14.452 Economic Growth: Lecture 7, Neoclassical Endogenous Growth

Daron Acemoglu

MIT

October-December 2025

First-Generation Models of Endogenous Growth

- Models so far: no sustained long-run growth; relatively little to say about sources of technology differences.
- Models in which technology evolves as a result of firms' and workers' decisions are most attractive in this regard.
- But sustained economic growth is possible in the neoclassical model as well:
  - AK model before: relaxed Assumption 2 and prevented diminishing returns to capital.
  - Capital accumulation could act as the engine of sustained economic growth.
- Neoclassical version of the AK model:
  - Very tractable and applications in many areas.
  - Shortcoming: capital is essentially the only factor of production, asymptotically share of income accruing to it tends to 1.
- Two-sector endogenous growth models behave very similarly to the baseline AK model, but avoid this.

Demographics, Preferences and Technology I

- Focus on balanced economic growth, i.e. consistent with the Kaldor facts.
- Thus CRRA preferences as in the canonical neoclassical growth model.
- Economy admits an infinitely-lived representative household, household size growing at the exponential rate n.
- Preferences

$$U = \int_0^{\infty} \exp\left(-\left(\rho - n\right)t\right)\left[\frac{c\left(t\right)^{1-\theta} - 1}{1-\theta}\right]dt. \tag{1}$$

- Labor is supplied inelastically.
- Flow budget constraint,

$$\dot{a}\left(t\right) = \left(r\left(t\right) - n\right)a\left(t\right) + w\left(t\right) - c\left(t\right), \tag{2}$$

Demographics, Preferences and Technology II

- No-Ponzi game constraint:

$$\lim_{t\to\infty}\left\{a(t)\exp\left[-\int_0^t [r(s)-n]\,ds\right]\right\}\geq 0. \tag{3}$$

- Euler equation:

$$\frac{\dot{c}(t)}{c(t)}=\frac{1}{\theta}(r(t)-\rho). \tag{4}$$

- Transversality condition,

$$\lim_{t\to\infty}\left\{a(t)\exp\left[-\int_0^t [r(s)-n]\,ds\right]\right\}=0. \tag{5}$$

- Problem is concave, solution to these necessary conditions is in fact an optimal plan.
- Final good sector similar to before, but Assumptions 1 and 2 are *not* satisfied.

Demographics, Preferences and Technology III

- More specifically,
$$Y(t) = AK(t),$$
with $A > 0$.
- Does not depend on labor, thus $w(t)$ in (2) will be equal to zero.
- Defining $k(t) \equiv K(t)/L(t)$ as the capital-labor ratio,
$$\begin{aligned} y(t) &\equiv \frac{Y(t)}{L(t)} \\ &= Ak(t). \end{aligned} \tag{6}$$
- Notice output is only a function of capital, and there are no diminishing returns
- But introducing diminishing returns to capital does not affect the main results in this section.

Demographics, Preferences and Technology IV

- More important assumption is that the Inada conditions embedded in Assumption 2 are no longer satisfied,

$$\lim_{k \to \infty} f'(k) = A > 0.$$

- Conditions for profit-maximization are similar to before, and require $R(t) = r(t) + \delta$.
- From (6) the marginal product of capital is A, thus $R(t) = A$ for all t,

$$r(t) = r = A - \delta, \text{ for all } t. \tag{7}$$

Equilibrium I

- A competitive equilibrium of this economy consists of paths $[c(t), k(t), w(t), R(t)]_{t=0}^{\infty}$, such that the representative household maximizes (1) subject to (2) and (3) given initial capital-labor ratio $k(0)$ and $[w(t), r(t)]_{t=0}^{\infty}$ such that $w(t) = 0$ for all t, and $r(t)$ is given by (7).
- Note that $a(t) = k(t)$.
- Using the fact that $r = A - \delta$ and $w = 0$, equations (2), (4), and (5) imply

