Diversity Preferences, Affirmative Action and Choice Rules

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Abstract

Many institutions implement affirmative action policies for hiring individuals or allocating resources, indicating a preference for diversity as well as match quality. I introduce a framework to analyze diversity preferences and their effect on the affirmative action policies and choice rules adopted by institutions. I characterize the choice rules that can be rationalized by diversity preferences and demonstrate that the rule used to allocate government positions in India cannot be rationalized. I show that if institutions evaluate diversity using marginal (*i.e.,* not cross-sectional) distribution of identities, then choices induced by their preferences cannot satisfy the substitutes condition, which is crucial for the existence of competitive equilibria and stable allocations. I characterize a class of choice rules that satisfy the substitutes condition and are rationalizable by preferences that evaluate diversity and quality separately and identify the preferences that induce some widely used choice rules. My framework and results provide a systematic way of evaluating the diversity preferences behind the choices made by institutions.

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1. Introduction

Institutions in charge of allocating resources or hiring individuals make their decisions based on multiple criteria, such as the quality of the individuals they hire, the benefits individuals receive from the allocated resource or the socio-economic characteristics of individuals who are hired or allocated the resource. For example, school districts in Chicago and Boston as well as universities in Brazil prefer schools to have a diverse student body (Dur, Kominers, Pathak, and Sönmez, 2018; Dur, Pathak, and Sönmez, 2020; Aygun and Bó, 2021), medical authorities prefer the allocation of scarcely available treatments to consider values such as equity and diversity while making sure medical workers themselves are able to receive the treatment (Pathak, Sönmez, Ünver, and Yenmez, 2021; Akbarpour, Budish, Dworczak, and Kominers, 2021) and Indian government uses protections for historically discriminated groups when allocating government positions (Aygün and Turhan, 2017; Sönmez and Yenmez, 2022a). In all these settings, individuals are heterogeneous in two domains: their identities (*e.g.*, a student's socio-economic status, a patient's healthcare worker status or the caste of a government position applicant) and scores that denote their quality or suitability to the resource (*e.g.*, exam scores in school choice and government job allocation, index of clinical need in medical resource allocation). Moreover, these institutions implement affirmative action programs aimed at increasing the representation of individuals from certain underrepresented groups. These programs suggest that these institutions prefer to allocate the resource not only to the highest quality or most suitable individuals, but also care about values such as diversity and equity.

The goal of this paper is to present a theory of diversity preferences. This allows us to understand the relationship between (i) how institutions evaluate diversity and consider the trade-offs between diversity and quality, and (ii) how they determine the allocation rules and affirmative action programs. To this end, I develop a model where individuals are heterogeneous in two domains, their (possibly multidimensional) identity, and quality. An institution chooses whom to hire or allocate a resource to. In most of my applications, the institution uses a mechanism that determines how the resource will be allocated, which I refer to as the choice rule. However, one can also consider settings where these choices are based on observations of the choices of the institution on different instances (for example, different sets of applicants), as in standard consumer theory models.

I commence the analysis by studying when do the choice rules adopted by the institution can be rationalized by diversity preferences. I establish a connection between the problem of choosing sets of individuals with different identities and qualities and standard consumer theory. This allows me to leverage existing results from consumer theory to gain insights about diversity preferences. First, I analyze a setting where the quality domain does not necessarily rank all agents.[1] I characterize the choice rules that can be rationalized by a preference relation (or a utility function) over the quality and the diversity of the chosen individuals. Invoking Theorem 1 in Richter (1966), I give a necessary and sufficient condition under which the choice rule is rationalizable. This condition is the *congruence axiom* of Richter (1966): if a set of individuals I is indirectly revealed preferred to another set I' under the choice rule, then I' cannot be revealed preferred to I. Next, I assume that the individuals are ordered according to their quality (denoted by their scores), as in many of the applications this paper considers. If the institution does not value diversity or is not concerned with the identities of the individuals, then they choose the individuals with the highest scores. However, various policies ranging from explicit quotas and priority bonuses in centralized allocation systems to hiring policies that mandate representation from diverse groups for individual companies allow lower scoring individuals to be chosen before higher scoring individuals from other groups. I then characterize the choice rules that can be rationalized by a utility function that is increasing in the quality of individuals. These are the choice rules that a rational decision maker can adopt under diversity preferences in order to increase the representation of certain groups. This characterization is obtained by using a result from Nishimura, Ok, and Quah (2016), who give a generalization of Richter's congruence axiom that characterizes the choice rules that are rationalized by utility function that is increasing with respect to a given exogenously given preorder. It turns out that both conditions are fairly intuitive and one might think that this characterization is unnecessary, as they would be satisfied in any context. Unfortunately, this is not the case. I show that the main choice rule that has been used in India for between 1995 and 2020 to match individuals to government jobs fails both conditions. This indicates that the rule wouldn't be chosen by a rational decision maker, building on and complementing the analysis of Sönmez and Yenmez (2022a) who show other important shortcomings of this choice rule and illustrating the practicality of the characterization.

Even though the characterizations of rationality and monotonicity allow us to evaluate mechanisms, they place few restrictions on the choice rule and therefore the rationalizing preferences. Next, I study an important property of choice rules, the (gross) substitutes condition. Substitutes is an important theoretical property of choice rules and has proved to be crucial for existence of competitive equilibria and stable matching (Kelso and Crawford, 1982; Roth, 1984; Gul and Stacchetti, 1999; Hatfield and Milgrom, 2005). A preference for diversity may induce complementarities across individuals with different types and the choice

[1]For example, the quality domain can represent to different qualities individuals have or jobs they can perform, rather than the exam score of a student or clinical need of a patient.

rules that incorporate different constraints and are compatible with the substitutes conditions have been studied extensively. However, these studies do not consider the multidimensional and overlapping nature of identities, which has been introduced to market design recently (Kurata, Hamada, Iwasaki, and Yokoo, 2017; Aygun and Bó, 2021; Sönmez and Yenmez, 2022a) and are important in many settings.

To this end, I develop a model where each individual belongs to a group in each dimension. For example, there can be three dimensions constituting gender, race and income, where an individual's identity is given by the groups they belong in each of these dimensions. When identities are multidimensional, they can also be overlapping: individuals may belong to the same group in some the dimensions and different groups in some others. I study the relationship between the way multidimensional/overlapping identities are evaluated by an institution and how its allocation/hiring decisions (*i.e.,* choice rules) satisfy the crucial substitutes conditions. In particular, I focus on preferences that evaluate diversity by considering the marginal, and not the cross-sectional distribution of identities. For example, a company that prefers to have a diverse workforce but considers the fraction of woman workers and the fraction of workers from underrepresented minority groups, but not the fraction of woman from underrepresented groups, evaluates diversity only through marginal distributions of identities. Institutions and companies often attach a great deal of value to diversity, incorporate it explicitly into their hiring practices and even publish reports evaluating the diversity of their workforce. However, many of these reports (such as yearly diversity reports of Microsoft, Apple and MIT) only include the marginal distribution of their workforce, that is, they report the percentage of workers who belong to underrepresented groups (*e.g.*, black, asian or hispanic) and gender separately, but do not report, for example, the percentage of woman from underrepresented groups.[2] Similarly, many affirmative action programs aimed at widening representation in legislatures have quotas for both woman and minorities, but these have been typically evolved separately and work independently (Hughes, 2018). Indeed, the large literature on *intersectionality* study how different identities combine to create different modes of discrimination and privilege, with a particular focus on the experience of black woman in the United States (see, *e.g.* Crenshaw (2013)). However, until recently, such considerations have been neglected in the market design literature.

In my model, an institution does not consider intersectionality if they are indifferent be-

[2] Apple (Apple Inclusion & Diversity) and Microsoft (Microsoft Global Diversity & Inclusion Report 2020) report the fraction of employees that belong to different races and genders, while MIT does the same for its student body (MIT Diversity Dashboard). An exception is Google (Google Diversity Annual Report 2020), where the percentage of each gender is reported separately within each race in the intersectional hiring section of their annual report. See Figures 1 and 2 in Appendix B for examples of different diversity reporting practices.

tween two sets of individuals with the same marginal distributions of identities. Naturally, a special case of such preferences is an institution that does not care about the diversity of the individuals at all and is indifferent between all allocations. To rule out such cases, I require that the institution has a non-trivial preference for diversity by assuming the most preferred distributions are not the ones that do not include any individuals from a group. This mild assumption, which I call interior optimum, makes sure diversity plays at least a minimal role in hiring decisions by indicating a preference for representing each group with at least one individual. I first analyze a setting where individuals are identical in the quality domain and show that if the preferences of an institution satisfy interior optimum and do not consider intersectionality, then the choice rule induced by their preferences does not satisfy the substitutes condition of Roth (1984) (Proposition 3). Next, I allow agents to have different scores that denote their quality. When the institution prefers higher quality individuals to lower quality ones (*monotonicity*), the scores can be viewed as (inverse) salaries, and the model is closely related to the classical job matching model of Kelso and Crawford (1982). In Proposition 4, I show that when the preferences of an institution does not take intersectionality into account and satisfies monotonicity and interior optimum, then the choice rule induced by their preferences does not satisfy the gross substitutes condition of Kelso and Crawford (1982), extending the first result. Both results show that when firms and institutions choose workers or allocate resources and value diversity, intersectionality emerges as an important consideration for selection or allocation procedures to satisfy (gross) substitutes.

Finally, I present a framework to study the choice rules that satisfy gross substitutes and treat diversity and quality domains separately. That is, the choice between two individuals can depend on their quality and the representation of their groups, but not the quality of other individuals or representation of other groups. I show that a choice rule is rationalizable by a utility function that is additively separable in quality and diversity domains (where the utility is increasing in quality and concave in representation of each group) if and only if it satisfies the gross substitutes condition and an adaptation of the acyclicity condition of Tversky (1964). I then map existing choice rules such as quotas and reserves to this framework. I show how they can be rationalized (in the case of quotas and certain types of reserves) and how they can fail the conditions and therefore cannot be rationlized by preferences that treat quality and diversity domains separately.

