On Concave Functions over Lotteries*

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1 Introduction

This note discusses functions that are concave but not necessarily differentiable, and when their maxima do and do not exist. It provides counter-examples to illustrate why various conditions are needed, and points out errors in Machina [1984] and Maccheroni [2002]'s analyses of utility functions over lotteries, and in the statement of Theorem 2 in Frankel and Kamenica [2019]'s proposed measure of information and uncertainty. One source of these errors was the misapplication of the separating hyperplane theorem to infinite-dimensional lotteries, so we also give a proof of the relevant result.

2 Concavity and Local Utility

We take X to be a compact set and $\mathcal{F}$ to be the lotteries on X. We will be interested in the properties of concave functions $V : \mathcal{F} \to \mathbb{R}$, which we will interpret as representing an agent's preferences. We say that a continuous function $w(x)$ on a compact set X is a *local utility function* for V at $F \in \mathcal{F}$ if for all $\int w(x)d\tilde{F}(x) \geqslant V(\tilde{F})$ and $\int w(x)dF(x) = V(F)$. We say that $V(F)$ has a local expected utility if it has a local utility function

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at each $F \in \mathcal{F}$. We let $\mathcal{W}_V(F) \subseteq C(X)$ denote the set of local expected utilities of V at F and, when V has a local expected utility, we denote $\mathcal{W}_V = \bigcup_{F \in \mathcal{F}} \mathcal{W}_V(F)$.

It is immediate to see that having a local expected utility implies concavity.

Lemma 1. *If $V(F)$ has local expected utility then $V(F)$ is concave.*

Proof. Take $\overline{F} = \lambda F + (1 - \lambda \tilde{F})$ and w such that $\int w(x) d\overline{F}(x) \geqslant V(F)$ with $\int w(x) d\overline{F}(x) = V(\overline{F})$. Then $V(\overline{F}) = \int w(x) d\overline{F}(x) = \lambda \int w(x) dF + (1 - \lambda) \int w(x) d\tilde{F} \geqslant \lambda V(F) + (1 - \lambda) V(\tilde{F})$. □

The converse is not true. For example for lotteries $(p, 1-p)$ on $\{0, 1\}$, the entropy $V(p) = -p \log p - (1-p) \log(1-p)$ is concave but has infinite slope on the boundaries, so there is not a local expected utility function at those points. Nevertheless, this V is upper semi-continuous and so attains a maximum on compact sets. Moreover, there is a sense in which it "almost" has a local expected utility, as it is the envelope of a family of expected utility functions.

Formally, we define *extended adversarial representations* as

$$V(F) \equiv \inf_{y \in Y} \int u(x, y) dF(x)$$

where $u(x, y)$ is continuous with X compact and Y a separable metric space. These preferences are concave and upper semi-continuous. We next show that the converse is also true.

Theorem 1. *$V(F)$ arises from a extended adversarial representation if and only if it is upper semi-continuous and concave.*

The idea here is that we can get "near" tangent hyperplanes even at boundary points by taking hyperplanes that pass through a point above but near the concave function. Where the function has infinite slope as we take closer points we get steeper separating hyperplanes which is why we must use the inf rather than the min.

Notice that $V(F)$ arising from an extended adversarial representation need not be continuous. Even in finite dimensional spaces, there are concave functions that fail to be lower semi-continuous; we give an example below. However, the restriction of concave and upper semi-continuous functions from $\mathcal{F}$ to a finite-dimensional subset of it is continuous, as we show below. Thus when we consider an arbitrary finite $X_0 \subseteq X$, the restriction of V to the space of lotteries over X_0 is continuous.

Proof. First we show that extended adversarial implies concave. First concavity. Consider $F, \tilde{F}$ with $0 \leqslant \lambda \leqslant 1$ and y^n such that $\int u(x, y^n)dF(x) \to V(\lambda F + (1-\lambda)\tilde{F})$. Consider that $V(F) \leqslant \int u(x, y^n)dF(x)$ and $V(\tilde{F}) \leqslant \int u(x, y^n)d\tilde{F}(x)$ so that $\lambda V(F) + (1-\lambda)V(\tilde{F}) \leqslant \int u(x, y^n)d(\lambda F + (1-\lambda)\tilde{F})(x) \to V(\lambda F + (1-\lambda)\tilde{F})$.

