14.452 Economic Growth: Lectures 2 and 3, Solow Growth Model: Theory and Data

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Solow Growth Model

- Develop a simple framework for the *proximate* causes and the mechanics of economic growth and cross-country income differences.
- Solow-Swan model named after Robert (Bob) Solow and Trevor Swan, or simply the *Solow model*
- Before Solow growth model, the most common approach to economic growth built on the Harrod-Domar model.
- Harrod-Domar mdel emphasized potential dysfunctional aspects of growth: e.g, how growth could go hand-in-hand with increasing unemployment.
- Solow model demonstrated why the Harrod-Domar model was not an attractive place to start.
- At the center of the Solow growth model is the *neoclassical* aggregate production function.

Households and Production I

- Closed economy, with a unique final good.
- Discrete time running to an infinite horizon, time is indexed by $t = 0, 1, 2, ...$
- Households save a constant exogenous fraction s of disposable income (no explicit optimization, as in basic Keynesian models).
- All firms have access to the same production function: economy admits a **representative firm**, with a representative (or aggregate) production function.
- Aggregate production function for the unique final good is

$$Y(t) = F[K(t), L(t), A(t)] \tag{1}$$

- Assume capital is the same as the final good of the economy, but used in the production process of more goods.
- $A(t)$ is a *technological shifter* of the production function (1).
- **Major assumption**: technology is **free**; it is publicly available as a non-excludable, non-rival good.

Some Assumptions

Assumption 1 **(Continuity, Differentiability, Positive and Diminishing Marginal Products, and Constant Returns to Scale)** The production function $F : \mathbb{R}^3_+ \to \mathbb{R}_+$ is twice continuously differentiable in K and L, and satisfies

$$F_K(K, L, A) \equiv \frac{\partial F(\cdot)}{\partial K} > 0, \quad F_L(K, L, A) \equiv \frac{\partial F(\cdot)}{\partial L} > 0,$$

$$F_{KK}(K, L, A) \equiv \frac{\partial^2 F(\cdot)}{\partial K^2} < 0, \quad F_{LL}(K, L, A) \equiv \frac{\partial^2 F(\cdot)}{\partial L^2} < 0.$$

Moreover, F exhibits constant returns to scale in K and L.

- Assume F exhibits *constant returns to scale* in K and L. I.e., it is *linearly homogeneous* (homogeneous of degree 1) in these two variables.

Review

Definition Let K be an integer. The function $g : \mathbb{R}^{K+2} \to \mathbb{R}$ is homogeneous of degree m in $x \in \mathbb{R}$ and $y \in \mathbb{R}$ if and only if

$$g(\lambda x, \lambda y, z) = \lambda^m g(x, y, z) \text{ for all } \lambda \in \mathbb{R}_+ \text{ and } z \in \mathbb{R}^K.$$

Theorem **(Euler's Theorem)** Suppose that $g : \mathbb{R}^{K+2} \to \mathbb{R}$ is continuously differentiable in $x \in \mathbb{R}$ and $y \in \mathbb{R}$, with partial derivatives denoted by g_x and g_y and is homogeneous of degree m in x and y. Then

$$\begin{aligned} mg(x, y, z) &= g_x(x, y, z)\, x + g_y(x, y, z)\, y \\ \text{for all } x &\in \mathbb{R},\ y \in \mathbb{R} \text{ and } z \in \mathbb{R}^K. \end{aligned}$$

Moreover, $g_x(x, y, z)$ and $g_y(x, y, z)$ are themselves homogeneous of degree $m - 1$ in x and y.

Market Structure, Endowments and Market Clearing I

- We will assume that markets are competitive, so ours will be a prototypical *competitive general equilibrium model*.
- Households own all of the labor, which they supply inelastically.
- Endowment of labor in the economy, $\bar{L}(t)$, and all of this will be supplied regardless of the price.
- The *labor market clearing* condition can then be expressed as:

$$L(t) = \bar{L}(t)$$

  for all t, where $L(t)$ denotes the demand for labor (and also the level of employment).
- More generally, should be written in complementary slackness form.
- In particular, let the *wage rate* at time t be $w(t)$, then the labor market clearing condition takes the form

$$L(t) \leq \bar{L}(t), w(t) \geq 0 \text{ and } \left(L(t) - \bar{L}(t)\right) w(t) = 0$$

Market Structure, Endowments and Market Clearing II

- But Assumption 1 and competitive labor markets make sure that wages have to be strictly positive.
- Households also own the capital stock of the economy and rent it to firms. Take initial holdings, $K(0)$, as given
- Denote the *rental price of capital* at time t be $R(t)$.
- Capital market clearing condition:

$$K^s(t) = K^d(t)$$

- Assume capital depreciates, with "exponential form," at the rate δ: out of 1 unit of capital this period, only $1-\delta$ is left for next period.
- Then, the *interest rate* faced by the household will be $r(t) = R(t) - \delta$.
- Why is it enough to keep track of the interest rate rather than other intertemporal prices?

