14.452 Economic Growth: Lecture 6, Overlapping Generations

Daron Acemoglu

MIT

October-December 2024

Growth with Overlapping Generations

- In many situations, the assumption of a *representative household* is not appropriate because
  1. households do not have an infinite planning horizon
  2. new households arrive (or are born) over time.
- New economic interactions: decisions made by older "generations" will affect the prices faced by younger "generations".
- *Overlapping generations models*
  1. Capture potential interaction of different generations of individuals in the marketplace;
  2. Provide tractable alternative to infinite-horizon representative agent models;
  3. Some key implications different from neoclassical growth model;
  4. Dynamics in some special cases quite similar to Solow model rather than the neoclassical model;
  5. Generate new insights about the role of national debt and Social Security in the economy.

Welfare Theorems I

- Let us start by revisiting welfare theorems before we delve into overlapping generations.
- There should be a close connection between Pareto optima and competitive equilibria.
- Start with $\mathcal{H}$ finite (finite number of consumers), but we allow an infinite number of commodities.
- Results here have analogs for economies with a continuum of commodities, but focus on countable number of commodities.
- Let commodities be indexed by $j \in \mathbb{N}$ and $x^i \equiv \left\{x_j^i\right\}_{j=0}^{\infty}$ and $\omega^i \equiv \left\{\omega_j^i\right\}_{j=0}^{\infty}$ denote consumption and endowment bundles of household i.
- Most relevant interpretation for us is that at each date $j = 0, 1, ...,$ each individual consumes a finite dimensional vector of products.
- Also use the notation $\mathbf{x} \equiv \left\{x^i\right\}_{i \in \mathcal{H}}$ and $\boldsymbol{\omega} \equiv \left\{\omega^i\right\}_{i \in \mathcal{H}}$ to describe the entire consumption allocation and endowments in the economy.

Welfare Theorems II

- Production side: finite number of firms represented by $\mathcal{F}$
- Each firm $f \in \mathcal{F}$ is characterized by production set Y^f, specifies levels of output firm f can produce from specified levels of inputs.
- E.g., if there were only labor and a final good, Y^f would include pairs $(-l, y)$ such that with labor input l the firm can produce at most y.
- Let $\mathbf{Y} \equiv \prod_{f \in \mathcal{F}} Y^f$ represent the aggregate production set and $\mathbf{y} \equiv \{y^f\}_{f \in \mathcal{F}}$ such that $y^f \in Y^f$ for all f, or equivalently, $\mathbf{y} \in \mathbf{Y}$.
- Ownership structure of firms: if firms make profits, they should be distributed to some agents
- Assume there exists a sequence of numbers (profit shares) $\boldsymbol{\theta} \equiv \left\{\theta_f^i\right\}_{f \in \mathcal{F}, i \in \mathcal{H}}$ such that $\theta_f^i \geq 0$ for all f and i, and $\sum_{i \in \mathcal{H}} \theta_f^i = 1$ for all $f \in \mathcal{F}$.
- θ_f^i is the share of profits of firm f that will accrue to household i.

Welfare Theorems III

- An economy $\mathcal{E}$ is described by $\mathcal{E} \equiv (\mathcal{H}, \mathcal{F}, \mathbf{u}, \boldsymbol{\omega}, \mathbf{Y}, \mathbf{X}, \boldsymbol{\theta})$.
- An allocation is $(\mathbf{x}, \mathbf{y})$ such that $\mathbf{x}$ and $\mathbf{y}$ are feasible, that is, $\mathbf{x} \in \mathbf{X}$, $\mathbf{y} \in \mathbf{Y}$, and $\sum_{i \in \mathcal{H}} x_j^i \leq \sum_{i \in \mathcal{H}} \omega_j^i + \sum_{f \in \mathcal{F}} y_j^f$ for all $j \in \mathbb{N}$.
- A price system is a sequence $p \equiv \{p_j\}_{j=0}^{\infty}$, such that $p_j \geq 0$ for all j.
- We can choose one of these prices as the numeraire and normalize it to 1.
- Also define $p \cdot x$ as the inner product of p and x, i.e., $p \cdot x \equiv \sum_{j=0}^{\infty} p_j x_j$.

Definition Household $i \in \mathcal{H}$ is *locally non-satiated* if at each x^i, $u^i\left(x^i\right)$ is strictly increasing in at least one of its arguments at x^i and $u^i\left(x^i\right) < \infty$.

Welfare Theorems IV

Definition A competitive equilibrium for the economy $\mathcal{E} \equiv (\mathcal{H}, \mathcal{F}, \mathbf{u}, \boldsymbol{\omega}, \mathbf{Y}, \mathbf{X}, \boldsymbol{\theta})$ is given by an allocation $\left(\mathbf{x}^* = \{x^{i*}\}_{i \in \mathcal{H}}, \mathbf{y}^* = \{y^{f*}\}_{f \in \mathcal{F}}\right)$ and a price system p^* such that

1. The allocation $(\mathbf{x}^*, \mathbf{y}^*)$ is feasible and market clearing, i.e., $x^{i*} \in X^i$ for all $i \in \mathcal{H}$, $y^{f*} \in Y^f$ for all $f \in \mathcal{F}$ and
$$\sum_{i \in \mathcal{H}} x_j^{i*} = \sum_{i \in \mathcal{H}} \omega_j^i + \sum_{f \in \mathcal{F}} y_j^{f*} \text{ for all } j \in \mathbb{N}.$$
2. For every firm $f \in \mathcal{F}$, y^{f*} maximizes profits, i.e.,
$$p^* \cdot y^{f*} \geq p^* \cdot y \text{ for all } y \in Y^f.$$
3. For every consumer $i \in \mathcal{H}$, x^{i*} maximizes utility, i.e.,
$$u^i\left(x^{i*}\right) \geq u^i(x) \text{ for all } x \text{ s.t. } x \in X^i \text{ and } p^* \cdot x \leq p^* \cdot x^{i*}.$$

Welfare Theorems V

- Establish existence of competitive equilibrium with finite number of commodities and standard convexity assumptions is straightforward.
- With infinite number of commodities, somewhat more difficult and requires more sophisticated arguments.