To show continuity, let $F^n \to F$,and choose y^m such that $V(F) > \int u(x, y^m)dF(x) - 1/m$. Then

$$V(F^n) \leqslant \int u(x, y^m)dF^n(x),$$

so

$$\lim V(F^n) \leqslant \lim \int u(x, y^m)dF^n(x) = \int u(x, y^m)dF(x) < V(F) + 1/m.$$

Hence V is upper semi-continuous.

To go the other way we use a version of the separating hyperplane theorem. Theorem 4 at the end of these notes shows that for each $v > V(F)$ there is a continuous function $w(x)$ such that $\int w(x)d\tilde{F} > V(\tilde{F})$ for all $\tilde{F} \in \mathcal{F}$. Now take Y to be the subset of continuous functions in the sup norm for which $\int y(x)d\tilde{F}(x) > V(\tilde{F})$ for all $\tilde{F}$. This is a separable metric space, since it is an open subset of separable space of all continuous function in that norm. Since for every (v, F) there exists a $y \in Y$ such that $v \geqslant \int y(x)dF(x) > V(F)$, we see that $V(F) \equiv \inf_{y \in Y} \int u(x, y)dF(x)$. □

A concave function that fails to be lower semi-continuous

Theorem 1 only delivers concavity and upper semi-continuity. Concave functions need not be lower semi-continuous, even in finite dimensions, as shown by example in Rockafellar [1970] Chapter 10. Here is a version of that example: Consider the linearly homogeneous function $u(p, q) = -(q+p)^2 p^{-1}$ for $p > 0$. We have

$$Du = \begin{bmatrix} ((q+p)p^{-1})^2 - 2(q+p)p^{-1} \\ -2(q+p)p^{-1} \end{bmatrix}$$

$$D^2u = \begin{bmatrix} 2\left((q+p)p^{-1} - 1\right)\left(p^{-1} - (q+p)p^{-2}\right) & 2(q+p)p^{-2} - 2p^{-1} \\ 2(q+p)p^{-2} - 2p^{-1} & -2p^{-1} \end{bmatrix}$$

$$= \begin{bmatrix} -2p^{-1}\left((q+p)p^{-1} - 1\right)^2 & 2p^{-1}\left((q+p)p^{-1} - 1\right) \\ 2p^{-1}\left((q+p)p^{-1} - 1\right) & -2p^{-1} \end{bmatrix}$$

which has a non-negative diagonal. Because u is linearly homogeneous, the determinant of D^2u is 0, so it is negative semi-definite, and hence u is concave.

Now restrict u to the convex set $0 \leqslant q \leqslant \sqrt{p} - p$ on $0 < p \leqslant 1$. Define $u(0,0) = 0$ and observe that convex combinations of $(1-\lambda)0 + \lambda F$ have $u((1-\lambda)0 + \lambda F) = \lambda u(F) = (1-\lambda)u(0) + \lambda u(F)$ so $u(p,q)$ on $0 \leqslant q \leqslant \sqrt{p} - p$ on $0 < p \leqslant 1$. Consider, however the sequence $F^n = (p^n, \sqrt{p^n} - p^n)$ with $p^n > 0$ and $p^n \to 0$. Certainly $F^n \to 0$, but $u(F^n) = -1$, so the function is not lower semi-continuous.

In the last example, the boundary of the domain is non-linear, causing the failure of lower semi-continuity. In studying preference over lotteries, the convex sets on which utility is defined are simplices and this is enough to obtain lower semi-continuity in the finite-dimensional case.

Theorem 2. *Consider the space $\overline{\mathcal{F}}$ of all convex combinations of N lotteries $\overline{F}^1, ..., \overline{F}^N$, and suppose that $V(F)$ is concave. Then $V(F)$ restricted to $\overline{\mathcal{F}}$ is lower semi-continuous, and in particular if $V(F)$ is upper semi-continuous then it is continuous.*

When the probability distributions $\overline{F}^1, ..., \overline{F}^N$ coincide with the point-mass measures over N outcomes, the set $\overline{\mathcal{F}}$ is a probability simplex. The result holds for "generalized simplicies" that are formed by linearly combining N arbitrary lotteries.

Proof. (*Adapted from Chapter 10 in Rockafellar [1970].*) There is a subset of $\{\overline{F}^1, ..., \overline{F}^N\}$ that consists of extremal points and whose convex hull is equal to $\overline{\mathcal{F}}$, so w.l.o.g. can assume that $\overline{F}^1, ..., \overline{F}^N$ are extremal and in particular affinely independent. Hence we may think of points being identified with vectors p on the n-dimensional simplex and we write $\overline{p}^i$ for the basis vectors.