Firm Optimization I

- Only need to consider the problem of a *representative firm*:

$$\max_{L(t)\geq 0, K(t)\geq 0} F[K(t), L(t), A(t)] - w(t)\,L(t) - R(t)\,K(t)\,.$$

- Since there are no irreversible investments or costs of adjustments, the production side can be represented as a static maximization problem.
- Equivalently, *cost minimization problem.*
- Features worth noting:
  1. Problem is set up in terms of aggregate variables.
  2. Nothing multiplying the F term, price of the final good has normalized to 1.
  3. Already imposes competitive factor markets: firm is taking as given $w(t)$ and $R(t)$.
  4. Concave problem, since F is concave.

Firm Optimization II

- Since F is differentiable, first-order necessary conditions imply:

$$w(t) = F_L[K(t), L(t), A(t)], \tag{2}$$

and

$$R(t) = F_K[K(t), L(t), A(t)]. \tag{3}$$

- Note also that in (2) and (3), we used $K(t)$ and $L(t)$, the amount of capital and labor used by firms.
- In fact, solving for $K(t)$ and $L(t)$, we can derive the capital and labor demands of firms in this economy at rental prices $R(t)$ and $w(t)$.
- Thus we could have used $K^d(t)$ instead of $K(t)$, but this additional notation is not necessary.

Firm Optimization III

Proposition Suppose Assumption 1 holds. Then in the equilibrium of the Solow growth model, firms make no profits, and in particular,

$$Y(t) = w(t) L(t) + R(t) K(t).$$

- **Proof:** Follows immediately from Euler Theorem for the case of $m = 1$, i.e., constant returns to scale.
- Thus firms make no profits, so ownership of firms does not need to be specified.

Second Key Assumption

Assumption 2 **(Inada conditions)** F satisfies the Inada conditions

$$\begin{aligned} \lim_{K \to 0} F_K(\cdot) &= \infty \text{ and } \lim_{K \to \infty} F_K(\cdot) = 0 \text{ for all } L > 0 \text{ all } A \\ \lim_{L \to 0} F_L(\cdot) &= \infty \text{ and } \lim_{L \to \infty} F_L(\cdot) = 0 \text{ for all } K > 0 \text{ all } A. \end{aligned}$$

- Important in ensuring the existence of *interior equilibria.*
- It can be relaxed quite a bit, though useful to get us started.

Production Functions

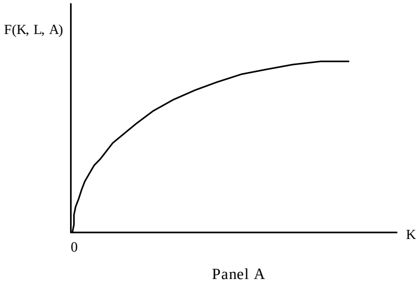


Figure: Production functions and the marginal product of capital. The example in Panel A satisfies the Inada conditions in Assumption 2, while the example in Panel B does not.

Fundamental Law of Motion of the Solow Model I

- Recall that K depreciates exponentially at the rate δ, so

$$K(t+1) = (1-\delta) K(t) + I(t), \tag{4}$$

where $I(t)$ is investment at time t.

- From national income accounting for a closed economy,

$$Y(t) = C(t) + I(t), \tag{5}$$

- *Behavioral rule* of the constant saving rate simplifies the structure of equilibrium considerably.
- Note not derived from the maximization of utility function: welfare comparisons have to be taken with a grain of salt.

