + \sigma_{q,t}) - \frac{d\zeta_t}{N_t} \right) dt + \frac{\psi_t}{\eta_t} (\sigma + \sigma_{q,t}) dW_t.$$

and banks' asset pricing condition implies

$$-g - \mu_{q,t} - \sigma \sigma_{q,t} = \frac{a}{q_t} - (r - \phi_L) + \sigma_{\theta,t} (\sigma + \sigma_{q,t}).$$

By Ito's product rule,

$$\begin{aligned} \frac{d\eta_t}{\eta_t} = & \left(\frac{a}{q_t} + (1 - \psi_t)(g - \underline{g}) - \left(\frac{\psi_t}{\eta_t} - 1 \right) (\sigma_{\theta,t} + \sigma + \sigma_{q,t})(\sigma + \sigma_{q,t}) \right) dt \\ & - d\Xi_t + \left(\frac{\psi_t}{\eta_t} - 1 \right) (\sigma + \sigma_{q,t}) dW_t. \end{aligned}$$

Otherwise, the leverage constraint L binds, hence $\psi_t = (1 + L)\eta_t$. The law of motion for bank equity is

$$\frac{dN_t}{N_t} = \left(r - \phi_L + \frac{\psi_t}{\eta_t} (\mathbb{E}[dr_{y,t}] - (r - \phi_L)) - \frac{d\zeta_t}{N_t} \right) dt + \frac{\psi_t}{\eta_t} (\sigma + \sigma_{q,t}) dW_t$$

and households' asset pricing condition implies

$$\mu_{q,t} = r - \frac{a}{q_t} - \underline{g} - \sigma \sigma_{q,t},$$

or equivalently

$$-g - \mu_{q,t} - \sigma \sigma_{q,t} = \frac{a}{q_t} - r - (g - \underline{g}),$$

which we may use to write

$$\begin{aligned} \mathbb{E}[dr_{y,t}] - (r - \phi_L) &= \frac{a}{q_t} + g + r - \frac{a}{q_t} - \underline{g} - \sigma \sigma_{q,t} + \sigma \sigma_{q,t} - (r - \phi_L) \\ &= \frac{a - \underline{a}}{q_t} + (g - \underline{g}) + \phi_L \\ \frac{d(1/(q_t Y_t))}{1/(q_t Y_t)} &= \left((\sigma + \sigma_{q,t})^2 - \psi_t (g - \underline{g}) + \frac{a}{q_t} - r \right) dt - (\sigma + \sigma_{q,t}) dW_t. \end{aligned}$$

By Ito's product rule,

$$\begin{aligned} \frac{d\eta_t}{\eta_t} &= \left(\frac{\psi_t}{\eta_t} \left(\frac{a - \underline{a}}{q_t} + (g - \underline{g}) + \phi_L \right) - \phi_L + (\sigma + \sigma_{q,t})^2 - \psi_t (g - \underline{g}) + \frac{a}{q_t} - \frac{\psi_t}{\eta_t} (\sigma + \sigma_{q,t})^2 \right) dt \\ &\quad - d\Xi_t + \left(\frac{\psi_t}{\eta_t} - 1 \right) (\sigma + \sigma_{q,t}) dW_t \\ &= \frac{\psi_t}{\eta_t} \frac{a}{q_t} - \left(\frac{\psi_t}{\eta_t} - 1 \right) \left(\frac{a}{q_t} - \phi_L \right) + \psi_t \left(\frac{1}{\eta_t} - 1 \right) (g - \underline{g}) - \left(\frac{\psi_t}{\eta_t} - 1 \right) (\sigma + \sigma_{q,t})^2 dt \\ &\quad - d\Xi_t + \left(\frac{\psi_t}{\eta_t} - 1 \right) (\sigma + \sigma_{q,t}) dW_t \end{aligned}$$

Plugging in for ψ_t completes the proof. □

Equilibrium consists of a law of motion for η_t and asset allocations and prices as functions of η . The asset prices are $q(\eta)$, $\theta(\eta)$, and the flow allocation is $\psi(\eta)$; we suppress time subscripts to reduce notation unless necessary for clarity. We solve for equilibrium by converting the equilib-

rium conditions into a system of differential equations (“ODE”) in the asset prices q and θ . Given $q(\eta), q'(\eta)$ and $\theta(\eta), \theta'(\eta)$ We can get equilibrium returns and allocations to get $q''(\eta), \theta''(\eta)$. We solve the ODE using appropriate boundary conditions.

Proposition 16. *The equilibrium domain of the functions $q(\eta), \theta(\eta)$, and $\psi(\eta)$ is an interval $[\underline{\eta}, \bar{\eta}]$. The function $q(\eta)$ is increasing, $\theta(\eta)$ is decreasing, and the following boundary conditions hold: (i) $\theta(\bar{\eta}) = 1$; (ii) $\theta(\underline{\eta}) = 1 + \gamma$; (iii) $q'(\bar{\eta}) = 0$; (iv) $\theta'(\bar{\eta}) = 0$; (v) $\theta'(\underline{\eta})\underline{\eta} + \theta(\underline{\eta}) = 1$. Over $[\underline{\eta}, \bar{\eta}]$, $\theta_t \geq 1$ and $d\zeta_t = 0$. At $\bar{\eta}$, $d\zeta_t > 0$, and at $\underline{\eta}$, $d\zeta_t < 0$, creating regulated barriers for the process η_t .*

The first two boundary conditions reflect optimal policy for dividends/equity issuance. The conditions for $q'(\bar{\eta})$ and $\theta'(\bar{\eta})$ are smooth pasting conditions. If the price function and marginal value of bank equity are twice-continuously differentiable, then equations (24) and (28) are functions of η :

$$\frac{dq_t}{q_t} = \mu_q(\eta_t)dt + \sigma_q(\eta_t)dW_t, \quad \frac{d\theta_t}{\theta_t} = \mu_\theta(\eta_t)dt + \sigma_\theta(\eta_t)dW_t, \quad (64)$$

where the drift and variance terms are determined by the derivatives of $q(\eta)$ and $\theta(\eta)$. (For the remainder of the paper, the dependence on the state-variable η_t is suppressed for notational ease.)

Boundary Conditions. Here we derive the boundary condition for costly equity issuance. Since $\underline{\eta}$ is a reflecting barrier, the boundary condition for $\theta'(\underline{\eta})$ is

$$\theta(\underline{\eta}) + \theta'(\underline{\eta})\underline{\eta} = 1. \quad (65)$$

Since $\theta(\underline{\eta}) = 1 + \gamma$, this simplifies to

$$\theta'(\underline{\eta}) = -\frac{\gamma}{\underline{\eta}}.$$

We now consider dividend subsidies. When banks issue a dollar of dividends, the government also pays out χ dollars and funds these payouts by lump sum taxes on households. We model this policy by modifying banks' objective function to be

$$\mathbb{E} \left[\int_0^\infty e^{-rt} ((1 + \chi) d\zeta_t^b \mathbb{1}_{d\zeta_t^b \geq 0} + d\zeta_t^b \mathbb{1}_{d\zeta_t^b < 0}) \right].$$

This objective function implies $\theta(\bar{\eta}) = 1 + \chi$. The smooth-pasting condition becomes

$$\theta'(\bar{\eta}) = \frac{1}{\bar{\eta}} (1 - (1 + \chi)) = \frac{\chi}{\bar{\eta}}$$

since it still costs banks one unit of equity to pay dividends.

We now suppose the government provides Tail Risk Insurance. The boundary condition $\theta(\underline{\eta})$ changes to

$$\theta(\underline{\eta}) = -\theta'(\underline{\eta})\underline{\eta}. \quad (66)$$

To derive this boundary condition, note that tail risk insurance is equivalent to a lower reflecting

barrier with zero cost of reflection, hence

$$\theta(\underline{\eta}) + \theta'(\underline{\eta})\underline{\eta} = 0. \quad (67)$$

Re-arranging yields (66). Note that $\theta' < 0$, hence this condition is positive.

Finally, we consider a policy of Dividend Restrictions. We model this by assuming banks cannot issue dividends until $\eta = \bar{\eta}^{DR}$. Because banks are restricted from issuing dividends, it is possible that $\theta(\bar{\eta}^{DR}) < 1$. Since $\bar{\eta}^{DR}$ is still a reflecting barrier, the boundary condition for $\theta'(\bar{\eta}^{DR})$ is

$$\theta(\bar{\eta}^{DR}) + \theta'(\bar{\eta}^{DR})\bar{\eta}^{DR} = 1. \quad (68)$$

Note that without a restriction $\theta(\bar{\eta}) = 1$ and therefore $\theta'(\bar{\eta}) = 0$.

C.2 Numerically Comparing the Log-utility and Linear Models

We compare the linear model to the log utility model. The dynamics of the models are quite similar, with some differences in levels for key variables, and solving the linear model requires adjusting some of the parameters, e.g., we require that $r - \phi_L - g > 0$ since the frictionless asset price would be $\frac{1}{r - \phi_L - g}$, which must be well-defined. We adjust the parameters only slightly, setting $r = 4\%$ and $\phi_L = .005 = r\beta_1$. We then solve the linear model and compare to the original model (log-log) and to the model with log utility over consumption and linear utility over deposits (log-linear). Hence, we call the risk-neutral model the “linear-linear model.” In the log-linear model, we set β_1 equal

to the average deposit spread in the baseline model (which is 0.59%).