Definition A feasible allocation $(\mathbf{x}, \mathbf{y})$ for economy $\mathcal{E} \equiv (\mathcal{H}, \mathcal{F}, \mathbf{u}, \boldsymbol{\omega}, \mathbf{Y}, \mathbf{X}, \boldsymbol{\theta})$ is *Pareto optimal* if there exists no other feasible allocation $(\hat{\mathbf{x}}, \hat{\mathbf{y}})$ such that $\hat{x}^i \in X^i$, $\hat{y}^f \in Y^f$ for all $f \in \mathcal{F}$,

$$\sum_{i \in \mathcal{H}} \hat{x}_j^i \leq \sum_{i \in \mathcal{H}} \omega_j^i + \sum_{f \in \mathcal{F}} \hat{y}_j^f \text{ for all } j \in \mathbb{N},$$

and

$$u^i\left(\hat{x}^i\right) \geq u^i\left(x^i\right) \text{ for all } i \in \mathcal{H}$$

with at least one strict inequality.

Welfare Theorems VI

Theorem **(First Welfare Theorem I)** Suppose that $(\mathbf{x}^*, \mathbf{y}^*, p^*)$ is a competitive equilibrium of economy $\mathcal{E} \equiv (\mathcal{H}, \mathcal{F}, \mathbf{u}, \boldsymbol{\omega}, \mathbf{Y}, \mathbf{X}, \boldsymbol{\theta})$ with $\mathcal{H}$ finite. Assume that all households are locally non-satiated. Then $(\mathbf{x}^*, \mathbf{y}^*)$ is Pareto optimal.

Proof of First Welfare Theorem I

- To obtain a contradiction, suppose that there exists a feasible $(\hat{\mathbf{x}}, \hat{\mathbf{y}})$ such that $u^i\left(\hat{x}^i\right) \geq u^i\left(x^i\right)$ for all $i \in \mathcal{H}$ and $u^i\left(\hat{x}^i\right) > u^i\left(x^i\right)$ for all $i \in \mathcal{H}'$, where $\mathcal{H}'$ is a non-empty subset of $\mathcal{H}$.
- Since $(\mathbf{x}^*, \mathbf{y}^*, p^*)$ is a competitive equilibrium, it must be the case that for all $i \in \mathcal{H}$,

$$\begin{aligned} p^* \cdot \hat{x}^i &\geq p^* \cdot x^{i*} \\ &= p^* \cdot \left(\omega^i + \sum_{f \in \mathcal{F}} \theta_f^i y^{f*}\right) \end{aligned} \tag{1}$$

and for all $i \in \mathcal{H}'$,

$$p^* \cdot \hat{x}^i > p^* \cdot \left(\omega^i + \sum_{f \in \mathcal{F}} \theta_f^i y^{f*}\right). \tag{2}$$

Proof of First Welfare Theorem II

- Second inequality follows immediately in view of the fact that x^{i*} is the utility maximizing choice for household i, thus if $\hat{x}^i$ is strictly preferred, then it cannot be in the budget set.
- First inequality follows with a similar reasoning. Suppose that it did not hold.
- Then by the hypothesis of local-satiation, u^i must be strictly increasing in at least one of its arguments, let us say the j'th component of x.
- Then construct $\hat{x}^i(\varepsilon)$ such that $\hat{x}^i_j(\varepsilon) = \hat{x}^i_j$ and $\hat{x}^i_{j'}(\varepsilon) = \hat{x}^i_{j'} + \varepsilon$.
- For $\varepsilon \downarrow 0$, $\hat{x}^i(\varepsilon)$ is in household i's budget set and yields strictly greater utility than the original consumption bundle x^i, contradicting the hypothesis that household i was maximizing utility.
- Note local non-satiation implies that $u^i(x^i) < \infty$, and thus the right-hand sides of (1) and (2) are finite.

Proof of First Welfare Theorem III

- Now summing over (1) and (2), we have

$$\begin{aligned} p^* \cdot \sum_{i \in \mathcal{H}} \hat{x}^i &> p^* \cdot \sum_{i \in \mathcal{H}} \left(\omega^i + \sum_{f \in \mathcal{F}} \theta_f^i y^{f*} \right), \\ &= p^* \cdot \left(\sum_{i \in \mathcal{H}} \omega^i + \sum_{f \in \mathcal{F}} y^{f*} \right), \end{aligned} \tag{3}$$