$$\dot{k}(t) = (A - \delta - n)k(t) - c(t) \tag{8}$$

$$\frac{\dot{c}(t)}{c(t)} = \frac{1}{\theta}(A - \delta - \rho), \tag{9}$$

$$\lim_{t\to\infty} k(t) \exp(-(A - \delta - n)t) = 0. \tag{10}$$

Equilibrium II

- The important result immediately follows from (9).
  - Since the right-hand side is constant, there must be a constant rate of consumption growth (as long as $A - \delta - \rho > 0$).
  - Growth of consumption is independent of the level of capital stock per person, $k(t)$.
  - No transitional dynamics in this model.
- To develop, integrate (9) starting from some $c(0)$, to be determined from the lifetime budget constraint,

$$c(t) = c(0) \exp\left(\frac{1}{\theta}(A - \delta - \rho)t\right). \tag{11}$$

- Need to ensure that the transversality condition is satisfied and ensure positive growth $(A - \delta - \rho > 0)$. Impose:

$$A > \rho + \delta > (1 - \theta)(A - \delta) + \theta n + \delta. \tag{12}$$

Equilibrium Characterization

- Substitute for $c(t)$ from equation (11) into equation (8),

$$\dot{k}(t) = (A-\delta-n)k(t) - c(0)\exp\left(\frac{1}{\theta}(A-\delta-\rho)t\right), \tag{13}$$

- The solution to this differential equation combined with the transversality condition gives (see book for details):

$$\begin{aligned} k(t) &= \left[(A-\delta)(\theta-1)\theta^{-1}+\rho\theta^{-1}-n\right]^{-1} \\ &\quad \times \left[c(0)\exp\left(\theta^{-1}(A-\delta-\rho)t\right)\right] \\ &= k(0)\exp\left(\theta^{-1}(A-\delta-\rho)t\right), \end{aligned} \tag{14}$$

- Second line follows from the fact that the boundary condition has to hold for capital at $t=0$.
- Hence capital and output grow at the same rate as consumption.
- This also pins down the initial level of consumption as

$$c(0) = \left[(A-\delta)(\theta-1)\theta^{-1}+\rho\theta^{-1}-n\right]k(0). \tag{15}$$

Equilibrium Savings Rate

- No transitional dynamics: growth rates of consumption, capital and output are constant and given in (9).
- Growth is not only sustained, but also endogenous in the sense of being affected by underlying parameters.
  - E.g., an increase in ρ, will reduce the growth rate.
- Saving rate=total investment (increase in capital plus replacement investment) divided by output:

$$\begin{aligned} s &= \frac{\dot{K}(t) + \delta K(t)}{Y(t)} \\ &= \frac{\dot{k}(t)/k(t) + n + \delta}{A} \\ &= \frac{A - \rho + \theta n + (\theta - 1)\delta}{\theta A}, \end{aligned} \tag{16}$$

- Last equality exploited $\dot{k}(t)/k(t) = (A - \delta - \rho)/\theta$.

Equilibrium Characterization: Summary

- Saving rate, constant and exogenous in the basic Solow model, is again constant.
- But is now a function of parameters, also those that determine the equilibrium growth rate of the economy.

Proposition Consider the above-described AK economy, with a representative household with preferences given by (1), and the production technology given by (6). Suppose that condition (12) holds. Then, there exists a unique equilibrium path in which consumption, capital and output all grow at the same rate $g^* \equiv (A - \delta - \rho)/\theta > 0$ starting from any initial positive capital stock per worker $k(0)$, and the saving rate is endogenously determined by (16).

Pareto Optimality

- Since all markets are competitive, there is a representative household, and there are no externalities, the competitive equilibrium will be Pareto optimal.
- Can be proved either using First Welfare Theorem type reasoning, or by directly constructing the optimal growth solution.

Proposition Consider the above-described AK economy, with a representative household with preferences given by (1), and the production technology given by (6). Suppose that condition (12) holds. Then, the unique competitive equilibrium is Pareto optimal.