Figure 15 plots asset holdings, risk premium, the stationary distribution, and welfare across these economies. Figure 16 plots the evolution of η and q . As is clear, the dynamics of the models are very similar with the main, and inconsequential, difference being that μ_q is shifted across the models.¹² The asset price q is also shifted higher in the linear-linear model. Most importantly, the welfare result that the welfare function is decreasing in η holds across the models. Risk aversion appears to mute the size of that result, as evidenced by a flatter welfare function compared to the linear-linear model. In sum, the reader should not worry that introducing risk aversion for the Brownian model is driving the results, nor that imposing risk-neutrality in the two-state model is a knife-edge case.

[Figure 15 about here.]

[Figure 16 about here.]

C.3 *Robustness in the Linear Brownian Model*

In this section, we first show the robustness of our main result that welfare is a decreasing function to different levels of ϕ_L and γ , which determine how the degree to which banks have to sell the tree to households. In Figure 17 we vary the liquidity premium ϕ_L . A lower liquidity premium leads to larger fire sales and can therefore lead to an increasing welfare function. Figure 18 varies the issuance cost. As is true in the baseline log-log model, a higher issuance cost decreases welfare and can lead to an increasing welfare function because fire sales are worsened when the financial friction is tighter.

[Figure 17 about here.]

[Figure 18 about here.]

We also show the results when output or productivity growth falls whenever households hold the tree. The log utility model is very tractable when output is given by an exogenous tree. However, the linear model has some computational advantages when output is *endogenous*, i.e., when the rate of output or the growth rate vary when the households hold the tree. We provide those results here.

We extend the model to include productivity losses when households hold the tree in the form of lost output and/or lower productivity growth. In particular, households only retain a fraction $\underline{a} \leq 1$ of dividends from shares of the tree, and the productivity of these shares grow at the rate $\underline{g} \leq g$. To obtain an equilibrium where banks sell the tree to households for low range of η (i.e., with fire sales), thus generating productivity losses from decreased intermediation, we lower the liquidity value of debt from 1% to 0.4% and we set $g = 1\%$.¹³ All else equal, productivity losses actually *reinforce* our baseline results. Figure 19 illustrates equilibrium when $\underline{a} < 1$. Notably, when $\underline{a}$ is lower, household welfare and its rate of decrease with respect to η are higher. The case of $\underline{g} < g$ is similar and is shown in Figure 20.

[Figure 19 about here.]

The reason for these results is that banks endogenously earn higher returns when $\underline{a}$ is lower. Because trees produce less output when held by households, the more that $\underline{a}$ falls, the more that η can fall before households are willing to buy the tree from banks. All else equal, fire sales

become less probable in equilibrium, which the top right panel in Figure 19 confirms. Because fire sales amplify the severity of adverse shocks, a lower likelihood of fire sales induces banks to borrow more since banks expect less downside risk. With higher leverage, banks earn higher Sharpe ratios. The Sharpe ratio also rises because the returns from dividends increase due to a lower q when $\underline{a} < 1$. The lower right panel shows that the Sharpe ratio globally increases as $\underline{a}$ falls. This outcome is an example of the *intermediation paradox* pointed out by Phelan (2016). Although productivity losses create adverse feedback between fire sales and output in bad times, the economy is counterintuitively more stable due to general equilibrium effects.

Due to these dynamics, welfare becomes more and more a decreasing function of η the lower $\underline{a}$ is. If banks can stave off fire sales through equity issuance, then high levels of equity become less valuable. The trade off between liquidity provision and equity shifts further in favor of liquidity provision as a result. Conversely, the greater the frequency with which households hold the risky asset, the greater the gains from bank equity. Indeed, when $\underline{a} = 1$ and banks sell up to nearly 50% of the tree in bad times, household welfare is generally increasing except for very low η . The reason is that fire sales eliminate the trade off between liquidity provision and equity. During fire sales, banks deleverage because they are net worth constrained. Increasing the banking sector's equity level in bad times loosens financial frictions, so banks become willing to borrow more and increase liquidity provision.

[Figure 20 about here.]

These results suggest that an economy may be more vulnerable to financial instability when banks primarily provide liquidity. Large productivity losses are stabilizing because they raise the

expected returns earned by banks due to general equilibrium effects. Higher returns help banks recapitalize faster, which promotes long-run stability. Without this endogenous “hedge” when bank equity is low, banks have to deleverage through fire sales sooner, making financial crises more likely. This conclusion also implies that the main source of inefficiency from a distressed banking sector is not capital misallocation but depressed liquidity provision.

Finally, Figure 21 confirms that tail risk insurance improves welfare.

[Figure 21 about here.]

D ANALYTICAL MODEL WITH UPSIDE RISK

In this section, we extend the baseline analytical model to also include upside risk and demonstrate that there exists a mapping between the baseline model and this extension. For simplicity, we assume a constant liquidity value of deposits.

D.1 Set Up

Technology, Environment, Markets, and Optimization There are now three states in the model, $s \in \{1, D, U\}$. In state $s = 1$, there is a probability λ_D of transitioning to state $s = D$, a probability λ_U of transitioning to state $s = U$, and a probability $1 - \lambda_D - \lambda_U$ of no transition. Once the transition occurs, the economy remains permanently in that state. We modify Assumption 1 to be

Assumption 6. *The growth rates satisfy $g_U > g_1 > g_D$.*

Under this assumption, $s = U$ is an “up” state, and $s = D$ is a “down” state.

Returns, the evolution of the risky asset price, and the available financial markets are unchanged. We also assume that households and banks solve the same problems as before.

Portfolio Choice By construction, the economies in states $s = U$ and $s = D$ are deterministic, with banks financing themselves entirely with debt. Thus, we focus on banks' decision-making in state $s = 1$. Banks choose $y_{t,1}$ shares in the tree and issue dividends at rate $d\zeta_{t,1}^b$ subject to the transition equations

$$dn_{t,1}^b = (n_{t,1}^b r_d + q_{t,1} y_{t,1} (dr_{y,t,1} - r_d) dt - d\zeta_{t,1}^b, \quad (69)$$

$$n_{t,D}^b = n_{t,1}^b + y_{t,1} (q_{t,D} - q_{t,1}), \quad (70)$$

$$n_{t,U}^b = n_{t,1}^b + y_{t,1} (q_{t,U} - q_{t,1}). \quad (71)$$

The only change is the additional equation (71), which captures the change in bank equity after a transition to state $s = U$.

It is again convenient to replace the control $y_{t,1}$ with $x_{t,1} = q_{t,1} y_{t,1} / n_{t,s}^b$, which is the share of a bank's equity invested in the risky asset. The Hamilton-Jacobi-Bellman (HJB) equation is

$$\begin{aligned} r\theta_{t,1}n_{t,1} &= \max_{x_{t,s}, d\zeta_{t,1}^b \geq 0} d\zeta_{t,1} + \mathbb{E}[d(\theta_{t,1}n_{t,1})] \\ &= \max_{x_{t,s}, d\zeta_{t,1}^b \geq 0} d\zeta_{t,1} + \theta_{t,1}(n_{t,1}(r_d + x_{t,1}(dr_{y,t,1} - r_d)) - d\zeta_{t,1}) + \frac{\partial(\theta_{t,1}n_{t,1})}{\partial t} \\ &\quad + \lambda_D \left(\theta_{t,D}n_{t,1} \left(1 + x_{t,1} \frac{q_{t,D} - q_{t,1}}{q_{t,1}} \right) - \theta_{t,1}n_{t,1} \right) \\ &\quad + \lambda_U \left(\theta_{t,U}n_{t,1} \left(1 + x_{t,1} \frac{q_{t,U} - q_{t,1}}{q_{t,1}} \right) - \theta_{t,1}n_{t,1} \right). \end{aligned}$$

The HJB has first-order conditions (FOCs)

$$(x_{t,s}) : \quad 0 = \theta_{t,1} n_{t,1} (dr_{y,t,1} - r_d) + \lambda_D \theta_{t,D} n_{t,1} \frac{q_{t,D} - q_{t,1}}{q_{t,1}} + \lambda_U \theta_{t,U} n_{t,1} \frac{q_{t,U} - q_{t,1}}{q_{t,1}}$$

$$(d\zeta_{t,1}) : \quad 0 \geq 1 - \theta_{t,1} \text{ subject to the additional constraint } d\zeta_{t,1} \geq 0$$

Re-arranging the first FOC yields

$$dr_{y,t,1} - r_d = -\lambda_D \frac{\theta_{t,D}}{\theta_{t,1}} \frac{q_{t,D} - q_{t,1}}{q_{t,1}} - \lambda_U \frac{\theta_{t,U}}{\theta_{t,1}} \frac{q_{t,U} - q_{t,1}}{q_{t,1}}. \quad (72)$$

The second FOC yields the same bang-bang control.

D.2 Equilibrium

Observe that the condition relating the evolution of $\theta_{t,1}$ to the liquidity value of debt ϕ_L derived from banks' HJB is identical. To see this, one only needs to plug (72) into the HJB. Additionally, banks' new asset pricing condition (72) can be mapped into the form stated in the main text (9).