- Second line uses the fact that the summations are finite, can change the order of summation, and that by definition of shares $\sum_{i \in \mathcal{H}} \theta_f^i = 1$ for all f.
- Finally, since $\mathbf{y}^*$ is profit-maximizing at prices p^*, we have that

$$p^* \cdot \sum_{f \in \mathcal{F}} y^{f*} \geq p^* \cdot \sum_{f \in \mathcal{F}} y^f \text{ for any } \left\{ y^f \right\}_{f \in \mathcal{F}} \text{ with } y^f \in Y^f \text{ for all } f \in \mathcal{F} \tag{4}$$

Proof of First Welfare Theorem IV

- However, by market clearing of $\hat{x}^i$ (Definition above, part 1), we have

$$\sum_{i\in\mathcal{H}} \hat{x}^i_j = \sum_{i\in\mathcal{H}} \omega^i_j + \sum_{f\in\mathcal{F}} \hat{y}^f_j,$$

- Therefore, by multiplying both sides by p^* and exploiting (4),

$$\begin{aligned} p^* \cdot \sum_{i\in\mathcal{H}} \hat{x}^i_j &\leq p^* \cdot \left(\sum_{i\in\mathcal{H}} \omega^i_j + \sum_{f\in\mathcal{F}} \hat{y}^f_j \right) \\ &\leq p^* \cdot \left(\sum_{i\in\mathcal{H}} \omega^i_j + \sum_{f\in\mathcal{F}} y^{f*}_j \right), \end{aligned}$$

- Contradicts (3), establishing that any competitive equilibrium allocation $(\mathbf{x}^*, \mathbf{y}^*)$ is Pareto optimal.

Welfare Theorems VI

- Proof of the First Welfare Theorem based on two intuitive ideas.
    1. If another allocation Pareto dominates the competitive equilibrium, then it must be non-affordable in the competitive equilibrium.
    2. Profit-maximization implies that any competitive equilibrium already contains the maximal set of affordable allocations.
- Note it makes no convexity assumption.
- Also highlights the importance of the feature that the relevant sums exist and are finite.
    - Otherwise, the last step would lead to the conclusion that "$\infty < \infty$".
- That these sums exist followed from two assumptions: finiteness of the number of individuals and non-satiation.

Welfare Theorems VII

Theorem **(First Welfare Theorem II)** Suppose that $(\mathbf{x}^*, \mathbf{y}^*, p^*)$ is a competitive equilibrium of the economy $\mathcal{E} \equiv (\mathcal{H}, \mathcal{F}, \mathbf{u}, \boldsymbol{\omega}, \mathbf{Y}, \mathbf{X}, \boldsymbol{\theta})$ with $\mathcal{H}$ countably infinite. Assume that all households are locally non-satiated and that $p^* \cdot \omega^* = \sum_{i \in \mathcal{H}} \sum_{j=0}^{\infty} p_j^* \omega_j^i < \infty$. Then $(\mathbf{x}^*, \mathbf{y}^*, p^*)$ is Pareto optimal.

- **Proof:**
  - Same as before but now local non-satiation does not guarantee summations are finite (3), since we sum over an infinite number of households.
  - But since endowments are finite, the assumption that $\sum_{i \in \mathcal{H}} \sum_{j=0}^{\infty} p_j^* \omega_j^i < \infty$ ensures that the sums in (3) are indeed finite.

Welfare Theorems VIII

- Second Welfare Theorem (converse to First): whether or not $\mathcal{H}$ is finite is not as important as for the First Welfare Theorem.
- But requires assumptions such as the convexity of consumption and production sets and preferences, and additional requirements because it contains an "existence of equilibrium argument".
- But with an infinite horizon (infinite number of commodities), additional technical issues arise. One has to make sure that preferences don't depend too much on things that will happen in the future.
- Discounting ensures that, and the Second Welfare Theorem holds in all of the economies we will focus on and discourse, but I will not go through its statement, which requires more notation and care.

Problems of Infinity I

- Let me now illustrate the difficulty of the welfare theorem in an infinite horizon economy with a more specific example (similar to overlapping generations models).
- Static economy with countably infinite number of households, $i \in \mathbb{N}$
- Countably infinite number of commodities, $j \in \mathbb{N}$.
- All households behave competitively (alternatively, there are M households of each type, M is a large number).
- Household i has preferences:

$$u_i = c_i^i + c_{i+1}^i,$$

- c_j^i denotes the consumption of the jth type of commodity by household i.
- Endowment vector ω of the economy: each household has one unit endowment of the commodity with the same index as its index.
- Choose the price of the first commodity as the numeraire, i.e., $p_0 = 1$.

Problems of Infinity II

Proposition In the above-described economy, the price vector $\bar{p}$ such that $\bar{p}_j = 1$ for all $j \in \mathbb{N}$ is a competitive equilibrium price vector and induces an equilibrium with no trade, denoted by $\bar{x}$.

- **Proof:**
  - At $\bar{p}$, each household has income equal to 1.
  - Therefore, the budget constraint of household i can be written as

$$c_i^i + c_{i+1}^i \leq 1.$$

  - This implies that consuming own endowment is optimal for each household,
  - Thus $\bar{p}$ and no trade, $\bar{x}$, constitute a competitive equilibrium.

Problems of Infinity III

- However, this competitive equilibrium is not Pareto optimal. Consider alternative allocation, $\tilde{x}$:
  - Household $i = 0$ consumes its own endowment and that of household 1.
  - All other households, indexed $i > 0$, consume the endowment of than neighboring household, $i + 1$.
  - All households with $i > 0$ are as well off as in the competitive equilibrium $(\bar{p}, \bar{x})$.
  - Individual $i = 0$ is strictly better-off.