The Role of Policy I

- Suppose there is an effective tax rate of τ on the rate of return from capital income, so budget constraint becomes:

$$\dot{a}(t) = ((1-\tau)\, r(t) - n) a(t) + w(t) - c(t). \tag{17}$$

- Repeating the analysis above this will adversely affect the growth rate of the economy, now:

$$g = \frac{(1-\tau)(A-\delta)-\rho}{\theta}. \tag{18}$$

- Moreover, saving rate will now be

$$s = \frac{(1-\tau)A - \rho + \theta n - (1-\tau-\theta)\delta}{\theta A}, \tag{19}$$

which is a decreasing function of τ if $A - \delta > 0$.

The Role of Policy II

- In contrast to Solow, constants saving rate responds endogenously to policy.
- Since saving rate is constant, differences in policies will lead to permanent differences in the rate of capital accumulation.
  - In the baseline neoclassical growth model even large differences in distortions could only have limited effects on differences in income per capita.
  - Here even small differences in τ can have very large effects.
- Consider two economies, with tax rates on capital income τ and $\tau' > \tau$, and exactly the same otherwise.
- For any $\tau' > \tau$,

$$\lim_{t \to \infty} \frac{Y(\tau', t)}{Y(\tau, t)} = 0,$$

The Role of Policy III

- Why then focus on standard neoclassical if AK model can generate arbitrarily large differences?
  1. AK model, with no diminishing returns and the share of capital in national income asymptoting to 1, is not a good approximation to reality.
  2. Relative stability of the world income distribution in the post-war era makes it more attractive to focus on models in which there is a stationary world income distribution.

The Two-Sector AK Model I

- Model before creates another factor of production that accumulates linearly, so equilibrium is again equivalent to the one-sector AK economy.
- Thus, in some deep sense, the economies of both sections are one-sector models.
- Also, potentially blur key underlying characteristic driving growth.
- What is important is not that production technology is AK, but that the *accumulation technology* is linear.
- Preference and demographics are the same as in the model of the previous section, (1)-(5) apply as before
- No population growth, i.e., $n = 0$, and L is supplied inelastically.
- Rather than a single good used for consumption and investment, now two sectors.

The Two-Sector AK Model II

- Sector 1 produces consumption goods with the following technology

$$C(t) = B(K_C(t))^{\alpha} L_C(t)^{1-\alpha}, \tag{20}$$

- Cobb-Douglas assumption here is quite important in ensuring that the share of capital in national income is constant
- Capital accumulation equation:

$$\dot{K}(t) = I(t) - \delta K(t),$$

- $I(t)$ denotes investment. Investment goods are produced with a different technology,

$$I(t) = AK_I(t). \tag{21}$$

- Extreme version of an assumption often made in two-sector models: investment-good sector is more capital-intensive than the consumption-good sector.

The Two-Sector AK Model III

- Market clearing implies:

$$K_C(t) + K_I(t) \leq K(t),$$

$$L_C(t) \leq L,$$

- An equilibrium is defined similarly, but also features an allocation decision of capital between the two sectors.
- Also, there will be a relative price between the two sectors which will adjust endogenously.
- Both market clearing conditions will hold as equalities, so letting $\kappa(t)$ denote the share of capital used in the investment sector

$$K_C(t) = (1 - \kappa(t)) K(t) \text{ and } K_I(t) = \kappa(t) K(t).$$

- From profit maximization, the rate of return to capital has to be the same when it is employed in the two sectors.

The Two-Sector AK Model IV

- Let the price of the investment good be denoted by $p_I(t)$ and that of the consumption good by $p_C(t)$, then

$$p_I(t)\,A = p_C(t)\,\alpha B\left(\frac{L}{(1-\kappa(t))\,K(t)}\right)^{1-\alpha}. \tag{22}$$

- Define a steady-state (a balanced growth path) as an equilibrium path in which $\kappa(t)$ is constant and equal to some $\kappa \in [0,1]$.
- Moreover, choose the consumption good as the numeraire, so that $p_C(t) = 1$ for all t.
- Then differentiating (22) implies that at the steady state:

$$\frac{\dot{p}_I(t)}{p_I(t)} = -(1-\alpha)\,g_K, \tag{23}$$

- g_K is the steady-state (BGP) growth rate of capital.