Substitute the risky asset price and the marginal value of bank equity in states $s = D$ and $s = U$ into (72).

$$dr_{y,t,1} - r_d = -\lambda_D \frac{1}{\theta_{t,1}} \frac{q_D - q_{t,1}}{q_{t,1}} - \lambda_U \frac{1}{\theta_{t,1}} \frac{q_U - q_{t,1}}{q_{t,1}},$$

where

$$q_D = \frac{1}{r - \phi_L - g_D},$$

$$q_U = \frac{1}{r - \phi_L - g_U}.$$

Re-arranging yields

$$dr_{y,t,1} - r_d = \frac{-\lambda_D q_D - \lambda_U q_U + (\lambda_D + \lambda_U) q_{t,1}}{\theta_{t,1} q_{t,1}} = -\frac{\lambda}{\theta_{t,1}} \left(\frac{\bar{q}_2 - q_{t,1}}{q_{t,1}} \right),$$

where

$$\lambda = \lambda_D + \lambda_U, \quad \bar{q}_2 = \frac{1}{\lambda_D + \lambda_U} (\lambda_D q_D + \lambda_U q_U).$$

Under this view, the risk premium essentially is the same, where λ remains the probability that the economy transitions away from $s = 1$ (but now to one of two different states), and the risk premium is still proportional to the expected change in the risky asset price. Further, $\bar{q}_2$ is the expected risky asset price conditional on a transition away from state $s = 1$. To recover the baseline model's $s = 2$ price, set $\lambda_U = 0$. As long as $\bar{q}_2 < q_{t,1}$, then the risk premium is positive. Adding upside risk merely increases the expected asset price when the economy transitions away from state $s = 1$.

Explicitly, given the parameters $\lambda_D, \lambda_U, g_D, g_U$, this extended model is isomorphic to the base-

line model when $\lambda = \lambda_D + \lambda_U$ and g_2 satisfies

$$g_2 = (r - \phi_L) \left(1 - \frac{(\lambda_D + \lambda_U)^2}{\lambda_D \lambda_U} \right) + (\lambda_D + \lambda_U) \left(\frac{g_D}{\lambda_D} + \frac{g_U}{\lambda_U} \right),$$

which follows from setting $q_2 = \bar{q}_2$:

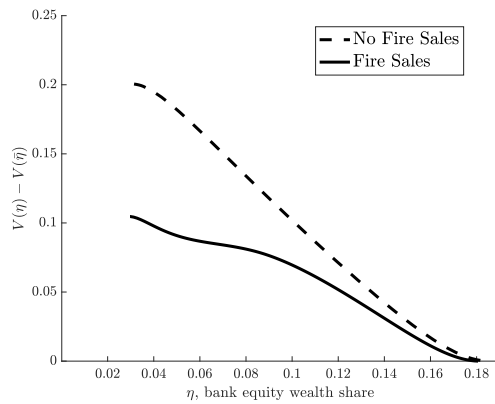
$$\begin{aligned} \frac{1}{r - \phi_L - g_2} &= \frac{1}{\lambda_D + \lambda_U} \left(\frac{\lambda_D}{r - \phi_L - g_D} + \frac{\lambda_U}{r - \phi_L - g_U} \right), \\ r - \phi_L - g_2 &= \frac{\lambda_D + \lambda_U}{\lambda_D} (r - \phi_L - g_D) + \frac{\lambda_D + \lambda_U}{\lambda_U} (r - \phi_L - g_U). \end{aligned}$$

Re-arranging yields the condition on g_2 . The parameters $\lambda_D, \lambda_U, g_D, g_U$ also need to imply a g_2 such that $g_2 < g_1$ still, as we assumed in the main text. In other words, our model's results hold as long as the downside risk is sufficiently larger than the upside risk.

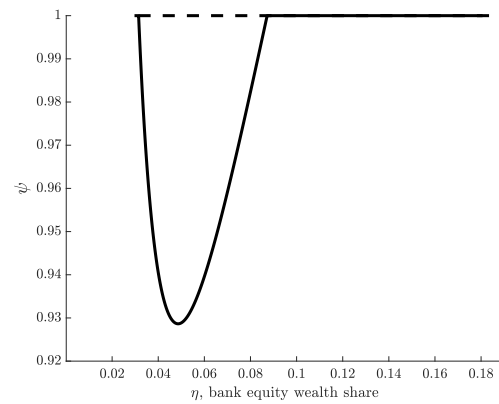
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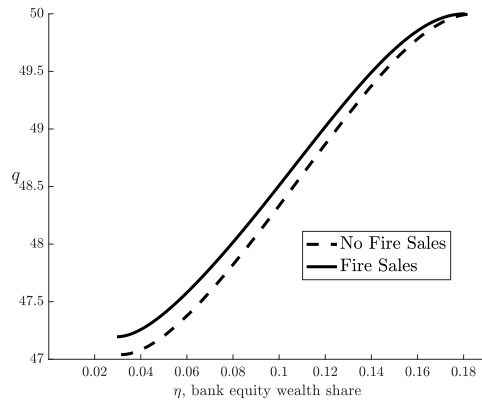
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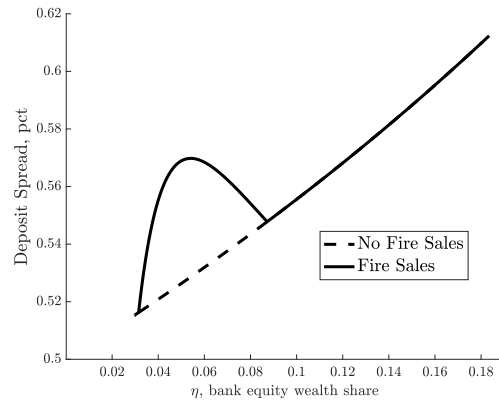
(a) Welfare $V(\eta) - V(\bar{\eta})$



(b) Bank asset holdings ψ

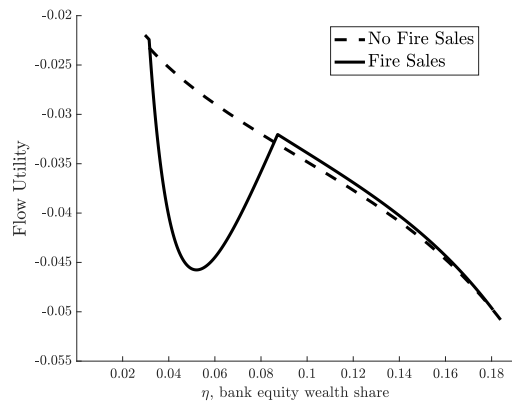


(c) Asset price q

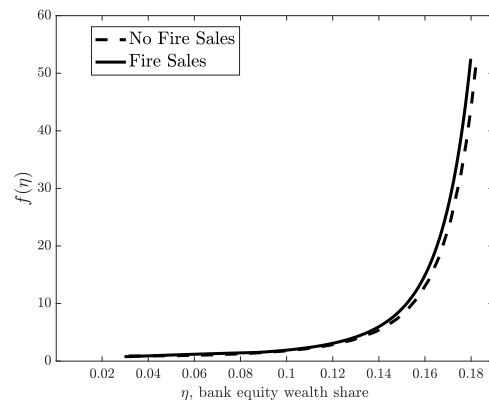


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Figure 1: Welfare is decreasing in aggregate bank equity η . Dashed lines show economy with no fire sales ($\psi = 1$) and solid lines show economy with fire sales ($\psi \leq 1$).

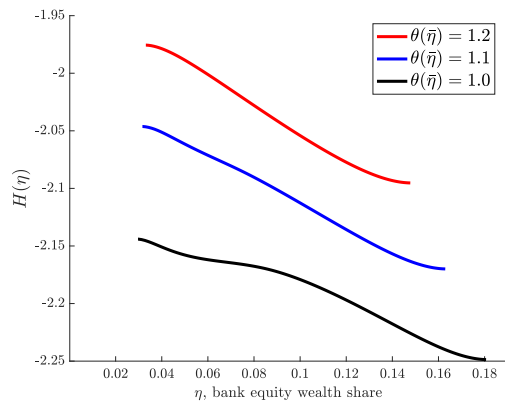


(a) Flow Utility

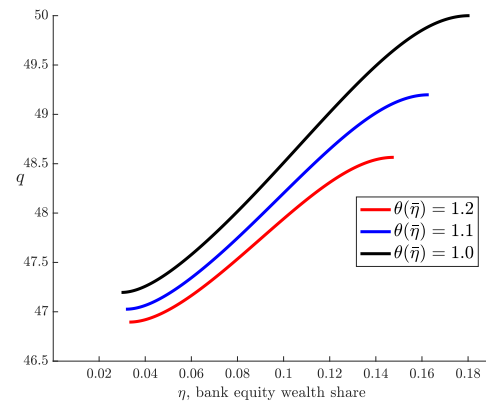


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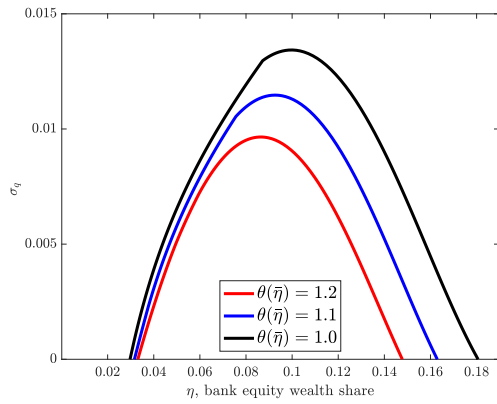
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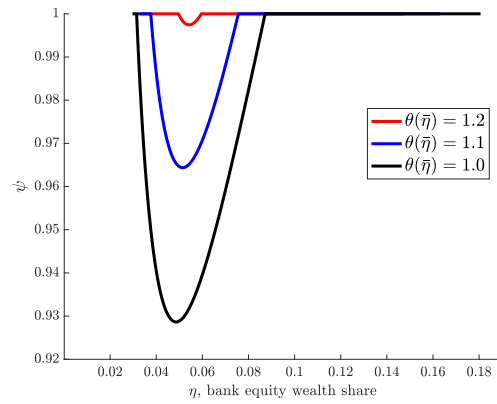
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(b) Asset Price