Proposition In the above-described economy, the competitive equilibrium at $(\bar{p}, \bar{x})$ is not Pareto optimal.

Problems of Infinity IV

- Source of the problem must be related to the infinite number of commodities.
- Extended version of the First Welfare Theorem covers infinite number of commodities, but only assuming $\sum_{j=0}^{\infty} p_j^* \omega_j < \infty$ (written with the aggregate endowment ω_j).
- Here theonly endowment is labor, and thus $p_j^* = 1$ for all $j \in \mathbb{N}$, so that $\sum_{j=0}^{\infty} p_j^* \omega_j = \infty$ (why?).
- This abstract economy is "isomorphic" to the baseline overlapping generations model.
- The Pareto suboptimality in this economy will be the source of potential inefficiencies in overlapping generations model.

Problems of Infinity V

- Second Welfare Theorem did not assume $\sum_{j=0}^{\infty} p_j^* \omega_j < \infty$.
- Instead, it used convexity of preferences, consumption sets and production possibilities sets.
- This exchange economy has convex preferences and convex consumption sets:
  - Pareto optima must be decentralizable by some redistribution of endowments.

Proposition In the above-described economy, there exists a reallocation of the endowment vector $\boldsymbol{\omega}$ to $\tilde{\boldsymbol{\omega}}$, and an associated competitive equilibrium $(\bar{p}, \tilde{x})$ that is Pareto optimal where $\tilde{x}$ is as described above, and $\bar{p}$ is such that $\bar{p}_j = 1$ for all $j \in \mathbb{N}$.

Proof of Proposition

- Consider the following reallocation of ω: endowment of household $i \geq 1$ is given to household $i-1$.
  - At the new endowment vector $\tilde{\omega}$, household $i=0$ has one unit of good $j=0$ and one unit of good $j=1$.
  - Other households i have one unit of good $i+1$.
- At the price vector $\bar{p}$, household 0 has a budget set

$$c_0^0 + c_1^1 \leq 2,$$

  thus chooses $c_0^0 = c_1^0 = 1$.
- All other households have budget sets given by

$$c_i^i + c_{i+1}^i \leq 1,$$

- Thus it is optimal for each household $i > 0$ to consume one unit of the good c_{i+1}^i
- Thus $\tilde{x}$ is a competitive equilibrium.

The Baseline Overlapping Generations Model

- Time is discrete and runs to infinity.
- Each individual lives for two periods.
- Individuals born at time t live for dates t and $t+1$.
- Assume a general (separable) utility function for individuals born at date t,

$$U(t) = u(c_1(t)) + \beta u(c_2(t+1)), \tag{5}$$

- $u : \mathbb{R}_+ \to \mathbb{R}$ satisfies the usual Assumptions on utility.
- $c_1(t)$: consumption of the individual born at t when young (at date t).
- $c_2(t+1)$: consumption when old (at date $t+1$).
- $\beta \in (0,1)$ is the discount factor.

Demographics, Preferences and Technology I

- Exponential population growth,

$$L(t) = (1+n)^t L(0). \tag{6}$$

- Production side same as before: competitive firms, constant returns to scale aggregate production function, satisfying Assumptions 1 and 2:

$$Y(t) = F(K(t), L(t)).$$

- Factor markets are competitive.
- Individuals can only work in the first period and supply one unit of labor inelastically, earning $w(t)$.

Demographics, Preferences and Technology II

- Assume that $\delta = 1$.
- $k \equiv K/L$, $f(k) \equiv F(k, 1)$, and the (gross) rate of return to saving, which equals the rental rate of capital, is

$$1 + r(t) = R(t) = f'(k(t)), \tag{7}$$

- As usual, the wage rate is

$$w(t) = f(k(t)) - k(t) f'(k(t)). \tag{8}$$

Consumption Decisions I

- Savings by an individual of generation t, $s(t)$, is determined as a solution to

$$\max_{c_1(t), c_2(t+1), s(t)} u(c_1(t)) + \beta u(c_2(t+1))$$

subject to

$$c_1(t) + s(t) \leq w(t)$$

and

$$c_2(t+1) \leq R(t+1) s(t),$$

- Old individuals rent their savings of time t as capital to firms at time $t+1$, and receive gross rate of return $R(t+1) = 1 + r(t+1)$
- Second constraint incorporates notion that individuals only spend money on their own end of life consumption (no altruism or bequest motive).

Consumption Decisions II

- No need to introduce $s(t) \geq 0$, since negative savings would violate second-period budget constraint (given $c_2(t+1) \geq 0$).
- Since $u(\cdot)$ is strictly increasing, both constraints will hold as equalities.
- Thus first-order condition for a maximum can be written in the familiar form of the consumption Euler equation,

$$u'(c_1(t)) = \beta R(t+1) u'(c_2(t+1)). \tag{9}$$

- Problem of each individual is strictly concave, so this Euler equation is sufficient.
- Solving for consumption and thus for savings,

$$s(t) = s(w(t), R(t+1)), \tag{10}$$

Consumption Decisions III

- $s : \mathbb{R}^2_+ \rightarrow \mathbb{R}$ is strictly increasing in its first argument and may be increasing or decreasing in its second argument.
- Total savings in the economy will be equal to

$$S(t) = s(t) L(t),$$

- $L(t)$ denotes the size of generation t, who are saving for time $t+1$.
- Since capital depreciates fully after use and all new savings are invested in capital,

$$K(t+1) = L(t) s(w(t), R(t+1)). \tag{11}$$

Equilibrium I

Definition A competitive equilibrium can be represented by a sequence of aggregate capital stocks, individual consumption and factor prices, $\{K(t), c_1(t), c_2(t), R(t), w(t)\}_{t=0}^{\infty}$, such that the factor price sequence $\{R(t), w(t)\}_{t=0}^{\infty}$ is given by (7) and (8), individual consumption decisions $\{c_1(t), c_2(t)\}_{t=0}^{\infty}$ are given by (9) and (10), and the aggregate capital stock, $\{K(t)\}_{t=0}^{\infty}$, evolves according to (11).