Fundamental Law of Motion of the Solow Model II

- Since the economy is closed (and there is no government spending),

$$S(t) = I(t) = Y(t) - C(t).$$

- Individuals are assumed to save a constant fraction s of their income,

$$S(t) = sY(t), \tag{6}$$

$$C(t) = (1-s)Y(t) \tag{7}$$

- Implies that the supply of capital resulting from households' behavior can be expressed as

$$K^s(t) = (1-\delta)K(t) + S(t) = (1-\delta)K(t) + sY(t).$$

Fundamental Law of Motion of the Solow Model III

- Setting supply and demand equal to each other, this implies $K^s(t) = K(t)$.
- We also have $L(t) = \bar{L}(t)$.
- Combining these market clearing conditions with (1) and (4), we obtain *the fundamental law of motion* the Solow growth model:

$$K(t+1) = sF[K(t), L(t), A(t)] + (1-\delta) K(t). \tag{8}$$

- Nonlinear *difference equation*.
- Equilibrium of the Solow growth model is described by this equation together with laws of motion for $L(t)$ (or $\bar{L}(t)$) and $A(t)$.

Definition of Equilibrium I

Definition In the basic Solow model for a given sequence of $\{L(t), A(t)\}_{t=0}^{\infty}$ and an initial capital stock $K(0)$, an equilibrium path is a sequence of capital stocks, output levels, consumption levels, wages and rental rates $\{K(t), Y(t), C(t), w(t), R(t)\}_{t=0}^{\infty}$ such that $K(t)$ satisfies (8), $Y(t)$ is given by (1), $C(t)$ is given by (7), and $w(t)$ and $R(t)$ are given by (2) and (3).

- Note an equilibrium is defined as an entire path of allocations and prices: *not* a static object.
- Make some further assumptions, which will be relaxed later:
  1. There is no population growth; total population is constant at some level $L > 0$. Since individuals supply labor inelastically, $L(t) = L$.
  2. No technological progress, so that $A(t) = A$.

Preliminaries

- Define the capital-labor ratio of the economy as

$$k(t) \equiv \frac{K(t)}{L}, \tag{9}$$

- Using the constant returns to scale assumption, we can express output (income) per capita, $y(t) \equiv Y(t)/L$, as

$$\begin{aligned} y(t) &= F\left[\frac{K(t)}{L}, 1, A\right] \\ &\equiv f(k(t)). \end{aligned} \tag{10}$$

- Note that $f(k)$ here depends on A, so I could have written $f(k, A)$; but A is constant and can be normalized to $A = 1$.
- From Euler Theorem,

$$\begin{aligned} R(t) &= f'(k(t)) > 0 \text{ and} \\ w(t) &= f(k(t)) - k(t) f'(k(t)) > 0. \end{aligned} \tag{11}$$

- Both are positive from Assumption 1.

Equilibrium Without Population Growth and Technological Progress

- The per capita representation of the aggregate production function enables us to divide both sides of (8) by L to obtain:

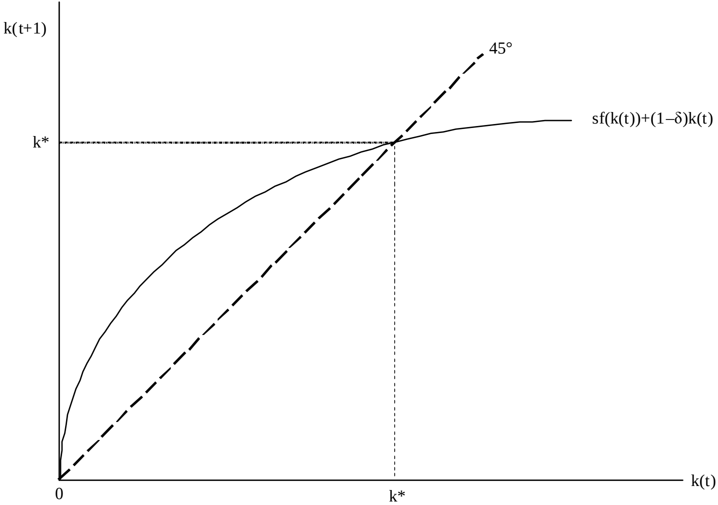
$$k(t+1) = sf(k(t)) + (1-\delta)k(t). \tag{12}$$

- Since it is derived from (8), it also can be referred to as the *equilibrium difference equation* of the Solow model
- The other equilibrium quantities can be obtained from the capital-labor ratio $k(t)$.

Definition A steady-state equilibrium without technological progress and population growth is an equilibrium path in which $k(t) = k^*$ for all t.