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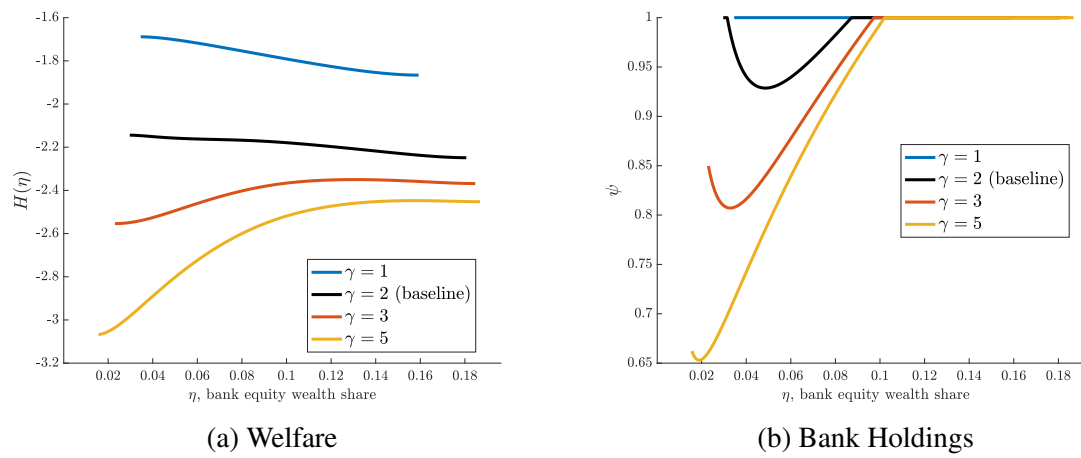
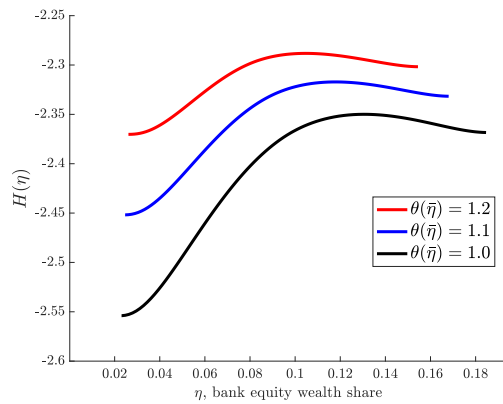
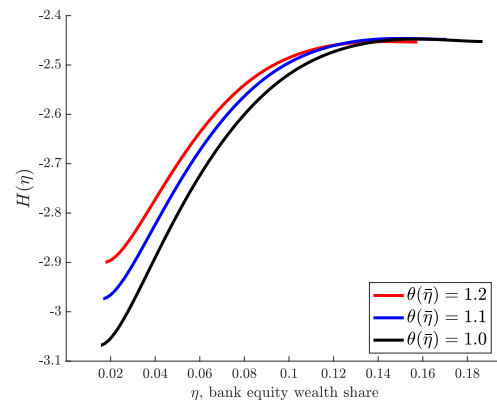


Figure 4: Varying issuance costs. Welfare is an increasing function of η when issuance costs are sufficiently high, implying fire sales are sufficiently severe.

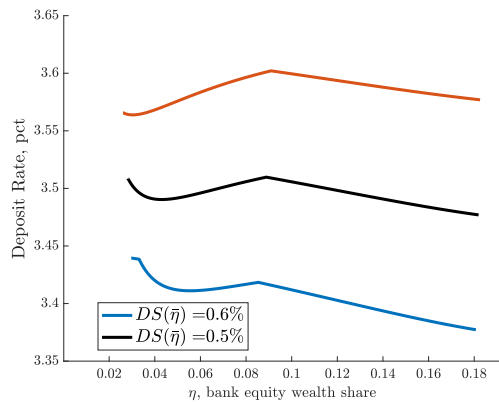


(a) $\gamma = 3$

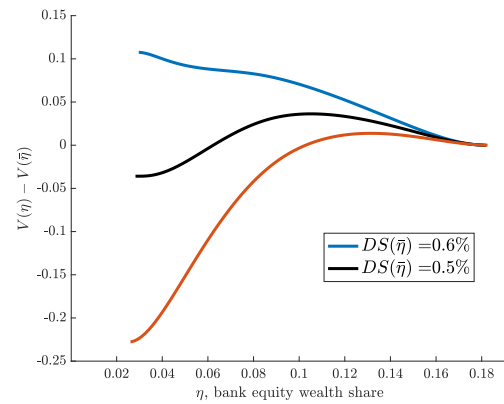


(b) $\gamma = 5$

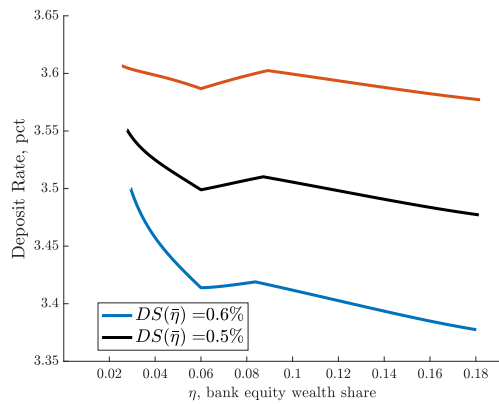
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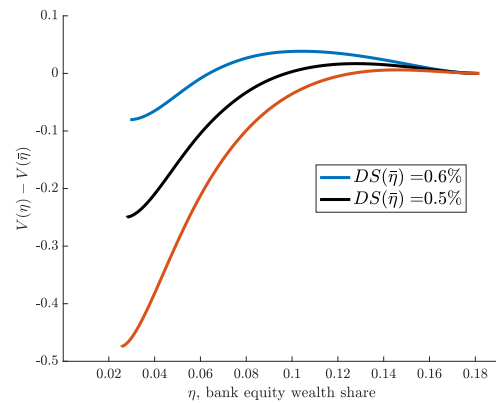
(a) Deposit Rate (no liquidity adjustment)



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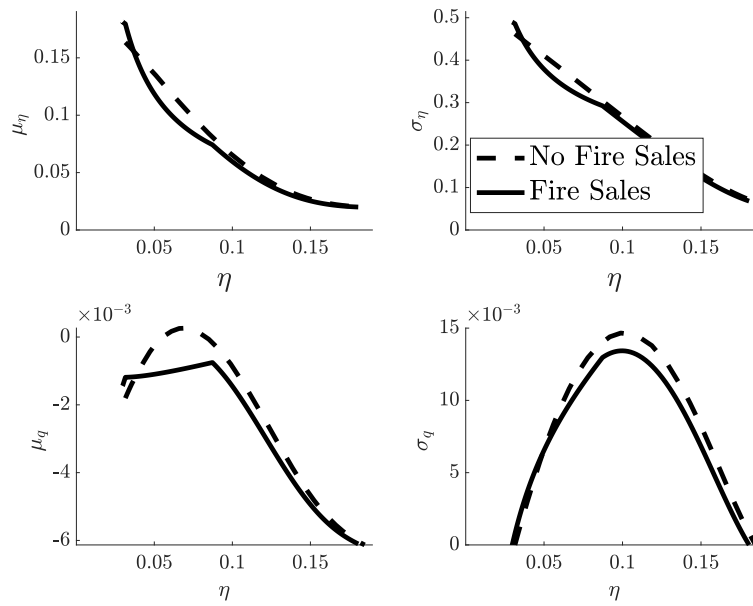
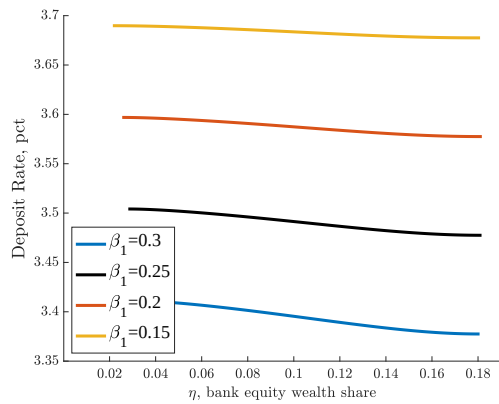
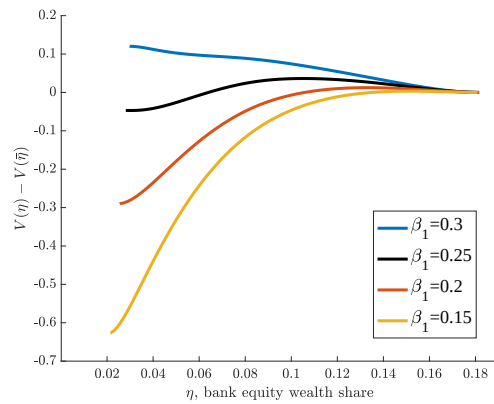