This paper contributes the study of affirmative action and diversity concerns in market design. On the theoretical side, I establish a connection between decision theory and market design, characterize when the choice rules that can be rationalized by diversity preferences and when they satisfy gross substitutes and can be rationalized by a utility function that treats diversity and quality separately. On the applied side, I show that the mechanism used to match workers to government jobs in India cannot be rationalized by diversity preferences, considering intersectionality is crucial for the choice rule to satisfy the substitutes condition when identities are multidimensional and identify the preferences that induce some well-known choice rules such as quotas and reserves.

Related Literature This paper is contributes to the large literature on matching with affirmative action and diversity concerns initiated by Abdulkadiroğlu and Sönmez (2003) and Abdulkadiroğlu (2005). Most papers in this literature consider models where agents do not have multidimensional and overlapping identities. For example, Kojima (2012) studies quota policies and shows how affirmative action policies that place an upper bound on the enrollment of non-minority students may harm all minority students, Hafalir, Yenmez, and Yildirim (2013) introduce the alternative and more efficient minority reserve policies and Ehlers, Hafalir, Yenmez, and Yildirim (2014) generalize reserves to accommodate policies that have floors and ceilings for minority admission. Dur et al. (2018) and Dur et al. (2020) study reserves in Boston and Chicago public schools and show that the precedence order (the order in which reserve and non-reserve portions of the positions are processed) is important for the allocation. Kamada and Kojima (2017), Kamada and Kojima (2018) and Goto, Kojima, Kurata, Tamura, and Yokoo (2017) study stability and efficiency in more general matching-with-constraints models.

Another strand of literature that this paper builds upon is the study of (gross) substitutes conditions of Kelso and Crawford (1982) and Roth (1984). These conditions and their generalizations play an important role in characterizing the existence of competitive equilibria Gul and Stacchetti (1999); Milgrom (2000) and existence and structure of stable allocations Hatfield and Milgrom (2005); Hatfield and Kojima (2010, 2008); Aygün and Sönmez (2013). A number of papers have studied the choice rules that satisfy substitutability. Echenique and Yenmez (2015) characterize the choice rules that regard students as substitutes under some additional axioms. Kojima, Sun, and Yu (2020a) characterize all feasibility constraints that preserve the substitutes condition, which they call the generalized interval constraints. Kojima, Sun, and Yu (2020b) complements the analysis of their previous paper by characterizing when softer pecuniary transfer policies (rather than compulsory constraints) preserve substitutes conditions.[3] My results complement these earlier works by explicitly considering multidimensional and overlapping structure of types of individuals, adopting the alternative approach of formally studying the preferences and characterizing choice rules that satisfy gross substitutes and can be rationalized by additively separable utility functions.

[3]Both papers build on and contribute to the theory of discrete convex analysis, see Murota (1998) and Murota et al. (2016).

This paper builds on the literature that studies affirmative action with multidimensional and overlapping identities. Kurata et al. (2017) propose a mechanism called Deferred Acceptance for Overlapping Types that is strategy-proof and obtains the student-optimal matching within all stable matchings when students have strict preferences over reserve and non-reserve positions. Sönmez and Yenmez (2022a) give a detailed account of and analyze the affirmative action policies and the mechanisms that assign government positions to individuals as a function of their merit and socio-economic status in India, where protections for different domains such as gender (woman) and underrepresented socio-economic groups (scheduled castes) play an important role. Clearly, there is considerable overlap within these domains; an individual can be a woman and also belong to a scheduled caste, and therefore be eligible for both kinds of protections. Sönmez and Yenmez (2022a) formalize the main mechanism that has been used in India to match workers to government jobs between 1995 and 2020 (the SCI-AKG choice rule) and show that a lack of consideration for the overlapping nature of the protected identities has led to violations of important properties such as no justified envy and incentive compatibility. These issues has resulted in many court cases across the country and led to the subsequent termination of the mechanism. Sönmez and Yenmez (2022b) allow allocation of multiple resources and generalizes their earlier paper. I show that the SCI-AKG mechanism cannot be rationalized, adding to the deficiencies this rule exhibits. Aygun and Bó (2021) study the affirmative action policies with multidimensional privileges (where students can qualify for affirmative action in two different dimensions, income and racial minority status). They show that the way overlapping identities are treated in university admissions in Brazil can cause unfairness and incentive compatibility issues (*e.g.*, minority students becoming worse off by declaring they are minorities) and using the admissions data for the year 2013, document that this is the case for more than 54 percent of the programs. They then propose an incentive-compatible and fair mechanism that solves these issues.

This paper is also related to my prior work. Çelebi and Flynn (2022a) and Çelebi and Flynn (2022b) consider a mechanism designer with an additively separable utility function in quality and diversity domains and study the optimal design of affirmative action policies and coarsenings that map some underlying score to priorities, respectively. The analysis in Section 5 complements those papers, by analyzing when do choice rules adopted by institutions are rationalizable by a utility function that is additively separable in quality and diversity domains.[4]

Finally, in a recent paper, Carvalho, Pradelski, and Williams (2022) study the repre-

[4]Chan and Eyster (2003) and Ellison and Pathak (2021) also model the preferences of a school district with a utility function separable across diversity and quality domains.

sentativeness of the affirmative action policies that do not consider intersectionality in a model with continuum of applicants with heterogeneous qualities and overlapping identities. They show that for generic distributions policies that do not consider intersectionality cannot yield a representative outcome. Even though the models are quite different, the results in this paper complement theirs by focusing on the properties, and not the outcomes, of non-intersectional policies.

Outline Section 2 introduces the main model and notation. Section 3 characterizes choice rules that can be rationalized by a utility function and when does that utility function is increasing in the scores of individuals. Section 4 formally defines intersectionality and (gross) substitutes and shows that preferences must consider intersectionality for the induced choice rule to satisfy the (gross) substitutes condition. Section 5 characterizes the choice rules that are rationalizable by an utility function which is additively separable in diversity and quality domains. Section 6 concludes. The proofs of all results are in Appendix A.

2. Model

There are N dimensions that represent different identities of individuals. For each $l \in \{1, \ldots, N\}$, Θ_l denotes the finite set of possible groups to which an individual can belong in dimension l, $\Theta = \Theta_1 \times \ldots \times \Theta_N$ and $|\Theta_l| > 1$ for all l.[5] Each individual has a score $s \in \mathcal{S}$, where $\mathcal{S}$ is a finite set of possible scores.[6] $T = \Theta \times \mathcal{S}$ denotes all possible *types* of individuals.

For each individual i, $\theta_l(i)$ denotes the group of i in dimension l, $\theta(i) = (\theta_1(i), \ldots, \theta_N(i))$ denotes the groups to which i belongs in all dimensions which I refer as the *identity* of i. $s(i)$ denotes the score of individual i and $t(i) = (\theta(i), s(i))$ denotes the *type* of i.

For each set I of individuals, $M_l(I)$ returns the number of individuals that belong to each group in dimension l and $M(I) = (M_1(I), \ldots, M_N(I))$.[7] $M(I)$ is the *marginal distribution* of I, as it returns the number of individuals that belong to each group in each dimension, but does not have any information about the cross-sectional distribution of groups. $N_\theta(I)$ denotes the number of individuals with identity θ in I and $s(I)$ denotes the vector of scores of individuals in I.

There is an institution that chooses q individuals from $\mathcal{I}$, which denotes the set of all

[5] For example, Θ_1 can denote gender where $\Theta_1 = \{\text{men, woman}\}$ and Θ_2 can denote income where $\Theta_2 = \{\text{rich, middle class, poor}\}$.

[6] Even though $\mathcal{S}$ is not necessarily an ordered set, I use the terminology of the score as the order will be assumed for the most of the paper.

[7] Continuing the example, if $I = \{i, j\}$ where $\theta_1(i) = \theta_1(j) = \textit{men}$, $\theta_2(i) = \textit{rich}$ and $\theta_2(j) = \textit{poor}$, then $M_1(I) = (2, 0)$, $M_2(I) = (1, 0, 1)$ and $M(I) = \{(2, 0), (1, 0, 1)\}$.

individuals. For each I with $|I| \geq q$, 2^I_q denotes all q element subsets of I. A choice rule is a correspondence $C : 2^{\mathcal{I}} \to 2^{2^{\mathcal{I}}}$ such that if $I \in C(I')$, then (i) $I \subseteq I'$ (if I is a chosen set from I', then it must be included in I), (ii) $I \in 2^{\mathcal{I}}_q$ whenever $|I| \geq q$ (the institution fills its capacity whenever there are enough individuals) and (iii) $I = I'$ whenever $|I| < q$ (the institution chooses all individuals if there are not enough individuals to fill the capacity.).[8] Throughout the paper, I assume that the choices of the institution only depends on the scores and groups of the individuals. Therefore, if $t(i) = t(j)$ and $I^* \cup \{i\} \in C(I)$ with $j \notin I^*$, then $I^* \cup \{j\} \in C(I)$.

The relation $\succeq$ denotes the preferences of the institution over $\mathcal{I}$. A choice function C is induced by $\succeq$ if it returns the set of $\succeq$-maximal elements in 2^I_q. The choice function induced by $\succeq$ is denoted by $C_{\succeq}$. Formally,

$$C_{\succeq}(I) = \begin{cases} \{I' \in 2^I_q : I' \succeq I'' \text{ for all } I'' \in 2^I_q\} & \text{if } |I > q| \\ I & \text{if } |I \leq q| \end{cases}$$

3. Rationality and Monotonicity

I begin the analysis by studying the rationality of a choice rule. Let $\prod_1^q T = \mathcal{T}$ denote all possible type distributions q individuals can have. For $|I| = q$, $\tau(I) \in \mathcal{T}$ denotes the types of individuals in I.[9] Note that $\tau \in \mathcal{T}$ is enough to determine the choices, as if $\hat{I} \in C(I)$ and $\tau(\hat{I}) = \tau(\tilde{I})$, then $I' \in C(I)$. I now define a *choice cycle*, which is the main axiom that characterizes the rationality of a choice rule.