- The economy will tend to this steady state equilibrium over time (but never reach it in finite time).

Steady-State Capital-Labor Ratio



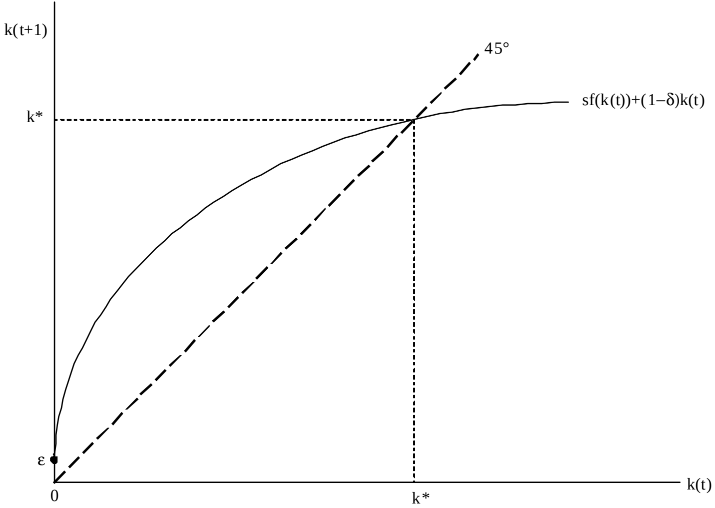
Equilibrium Without Population Growth and Technological Progress II

- Thick curve represents (12) and the dashed line corresponds to the 45° line.
- Their (positive) intersection gives the steady-state value of the capital-labor ratio k^*,

$$\frac{f\left(k^*\right)}{k^*} = \frac{\delta}{s}. \tag{13}$$

- There is another intersection at $k = 0$, because the figure assumes that $f\left(0\right) = 0$.
- Will ignore this intersection throughout:
  1. If capital is not essential, $f\left(0\right)$ will be positive and $k = 0$ will cease to be a steady state equilibrium
  2. This intersection, even when it exists, is an *unstable point*
  3. It has no economic interest for us.

Equilibrium Without Population Growth and Technological Progress III



Consumption and Investment in Steady State

- Alternative visual representation: intersection between δk and the function $sf(k)$, which shows consumption and investment:

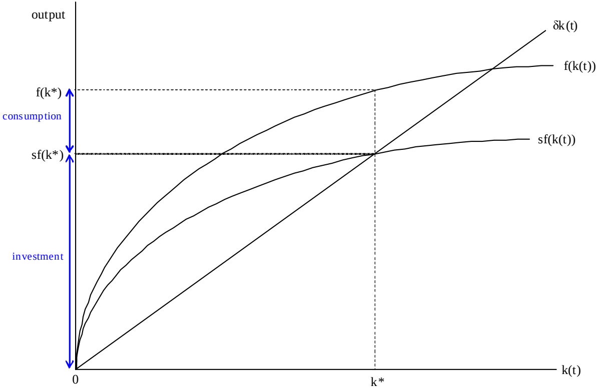


Figure: Investment and consumption in steady state

Equilibrium Without Population Growth and Technological Progress V

Proposition Consider the basic Solow growth model and suppose that Assumptions 1 and 2 hold. Then there exists a unique steady state equilibrium where the capital-labor ratio $k^* \in (0, \infty)$ is given by (13), per capita output is given by

$$y^* = f(k^*) \tag{14}$$

and per capita consumption is given by

$$c^* = (1 - s) f(k^*) . \tag{15}$$

Proof

- The preceding argument establishes that any k^* that satisfies (13) is a steady state.
- To establish existence, note that from Assumption 2 (and from L'Hospital's rule), $\lim_{k\to 0} f(k)/k = \infty$ and $\lim_{k\to\infty} f(k)/k = 0$.
- Moreover, $f(k)/k$ is continuous from Assumption 1, so by the Intermediate Value Theorem there exists k^* such that (13) is satisfied.
- To see uniqueness, differentiate $f(k)/k$ with respect to k, which gives

$$\frac{\partial [f(k)/k]}{\partial k} = \frac{f'(k)k - f(k)}{k^2} = -\frac{w}{k^2} < 0, \tag{16}$$

  where the last equality uses (11).
- Since $f(k)/k$ is everywhere (strictly) decreasing, there can only exist a unique value k^* that satisfies (13).
- Equations (14) and (15) then follow by definition.