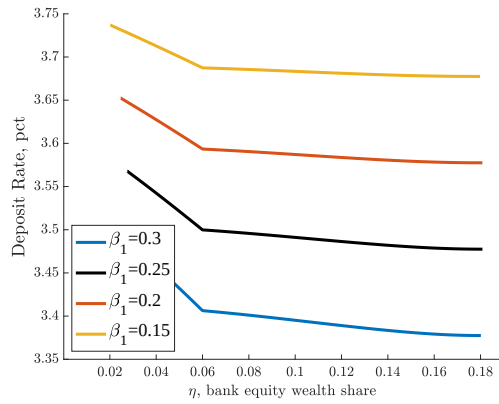
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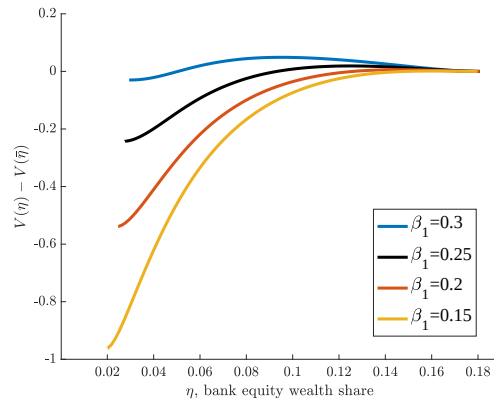
(a) Deposit Rate (no liquidity adjustment)



(b) Welfare (no liquidity adjustment)

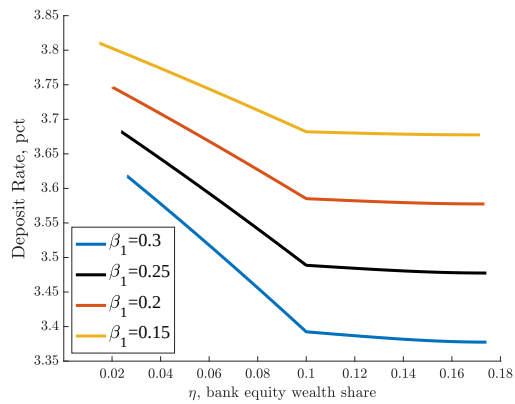


(c) Deposit Rate (with liquidity adjustment)

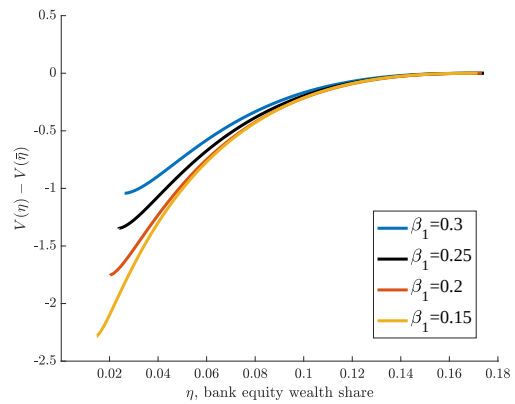


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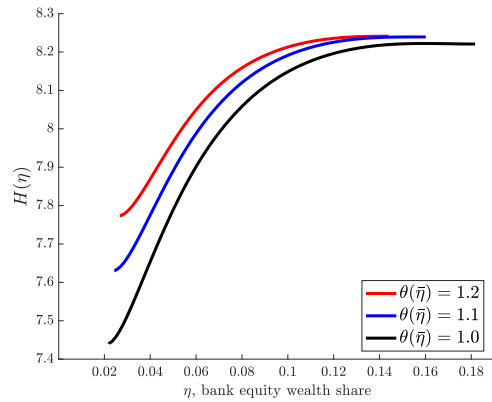


(a) Deposit Rate (with aggressive adjustment)

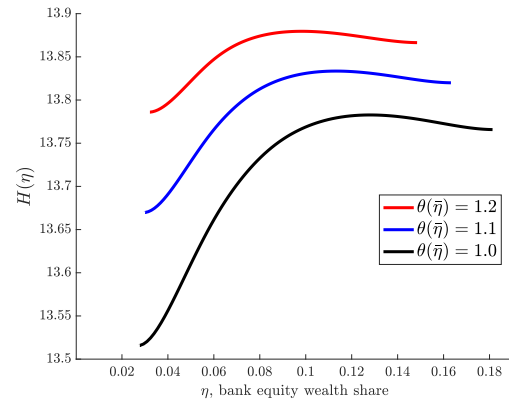


(b) Welfare (with aggressive adjustment)

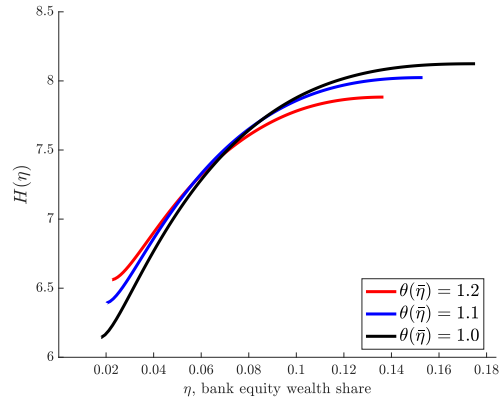
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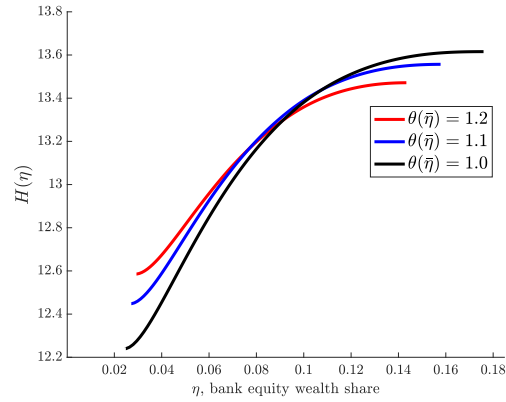
(a) $DS(\bar{\eta}) = 3\%$, baseline liquidity adjustment



(b) $DS(\bar{\eta}) = 5\%$, baseline liquidity adjustment

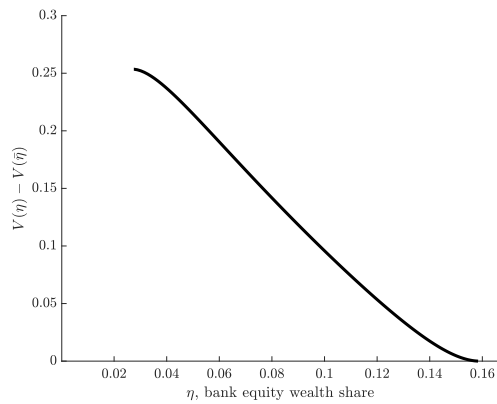


(c) $DS(\bar{\eta}) = 3\%$, aggressive liquidity adjustment

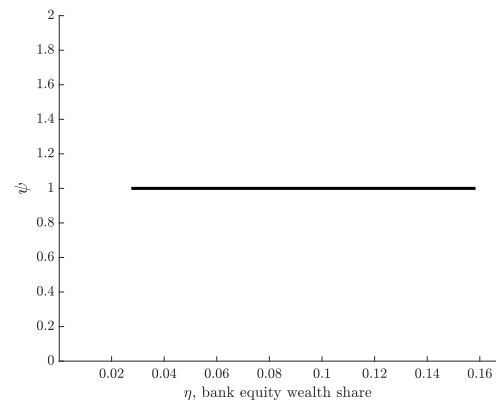


(d) $DS(\bar{\eta}) = 5\%$, aggressive liquidity adjustment

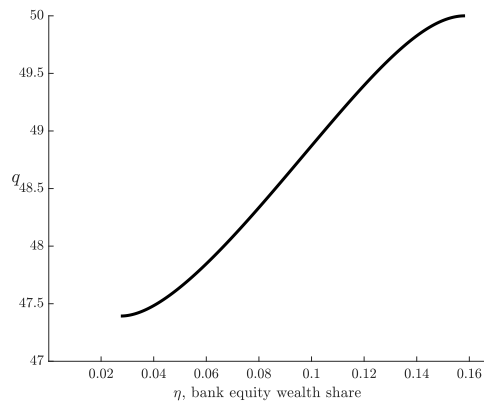
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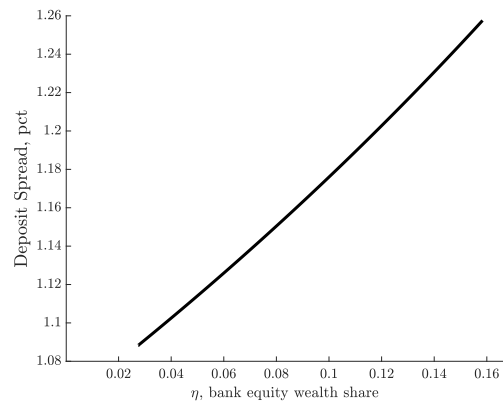
(a) Welfare $V(\eta) - V(\bar{\eta})$