Now consider $\tilde{p} \in \overline{\mathcal{F}}$. Our goal is to prove that for any sequence $p^n \to \tilde{p}$ we have $\liminf V(p^n) \leqslant V(\tilde{p})$. Suppose $\tilde{p}$ is not extremal and consider some particular p^n and define $\lambda \equiv \max\{\lambda \geqslant 0 | \lambda\tilde{p} \leqslant p^n\}$. If $\lambda = 0$ choose i such that $p_i^n = 0$ and $\tilde{p}_i > 0$ otherwise choose i such that $\lambda\tilde{p}_i = p_i^n$ and $\tilde{p}_i > 0$. Consider then the set $\{\overline{p}^1, ..., \overline{p}^N, \tilde{p}\} - \overline{p}_i$. Since $\tilde{p}_i > 0$ this set is affinely independent. We claim in addition that p^n is a convex combination of these vectors. If $\lambda = 0$ since $p_i^n = 0$ we have p^n already a convex combination of $\{\overline{p}^1, ..., \overline{p}^N\} - \overline{p}_i$. Otherwise since $p_j^n - \lambda\tilde{p}_j \geqslant 0$ we may write $p^n = \lambda\tilde{p} + \sum_j (p_j^n - \lambda\tilde{p}_j)\overline{p}^j$ since this is the same as $p_j^n = \lambda\tilde{p}_j + (p_j^n - \lambda\tilde{p}_j)$.

Consider affinely independent sets of the form $(\tilde{p}, \tilde{p}^1, ..., \tilde{p}^{n-1})$. We showed that if $\tilde{p}$ is not extremal then p^n there exists a set of this form, so p^n lies in the convex hull of the set, and if $\tilde{p}$ is extremal this is true by taking the remaining $n-1$ vectors to be the remaining basis vectors. Since there are at most n sets of this form it follows that there is a subseqence p^m converging to $\tilde{p}$ that lies entirely in such a set. Clearly $\liminf V(p^n) \leqslant \liminf V(p^m)$, so it suffices to prove $\liminf V(p^m) \leqslant V(\tilde{p})$.

Consider then that we can write $p^m = \gamma^m \tilde{p} + \sum_{i=1}^{n-1} \gamma_i^m \tilde{p}^i$ and that $\gamma^m \to 1, \gamma_k^m \to 0$. Then

$$V(p^m) = V(\gamma^m \tilde{p} + \sum_{i=1}^{n-1} \gamma_i^m V(\tilde{p}^i) \geqslant \gamma^m V(\tilde{p}) + \sum_{i=1}^{n-1} \gamma_i^m V(\tilde{p}^i) \to V(\tilde{p})$$

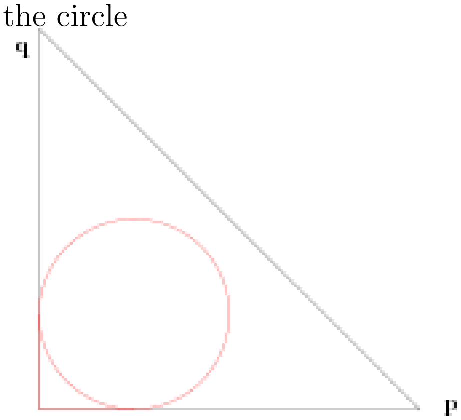
which was our goal. □

3 A "Snowcone" Counterexample

This section exhibits a utility function over lotteries with three outcomes that is convex, continuous, and can be represented by a concave function. Moreover, there is a deterministic outcome that is preferred to all other lotteries with the property that the independence axiom is satisfied with respect to that certain outcome. However, the associated preferences over lotteries cannot be represented as the minimum of affine functions. This directly contradicts Theorem 1 in Maccheroni [2002], and as we explain below it is not consistent with the statements of Theorem 2 in Machina [1984] and Theorem 2 in Frankel and Kamenica [2019]

A lottery between three outcomes is given by the simplex $0 \leqslant p, q \leqslant 1$ and $p+q \leqslant 1$. It is convenient to denote points (p, q) as F and also to consider polar coordinates $\theta(F), R(F)$ and conversely $F(r, \theta)$.