Non-Existence and Non-Uniqueness

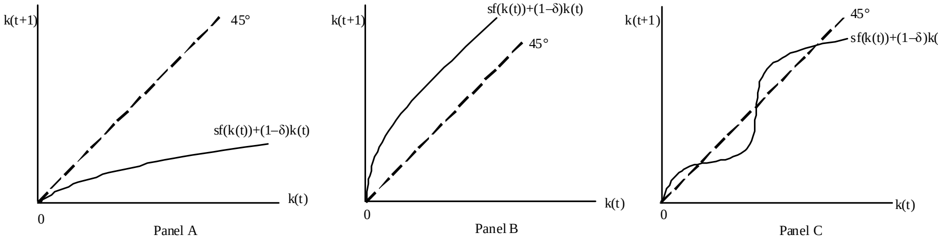


Figure: Examples of nonexistence and nonuniqueness of interior steady states when Assumptions 1 and 2 are not satisfied.

Equilibrium Without Population Growth and Technological Progress VI

- Comparative statics with respect to s, a and δ straightforward for k^* and y^*.
- But c^* will not be monotone in the saving rate (think, for example, of $s = 1$).
- In fact, there will exist a specific level of the saving rate, s_{gold}, referred to as the "golden rule" saving rate, which maximizes c^*.
- But cannot say whether the golden rule saving rate is "better" than some other saving rate.
- Write the steady state relationship between c^* and s and suppress the other parameters:

$$\begin{aligned} c^*(s) &= (1-s) f(k^*(s)), \\ &= f(k^*(s)) - \delta k^*(s), \end{aligned}$$

- The second equality exploits that in steady state $sf(k) = \delta k$.

Equilibrium Without Population Growth and Technological Progress X

- Differentiating with respect to s,

$$\frac{\partial c^*(s)}{\partial s} = \left[f'(k^*(s)) - \delta\right] \frac{\partial k^*}{\partial s}. \tag{17}$$

- s_{gold} is such that $\partial c^*(s_{gold})/\partial s = 0$. The corresponding steady-state golden rule capital stock is defined as k^*_{gold}.

Proposition In the basic Solow growth model, the highest level of steady-state consumption is reached for s_{gold}, with the corresponding steady state capital level k^*_{gold} such that

$$f'(k^*_{gold}) = \delta. \tag{18}$$

The Golden Rule

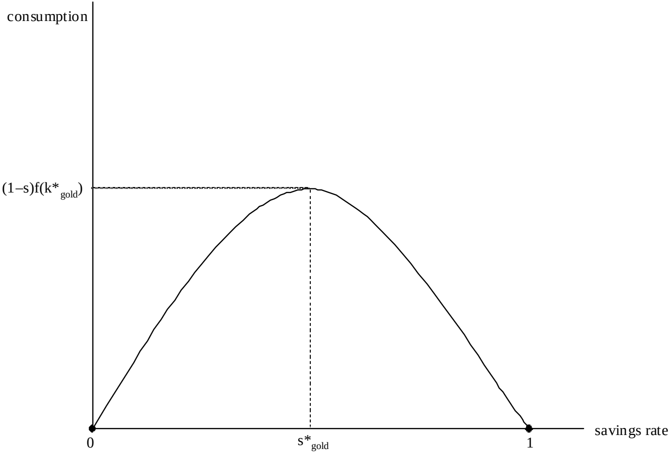


Figure: The "golden rule" level of savings rate, which maximizes steady-state consumption.

Dynamic Inefficiency

- When the economy is below k^*_{gold}, higher saving will increase consumption; when it is above k^*_{gold}, steady-state consumption can be increased by saving less.
- In the latter case, capital-labor ratio is too high so that individuals are investing too much and not consuming enough (*dynamic inefficiency)*.
- But no utility function, so statements about "inefficiency" have to be considered with caution.
- Such dynamic inefficiency will not arise once we endogenize consumption-saving decisions.