Definition 1. *$I_1, \ldots, I_n, I_1$ is a choice cycle if*

- *for each $i < n$, there exists an $\hat{I}_i$ such that $I_i \in C(\hat{I}_i)$ and $I_{i+1} \subset \hat{I}_i$*
- *there exists $\hat{I}_n$ such that $I_n \in C(\hat{I}_n)$, $I_1 \subset \hat{I}_n$ and $I_1 \notin C(\hat{I}_n)$.*

A choice cycle is a violation of *congruence* axiom of Richter (1966). If there is a choice cycle under C, then the institution has chosen I_1 when I_2 was available, I_2 when I_3 is available, ..., I_{n-1} when I_n is available. Therefore, I_1 is indirectly (weakly) revealed preferred to I_n. The fact that I_n being chosen when I_1 is available and is not chosen means that I_n is directly (strictly) revealed preferred to I_1, which contradicts the rationality of the choice rule as I_1 was indirectly revealed preferred to I_n.

A function $U : \mathcal{T} \to \mathbb{R}$ rationalizes C if $C(I) = \{I' : U(\tau(I')) = \max_{\hat{I} \in 2^I_q} U(\tau(\hat{I}))\}$. In other words, the choice rule is rationalized by U if it always chooses the U-maximal sets of

[8]These are referred as capacity filling choice rules in the literature.
[9]Formally, $\tau(I) = (t(i_1), \ldots, t(i_q))$.

individuals. The following proposition shows that the preferences of the institution can be rationalized by some U if and only if C does not admit a choice cycle.

Proposition 1. *C does not admit a choice cycle if and only if there exists a function U (and a preference relation $\succeq$) that rationalizes C.*

Proposition 1 characterizes the rationalizable choice rules, but does not put any restrictions on the utility function. In particular, $\mathcal{S}$ may not even be an ordered set. This will be the case if scores are multidimensional or they can represent the tasks the individual can perform so that it is not always possible to rank two different individuals. However, in most of the applications I consider, the scores are from an ordered set. Therefore, for the rest of the paper, I will assume that $\mathcal{S} \subset [0, 1]$. In this setting, it is reasonable to expect U to be increasing in $\mathcal{S}$, which I refer as *monotonic* utility functions. We say that $I \trianglerighteq I'$ ($I \triangleright I'$) if I is obtained from I' by (strictly) increasing the scores of some individuals. Formally, $I \trianglerighteq I'$ if there exists a bijection $h : I \to I'$ such that $\theta(i) = \theta(h(i))$, $s(i) \geq s(h(i))$ for all $i \in I$ and $I \triangleright I'$ if at least one of the inequalities is strict. U is increasing in $\mathcal{S}$ if $I \triangleright I'$ implies $U(I) > U(I')$. It turns out that the following generalization of choice cycles characterize the choices that can be rationalized by a monotonic utility function.

Definition 2. *$I_1, \ldots, I_n, I_1$ is a score-choice cycle if*

- *for each $i < n$, either (i) there exists an $\hat{I}_i$ such that $I_i \in C(\hat{I}_i)$ and $I_{i+1} \subset \hat{I}_i$ or (ii) $I_i \trianglerighteq I_{i+1}$.*
- *either (i) there exists $\hat{I}_n$ such that $I_n \in C(\hat{I}_n)$, $I_1 \subset \hat{I}_n$ and $I_1 \notin C(\hat{I}_n)$ or (ii) $I_n \triangleright I_1$.*

A score-choice cycle has additional requirement that the choice rule must prefer higher scoring individuals to lower scoring ones. It is a generalization of congruence axiom and is introduced by Nishimura et al. (2016) to generalize the rationalizability result of Richter (1966) to settings where an exogenous order is present. Applying their result, we can characterize the choice rules that are rationalizable by a *monotonic* utility function.

Proposition 2. *C does not admit a score-choice cycle if and only if there exists a function U that rationalizes C and is increasing in $\mathcal{S}$.*

One might think that lack of choice and score-choice cycles are fairly weak requirements and would be satisfied in all circumstances, which would make the analysis here trivial. However, the choice rule mandated by the Supreme Court of India and have been used in India for 25 years fails these properties. I now define this supreme court mandated choice rule (C_S) when there are two groups in two dimensions. $\Theta_1 = \{g, r\}$ where g denotes the general population and r denotes reserve eligible population (scheduled castes in their setting). $\Theta_2 = \{m, w\}$ denotes the gender of the individuals. I^g denotes the set of general category individuals who are not eligible for reserves. In this simpler setting, the choice rule C_S is characterized by 4 numbers, (q, r, r_w, o_w). q denotes the total number of positions the institution is looking to fill. $r \leq q$ is the number of reserve positions that are only open to reserve eligible individuals, while the remaining $q - r$ positions are open to all individuals. r_w is the number of reserve positions that are protected for woman, where $r_w \leq r$. Similarly, o_w is the number of open positions protected for woman, where $o_w \leq q - r$. In this simpler setting, C_S proceeds as follows.[10]

Supreme Court Mandated Choice Rule C_S

Step 1: Define I^m as the *meritorious reserve candidates.* I^m is the set of reserve eligible candidates who are among the $q - r$ highest scoring individuals in the population.

Step 2: Allocate o_w positions to the highest scoring woman in $I^g \cup I^m$.

Step 3: Allocate remaining $q - r - o_w$ positions to highest scoring individuals who are in the set $I^g \cup I^m$ and are not already allocated to a position.

Step 4: Allocate r positions to r_w highest scoring woman in I^r who are not already allocated to a position.[11]

Step 5: Allocate $r - o_r$ positions to highest scoring individuals in I^r who are not already allocated to a position.

As detailed in Sönmez and Yenmez (2022a), C_S has many shortcomings stemming from the concept of meritorious reserve candidates. If a reserve eligible candidate is not amongst the meritorious reserve candidates, then she is not considered for the open positions that are protected for woman. As a result, C_S does not satisfy *no justified envy*, which means that a reserve-eligible individual can score higher than a reserve-ineligible individual and fail receive a position while the reserve-ineligible individual receives one. As protections are adopted to help the reserve eligible individuals, this goes against the philosophy of affirmative action. Moreover, if there is a woman who does not have a high enough score to be a meritorious reserve candidate, but has a high enough score to receive an open category position protected for woman, then she can receive a position by not disclosing their reserve eligibility, violating *incentive compatibility.* Although these deficiencies are extremely important and led to the subsequent termination of the mechanism, they do not imply that the decision-maker is irrational. Next example shows that C_S admits a score-choice cycle and therefore, cannot

[10]See Sönmez and Yenmez (2022a) for the definition of the choice rule in more general settings.

[11]If at any point there fewer than o_w or r_w woman, then these positions are allocated to highest scoring men who are considered in that stage.

be rationalized by a monotonic utility function. Moreover, similar to other two problems, these cycles exist because of the way meritorious reserve candidates are processed and gives yet another reason why the rule is rescinded.[12]

Example 1. $I = \{m_1^g, w_1^r, m_1^r, w_1^g\}$, where m (w) are men (woman) and superscripts denote the groups individuals belong. Capacity and reserves are $q = 3$, $r = 1$, $o_w = 1$ and $r_w = 1$. The scores of individuals are given by

$$s(m_1^g) > s(w_1^r) > s(m_1^r) > s(w_1^g)$$

As there are two open positions and w_1^r is the second highest scoring individual, $I_m = \{w_1^r\}$. In the first stage, m_1^g and w_1^r are chosen for the open positions. In the second stage, the only remaining reserve eligible individual, m_1^r, is chosen as there is no reserve eligible woman candidate. Therefore, $\{m_1^g, w_1^r, m_1^r\}$ is chosen when $\{m_1^g, w_1^r, w_1^g\}$ is available.

Now consider a setting where individual m_1^r is replaced by $\tilde{m}_1^r$ and $s(\tilde{m}_1^r) \in (s(m_1^g), s(w_1^r))$. Thus, $\{m_1^g, w_1^r, \tilde{m}_1^r\} \triangleright \{m_1^g, w_1^r, m_1^r\}$. In this case, w_1^r becomes the third highest scoring individual and $I_m = \{\tilde{m}_1^r\}$. Since one of the open positions is reserved for woman, in the first stage m_1^g and w_1^g (who is the only woman eligible at this stage) are chosen for the open positions. In the second stage, the only remaining reserve eligible woman, w_1^r is chosen. Thus, $\{m_1^g, w_1^g, w_1^r\}$ is chosen when $\{m_1^g, w_1^r, \tilde{m}_1^r\}$ is available, creating the following score-choice cycle:

$$\{m_1^g, w_1^g, w_1^r\}, \{m_1^g, w_1^r, \tilde{m}_1^r\}, \{m_1^g, w_1^r, m_1^r\}, \{m_1^g, w_1^g, w_1^r\}$$

△

This example shows that C_S cannot be rationalized by preferences that are increasing in scores. Indeed, m_1^r was chosen at first, but was not chosen after his score has increased. This also shows that C_S does not *respect improvements*, which requires a chosen agent to be chosen after an increase of her score. Respecting improvements and stability are the two main conditions that characterize the Deferred Acceptance Mechanism (Balinski and Sönmez, 1999).The next example shows that the shortcomings of this choice rule are even greater and it cannot be rationalized at all.

Example 2. $I = \{m_1^g, m_2^g, w_1^r, w_2^r, w_1^g\}$. Capacity and reserves are $q = 3$, $r = 1$, $o_w = 1$ and

[12]One important detail about C_S it is not capacity filling, it can leave some positions empty if there are not enough individuals who belong to the groups those positions are reserved for. This creates some further problems when there are not enough individuals from each group. However, if there are enough individuals from each group, then C_S always allocates all positions. Therefore, my results show that even if availability of individuals from each group is not an issue and C_S allocates all the positions, it is still not rationalizable.