The Two-Sector AK Model V

- Euler equation (4) still holds, but interest rate has to be for *consumption-denominated loans*, $r_C(t)$.
- I.e., the interest rate that measures how many units of consumption good an individual will receive tomorrow by giving up one unit of consumption today.
- Relative price of consumption goods and investment goods is changing over time, thus:
  - By giving up one dollar, the individual will buy $1/p_I(t)$ units of capital goods.
  - This will have an instantaneous return of $r_I(t)$.
  - Individual will get back the one unit of capital, which has experienced a change in its price of $\dot{p}_I(t)/p_I(t)$.
  - Finally, he will have to buy consumption goods, whose prices changed by $\dot{p}_C(t)/p_C(t)$.

The Two-Sector AK Model VI

- Therefore,

$$r_C(t) = \frac{r_I(t)}{p_I(t)} + \frac{\dot{p}_I(t)}{p_I(t)} - \frac{\dot{p}_C(t)}{p_C(t)}. \tag{24}$$

- Given our choice of numeraire, we have $\dot{p}_C(t)/p_C(t) = 0$.
- Moreover, $\dot{p}_I(t)/p_I(t)$ is given by (23).
- Finally,

$$\frac{r_I(t)}{p_I(t)} = A - \delta$$

given the linear technology in (21).

- Therefore, we have

$$r_C(t) = A - \delta + \frac{\dot{p}_I(t)}{p_I(t)}.$$

The Two-Sector AK Model VII

- In steady state, from (23):

$$r_C = A - \delta - (1-\alpha)\, g_K .$$

- From (4), this implies a consumption growth rate of

$$g_C \equiv \frac{\dot{C}(t)}{C(t)} = \frac{1}{\theta}\left(A - \delta - (1-\alpha)\, g_K - \rho\right). \tag{25}$$

- Finally, differentiate (20) and use the fact that labor is always constant to obtain

$$\frac{\dot{C}(t)}{C(t)} = \alpha \frac{\dot{K}_C(t)}{K_C(t)},$$

- From the constancy of $\kappa(t)$ in steady state, implies the following steady-state relationship:

$$g_C = \alpha g_K .$$

The Two-Sector AK Model VIII

- Substituting this into (25), we have

$$g_K^* = \frac{A-\delta-\rho}{1-\alpha\left(1-\theta\right)} \tag{26}$$

and

$$g_C^* = \alpha\frac{A-\delta-\rho}{1-\alpha\left(1-\theta\right)}. \tag{27}$$

- Because labor is being used in the consumption good sector, there will be positive wages.
- Since labor markets are competitive,

$$w\left(t\right) = \left(1-\alpha\right)p_C\left(t\right)B\left(\frac{\left(1-\kappa\left(t\right)\right)K\left(t\right)}{L}\right)^{\alpha}.$$

The Two-Sector AK Model IX

- Therefore, in the balanced growth path,

$$\begin{aligned} \frac{\dot{w}(t)}{w(t)} &= \frac{\dot{p}_C(t)}{p_C(t)} + \alpha \frac{\dot{K}(t)}{K(t)} \\ &= \alpha g_K^*, \end{aligned}$$

- Thus wages also grow at the same rate as consumption.

Proposition In the above-described two-sector neoclassical economy, starting from any $K(0) > 0$, consumption and labor income grow at the constant rate given by (27), while the capital stock grows at the constant rate (26).

- Can do policy analysis as before

The Two-Sector AK Model X

- Different from the neoclassical growth model, there is continuous *capital deepening*.
- Capital grows at a faster rate than consumption and output. Whether this is a realistic feature is debatable:
  - Kaldor facts include constant capital-output ratio as one of the requirements of balanced growth.
  - For much of the 20th century, capital-output ratio has been constant, but it has been increasing steadily over the past 30 years.
  - Part of the increase is because of relative price adjustments that have only been performed in the recent past.