- Steady-state equilibrium defined as usual: an equilibrium in which $k \equiv K/L$ is constant.
- To characterize the equilibrium, divide (11) by $L(t+1) = (1+n)L(t)$,

$$k(t+1) = \frac{s(w(t), R(t+1))}{1+n}.$$

Equilibrium II

- Now substituting for $R(t+1)$ and $w(t)$ from (7) and (8),

$$k(t+1) = \frac{s(f(k(t)) - k(t)f'(k(t)), f'(k(t+1)))}{1+n} \tag{12}$$

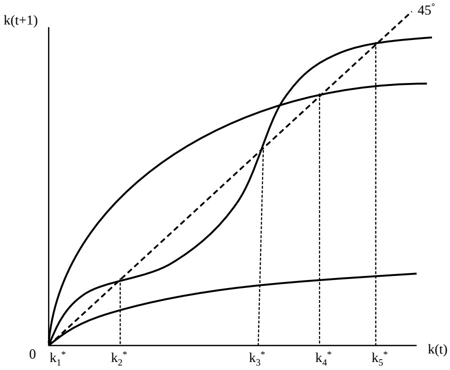
- This is the fundamental law of motion of the overlapping generations economy.
- A steady state is given by a solution to this equation such that $k(t+1) = k(t) = k^*$, i.e.,

$$k^* = \frac{s(f(k^*) - k^* f'(k^*), f'(k^*))}{1+n} \tag{13}$$

- Since the savings function $s(\cdot,\cdot)$ can take any form, the difference equation (12) can lead to quite complicated dynamics, and multiple steady states are possible.

Equilibrium III

- Possible patterns:



Restrictions on Utility and Production Functions I

- Suppose that the utility functions take the familiar CRRA form:

$$U(t) = \frac{c_1(t)^{1-\theta} - 1}{1-\theta} + \beta \left(\frac{c_2(t+1)^{1-\theta} - 1}{1-\theta} \right), \qquad (14)$$

  where $\theta > 0$ and $\beta \in (0, 1)$.
- Technology is Cobb-Douglas,

$$f(k) = k^{\alpha}$$

- The rest of the environment is as described above.
- The CRRA utility simplifies the first-order condition for consumer optimization,

$$\frac{c_2(t+1)}{c_1(t)} = (\beta R(t+1))^{1/\theta}.$$

Restrictions on Utility and Production Functions II

- This Euler equation can be alternatively expressed in terms of savings as

$$s(t)^{-\theta}\beta R(t+1)^{1-\theta} = (w(t) - s(t))^{-\theta}, \tag{15}$$

- Gives the following equation for the saving rate:

$$s(t) = \frac{w(t)}{\psi(t+1)}, \tag{16}$$

where

$$\psi(t+1) \equiv [1 + \beta^{-1/\theta} R(t+1)^{-(1-\theta)/\theta}] > 1,$$

- Ensures that savings are always less than earnings.

Restrictions on Utility and Production Functions III

- The impact of factor prices on savings is summarized by the following and derivatives:

$$s_w \equiv \frac{\partial s(t)}{\partial w(t)} = \frac{1}{\psi(t+1)} \in (0,1),$$
$$s_R \equiv \frac{\partial s(t)}{\partial R(t+1)} = \left(\frac{1-\theta}{\theta}\right)(\beta R(t+1))^{-1/\theta} \frac{s(t)}{\psi(t+1)}.$$

- Since $\psi(t+1) > 1$, we also have that $0 < s_w < 1$.
- Moreover, in this case $s_R > 0$ if $\theta < 1$, $s_R < 0$ if $\theta > 1$, and $s_R = 0$ if $\theta = 1$.
- Reflects counteracting influences of income and substitution effects. The substitution effects wins out when $\theta < 1$, and loses out when $\theta > 1$.
- Case of $\theta = 1$ (log preferences) is of special importance, as the income and substitution effects cancel out exactly.

Restrictions on Utility and Production Functions IV

- Equation (12) implies

$$k(t+1) = \frac{s(t)}{(1+n)} = \frac{w(t)}{(1+n)\psi(t+1)}, \tag{17}$$

- Or more explicitly,

$$k(t+1) = \frac{f(k(t)) - k(t) f'(k(t))}{(1+n)\left[1+\beta^{-1/\theta} f'(k(t+1))^{-(1-\theta)/\theta}\right]} \tag{18}$$

- The steady state then involves a solution to the following implicit equation:

$$k^* = \frac{f(k^*) - k^* f'(k^*)}{(1+n)\left[1+\beta^{-1/\theta} f'(k^*)^{-(1-\theta)/\theta}\right]}.$$