$r_w = 0$. The scores of individuals are given by

$$s(m_1^g) > s(m_2^g) > s(w_1^r) > s(w_2^r) > s(w_1^g)$$

As there are two open positions and m_1^g and m_2^g are the two highest scoring individuals, $I^m = \emptyset$ and only I^g are eligible for the open slots. Thus in the first stage, m_1^g and w_1^g are chosen. In the second stage, highest scoring reserve eligible individual, w_1^r is chosen. This means that the set $I_1 = \{m_1^g, w_1^r, w_1^g\}$ is chosen when $I_2 = \{m_1^g, w_1^r, w_2^r\}$ is available.

Now consider $\tilde{I} = \{m_1^g, w_1^r, w_2^r, w_1^g\}$, which removes m_2^g from the set of applicants. Then $I_m = \{w_1^r\}$ and in the first stage, m_1^g and w_1^r are chosen. In the second stage, the only remaining reserve eligible individual, w_2^r is chosen. Therefore, I_2 is chosen when I_1 is available and I_1, I_2, I_1 is a choice cycle. △

This example shows that the C_S rule not only violate monotonicity, but it also cannot be result of preferences of a rational decision-maker. Examples 1 and 2 provide evidence of shortcomings of the choice rule that has been used in allocating government jobs in India and show how my results can be used to determine the rationality of the choice rules and inform policymakers, assuming they have well defined preferences over the allocations.

As the examples demonstrate, the main problem with this mechanism is the fact that only high scoring reserve eligible individuals are considered for the open positions. In this simple setting with two groups in two dimensions, modifying the mechanism to consider all individuals for the open positions, would solve this issue.[13]

4. Intersectionality and (Gross) Substitutes

This section analyzes how institutions evaluate diversity and how this affects the properties of the choice rules they implement. If an institution evaluates diversity of a set of individuals by the marginal, but not the cross-sectional distributions of groups, then it does not consider intersectionality. I show that when the preferences do not consider intersectionality, then the choice rules induced by those preferences fail to satisfy the (gross) substitutes condition, an important property that is required for existence of a competitive equilibrium and a stable matching.

[13] Sönmez and Yenmez (2022a) propose the 2SMG mechanism that remedies the flaws of the previous mechanism in their model general model and was subsequently endorsed by the Supreme Court.

4.1. Homogeneous Individuals and the Substitutes Condition

I start the analysis with individuals who are homogeneous in terms of quality, but may belong to different socio-economic groups, assuming $|\mathcal{S}| = 1$ and suppressing any dependence to scores. Since $M(I)$ denotes the marginal, and not cross-sectional distribution of groups, if the preferences of the institution are based on $M(I)$, then they cannot incorporate intersectionality.

Definition 3. *$\succeq$ does not consider intersectionality if for all I and I' with $M(I) = M(I')$, $I \sim I'$.*

Definition 3 indicates that the institution considers only marginal, and not cross-sectional, distribution of groups when evaluating the diversity of a set of individuals. As described in the introduction, policies that reserve positions for different groups in different dimensions separately, or diversity statistics that only report the marginal distribution of individuals are examples of such preferences. Next, I state the substitutes condition.

Definition 4. *Let $\tilde{I} \subseteq I' \subset I$. C satisfies the substitutes condition if for all $\tilde{I}$ with $\tilde{I} \subseteq \hat{I} \in C(I)$, there exists $\bar{I}$ such that $\bar{I} \in C(I')$ and $\tilde{I} \subseteq \bar{I}$.*

This condition is the generalization of the substitutes condition of Roth (1984) to choice correspondences.[14] Substitutes condition states that whenever a set of individuals are chosen from a set I, then the same set of individuals are also chosen from any set $I' \subset I$. The following example illustrates the relationship between intersectionality and the substitutes condition.

Example 3. Assume $\succeq$ does not consider intersectionality. Let $q = 4$, $\Theta_1 = \{1, 2\}$ and $\Theta_2 = \{1, 2\}$, $\mathcal{I} = \{i_1, i_2, i_3, i_4, i_5, i_6, i_7, i_8\}$. Moreover, assume that the institution strictly prefers to have exactly 2 individuals from all 4 groups (the "most" diverse outcome when the institution form their preferences by the marginal distribution of groups) to any other distribution. The types of individuals are

	Θ_1	Θ_2
i_1, i_5	1	1
i_2, i_6	1	2
i_3, i_7	2	1
i_4, i_8	2	2

[14]Note that when C is a choice function (*i.e.*, $C(I)$ is singleton for all I), this condition is equivalent to the following: If $i \in C(I)$ and $I' \subseteq I$, then $i \in C(I')$. A similar generalization is employed by Kojima et al. (2020b) for a model with salaries.

The subset of agents $\hat{I} = \{i_1, i_5, i_4, i_8\}$ has two individuals from each group and therefore $\hat{I} \in C_{\succeq}(\mathcal{I})$. However, if the set of available agents are $\mathcal{I} \setminus \{i_1, i_5\}$, then the unique subset of agents that has two individuals from each group is $\tilde{I} = \{i_2, i_6, i_3, i_7\}$, and thus $C_{\succeq}(\mathcal{I} \setminus \{i_1, i_5\}) = \tilde{I}$. As $\hat{I} \in C(\mathcal{I})$ but $\hat{I} \notin C(\mathcal{I} \setminus \{i_1, i_5\})$, $C_{\succeq}$ does not satisfy the substitutes condition. △

As Example 3 illustrates, when preferences do not consider intersectionality and value diversity, then the substitutes condition might fail. As the institution evaluates diversity by marginal distributions, individuals $\{i_1, i_5\}$ and $\{i_4, i_8\}$ become *complements*: when $\{i_4, i_8\}$ is chosen, the institution already has two individuals from group 2 in both dimensions, and prefers individuals who belong to group 1 in these dimensions. Therefore, when $\{i_1, i_5\}$ is not available, choosing $\{i_4, i_8\}$ cannot be optimal as no two individuals from $\{i_2, i_3, i_6, i_7\}$ can complement $\{i_4, i_8\}$ to attain the preferred distribution.

It is possible that the preferences of the institution put no weight on diversity. This would, for example, be the case if the institution is indifferent between all possible group distributions and therefore indifferent between any two sets of individuals (with q elements) and substitutes condition is trivially satisfied. Therefore, I make a minimal assumption that makes diversity preferences non-trivial and assume that the most preferred diversity level of the institution does not completely exclude a group from being chosen. Formally, $M(I)$ is *at boundary* if I has no individual with some trait, *i.e.*, there exists $\hat{\theta} \in \Theta_l$ such that $\theta_l(i) \neq \hat{\theta}$ for all $i \in I$. Conversely, $M(I)$ is *interior* if it is not at boundary, *i.e.*, if I has at least one individual from each group. The following assumption states that when comparing groups of individuals, compositions that have no individuals from some group are not the most preferred ones.

Definition 5. *$\succeq$ satisfies **interior optimum** if for all I where $M(I)$ is at boundary, there exists I' where $M(I')$ is interior and $I' \succ I$.*

This is a reasonable assumption for diversity preferences: it requires that the institution values diversity and prefers to choose at least one individual from each group, but puts no other restrictions on how it values different compositions of individuals.

Proposition 3. *If $\succeq$ does not consider intersectionality and satisfies interior optimum then $C_{\succeq}$ does not satisfy the substitutes condition.*

Proposition 3 shows that the logic of Example 3 is indeed much more general: whenever the preferences of the institutions satisfies the mild interior optimum assumption, considering intersectionality is necessary to satisfy the substitutes condition. The proof of this result starts with an arbitrary $\succeq$ that satisfies the two assumptions of the proposition. It then proceeds to determine a particular distribution of groups, similar to the “most” diverse outcome in Example 3 and two sets of agents that complement each other the way $\{i_1, i_5\}$ and $\{i_4, i_8\}$ complements each other in Example 3.

4.2. Heterogeneous Qualities and Gross Substitutes

This section extends the analysis to the setting where, in addition to their socio-economic groups, each individual also has a score $s \in \mathcal{S} \subseteq [\underline{s}, \bar{s}]$ with $|\mathcal{S}| > 1$. I assume that holding their identities constant, the institution prefers higher quality agents to lower quality ones.

Definition 6. *$\succeq$ satisfies **monotonicity** if for all $I = \{i_1, \ldots, i_q\}$ and $I' = \{i'_1, \ldots, i'_q\}$ with $\theta(i_k) = \theta(i'_k)$ and $s(i) \geq s(i'_k)$ with at least one element strict, $I \succ I'$.*

Under monotonicity assumption, the scores in this model are analogous to (inverse) salaries in Kelso and Crawford (1982), where a higher salary is worse for the institution. Therefore, I adopt the following gross substitutes definition given in Kelso and Crawford (1982).

Definition 7. *Let $\tilde{I} \subseteq \hat{I} \in C(I)$. Define I' by (weakly) decreasing the scores of all $I \setminus \tilde{I}$. If C satisfies gross substitutes, then there exists $\bar{I}$ such that $\tilde{I} \subset \bar{I}$ and $\bar{I} \in C(I')$.*

Gross substitutes condition requires that if a set of individuals are chosen, and the scores of other individuals decrease, then that set of individuals must still be chosen. I also extend the definition of preferences that do not consider intersectionality to settings with heterogeneous qualities.

Definition 8. *$\succeq$ does not consider intersectionality if $\{s(I), \theta(I)\} = \{s(I'), (\tilde{I}')\}$ implies $I \sim I'$.*

With heterogeneous qualities, an institution does not consider intersectionality is indifferent between two sets of individuals whenever they have the same cross-sectional distribution of groups and the same scores. The following proposition shows that the the relationship between intersectionality and the substitutes condition generalizes to this setting. These results show that how diversity is evaluated is important for choice rule to satisfy certain properties, and in this particular case, seemingly unrelated criterion of intersectionality emerges as a critical consideration for the choice rules to satisfy the substitutes conditions.

Proposition 4. *If $\succ$ does not consider intersectionality, satisfies monotonicity and interior optimum, then $C_\succ$ does not satisfy gross substitutes.*

Proposition 4 is proved by making the appropriate adjustments to the proof of Proposition 3, where decreasing the scores of a set of individuals mirrors the effect of removing those individuals. It shows that whenever the institutions values higher scoring individuals and has a non-trivial preference for diversity, not considering intersectionality when evaluating diversity will cause failure of the gross substitutes condition.