We now construct a utility function using the 'snowcone" created by taking a circle tangent to the axes. For each θ the ray through θ intersects the circle at most twice; we take the indifference curve to by the larger of the two radii, that is the upper part of the circle



We extend this curve to a utility function over the entire simplex by radial scaling: Given a point F, let

$$V(F) = -\frac{r(F)}{\nu(\theta(F))}.$$

This is negative and takes on a maximum at the certain outcome $p = q = 0$; it assigns utility -1 to points on this indifference curve. It is linearly homogeneous: $V(\gamma F) = \gamma V(F)$ for $\gamma \geqslant 0$.

We say that the indifference curve $\nu(\theta)$ is convex if for any two points $F, \tilde{F}$ satisfying $u(F) = u(\tilde{F}) = -1$, any convex combination of those two points lies below the indifference curve. This is equivalent to $V(\lambda F + (1-\lambda)\tilde{F}) \geqslant -1$. Notice that this is true for the inscribed circle example since the unit sphere is a convex set.

We now prove if V has convex indifferecne curves it V is concave. First observe that

$$V\left(\frac{F}{-u(F)}\right) = V\left(\frac{F\nu(\theta(F))}{r(F)}\right) = \frac{\nu(\theta(F))}{r(F)}V(F) = -\frac{\nu(\theta(F))}{r(F)}\frac{r(F)}{\nu(\theta(F))} = -1.$$

Hence for any γ we have

$$V\left(\gamma\frac{F}{-u(F)} + (1-\gamma)\frac{\tilde{F}}{-u(\tilde{F})}\right) \geqslant -1$$

by the convexity of the indifference curve $\nu(\theta)$.

Fix $\lambda \in [0,1]$, and $F, \tilde{F} \in \mathcal{F}$, and define

$$\gamma = \frac{\lambda V(F)}{\lambda V F) + (1-\lambda)V(\tilde{F})}$$

and observe that

$$1-\gamma = \frac{(1-\lambda)V(\tilde{F})}{\lambda V(F) + (1-\lambda)V(\tilde{F})}$$

and that both γ and $(1-\gamma)$ since $0 \leqslant \lambda \leqslant 1$ and $V \leqslant 0$. Then

$$\begin{aligned}-1 \leqslant V\left(\gamma\frac{F}{-V(F)} + (1-\gamma)\frac{\tilde{F}}{-V(\tilde{F})}\right) &= V\left(-\frac{\lambda F + (1-\lambda)\tilde{F}}{\lambda V(F) + (1-\lambda)V(\tilde{F})}\right)\\ &= -\frac{1}{\lambda u(F) + (1-\lambda)u(\tilde{F})}V\left(\lambda F + (1-\lambda)\tilde{F}\right).\end{aligned}$$

Because $\lambda V(F)+(1-\lambda)V(\tilde{F})$ is negative, $V\left(\lambda F+(1-\lambda)\tilde{F}\right) \geqslant \lambda V(F)+(1-\lambda)V(\tilde{F})$.

Now consider the lottery $(1-\lambda)F+\lambda 0$. By homogeneity $V((1-\lambda)F+\lambda 0) = (1-\lambda)V(F)$, so F is preferred to $\tilde{F}$ if and only if $(1-\lambda)F+\lambda 0$ is preferred to $(1-\lambda)\tilde{F}+\lambda 0$, that is the independence axiom is satisfied with respect to mixtures with 0. Thus for any convex indifference curve we have constructed preferences that are convex, continuous, and have a deterministic outcome that is preferred to all other lotteries, where the independence axiom is satisfied with respect to that outcome, and the preferences can be represented as a concave function. Moreover along the axes $p=0$ and $q=0$ preferences are strictly declining in the other probability. We showed by example that this indifference curve can be tangent to the axis, and since it does not include zero, it must intersect the axis. Since there is strict preference along the y axis, every straight line passing through that point must contained points strictly preferred to the intersection point. Hence the collection of affine linear functions that dominate a utility representation of these preferences cannot have a minimum at that point.

Machina's induced preferences

Machina [1984] analyzes preferences over lotteries $\mathcal{F} = \Delta([0,1])$ that carry delayed risk. Concretely, consider an agent choosing first $F \in \mathcal{F}$ and then, before the outcome from F has been realized, an action $y \in Y$ from a set of feasible actions. Even if the agent has expected utility preferences over pairs of outcomes and actions, the induced preferences over lotteries is

$$V(F) = \max_{y \in Y} \int u(x,y)dF(x) \tag{1}$$

where u is the utility of the agent and where we assume that the maximum is attained.