Growth with Externalities I

- Romer (1986): model the process of "knowledge accumulation".
- Difficult in the context of a competitive economy.
- Solution: knowledge accumulation as a *byproduct* of capital accumulation.
- Technological spillovers: arguably crude, but captures that knowledge is a largely *non-rival* good.
- Non-rivalry does not imply knowledge is also non-excludable.
- But some of the important characteristics of "knowledge" and its role in the production process can be captured in a reduced-form way by introducing technological spillovers.

Preferences and Technology I

- No population growth (we will see why this is important).
- Production function with labor-augmenting knowledge (technology) that satisfies Assumptions 1 and 2.
- Instead of working with the aggregate production function, assume that the production side of the economy consists of a set $[0, 1]$ of firms.
- The production function facing each firm $i \in [0, 1]$ is

$$Y_i(t) = F(K_i(t), A(t) L_i(t)), \tag{28}$$

- $K_i(t)$ and $L_i(t)$ are capital and labor rented by a firm i.
- $A(t)$ is not indexed by i, since it is technology common to all firms.

Preferences and Technology II

- Normalize the measure of final good producers to 1, so market clearing conditions:

$$\int_0^1 K_i(t)\,di = K(t)$$

and

$$\int_0^1 L_i(t)\,di = L,$$

- L is the constant level of labor (supplied inelastically) in this economy.
- Firms are competitive in all markets, thus all hire the same capital to effective labor ratio, and

$$\begin{aligned} w(t) &= \frac{\partial F(K(t), A(t)L)}{\partial L} \\ R(t) &= \frac{\partial F(K(t), A(t)L)}{\partial K(t)}. \end{aligned}$$

Preferences and Technology III

- Key assumption: firms take $A(t)$ as given, but this stock of technology (knowledge) advances endogenously for the economy as a whole.
- Lucas (1988) develops a similar model, but spillovers work through human capital.
- Extreme assumption of sufficiently strong externalities such that $A(t)$ can grow continuously at the economy level. In particular,

$$A(t) = BK(t), \tag{29}$$

- Motivated by "learning-by-doing." Alternatively, could be a function of the cumulative output that the economy has produced up to now.
- Substituting for (29) into (28) and using the fact that all firms are functioning at the same capital-effective labor ratio, production function of the representative firm:

$$Y(t) = F(K(t), BK(t)L).$$

Preferences and Technology IV

- Using the fact that $F(\cdot,\cdot)$ is homogeneous of degree 1, we have

$$\frac{Y(t)}{K(t)} = F(1, BL) = \tilde{f}(L).$$

- Output per capita can therefore be written as:

$$y(t) \equiv \frac{Y(t)}{L} = \frac{Y(t)}{K(t)}\frac{K(t)}{L} = k(t)\tilde{f}(L),$$

- Again $k(t) \equiv K(t)/L$ is the capital-labor ratio in the economy.

Preferences and Technology V

- Normalized production function, now $\tilde{f}(L)$.
- We have

$$w(t) = K(t)\tilde{f}'(L) \tag{30}$$

and

$$R(t) = R = \tilde{f}(L) - L\tilde{f}'(L), \tag{31}$$

which is constant.

Equilibrium I

- An equilibrium is defined as a path $[C(t), K(t)]_{t=0}^{\infty}$ that maximize the utility of the representative household and $[w(t), R(t)]_{t=0}^{\infty}$ that clear markets.
- Important feature is that because the knowledge spillovers are external to the firm, factor prices are given by (30) and (31).
- I.e., they do not price the role of the capital stock in increasing future productivity.
- Since the market rate of return is $r(t) = R(t) - \delta$, it is also constant.
- Usual consumer Euler equation (e.g., (4) above) then implies that consumption must grow at the constant rate,

$$g_C^* = \frac{1}{\theta}\left(\tilde{f}(L) - L\tilde{f}'(L) - \delta - \rho\right). \tag{32}$$

Equilibrium II

- Capital grows exactly at the same rate as consumption, so the rate of capital, output and consumption growth are all g_C^*.
- Assume that

$$\tilde{f}(L) - L\tilde{f}'(L) - \delta - \rho > 0, \tag{33}$$

so that there is positive growth.