5. (Gross) Substitutes and Separability

The analysis in the previous sections has focused on the rationality of the institution and how it evaluates diversity, but it was silent on the possible trade-offs between quality and diversity domains. This section characterizes choice rules that satisfy gross substitutes and are induced by a class of preferences that treat diversity and quality domains separately. Separability is a reasonable property to study; in various settings, the contribution of an individual to an institution is independent from their identity and the institution prefers having a diverse body due to equity concerns, not because it would boost productivity. Moreover, I will show that many (but not all) of the choice rules adopted by different institutions satisfy these conditions and can be rationalized by the preferences I characterize.

For this section, I assume that if $I' \in C(I)$ and $I'' \in C(I)$, then I' is equivalent to I'', in other words, $\tau(I) = \tau(I')$. This means that although C is a choice correspondence as there can be many individuals with same type, it is actually a choice function if we restrict attention to equivalence classes $\mathcal{T}$. A choice rule C satisfies *gross substitutes** if C satisfies both the substitutes condition and the gross substitutes condition.

Let $R(I)$ denotes all agents who are not chosen in some $\hat{I} \in C(I)$. Formally, $i \in R(I)$ if there exists some $\hat{I} \in C(I)$ and $i \notin \hat{I}$. Given C, construct the following binary relation $>_C$, for $\theta(j) \neq \theta(k)$ and $|I| = q - 1$,

$$k \in R(I \cup \{j, k\}) \implies (s(j), \theta(j), N_{\theta(j)}(I \cup \{j\})) >_C (s(k), \theta(k), N_{\theta(k)}(I \cup \{k\}))$$

$>_C$ represents the revealed preference induced by C. For $\theta \neq \theta'$, $(s, \theta, n) >_C (s', \theta', n')$ states that an individual with identity θ and score s is chosen together with $n-1$ agents that shares her identity in favor of an individual with score s', identity θ' with n' other individuals who shares her identity. To economize on notation, let $Q = \{1, \ldots, q\}$ and $D = \Theta \times Q$ denote the set of all (θ, n) with generic element $d \in D$.

Definition 9. *A collection*

$$\begin{aligned}(s_1, d_1) &> (s'_1, d'_1)\\(s_2, d_2) &> (s'_2, d'_2)\\&\vdots\\(s_m, d_m) &> (s'_m, d'_m)\end{aligned}$$

is a cycle if $(s'_1, \ldots, s'_m)$ *is a permutation of* $(s_1, \ldots, s_m)$ *and* $(d'_1, \ldots, d'_m)$ *is a permutation of* $(d_1, \ldots, d_m)$

This definition is due to Tversky (1964) (see also Scott (1964); Adams (1965)) and is used it to characterize preferences that admit an additively separable utility representation. It implies transitivity, but is more restrictive. Existence of a cycle under $>_C$ means that the evaluation of diversity and quality domains are connected, since both $\{(s_i, d_i)\}_{i \leq m}$ and $\{(s'_i, d'_i)\}_{i \leq m}$ are formed from exactly same scores and diversity levels, but $\{(s_i, d_i)\}_{i \leq m}$ are revealed strictly preferred to $\{(s'_i, d'_i)\}_{i \leq m}$ for all i.

Definition 10. *C satisfies **Acyclicity** if there exists no cycles under $>_C$.*

Acyclicity of $>_C$ rules out any connection between the diversity and quality domains. We are now ready to characterize a general class of choice rules that can be induced by a utility function that is separable in diversity and quality domains.

Proposition 5. *If C satisfies gross substitutes* and acyclity, then there exists increasing u and concave $\{h_\theta\}_{\theta \in \Theta}$ such that*

$$U(I) = \sum_{i \in I} u(s(i)) + \sum_{\theta \in \Theta} h_\theta(N_\theta(I)) \tag{1}$$

where U rationalizes C.

For each increasing u and concave $\{h_\theta\}_{\theta \in \Theta}$, C_U satisfies gross substitutes and acyclity.

Proposition 5 shows that as long as the choice rules satisfy gross substitutes* and acyclicity, they can be induced by a utility function given in Equation 1. For a given set of individuals, this utility function evaluates the quality and diversity dimensions separately. The utility from quality dimension is given by $\sum_{i \in I} u(s(i))$, where u is the benefit of choosing an individual with score $s(i)$, which is increasing in $s(i)$. The utility from diversity of the chosen set is given by a set of functions $\{h_\theta\}_{\theta \in \Theta}$, each of which denotes the benefit of choosing a given number of individuals with each identity θ. Moreover, these functions are concave, which means that the marginal benefit of choosing an individual with a given identity is decreasing in the number of such individuals, representing a preference against choosing many individuals with the same identity. Conversely, if the preferences of the institution can be represented by a utility function given in Equation 1, then the choice rules induced by those preferences will satisfy gross substitutes.

A short sketch of the proof is useful for illustrating how these two conditions yield the structure in Proposition 5. First, the choice rule induces an incomplete binary relation $>_C$ over (s, θ, n) tuples. Under acyclicity, this incomplete binary relation can be extended to a complete preference relation which can be represented by an additively separable utility function $u(s) + h(\theta, n)$ over (s, θ, n) tuples. This is an application of the results in Tversky (1964). Next, $h_\theta(N_\theta(I))$ are constructed for each θ using h. At this point, u and $\{h_\theta\}_{\theta\in\Theta}$ represent C for decisions between any two individuals, but not necessarily for decisions over sets of individuals. I then show that under gross substitutes*, the information contained in these binary decisions is actually enough to pin down choices over sets of individuals, yielding the representation.

Many of the choice rules used in different contexts and studied in the literature can be mapped to this framework. For example, a quota policy that restricts admission of individuals with each type θ by some $k_\theta \geq 0$ (Kojima, 2012) can be rationalized by any strictly increasing u and $\{h_\theta\}_{\theta\in\Theta}$ given by

$$h_{\boldsymbol{\theta}}(\tilde{N}_\theta(I)) = \begin{cases} 0 & \text{if } N_\theta(I) \leq k_\theta \\ -q\hat{u} & \text{if } N_\theta(I) > k_\theta \end{cases}$$

where $\hat{u} = u(\overline{s}) - u(\underline{s})$. This indicates that the utility loss in the diversity dimension from going above the quota is larger than any utility gain from the quality dimension. Another example is a reserve policy (see Hafalir et al. (2013) and Dur et al. (2020)), that reserves r_θ of positions for individuals with each identity. These reserves and remaining open positions are then processed according to some order (which is called a precedence order in the literature) where in each step, highest scoring reserve eligible individuals are chosen. When open slots are processed after all reserve slots, then reserve policies can be rationalized by by any strictly increasing u and $\{h_\theta\}_{\theta\in\Theta}$ given by

$$h_\theta(\tilde{N}_\theta(I)) = \begin{cases} N_\theta(I)\hat{u} & \text{if } N_\theta(I) \leq k_\theta \\ k_\theta\hat{u} & \text{if } N_\theta(I) > k_\theta \end{cases}$$

where $\hat{u} = u(\overline{s}) - u(\underline{s})$. This indicates that the diversity utility is increasing and more important than any gains in the score dimension until the reserve is met and is constant after the reserve requirements are satisfied. However, as the following example shows, if open slots are processed before reserves, the choice rule may fail acyclicity and cannot be represented by an additively separable utility function.

Example 4. There are 3 positions to be allocated and two groups in one dimension, $\Theta = \{a, b\}$. There is one reserve position for each of the groups. The institution first processes the open position, then group a reserve and then group b reserve. We will consider two applicant sets, I and I'. The following table lists the score of each individual in these cases, where letters denote their groups.

	a_1	b_1	a_2	b_2
$s(I)$	3	2	1	1
$s(I')$	2	3	1	1

In each case, the open position goes to the individual with a score of 3 and the other individual from that group receives the reserve position. Under I, $C(I) = \{a_1, b_1, a_2\}$, which implies $(1, a, 2) >_C (1, b, 2)$ and $C(I') = \{a_1, b_1, b_2\}$, which implies $(1, b, 2) >_C (1, a, 2)$. This violates acyclicty and shows that this choice rule cannot be rationalized by a separable utility function. △

6. Conclusion

I introduce a model of diversity preferences that evaluate individuals in two domains: the diversity and the quality. I characterize the choice rules that can be rationalized by a utility function over these two domains and when this utility function is increasing in the quality domain. This characterization result allows us to determine whether the choice rule implemented by (or observed choices of) an institution can be rationalized by diversity preferences. I apply this result to show that the recently rescinded SCI-AKG choice rule which is used for allocating government jobs in India cannot be rationalized. I also study the choice rules that satisfy the (gross) substitutes condition, which is crucial for equilibrium existence and stability. First, I show that how diversity is evaluated is critical in this respect; if an institution does not consider intersectionality when evaluating diversity, then its choices do not satisfy the substitutes condition. Second, I characterize all choice rules that satisfy the substitutes condition and are rationalizable by preferences that evaluate diversity and quality separately and identify the preferences that induce some widely used choice rules. My results provide a systematic way of evaluating the diversity preferences behind the choices made institutions.

Appendices

A. Proofs

A.1. Proof of Proposition 1

Follows from Proposition 2 if we define $\triangleright$ and $\trianglerighteq$ as follows: $I \trianglerighteq I'$ if and only if $\tau(I) = \tau(I')$ and $\triangleright$ is the empty relation.

A.2. Proof of Proposition 2

Define the relation $>_C$ as follows: $I >_C I'$ if $I \in C(\hat{I})$ and $I' \subset \hat{I}$. Let $>_C \cup \trianglerighteq$ denote the union of $>_C$ and $\trianglerighteq$ and $\text{tran}(>_C \cup \trianglerighteq)$ denote the transitive closure of this relation. We say that C satisfies $\triangleright$-congruence if (i) $I\ \text{tran}(>_C \cup \trianglerighteq)I'$ and $I' \in C(\hat{I})$ imply $I \in C(\hat{I})$ for every $\hat{I}$ that contains I and (ii) $I\ \text{tran}(>_C \cup \trianglerighteq)I'$ imply not $I' \triangleright I$. The following lemma follows from the finiteness of $\mathcal{I}$.