Restrictions on Utility and Production Functions V

- Now using the Cobb-Douglas formula, steady state is the solution to the equation

$$(1+n)\left[1+\beta^{-1/\theta}\left(\alpha(k^*)^{\alpha-1}\right)^{(\theta-1)/\theta}\right]=(1-\alpha)(k^*)^{\alpha-1}. \quad (19)$$

- For simplicity, define $R^* \equiv \alpha(k^*)^{\alpha-1}$ as the marginal product of capital in steady-state, in which case, (19) can be rewritten as

$$(1+n)\left[1+\beta^{-1/\theta}\left(R^*\right)^{(\theta-1)/\theta}\right]=\frac{1-\alpha}{\alpha}R^*. \quad (20)$$

- Steady-state value of R^*, and thus k^*, can now be determined from equation (20), which always has a unique solution.
- To investigate the stability, substitute for the Cobb-Douglas production function in (18)

$$k(t+1)=\frac{(1-\alpha)\,k(t)^{\alpha}}{(1+n)\left[1+\beta^{-1/\theta}\left(\alpha k(t+1)^{\alpha-1}\right)^{-(1-\theta)/\theta}\right]}. \quad (21)$$

Restrictions on Utility and Production Functions VI

Proposition In the overlapping-generations model with two-period lived households, Cobb-Douglas technology and CRRA preferences, there exists a unique steady-state equilibrium with the capital-labor ratio k^* given by (19), this steady-state equilibrium is globally stable for all $k(0) > 0$.

- In this particular (well-behaved) case, equilibrium dynamics are very similar to the basic Solow model
- Figure shows that convergence to the unique steady-state capital-labor ratio, k^*, is monotonic.

Canonical Model I

- Even the model with CRRA utility and Cobb-Douglas production function is relatively messy.
- Many of the applications use log preferences ($\theta = 1$).
- Income and substitution effects exactly cancel each othe: changes in the interest rate (and thus in the capital-labor ratio of the economy) have no effect on the saving rate.
- Structure of the equilibrium is essentially identical to the basic Solow model.
- Utility of the household and generation t is,

$$U(t) = \log c_1(t) + \beta \log c_2(t+1), \tag{22}$$

- $\beta \in (0,1)$ (even though $\beta \geq 1$ could be allowed).
- Again $f(k) = k^{\alpha}$.

Canonical Model II

- Consumption Euler equation:

$$\frac{c_2\left(t+1\right)}{c_1\left(t\right)}=\beta R\left(t+1\right)$$

- Savings should satisfy the equation

$$s\left(t\right)=\frac{\beta}{1+\beta}w\left(t\right), \tag{23}$$

- Constant saving rate, equal to $\beta/\left(1+\beta\right)$, out of labor income for each individual.

Canonical Model III

- Combining this with the capital accumulation equation (12),

$$\begin{aligned} k(t+1) &= \frac{s(t)}{(1+n)} \\ &= \frac{\beta w(t)}{(1+n)(1+\beta)} \\ &= \frac{\beta(1-\alpha)[k(t)]^{\alpha}}{(1+n)(1+\beta)}, \end{aligned}$$

- Second line uses (23) and last uses that, given competitive factor markets, $w(t) = (1-\alpha)[k(t)]^{\alpha}$.
- There exists a unique steady state with

$$k^* = \left[\frac{\beta(1-\alpha)}{(1+n)(1+\beta)}\right]^{\frac{1}{1-\alpha}}. \quad (24)$$

- Equilibrium dynamics are identical to those of the basic Solow model and monotonically converge to k^*.

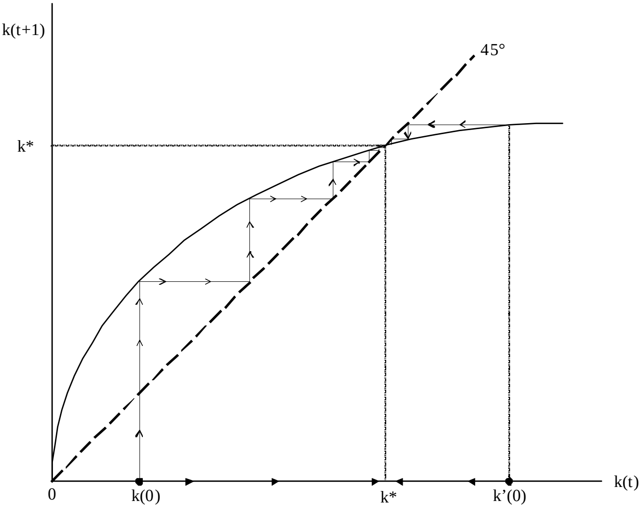


Figure: Equilibrium dynamics in the canonical overlapping generations model.

Canonical Model IV

Proposition In the canonical overlapping generations model with log preferences and Cobb-Douglas technology, there exists a unique steady state, with capital-labor ratio k^* given by (24). Starting with any $k(0) \in (0, k^*)$, equilibrium dynamics are such that $k(t) \uparrow k^*$, and starting with any $k'(0) > k^*$, equilibrium dynamics involve $k(t) \downarrow k^*$.

Overaccumulation I

- Compare the overlapping-generations equilibrium to the choice of a social planner wishing to maximize a weighted average of all generations' utilities.
- Suppose that the social planner maximizes

$$\sum_{t=0}^{\infty} \beta_S^t U(t)$$

- β_S is the discount factor of the social planner, which reflects how she values the utilities of different generations.