Theorem 2 in Machina [1984] states a converse of this fact, that is, if $V : \mathcal{F} \to \mathbb{R}$ is continuous and convex, then there exists a space of actions Y and a utility function $u(x,y)$ such that V can be represented as in equation 1 with the maximum being attained for every F. In particular, the set of actions is

$$Y = \left\{ y \in C([0,1]) : \forall F \in \mathcal{F}, V(F) \geqslant \int y(x)dF(X) \right\}.$$

and $u(x,y) = w(y)$. However the $V(F)$ in be the snowcone example above is continuous and convex, but does not have a representation in the form of a maximum as Machina asserts.

Measure of uncertainty and information

A similar statement to that of Machina is contained in Theorem 2 in Frankel and Kamenica [2019]. In this work, they define valid measures of uncertainty as measures of the extent of uncertainty in a decision cost of cost of uncertainty given a decision problem as follows. Fix a finite state space and let $F \in \mathcal{F}$ denote an arbitrary belief over X. A *decision problem* is a pair (u, Y) of an action space Y and a utility function $u(x, y)$ such that, for every belief F, the maximum $\max_{y \in Y} \int u(x, y) dF(x)$ is attained. They say that $V : \mathcal{F} \to \mathbb{R}$ is a *valid measure of uncertainty* if there exists a decision problem (u, Y) such that

$$V(F) = \int \max_{y \in Y} u(x, y) dF(x) - \max_{y \in Y} \int u(x, y) dF(x)$$

for every F. Their Theorem 2 then states that V is a valid measure of uncertainty *if and only if* it is concave and such that $V(\delta_x) = 0$ for all $x \in X$. The only if part is clearly true. However, the if part is also contradicted by the snowcone example above.

4 The Separating Hyperplane Theorem

Infering properties such as differentiability and concavity from a utility function rests on the separating hyperplane theorem, and one source of error has been mis-applying this theorem in the infinite dimensional context of lotteries. Here we give a careful proof of the separating hyperplane theorem in the context of lotteries. We use the Hahn decomposition Theorem 5.79 from Aliprantis and Border [2006]. For completeness and comparison we state that result in the form in which we use it.

Theorem 3. *If the hypograph of $V(F)$, that is the set in $\mathbb{R} \times \mathcal{F}$ consisting of $L = \{(v, F) | v \leqslant V(F)\}$, is closed and convex, then for each singleton set (v, F) with $v > V(F)$ by we may find a continuous linear functional separating (v, F). This means that there are numbers c_0, z and continuous function $w_1(x)$ such that for $\tilde{v}, \tilde{F} \in L$ we have $c_0 \tilde{v} + \int w_1(x) d\tilde{F}(x) < z$ and $c_0 v + \int w_1(x) dF(x) > z$.*

Crucially, this proposition is false if V can take extended real values, as shown by the following example from Bogachev and Smolyanov [2017]:

Consider the functional $p : \mathbb{R}^2 \to \overline{\mathbb{R}}^+$ e defined as $p(x_1, x_2) = 0$ if $x_2 > 0$ or $\text{x2} = 0$ and $x_1 \geqslant 0$; in otherwise $p(x_1, x_2) = \infty$. Set $E_1 = \{(x_1, 0) : x_1 \in \mathbb{R}\}$ and $f(x_1, 0) = x_1$. Then $f(x) \leqslant p(x)$ for all $x \in E_1$. But there is no no linear functional that extends f to all of $\mathbb{R}^2$ and satisfies the corresponding inequality, because there every line through through the point $(1, 0)$ intersects the set $\{(x_1, x_2) : p(x_1, x2_) < 1\}$.

Theorem 4. *For each $v > V(F)$ there exists a continuous function $w(x)$ such that $\int w(x) d\tilde{F} > V(\tilde{F})$ for all $\tilde{F} \in \mathcal{F}$.*

Proof. We analyze the space of signed measures $H \in \mathcal{M}$ on the Borel σ-sigma algebra of X. The Hahn decomposition theorem says that for any signed measure H the space X can be partitioned into two Borel sets A, B such that for Borel $E \subseteq A$ we have $H(E) \geqslant 0$ and for $E \subseteq B$ we have $H(E) \leqslant 0$. The Jordan decomposition further states that there are two positive (ordinary measures) H^+, H^- (uniquely defined) such that for any Hahn decomposition $H^+(B) = 0, H^-(A) = 0$ and $H = H^+ - H^-$. With this in mind, for any continuous function $w : Z \to \mathbb{R}$ and any signed measure we may define $\int w(x) dH(s) = \int w(x) dH^+(x) - \int w(x) dH^-(s)$ where these are ordinary integrals with respect to a signed measure. We may define the total variation $|H| = \int dH^+(x) + \int dH^-(x)$.