(b) Bank asset holdings ψ

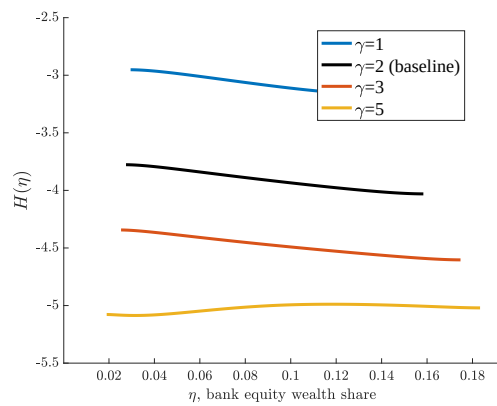


(c) Asset price q

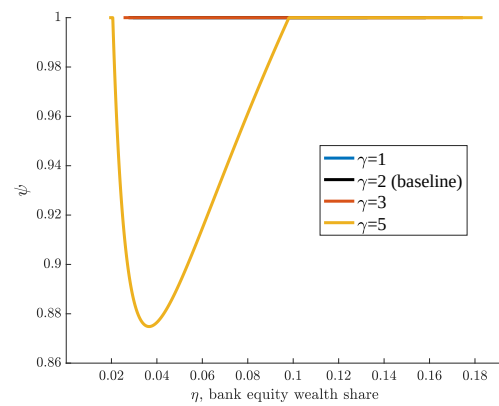


(d) Deposit Spread

Figure 11: Economy with deposit spreads calibrated to match empirical estimates (higher than baseline calibration). Welfare is decreasing in aggregate bank equity η .

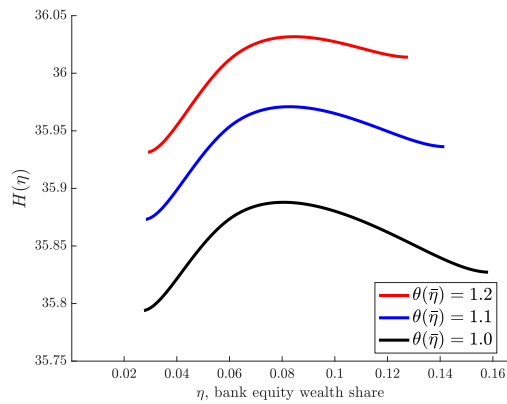


(a) Welfare

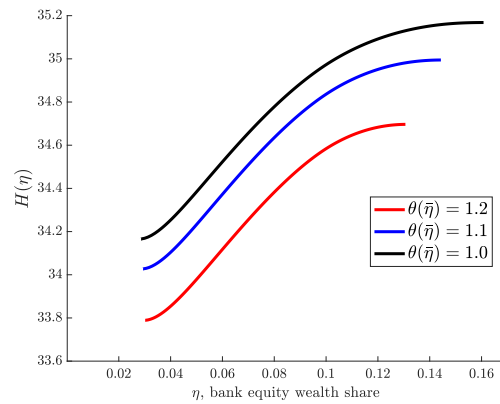


(b) Bank Holdings

Figure 12: Varying issuance costs with high deposit spreads. Welfare is an increasing function of η when issuance costs are sufficiently high, implying fire sales are sufficiently severe.

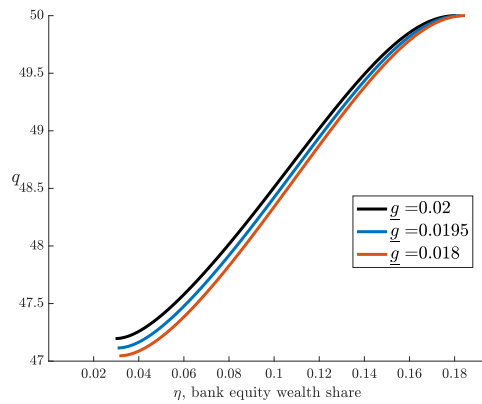


(a) Baseline liquidity adjustment

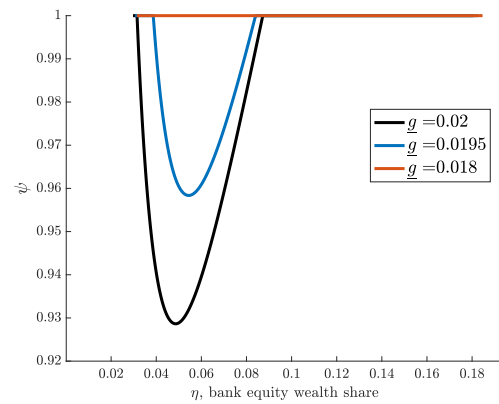


(b) Aggressive liquidity adjustment

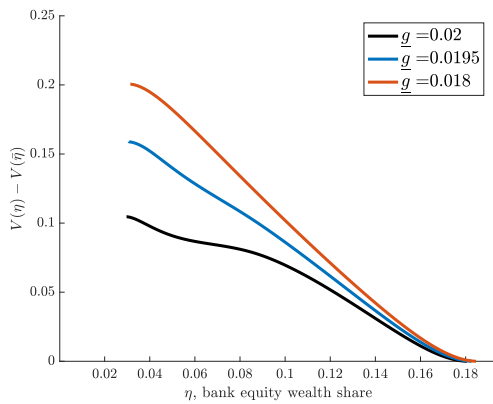
Figure 13: Dividend subsidy and welfare with exogenous liquidity adjustment. Panel (a) consider baseline adjustment for $\eta < 0.06$. Panel (b) considers aggressive adjustment for $\eta < 0.1$ at twice the rate. Dividend subsidy continues to improve welfare for baseline adjustment but not for aggressive adjustment.



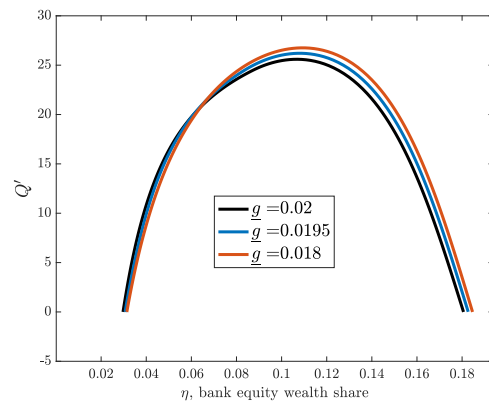
(a) Asset Price



(b) Bank Holdings ψ

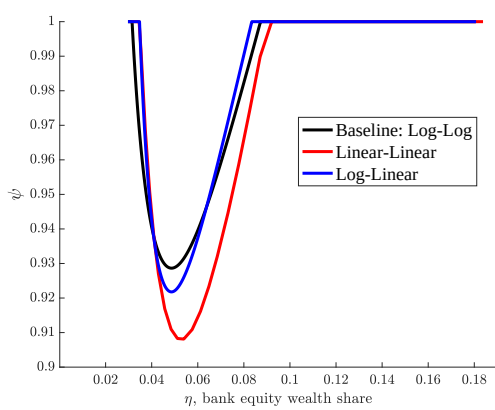


(c) Normalized Welfare

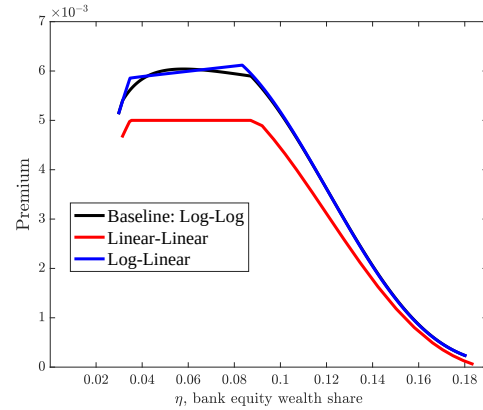


(d) Drift μ_η

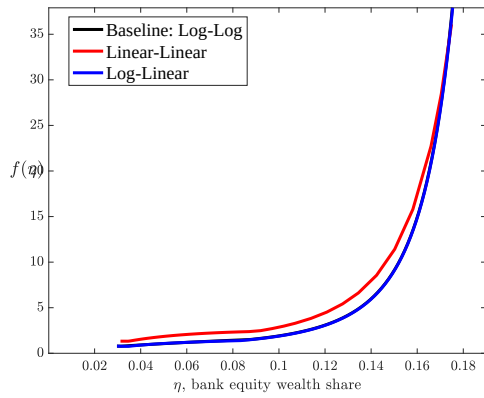
Figure 14: Robustness to different values of $\underline{g}$ if households get inferior tree growth $\underline{g} < g$.



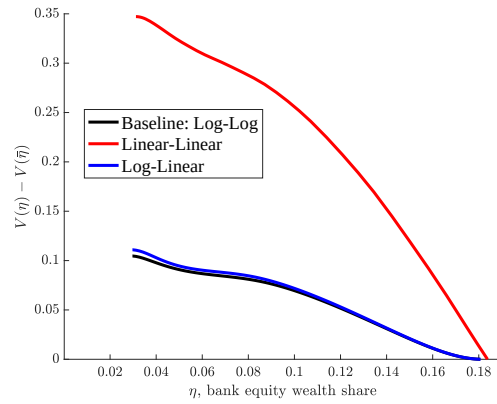
(a) Bank asset holdings ψ



(b) Risk Premium



(c) Stationary Distribution



(d) Normalized Welfare $V(\eta) - V(\bar{\eta})$

Figure 15: Comparing the Log utility and Risk-neutral model. Legend: black is Baseline log-log; red is linear-linear; blue is log-linear.