- But also that growth is not fast enough to violate the transversality condition,

$$(1-\theta)\left(\tilde{f}(L) - L\tilde{f}'(L) - \delta\right) < \rho. \tag{34}$$

Proposition Consider the above-described Romer model with physical capital externalities. Suppose that conditions (33) and (34) are satisfied. Then, there exists a unique equilibrium path where starting with any level of capital stock $K(0) > 0$, capital, output and consumption grow at the constant rate (32).

Equilibrium III

- Population must be constant in this model because of the *scale effect*.
- Since $\tilde{f}(L) - L\tilde{f}'(L)$ is always increasing in L (by Assumption 1), a higher population (labor force) L leads to a higher growth rate.
- The scale effect refers to this relationship between population and the equilibrium rate of economic growth.
- If population is growing, the economy will not admit a steady state and the growth rate of the economy will increase over time (output reaching infinity in finite time and violating the transversality condition).

Pareto Optimal Allocations I

- Given externalities, not surprising that the decentralized equilibrium is not Pareto optimal.
- The per capita accumulation equation for this economy can be written as

$$\dot{k}(t) = \tilde{f}(L)\,k(t) - c(t) - \delta k(t).$$

- The current-value Hamiltonian to maximize utility of the representative household is

$$\hat{H}(k, c, \mu) = \frac{c(t)^{1-\theta} - 1}{1-\theta} + \mu\left[\tilde{f}(L)\,k(t) - c(t) - \delta k(t)\right],$$

Pareto Optimal Allocations II

- Conditions for a candidate solution

$$\begin{aligned} \hat{H}_c(k, c, \mu) &= c(t)^{-\theta} - \mu(t) = 0 \\ \hat{H}_k(k, c, \mu) &= \mu(t)\left[\tilde{f}(L) - \delta\right] = -\dot{\mu}(t) + \rho\mu(t), \\ \lim_{t\to\infty}\left[\exp(-\rho t)\,\mu(t)\,k(t)\right] &= 0. \end{aligned}$$

- $\hat{H}$ strictly concave, thus these conditions characterize unique solution.

Pareto Optimal Allocations III

- Social planner's allocation will also have a constant growth rate for consumption (and output) given by

$$g_C^S = \frac{1}{\theta}\left(\tilde{f}\left(L\right) - \delta - \rho\right),$$

which is always greater than g_C^* as given by (32)—since $\tilde{f}\left(L\right) > \tilde{f}\left(L\right) - L\tilde{f}'\left(L\right)$.

- Social planner takes into account that by accumulating more capital, she is improving productivity in the future.

Proposition In the above-described Romer model with physical capital externalities, the decentralized equilibrium is Pareto suboptimal and grows at a slower rate than the allocation that would maximize the utility of the representative household.

Conclusions I

- Linearity of the models (most clearly visible in the AK model):
  - Removes transitional dynamics and leads to a more tractable mathematical structure.
  - Essential feature of any model that will exhibit sustained economic growth.
  - With strong concavity, especially consistent with the Inada, sustained growth will not be possible.
- But most models studied in this chapter do not feature technological progress:
  - Debate about whether the observed total factor productivity growth is partly a result of mismeasurement of inputs.
  - Could be that much of what we measure as technological progress is in fact capital deepening, as in AK model and its variants.

Conclusions II

- Important tension:
  - neoclassical growth model (or Solow growth model) have difficulty in generating very large income differences
  - models here suffer from the opposite problem.
  - Both a blessing and a curse: also predict an ever expanding world distribution.
- Issues to understand:
  1. Era of divergence is not the past 60 years, but the 19th century: important to confront these models with historical data.
  2. "Each country as an island" approach is unlikely to be a good approximation, much less so when we endogenize technology.