Denote the space of bounded continuous functions in the sup norm on X by $C(X)$. On $\mathcal{M} \times C(X)$ we define the operation $< H, w > \equiv \int w(x) dH(x)$. If this is continuous and linear in each argument and $< H, w >= 0$ for all $w \in C(X)$ if and only if $H = 0$ and for all $H \in \mathcal{M}$ if and only if $w = 0$ then $\mathcal{M}, C(X)$ is a dual pair. Continuity follows immediately from $< H, w > \int w(x) dH(x) \leqslant \|w\| \, \|H\|$ and this implies $< H, w >$ is jointly continuous in the product topology on $\mathcal{M} \times C(X)$.

It follows that $\mathcal{H}$ in the weak topology induced by $C(Z)$ is a locally convex topological space and its continuous linear functionals have the form $\int c(z) dH(z)$ for $c \in C(Z)$. Hence this topology relativizes to the subset of probability measures as the topology of weak convergence. That $< H, w >= 0$ for all H if $w = 0$ is obvious and only if $w = 0$ follows from considering that the Dirac delta functions $\delta(x)$ are in H and $\int w(x) d\delta(\hat{x})(x) = w(\hat{x})$. That $< H, w >= 0$ for all w if $H = 0$ is obvious but only if requires some work.

For any $H \neq 0$, e want to find a continuous function $w(x)$ such that $\int w(x) dH(x) \neq 0$. To this end, let A, B be a corresponding Hahn partition of X and write the Jordan decomposition $H = H^+ - H^-$. Assume without loss of generality that $H^+ \neq 0$. Because X is compact H^+, H^- are regular measures and in particular $H^+(A) = \sup_{K \subset A} H^+(K)$

and $H^-(B) = \sup_{K \subset B} H^-(K)$ where the supremum is over all compact subsets. Hence we can fix compact $K^+ \subset A$ such that $|H^+(A) - H^+(K^+)| \leqslant (1/3)H^+(A)$ and a compact $K^- \subset B$ such that $|H^-(A) - H^-(K)| \leqslant (1/3)H^+(A)$. As K^-, K^+ are disjoint and X is metric and K^+, K^- are close we may use Ursyohn's Lemma to find a continuous function $0 \leqslant w(x) \leqslant 1$ which is equal to 1 on K^+ and 0 on K^-. Now write the integral

$$\begin{aligned}
\int w(x)dH(x) &= \left[\int_{K^+} w(x)dH^+(x) + \int_{X-K^+} w(x)dH^+(x)\right] \\
&\quad - \left[\int_{K^-} w(x)dH^-(x) + \int_{K^+} w(x)dH^-(x) + \int_{X-K^+-K^-} w(x)dH^-(x)\right] \\
&= H^+(K^+) + \int_{X-K^+} w(x)dH^+(x) - \int_{X-K^+-K^-} w(x)dH^-(x) \\
&\geqslant (2/3)H^+(A) - (1/3)H^+(A) \geqslant (1/3)H^+(A) > 0.
\end{aligned}$$

Since $\mathcal{M}, C(X)$ is a dual pair $\mathcal{M}$ is locally convex with respect to the weak topology induces by $C(X)$ in the sup norm: relativized to the probability measures this is the same as the topology of weak convergence.

The hypograph of $V(F)$ is closed because V is upper semi-continuous. Hence by Theorem 3 for each compact (singleton) set (v, F) with $v > V(F)$ there are numbers c_0, z and continuous function $w_1(x)$ such that for $\tilde{v}, \tilde{F} \in L$ we have $c_0\tilde{v} + \int w_1(x)d\tilde{F}(x) < z$ and $c_0 v + \int w_1(x)dF(x) > z$. Applying the first to $\tilde{v}, F \in L$ we have $c_0\tilde{v} + \int w_1(x)dF(x) < z$ so that $c_0\tilde{v} < c_0 v$ implying since $v > \tilde{v}$ that $c_0 > 0$. Define $w(x) = -(w_1(x) - z)/c_0$. Observing that $(V(\tilde{F}), \tilde{F}) \in L$ the first inequality says $\int w(x)d\tilde{F} > V(\tilde{F})$ for all $\tilde{F}$ while the second implies $v \geqslant \int w(x)dF(x)$. □

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