Lemma 1. *C satisfies $\triangleright$-congruence if and only if C does not admit a score-choice cycle.*

Proof. Assume that C admits a score-choice cycle $I_1, \ldots, I_n, I_1$. Then for each $i \leq n-1$, either $I_i >_C I_{i+1}$ or $I_i \trianglerighteq I_{i+1}$. Thus, $I_1\ \text{tran}(>_C \cup \trianglerighteq)I_n$. Moreover, we also have either $I_n \triangleright I_1$ or $I_n \in C(\hat{I})$, $I_n \notin C(\hat{I})$ and $I_1 \subset \hat{I}$ for some $\hat{I}$, both of which cause failure of $\triangleright$-congruence.

Next, assume that C does not satisfy $\triangleright$-congruence and let I_1 and I_n denote the sets that cause the violation. Then $I_1\ \text{tran}(>_C \cup \trianglerighteq)I_n$. As $\mathcal{I}$ is finite, all q element subsets of $\mathcal{I}$ is also finite, which means that there exists $I_1, I_2, \ldots, I_n$ such that for each $i < n$, either (i) there exists an $\hat{I}_i$ such that $I_i \in C(\hat{I}_i)$ and $I_{i+1} \subset \hat{I}_i$ or (ii) $I_i \trianglerighteq I_{i+1}$. Moreover, we also have either either (i) there exists $\hat{I}_n$ such that $I_n \in C(\hat{I}_n)$, $I_1 \subset \hat{I}_n$ and $I_1 \notin C(\hat{I}_n)$ or (ii) $I_n \triangleright I_1$, which completes the proof. $\square$

The result then follows from Theorem 7 in Nishimura et al. (2016).

A.3. Proof of Proposition 3

Let $\mathcal{I}$ denote set of individuals where there are q individuals from each $\theta \in \Theta$ and $\succeq$ denote the preferences that does not take intersectionality into account. For the rest of the proof, I will refer individuals who belong to group j in dimension k as (j,k) individuals. Formally, i is a (j,k) individual if $\theta_k(i) = j$.

Without loss of generality, let the first dimension to be one of the dimensions with fewest available groups, *i.e.*, $|\Theta_1| \leq |\Theta_l|$ for all l. Let D^* denote the set of all optimal marginal distributions. Formally, $d \in D^*$ if there exists I' such that $I' \in C(\mathcal{I})$ and $M(I') = d$. Let d_1 denote an element of D^* with highest number of $(1,1)$ individuals and let m_{11} denote the number of $(1,1)$ individuals at d_1. Let D_1^* denote the set of all optimal group distributions where the number of $(1,1)$ individuals is m_{11}. Next, let d_{11}^* be a group distribution in D_1^* with the highest number of $(2,1)$ individuals. m_{21} denotes the number of $(2,1)$ individuals at d_{11}^*.

We say that a set of individuals I is *compatible with* marginal distributions d^* if there exists I' such that $M(I \cup I') = d^*$. If $M(I \cup I') = d^*$, then I' is a *complement* of I for d^*. Let $M_{ij}(I)$ denote the number of group i individuals in dimension j in I.

Lemma 2. *Let d denote a marginal distribution and let $M_{ij}(d)$ denote the number of group i individuals in dimension j. If $M_{ij}(I) \leq M_{ij}(d)$ for all i and j, then I is compatible with d.*

Proof. Note that since d is a marginal distribution, $|d| \equiv \sum_i M_{ij}(d)$ for all j. Moreover, $\sum_i M_{ij}(I) = |I|$ for all j.

First, if $|I| = |d|$, then $M_{ij}(I) \leq M_{ij}(d)$ implies $M_{ij}(I) = \theta_{ij}(d)$ and I is compatible with d. If $|I| < |d|$, then for each dimension i, there exists a group j such that $M_{ij}(I) < M_{ij}(d)$. Let t denote an individual who belongs to group j that satisfies this condition. Then the set $I \cup \{t\}$ still satisfies $M_{ij}(I) \leq M_{ij}(d)$ and repeating this procedure yields a $\tilde{I}$ such that $M_{ij}(\tilde{I}) = M_{ij}(d)$. Letting $I' = \tilde{I} \setminus I$, we have $M(I \cup I') = d^*$ and therefore I is compatible with d. □

Claim 1. $m_{11} < q$ *and* $m_{21} < q$.

Proof. If either $m_{11} = q$ or $m_{21} = q$, then there exists $I' \in C(I)$ with no $(1,2)$ or $(2,2)$ individuals, which is a contradiction to interior optimality since $M(I')$ is at boundary. □

We are now ready prove the result. There are two cases, either $m_{11} \leq m_{21}$ or $m_{11} > m_{21}$.

Case 1: $m_{11} \leq m_{21}$.

Claim 2. *There exists $I_{11} = \{i_1^{11}, \ldots, i_{m_{11}}^{11}\}$ where all $i \in I_{11}$ are $(1,1)$ and $(2,1)$ individuals and I_{11} is compatible with d_{11}^*.*

Proof. First, note that since $m_{11} < q$ and groups $(1,1)$ and $(2,1)$ have (weakly) more individuals at d_{11}^*, one can choose the groups of individuals in I_{11} to satisfy $M_{ij}(I_{11}) \leq M_{ij}(g_{11}^*)$ for all i and j. Then the result follows from Lemma 2. □

Let I' denote a complement of I_{11} at d_{11}^* that includes an individual who is neither a $(1,1)$ nor $(2,1)$ individual.[15] Take $j \in I_{11}$ and $k \in I'$, where k is not a $(1,1)$ or $(2,1)$ individual. Define $\tilde{j}$ and $\tilde{k}$ as

$$\theta_1(\tilde{j}) = \theta_1(j), \theta_\ell(\tilde{i}) = \theta_\ell(k) \text{ for all } \ell \neq 1$$

$$\theta_1(\tilde{k}) = \theta_1(k), \theta_\ell(\tilde{k}) = \theta_\ell(j) \text{ for all } \ell \neq 1$$

Let $\tilde{I}_{11} = I_{11} \setminus \{j\} \cup \tilde{j}$ and $I'' = I' \setminus \{k\} \cup \tilde{k}$. Note that I'' is a complement of $\tilde{I}_{11}$ at d_{11}^*.

Claim 3. *I' and I'' does not have any $(1,1)$ individuals. Moreover, I'' has $m_{21} - m_{11} + 1$ group $(2,1)$ individuals.*

Proof. The first part of the result follows from the fact that at any optimal I cannot have more than m_{11} $(1,1)$ individuals and I' and I'' are complements of I_{11} and $\tilde{I}_{11}$ at d_{11}^*. Second part follows from the fact that I'' is a complement of $\tilde{I}_{11}$ at d_{11}^* and $\tilde{I}_{11}$ has $m_{11} - 1$ $(2,1)$ individiuals. □

Let $\bar{I} = I_{11} \cup I' \cup \tilde{I}_{11} \cup I''$. First, note that $I_{11} \cup I' \in C(\bar{I})$ and $\tilde{I}_{11} \cup I'' \in C(\bar{I})$, since $M(I_{11} \cup I') = d_{11}^*$ and $M(\tilde{I}_{11} \cup I'') = d_{11}^*$.

Lemma 3. *There does not exist an $I^* \in C(\bar{I} \setminus I')$ such that $I_{11} \subset I^*$*

Proof. For a contradiction, assume such an I^* exists. Then it must be that, $\tilde{I}_{11} \cap I^* = \emptyset$, since otherwise there will be more than m_{11} $(1,1)$ individuals at I^*, which will be a contradiction. However, this means that $I^* = I_{11} \cup I''$. But I^* has $m_{21} + 1$ $(2,1)$ individuals and m_{11} $(1,1)$ individuals, which contradicts the optimality of I^* as d_{11}^* is a group distribution in D_1^* with the highest number of $(2,1)$ individuals and has m_{21} such individuals. Since $\tilde{I}_{11} \cup I''$ is available and optimal, this is a contradiction. □

The result then follows from the fact that I_{11} is chosen from $\bar{I}$, but not from $\bar{I} \setminus I'$.

Case 2: $m_{11} > m_{21}$ Let $n = m_{11} - m_{21}$.

Claim 4. *There exists $I_{12} = \{i_1^{11}, \ldots, i_{m_{21}}^{11}, i_1^1, \ldots, i_n^1\}$ where the first m_{21} elements are $(1,1)$ and $(2,1)$ individuals, rest are $(1,1)$ individuals and I_{12} is compatible with d_{11}^*.*

Proof. First, note that since $m_{21} < q$ and groups $(1,1)$ and $(2,1)$ have (weakly) more individuals at d_{11}^*, one can choose the groups of individuals in I_{12} to satisfy $M_{ij}(I_{12}) \leq M_{ij}(d_{11}^*)$ for all i and j. Then the result follows from Lemma 2. □

[15] This is possible since the preferences satisfy interior optimum and all individuals in I_{11} are both $(1,1)$ and $(2,1)$ indviduals.

Let I' denote a complement of I_{12} at d^*_{11} that includes an individual who is neither a $(1,1)$ nor $(2,1)$ individual.[16] Take $j \in I_{12}$ and $k \in I'$, where k is not a $(1,1)$ or $(2,1)$ individual. Define $\tilde{j}$ and $\tilde{k}$ as

$$\theta_2(\tilde{j}) = \theta_2(j), \theta_\ell(\tilde{i}) = \theta_\ell(k) \text{ for all } \ell \neq 1$$

$$\theta_2(\tilde{k}) = \theta_2(k), \theta_\ell(\tilde{k}) = \theta_\ell(j) \text{ for all } \ell \neq 1$$

Let $\tilde{I}_{12} = I_{12} \setminus \{j\} \cup \tilde{j}$ and $I'' = I' \setminus \{k\} \cup \tilde{k}$. Note that I'' is a complement of $\tilde{I}_{12}$ at d^*_{11}.