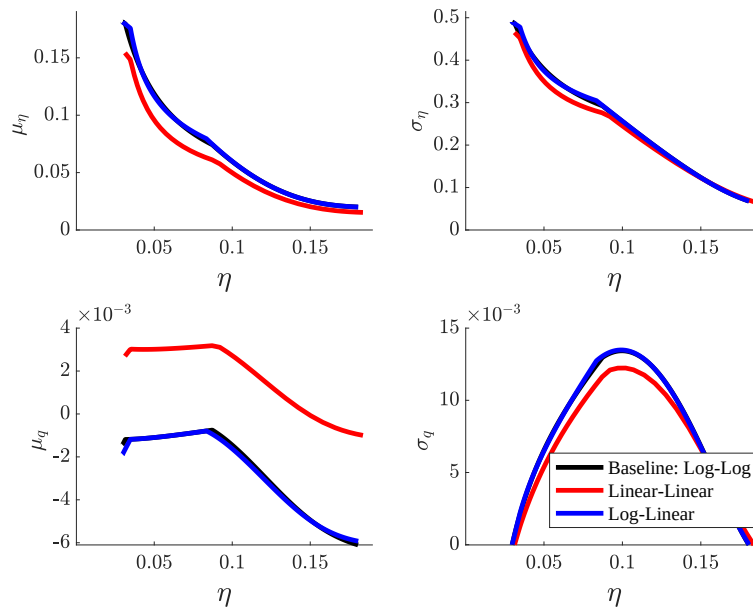
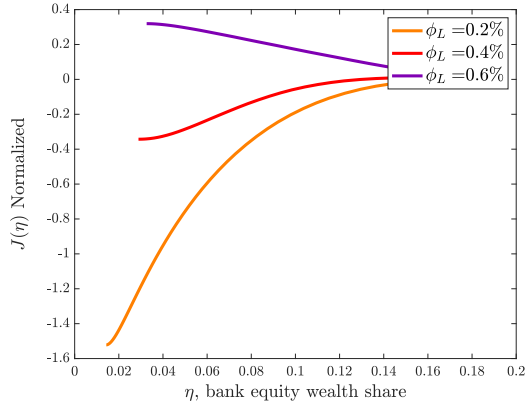
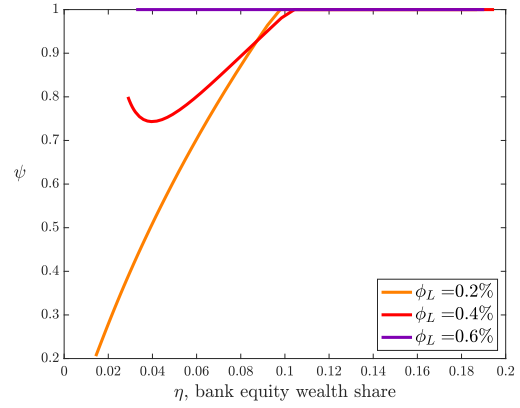


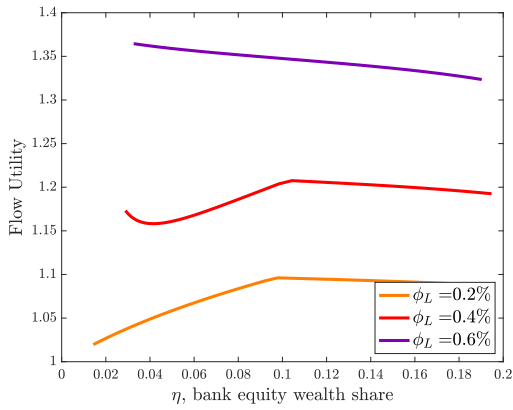
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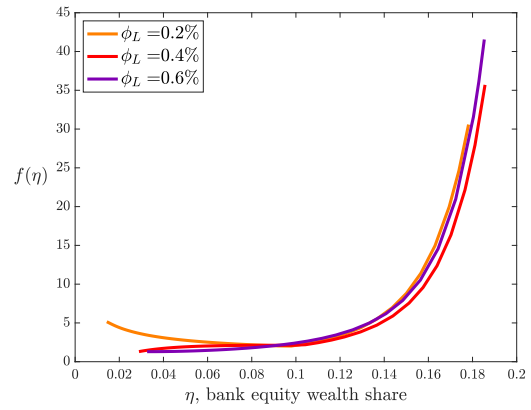
(a) Welfare $J(\eta) - J(\bar{\eta})$



(b) Bank Holdings

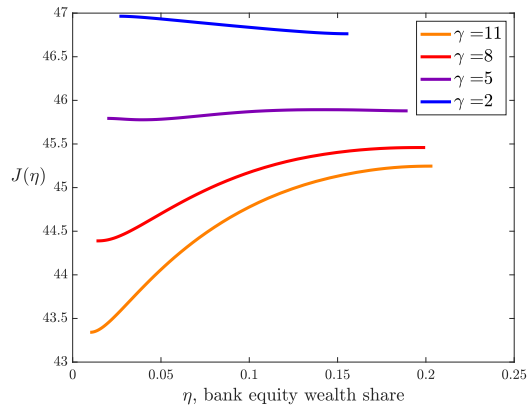


(c) Flow Utility

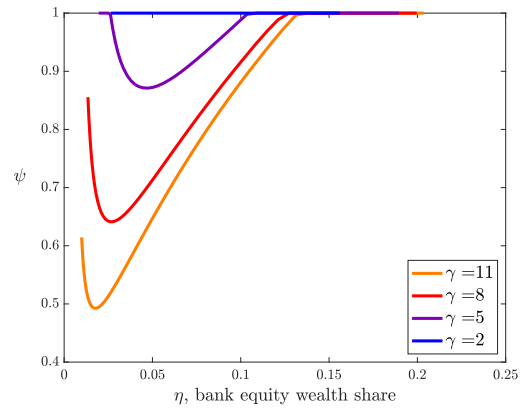


(d) Stationary Density

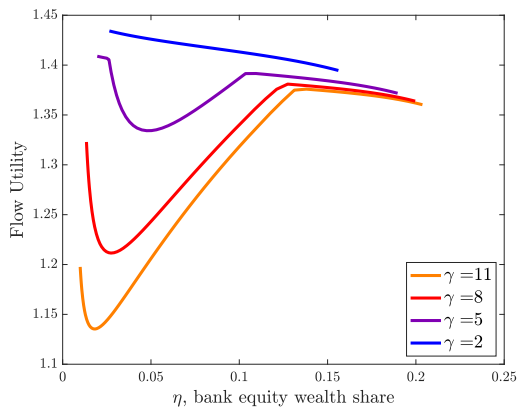
Figure 17: Robustness to different values of ϕ_L . We omit the case of $\phi_L = 1\%$ because the values for welfare and flow utility are much higher than the ones plotted here, which results in bad scaling of the figures. When $\phi_L = 0.1\%$, welfare becomes a mostly increasing function except for $\underline{a} = 0.8$. Welfare is scaled across economies.



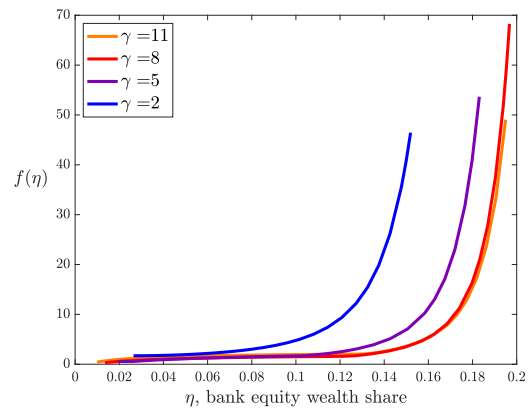
(a) Welfare



(b) Bank Holdings



(c) Flow Utility



(d) Stationary Density

Figure 18: Robustness to different values of γ . Large issuance costs reverse our main result because fire sales become more severe as banks become more constrained.