Overaccumulation II

- Substituting from (5), this implies:

$$\sum_{t=0}^{\infty} \beta_S^t \left(u \left(c_1 \left(t \right) \right) + \beta u \left(c_2 \left(t+1 \right) \right) \right)$$

subject to the resource constraint

$$F\left(K\left(t\right), L\left(t\right)\right) = K\left(t+1\right) + L\left(t\right) c_1\left(t\right) + L\left(t-1\right) c_2\left(t\right).$$

- Dividing this by $L\left(t\right)$ and using (6),

$$f\left(k\left(t\right)\right) = \left(1+n\right) k\left(t+1\right) + c_1\left(t\right) + \frac{c_2\left(t\right)}{1+n}.$$

Overaccumulation III

- Social planner's maximization problem then implies the following first-order necessary condition:

$$u'\left(c_1\left(t\right)\right)=\beta f'\left(k\left(t+1\right)\right)u'\left(c_2\left(t+1\right)\right).$$

- Since $R\left(t+1\right)=f'\left(k\left(t+1\right)\right)$, this is identical to (9).
- Not surprising: allocate consumption of a given individual in exactly the same way as the individual himself would do.
- No "market failures" in the over-time allocation of consumption at given prices.

Dynamic Inefficiency

- However, issues of dynamic and efficiency are still present.
- In particular, competitive equilibrium is **Pareto suboptimal** when $k^* > k_{gold}$, since reducing saving can increase consumption for every generation, where k_{gold} is defined as

$$f'\left(k_{gold}\right) = 1 + n.$$

- In steady state

$$\begin{aligned} f\left(k^*\right) - (1+n)k^* &= c_1^* + (1+n)^{-1} c_2^* \\ &\equiv c^*, \end{aligned}$$

- First line follows by national income accounting, and second defines c^*. Therefore

$$\frac{\partial c^*}{\partial k^*} = f'\left(k^*\right) - (1+n) < 0 \text{ iff } k^* > k_{gold}.$$

Overaccumulation IV

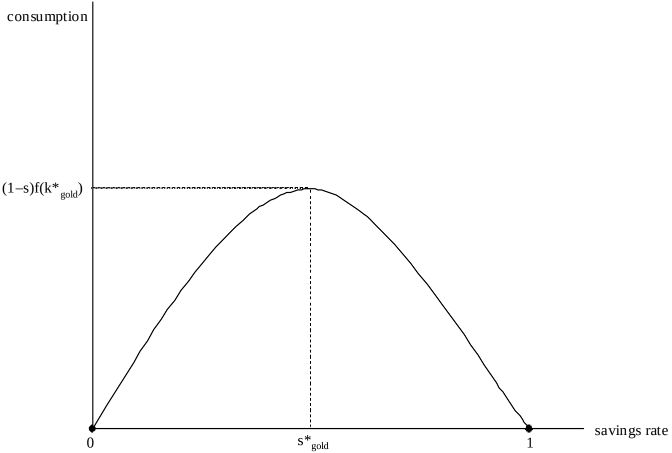


Figure: The "golden rule" level of savings rate, which maximizes steady-state consumption.

Overaccumulation V

- Now if $k^* > k_{gold}$, then $\partial c^* / \partial k^* < 0$: reducing savings can increase (total) consumption for everybody.
- More specifically, consider the following variation starting from steady state at time T: change next period's capital stock by $-\Delta k$, where $\Delta k > 0$, and from then on, we immediately move to a new steady state (clearly feasible) with the following consumption changes:

$$\begin{aligned} \Delta c(T) &= (1+n)\Delta k > 0 \\ \Delta c(t) &= -\left(f'(k^* - \Delta k) - (1+n)\right)\Delta k \text{ for all } t > T \end{aligned}$$

- The first expression reflects the direct increase in consumption due to the decrease in savings.
- In addition, since $k^* > k_{gold}$, for small enough Δk, $f'(k^* - \Delta k) - (1+n) < 0$, thus $\Delta c(t) > 0$ for all $t \geq T$.
- The increase in consumption for each generation can be allocated equally during the two periods of their lives, thus necessarily increasing the utility of all generations.

Overaccumulation VI

- If $k^* > k_{gold}$, the economy is referred to as *dynamically inefficient*—it involves overaccumulation.
- Another way of expressing dynamic inefficiency is that

$$r^* < n,$$

- Recall in infinite-horizon Ramsey economy, transversality condition required that $r > g + n$.
- Dynamic inefficiency arises because of the heterogeneity inherent in the overlapping generations model, which removes the transversality condition.

Pareto Optimality and Suboptimality in the OLG Model

Proposition In the baseline overlapping-generations economy, the competitive equilibrium is not necessarily Pareto optimal. More specifically, whenever $r^* < n$ and the economy is dynamically inefficient, it is possible to reduce the capital stock starting from the competitive steady state and increase the consumption level of all generations.

- Pareto inefficiency of the competitive equilibrium is the other side of the coin of *dynamic inefficiency.*

Interpretation

- Intuition for dynamic inefficiency:
  - Individuals who live at time t face prices determined by the capital stock with which they are working.
  - Pecuniary externality from the actions of previous generations affecting welfare of current generation.
  - Pecuniary externalities typically second-order and do not matter for welfare.
  - But not when an infinite stream of newborn agents joining the economy are affected.
  - It is possible to rearrange in a way that these pecuniary externalities can be exploited.