Claim 5. *I' and I'' does not have any $(2,1)$ individuals. Moreover, I'' has 1 group $(1,1)$ individual.*

Proof. First, note that $I_{12} \cup I'$ and $\tilde{I}_{12} \cup I''$ have m_{11} group $(1,1)$ individuals. First part then follows since any optimal I that has m_{11} group $(1,1)$ individuals cannot have more than m_{21} $(2,1)$ individuals and I' and I'' are complements of I_{12} and $\tilde{I}_{12}$ at d^*_{11}. Second part follows from the fact that I'' is a complement of $\tilde{I}_{12}$ at d^*_{11} and $\tilde{I}_{12}$ has $m_{11} - 1$ $(1,1)$ individuals. $\square$

Let $\bar{I} = I_{12} \cup I' \cup \tilde{I}_{12} \cup I''$. First, note that $I_{12} \cup I' \in C(\bar{I})$ and $\tilde{I}_{12} \cup I'' \in C(\bar{I})$, since $M(I_{12} \cup I') = d^*_{11}$ and $M(\tilde{I}_{12} \cup I'') = d^*_{11}$.

Lemma 4. *There does not exist an $I^* \in C(\bar{I} \setminus I')$ such that $I_{12} \subset I^*$*

Proof. First, note that since $\tilde{I}_{11} \cup I''$ is available and optimal, I^* must also be optimal. For a contradiction, assume such an I^* exists. Then it must be that, $\tilde{I}_{12} \cap I^* = \emptyset$.

To see why, assume there exist a $t \in \tilde{I}_{12} \cap I^*$. If $t = \tilde{k}$, then there are more than m_{21} group $(2,1)$ individuals and at least group m_{11} group $(1,1)$ individuals in I^*, which contradicts optimality of I^*. If $t \neq \tilde{k}$, then there are more than m_{11} group $(1,1)$ individuals in I^*, which contradicts optimality of I^*.

However, this means that $I^* = I_{11} \cup I''$. But then I^* has $m_{11} + 1$ $(1,1)$ individuals, which contradicts the optimality of I^* as d^*_{11}. $\square$

The result then follows from the fact that I_{11} is chosen from $\bar{I}$, but not from $\bar{I} \setminus I'$.

A.4. Proof of Proposition 4

The proof is follows from following the steps in the proof of Proposition 3 with minor modifications. Assume that all individuals have the highest scores, $\bar{s}$ and replicate the steps.

[16]This is possible since the preferences satisfy interior optimum and all individuals in I_{12} are both $(1,1)$ and $(2,1)$ indviduals.

For Case 1, consider I'_s where all individuals in I' have strictly lower scores than the highest score $\bar{s}$.

Let $\bar{I} = I_{11} \cup I' \cup \tilde{I}_{11} \cup I''$ and $\bar{I}_s = I_{11} \cup I'_s \cup \tilde{I}_{11} \cup I''$. First, note that $I_{11} \cup I' \in C(\bar{I})$ and $\tilde{I}_{11} \cup I'' \in C(\bar{I})$, since $M(I_{11} \cup I') = d^*_{11}$ and $M(\tilde{I}_{11} \cup I'') = d^*_{11}$.

Lemma 5. *There does not exist an $I^* \in C(\bar{I}_s)$ such that $I_{11} \subset I^*$.*

Proof. For a contradiction, assume such an I^* exists.

Claim 6. $I^* \cap I_s = \emptyset$.

Proof. For a contradiction, assume that $I^* \cap I_s \neq \emptyset$. Since $I^* \in C(\bar{I}_s)$, $I^* \succeq \tilde{I}_{11} \cup I''$. Define I^*_s by increasing the scores of all individuals in I^* to $\bar{s}$. From monotonicity, $I^*_s \succ I^* \succeq \tilde{I}_{11} \cup I''$, which is a contradiction as $\tilde{I}_{11} \cup I''$ is an optimal group distribution when all individuals have scores $\bar{s}$. □

Given Claim 6, following the same steps in Lemma 3 yields the result. □

The proof of Case 1 then follows from the fact that in $\bar{I}_s$ the scores of individuals in I_{11} are same as $\bar{I}$, scores of all other individuals are weakly lower than $\bar{I}$ and I_{11} is chosen from $\bar{I}$, but not from $\bar{I}_s$.

To prove the Case 2, Let $\bar{I} = I_{12} \cup I' \cup \tilde{I}_{12} \cup I''$ and $\bar{I}_s = I_{12} \cup I'_s \cup \tilde{I}_{12} \cup I''$. First, note that $I_{12} \cup I' \in C(\bar{I})$ and $\tilde{I}_{12} \cup I'' \in C(\bar{I})$, since $M(I_{12} \cup I') = d^*_{11}$ and $M(\tilde{I}_{12} \cup I'') = d^*_{11}$.

Lemma 6. *There does not exist an $I^* \in C(\bar{I}_s)$ such that $I_{12} \subset I^*$.*

Proof. For a contradiction, assume such an I^* exists.

Claim 7. $I^* \cap I_s = \emptyset$.

Proof. Follows from the same steps in Claim 6. □

Given Claim 7, following the same steps in Lemma 4 yields the result. □

The proof of Case 2 then follows from the fact that in $\bar{I}_s$ the scores of individuals in I_{12} are same as $\bar{I}$, scores of all other individuals are weakly lower than $\bar{I}$ and I_{12} is chosen from $\bar{I}$, but not from $\bar{I}_s$.

A.5. Proof of Proposition 5

Suppose that C satisfies gross substitutes* and acyclicty. First, I extend $>_C$ to include comparisons of individuals from same groups.

Lemma 7. *Suppose that gross substitutes and acyclicty are satisfied. Fix an s, θ and $n \geq 1$, suppose that for any $n' > n$, $>_C$ does not include*

$$(s, \theta, n') >_C (s, \theta, n'') \tag{2}$$

Define $\tilde{>}_C$ by adding $(s, \theta, n)\tilde{>}_C(s, \theta, n+1)$ to $>_C$. Then $\tilde{>}_C$ also satisfies acyclicity.

Proof. Assume for a contradiction $\tilde{>}_C$ does not satisfy acyclicity. This means that there exists a cycle under $\tilde{>}_C$ where $(s'_1, \ldots, s'_m)$ is a permutation of $(s_1, \ldots, s_m)$ and $(d'_1, \ldots, d'_m)$ is a permutation of $(d_1, \ldots, d_m)$.

Claim 8. *In this cycle, for some i, we have $(s_i, d_i) = (s, \theta, n)$ and $(s'_i, d'_i) = (s, \theta, n+1)$.*

Proof. Suppose that this is not the case. Then replacing $\tilde{>}_C$ with $>_C$ we still have a cycle as apart from $(s, \theta, n)\tilde{>}_C(s, \theta, n+1)$, the relations are the same. This contradicts that $>_C$ satisfies acyclicity and proves the result. □

Note that since $(s, \theta, n+1)$ is in the RHS of the cycle, there exists $(\hat{s}, \theta, n+1)$ in the LHS for some $\hat{s} \in \mathcal{S}$. Let $(\hat{s}, \theta, n+1)\tilde{>}_C(s', \theta', n')$ denote the element of the cycle that corresponds to $(\hat{s}, \theta, n+1)$. As $>_C$ does not include any relation characterized in Equation 2 for $n' > n$, $(\hat{s}, \theta, n+1)\tilde{>}_C(s', \theta', n')$ implies that $\theta' \neq \theta$, which in turn implies $(\hat{s}, \theta, n+1) >_C (s', \theta', n')$.

Claim 9. $(\hat{s}, \theta, n) >_C (s', \theta', n')$.

Proof. As $(\hat{s}, \theta, n+1) >_C (s', \theta', n')$, there exists I, j and k such that $s(j) = \hat{s}$, $\theta(j) = \theta$, $N_\theta(I \cup \{j\}) = n+1$, $s(k) = s'$, $\theta(k) = \theta'$, $N_{\theta'}(I \cup \{k\}) = n'$ and $k \in R(I \cup \{j, k\})$. Moreover, since $\theta(j) \neq \theta(k)$, we have that $(s(j), \theta(j), N_{\theta(j)}) >_C (s(k), \theta(k), N_{\theta(k)})$.

Now consider $\hat{I} = I \cup \{j, k, k'\}$ where $\theta(k) = \theta(k')$ and $s(k) = s(k')$. For a contradiction, assume that $j \in R(I \cup \{j, k, k'\})$. Let $I^* \in C(\hat{I})$ where $\{j, l\} \notin I^*$ for some other l. If $\theta(l) = \theta(k)$ and $s(l) = s(k)$, this means that $I^* = I \cup \{k\} \in C(\hat{I})$. However as I^* was available when the choice set was $I \cup \{j, k\}$, and $I \cup \{j\}$ was chosen, while I^* was not chosen, this creates a choice cycle, which is a contradiction. If $t(l) \neq t(k)$ or $s(l) \neq s(k)$, then $k \notin R(\hat{I})$. However, this violates gross substitutes as letting $I_{\theta(k)}$ denote all individuals in $\hat{I}$ with type $\theta(k)$, $I_{\theta(k)} \backslash \{k'\} \in C(\hat{I})$, but $I_{\theta(k)} \backslash \{k'\} \notin C(\hat{I} \backslash \{k'\})$. Therefore, $j \notin R(I \cup \{j, k, k'\})$

As $n \geq 1$, at I, there is another individual with type $\theta(j)$, which I denote by j'. Consider $(I \cup \{j, k, k'\} \setminus \{j'\}$ and let $I_{\theta(j)}$ denote all individuals in $I \cup \{j\} \setminus \{j'\}$. As $I_{\theta(j)}$ is chosen under $I \cup \{j, k, k'\}$, by gross substitutes*, it is also chosen from $I \cup \{j, k, k'\} \setminus \{j'\}$. This also implies that at least one of k and k' are rejected, which means that $(s(j), \theta(j), N_{\theta(j)} - 1) >_C (s(k), \theta(k), N_{\theta(k)})$, proving the claim. □

Now, we can replace $\hat{s}, \theta, n+1$ with $\hat{s}, \theta, n+1$ in the RHS and s, θ, n with $s, \theta, n+1$. Note that the line with $s, \theta, n > s, \theta, n+1$ is now $s, \theta, n+1 > s, \theta, n+1$. Removing this line, we still have a cycle on $\tilde{>}_C$, which does not use $s, \theta, n \tilde{>}_C s, \theta, n+1$, and therefore is also a cycle in $>_C$, which is a contradiction. This proves the result. □

Using Lemma 7 repeatedly, we arrive at an acyclic $\tilde{>}_C$ relation that satisfies $s, \theta, n > s, \theta, n+1$ for all $n \geq 1$, s and θ.