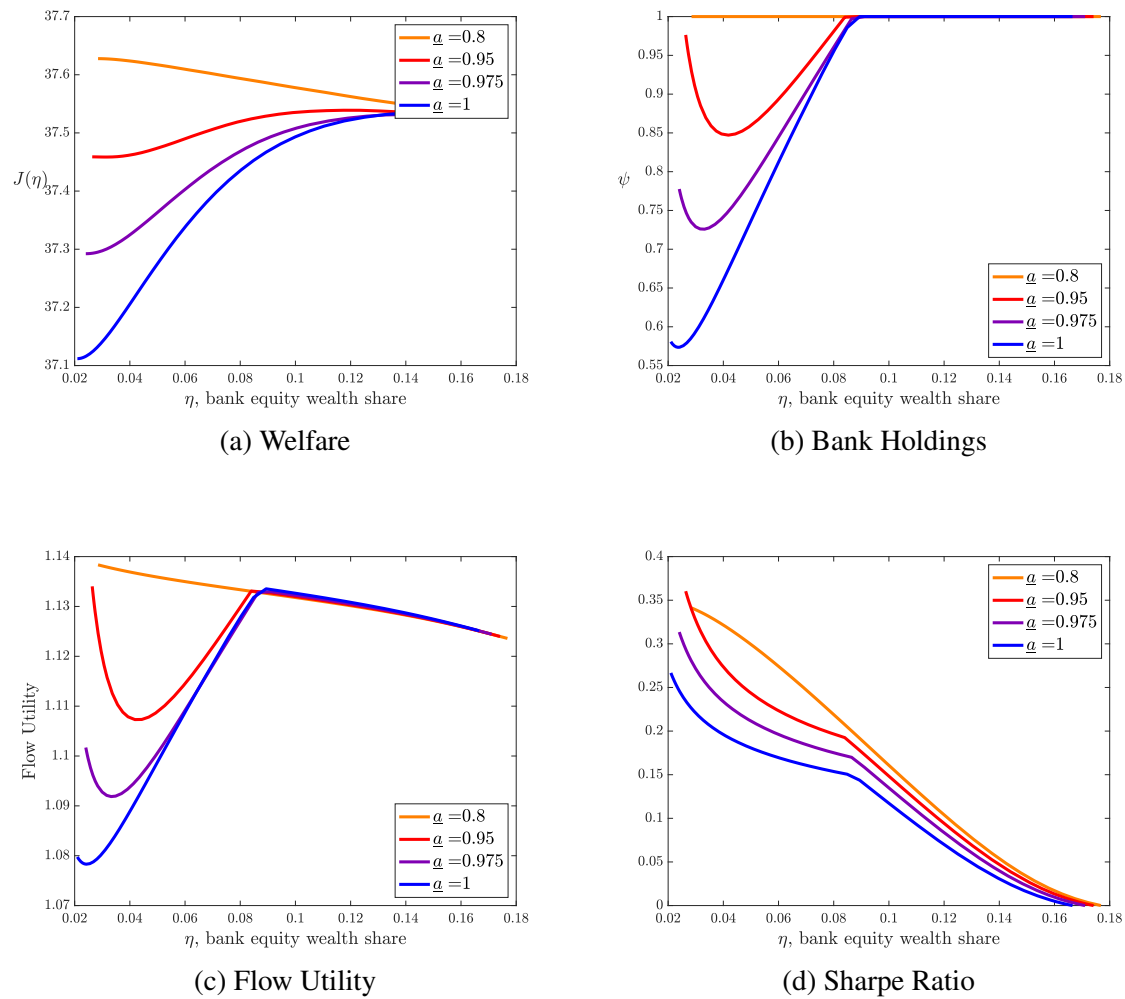
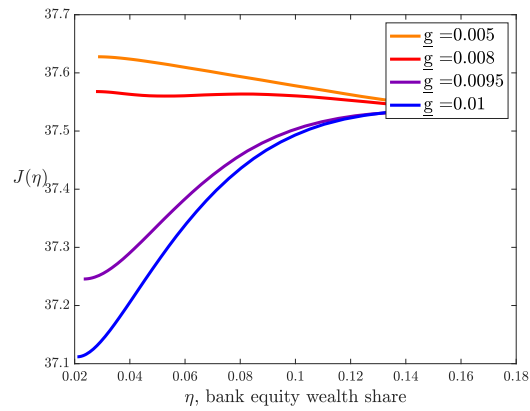
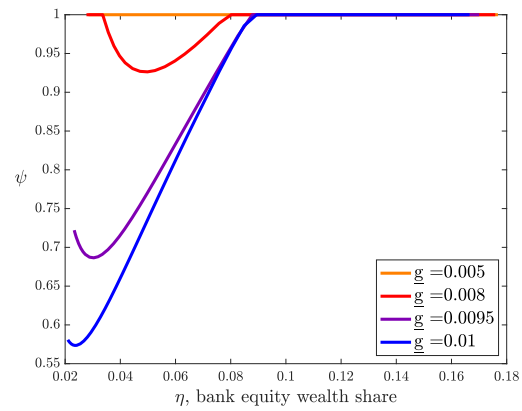


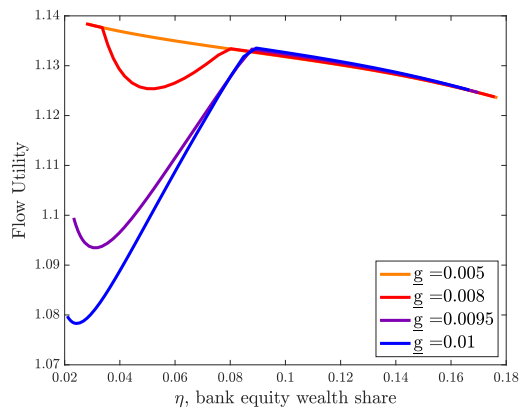
Figure 19: Welfare, Flow Utility, Tree Allocations, and Stationary Distribution when bank inter-mediation increases real output but not the growth rate of productivity ($\underline{a} \leq 1$ and $\underline{g} = g$).



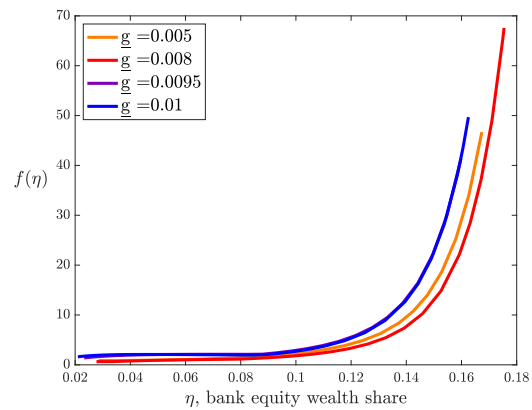
(a) Welfare



(b) Bank Holdings

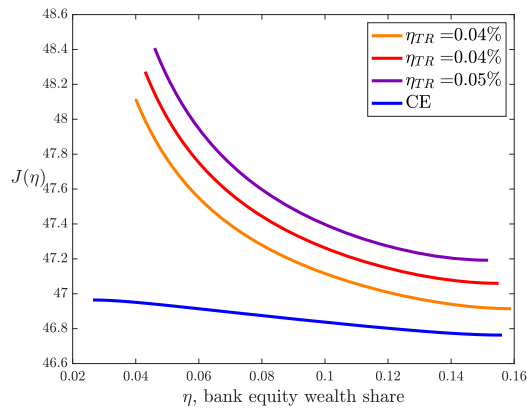


(c) Flow Utility

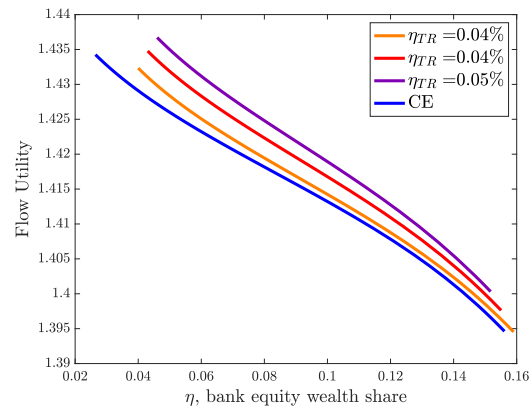


(d) Stationary Distribution

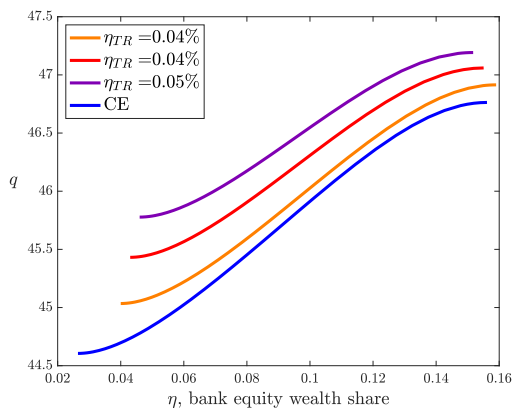
Figure 20: Robustness to different values of $\underline{g}$ if households get inferior tree growth $\underline{g} < g$.



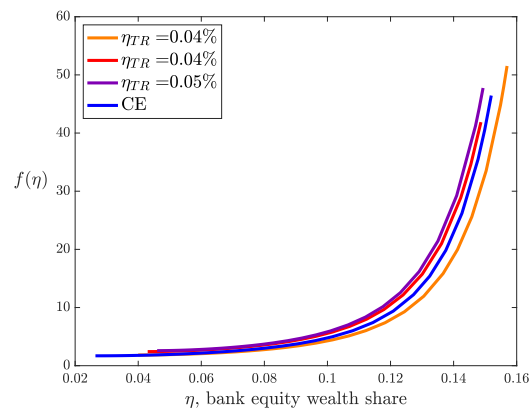
(a) Welfare



(b) Flow Utility



(c) Asset Price



(d) Stationary Density

Figure 21: These figures correspond to equilibrium with tail risk insurance. The government exogenously redistribute wealth from households to banks whenever η reaches η_{TR} . In these figures, we vary $\eta_{TR} = 4\%$, 4.3% , and 4.6% . The label “CE” is an abbreviation for the competitive equilibrium.