Further Intuition

- Complementary intuition:
  - Dynamic inefficiency arises from overaccumulation.
  - Results from current young generation needs to save for old age.
  - However, the more they save, the lower is the rate of return and may encourage to save even more.
  - Effect on future rate of return to capital is a pecuniary externality on next generation
  - If alternative ways of providing consumption to individuals in old age were introduced, overaccumulation could be ameliorated.

Role of Social Security in Capital Accumulation

- Social Security as a way of dealing with overaccumulation
- Fully-funded system: young make contributions to the Social Security system and their contributions are paid back to them in their old age.
- Unfunded system or a *pay-as-you-go:* transfers from the young directly go to the current old.
- Pay-as-you-go (unfunded) Social Security discourages aggregate savings.
- With dynamic inefficiency, discouraging savings may lead to a Pareto improvement.

Fully Funded Social Security I

- Government at date t raises some amount $d(t)$ from the young, funds are invested in capital stock, and pays workers when old $R(t+1)\,d(t)$.
- Thus individual maximization problem is,

$$\max_{c_1(t),c_2(t+1),s(t)} u(c_1(t)) + \beta u(c_2(t+1))$$

subject to

$$c_1(t) + s(t) + d(t) \leq w(t)$$

and

$$c_2(t+1) \leq R(t+1)(s(t) + d(t)),$$

for a given choice of $d(t)$ by the government.

- Notice that now the total amount invested in capital accumulation is $s(t) + d(t) = (1+n)\,k(t+1)$.

Fully Funded Social Security II

- No longer the case that individuals will always choose $s(t) > 0$.
- As long as $s(t)$ is free, whatever $\{d(t)\}_{t=0}^{\infty}$, the competitive equilibrium applies.
- When $s(t) \geq 0$ is imposed as a constraint, competitive equilibrium applies if given $\{d(t)\}_{t=0}^{\infty}$, privately-optimal $\{s(t)\}_{t=0}^{\infty}$ is such that $s(t) > 0$ for all t.

Fully Funded Social Security III

Proposition Consider a fully funded Social Security system in the above-described environment whereby the government collects $d(t)$ from young individuals at date t.

1. Suppose that $s(t) \geq 0$ for all t. If given the feasible sequence $\{d(t)\}_{t=0}^{\infty}$ of Social Security payments, the utility-maximizing sequence of savings $\{s(t)\}_{t=0}^{\infty}$ is such that $s(t) > 0$ for all t, then the set of competitive equilibria without Social Security are the set of competitive equilibria with Social Security.
2. Without the constraint $s(t) \geq 0$, given any feasible sequence $\{d(t)\}_{t=0}^{\infty}$ of Social Security payments, the set of competitive equilibria without Social Security are the set of competitive equilibria with Social Security.

- Moreover, even when there is the restriction that $s(t) \geq 0$, a funded Social Security program cannot lead to the Pareto improvement.

Unfunded Social Security I

- Government collects $d(t)$ from the young at time t and distributes to the current old with per capita transfer $b(t) = (1+n)\,d(t)$
- Individual maximization problem becomes

$$\max_{c_1(t), c_2(t+1), s(t)} u(c_1(t)) + \beta u(c_2(t+1))$$

subject to

$$c_1(t) + s(t) + d(t) \leq w(t)$$

and

$$c_2(t+1) \leq R(t+1)\,s(t) + (1+n)\,d(t+1),$$

for a given feasible sequence of Social Security payment levels $\{d(t)\}_{t=0}^{\infty}$.

- Rate of return on Social Security payments is n rather than $r(t+1) = R(t+1) - 1$, because unfunded Social Security is a pure transfer system.

Unfunded Social Security II

- Only $s(t)$—rather than $s(t)$ plus $d(t)$ as in the funded scheme—goes into capital accumulation.
- It is possible that $s(t)$ will change in order to compensate, but such an offsetting change does not typically take place.
- Thus unfunded Social Security reduces capital accumulation.
- Discouraging capital accumulation can have negative consequences for growth and welfare.
- In fact, empirical evidence suggests that there are many societies in which the level of capital accumulation is suboptimally low.
- But here reducing aggregate savings may be good when the economy exhibits dynamic inefficiency.

Unfunded Social Security III

Proposition Consider the above-described overlapping generations economy and suppose that the decentralized competitive equilibrium is dynamically inefficient. Then there exists a feasible sequence of unfunded Social Security payments $\{d(t)\}_{t=0}^{\infty}$ which will lead to a competitive equilibrium starting from any date t that Pareto dominates the competitive equilibrium without Social Security.

- Similar to way in which the Pareto optimal allocation was decentralized in the example economy above.
- Social Security is transferring resources from future generations to initial old generation.
- But with *no* dynamic inefficiency, any transfer of resources (and any unfunded Social Security program) would make some future generation worse-off.

Conclusions

- Overlapping generations of the more realistic than infinity-lived representative agents.
- Models with overlapping generations fall outside the scope of the First Welfare Theorem:
  - they were partly motivated by the possibility of Pareto suboptimal allocations.
- Equilibria may be "dynamically inefficient" and feature overaccumulation: unfunded Social Security can ameliorate the problem.
- Declining path of labor income important for overaccumulation, and what matters is not finite horizons but arrival of new individuals.
- Overaccumulation and Pareto suboptimality: pecuniary externalities created on individuals that are not yet in the marketplace.
- Not overemhasize dynamic inefficiency: major question of economic growth is why so many countries have so little capital.