Lemma 8. *There exists u and h such that $s, d \tilde{>}_C s', d'$ implies $u(s) + h(d) > u(s') + h(d')$.*

Proof. Follows from Fishburn Theorem 4.1. □

Claim 10. *$h(\theta, n) > h(\theta, n+1)$ for all θ and n.*

Proof. Immediate from the construction of $\tilde{>}_C$, which includes $(s, \theta, n) >_C (s, \theta, n+1)$ for all n. □

Now, define $h_\theta(n) = \sum_{i=1}^{n} h(\theta, n)$. From the previous claim, we know that h_θ is concave. The following Lemma finishes the proof of the first part.

Lemma 9. *$U(I)$ where*

$$U(I) = \sum_{i \in I} u(s(i)) + \sum_{\theta \in T} h_\theta(N_\theta(I))$$

rationalizes C.

Proof. For a contradiction, assume it does not rationalize C. Then there exists I and I' such that $\tau(I) \neq \tau(I')$, $U(I) > U(I')$, $I' \in C(\hat{I})$ for some $\hat{I}$ that includes I. Moreover, we can take I to be a maximizer of $U(\tau(I)) = \max_{\tilde{I} \in 2^{\hat{I}}_q} U(\tau(\tilde{I}))$, which exists by the finiteness of $\hat{I}$.

First, if there exists $i \in I \setminus I'$ and $j \in I' \setminus I$ such that $t(i) = t(j)$. If this is the case, Let $\tilde{I} = I' \setminus \{j\} \cup \{i\}$ Clearly, this does not change the outcome of the choices or the utility difference.[17] We can repeat this until there does not exist any $i \in I \setminus I'$ and $j \in I' \setminus I$ such that $t(i) = t(j)$, and denote $\tilde{I}$ as this updated set. Now, choose an arbitrary $i \in \tilde{I} \setminus I$. Since C satisfies gross substitutes*, there exists I_C such that $I_C \in C(I \cup \{i\})$ and $i \in I_C$. Thus, there exists $j \in I$ such that $j \notin i_C$. As $t(i) \neq t(j)$, this shows that $s(i), \theta(i), N_{\theta(I)}(I_C) >_C s(i), \theta(i), N_{\theta(I)}(I_C) + 1$, which implies that

$$u(s(i)) + h(\theta(i), N_{\theta(i)}(I_C)) > u(s(j)) + h(\theta(j), N_{\theta(j)}(I_C) + 1)$$

[17] Formally, the statement $\tau(I) \neq \tau(\tilde{I})$, $U(I) > U(\tilde{I})$, $\tilde{I} \in C(\hat{I})$ for some $\hat{I}$ that includes I still holds.

However, above equation indicates $U(I \cup \{i\} \setminus \{j\}) > U(I)$, which is a contradiction as I maximizes utility in $\hat{I}$, which includes $I \cup \{i\}$. □

To prove the second part, given $h_\theta(n)$, define $h(\theta, n) = h_\theta(n) - h_\theta(n-1)$. Assume for a contradiction there exists a cycle. This means that for each (s_i, d_i) and (s'_i, d'_i), $u(s_i)+h(d_i) > u(s'_i)+h(d'_i)$, which implies $\sum_i (s'_i, d'_i)$, $u(s_i)+h(d_i) > \sum_i u(s'_i)+h(d'_i)$, which is a contradiction (s'_i, d'_i) is a permutation of (s_i, d_i).

Next, for a contradiction, assume C_U violates gross substitutes*. Then there exists i, j, j' and a set of individuals I such that $s(j) > s(j')$, $\theta(j) = \theta(j')$, $i \in \hat{I}$ for some $\hat{I} \in C(I \cup \{i, j\})$ but either (i) there does not exist $\tilde{I} \in C(I \cup \{i, j'\})$ such that $i \in \tilde{I}$ or (ii) there does not exist $\tilde{I} \in C(I \cup \{i\})$ such that $i \in \tilde{I}$. Note that since U is increasing in scores, we have the following.

Claim 11. *If $i \in \hat{I}$ for some $\hat{I} \in C(I)$, $\theta(j) = \theta(i)$ and $s(j) > s(i)$, then there exists $j \in \hat{I}$.*

Both (i) and (ii) are proved in the same way. There are two subcases. Either for all $\tilde{I} \in C(I \cup \{i, j'\})$, there are fewer $\theta(i)$ individuals in $\tilde{I}$ compared to $\hat{I}$, or there exists a $\tilde{I} \in C(I \cup \{i, j'\})$ such that the number of $\theta(i)$ individuals in $\tilde{I}$ is weakly higher than $\hat{I}$. In the second subcase, from Claim 11, an individual who has same identity as i and weakly lower score must be selected, which means there exists $\tilde{I}' \in C(I \cup \{i, j'\})$ such that i is selected, which is a contradiction. In the first subcase, let $\tilde{I} \in C(I \cup \{i, j'\})$. Then there exists another type θ' such that θ' has a higher number of individuals in $\tilde{I}$ compared to $\hat{I}$. Let k and k' denote the lowest scoring θ' individuals chosen under $\hat{I}$ and $\tilde{I}$. Therefore, from Claim 11, $s(k) \geq s(k')$. Moreover, we have

$$\begin{aligned} s(i) + h_{\theta(i)}(N_{\theta(i)}(\tilde{I}) + 1) - h_{\theta(i)}(N_{\theta(i)}(\tilde{I})) &\geq s(i) + h_{\theta(i)}(N_{\theta(i)}(\hat{I})) - h_{\theta(i)}(N_{\theta(i)}(\hat{I}) - 1) \\ &\geq s(k) + h_{\theta(k)}(N_{\theta(k)}(\hat{I}) + 1) - h_{\theta(i)}(N_{\theta(i)}(\hat{I})) \\ &\geq s(k) + h_{\theta(k)}(N_{\theta(k)}(\tilde{I})) - h_{\theta(i)}(N_{\theta(i)}(\tilde{I}) - 1) \end{aligned}$$

where first and third inequalities hold by from concavity of h_θ for all θ and second holds as $i \in \hat{I}$. However, this shows that switching i with k' does not decrease utility, and therefore there exists $\tilde{I} \setminus \{k'\} \cup \{i\}$ is also chosen, which is a contradiction. This completes the proof.

B. Figures

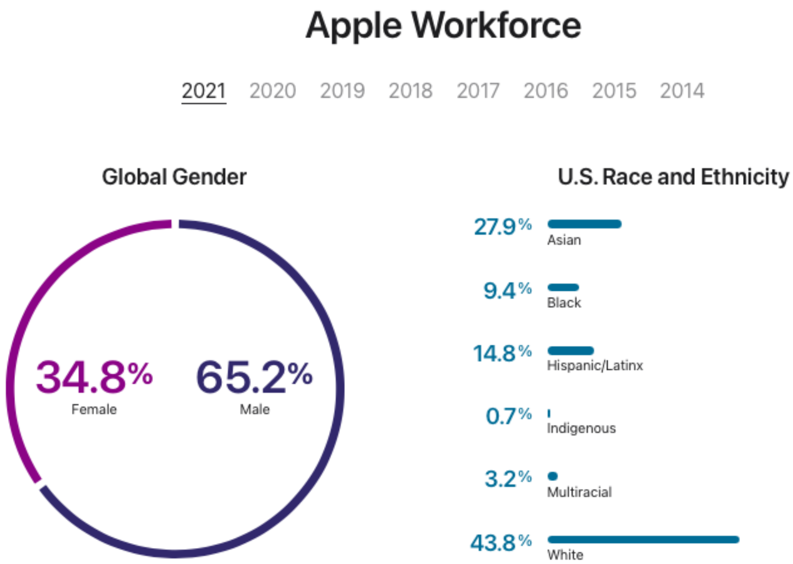

Fig. 1. **Apple Diversity Statistics.**

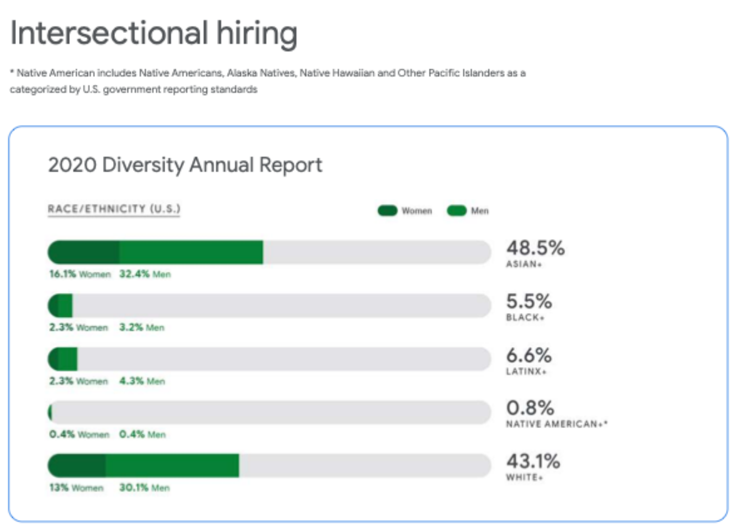

Fig. 2. **Google Diversity Statistics.**

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