Discrete-Time Solow Model Redux

- Per capita capital stock evolves according to

$$k(t+1) = sf(k(t)) + (1-\delta)k(t).$$

- The steady-state value of the capital-labor ratio k^* is given by

$$\frac{f(k^*)}{k^*} = \frac{\delta}{s}.$$

- Consumption is given by

$$C(t) = (1-s)Y(t)$$

- And factor prices are given by

$$\begin{aligned} R(t) &= f'(k(t)) > 0 \text{ and} \\ w(t) &= f(k(t)) - k(t)f'(k(t)) > 0. \end{aligned}$$

Transitional Dynamics

- *Equilibrium path*: not simply steady state, but entire path of capital stock, output, consumption and factor prices.
  - In engineering and physical sciences, equilibrium is point of rest of dynamical system, thus *the steady state equilibrium*.
  - In economics, non-steady-state behavior also governed by optimizing behavior of households and firms and market clearing.
- Need to study the "transitional dynamics" of the equilibrium difference equation (12) starting from an arbitrary initial capital-labor ratio $k(0) > 0$.
- Key question: whether economy will tend to steady state and how it will behave along the transition path.

Transitional Dynamics: Reminder

Simple Result About Stability

- Let $x(t), a, b \in \mathbb{R}$, then the unique steady state of the linear difference equation $x(t+1) = ax(t) + b$ is globally asymptotically stable (in the sense that $x(t) \to x^* = b/(1-a)$) if $|a| < 1$.
- Suppose that $g : \mathbb{R} \to \mathbb{R}$ is differentiable at the steady state x^*, defined by $g(x^*) = x^*$. Then, the steady state of the nonlinear difference equation $x(t+1) = g(x(t))$, x^*, is locally asymptotically stable if $|g'(x^*)| < 1$. Moreover, if $|g'(x)| < 1$ for all $x \in \mathbb{R}$, then x^* is globally asymptotically stable.

Transitional Dynamics in the Discrete Time Solow Model

Proposition Suppose that Assumptions 1 and 2 hold, then the steady-state equilibrium of the Solow growth model described by the difference equation (12) is globally asymptotically stable, and starting from any $k(0) > 0$, $k(t)$ monotonically converges to k^*.

Proof of Proposition: Transitional Dyamics I

- Let $g(k) \equiv sf(k) + (1-\delta)k$. First observe that $g'(k) > 0$ for all k.
- Next, from (12),
$$k(t+1) = g(k(t)), \tag{19}$$
with a unique steady state at k^*.
- From (13), the steady-state capital k^* satisfies $\delta k^* = sf(k^*)$, or
$$k^* = g(k^*). \tag{20}$$
- Recall that $f(\cdot)$ is concave and differentiable from Assumption 1 and satisfies $f(0) \geq 0$ from Assumption 2.

Proof of Proposition: Transitional Dyamics II

- For any strictly concave differentiable function,

$$f(k) > f(0) + kf'(k) \geq kf'(k), \tag{21}$$

- The second inequality uses the fact that $f(0) \geq 0$.
- Since (21) implies that $\delta = sf(k^*)/k^* > sf'(k^*)$, we have $g'(k^*) = sf'(k^*) + 1 - \delta < 1$. Therefore,

$$g'(k^*) \in (0, 1).$$

- The Simple Result then establishes local asymptotic stability.

Proof of Proposition: Transitional Dyamics III

- To prove global stability, note that for all $k(t) \in (0, k^*)$,

$$\begin{aligned} k(t+1) - k^* &= g(k(t)) - g(k^*) \\ &= -\int_{k(t)}^{k^*} g'(k)\, dk, \\ &< 0 \end{aligned}$$

- First line follows by subtracting (20) from (19), second line uses the fundamental theorem of calculus, and third line follows from the observation that $g'(k) > 0$ for all k.

Proof of Proposition: Transitional Dyamics IV

- Next, (12) also implies

$$\begin{aligned} \frac{k(t+1)-k(t)}{k(t)} &= s\frac{f(k(t))}{k(t)}-\delta \\ &> s\frac{f(k^*)}{k^*}-\delta \\ &= 0. \end{aligned}$$

  Moreover, for any $k(t) \in (0, k^* - \varepsilon)$, this is uniformly so.
- Second line uses the fact that $f(k)/k$ is decreasing in k (from (21) above) andlast line uses the definition of k^*.
- These two arguments together establish that for all $k(t) \in (0, k^*)$, $k(t+1) \in (k(t), k^*)$.
- An identical argument implies that for all $k(t) > k^*$, $k(t+1) \in (k^*, k(t))$.
- Therefore, $\{k(t)\}_{t=0}^{\infty}$ monotonically converges to k^* and is globally stable.

Transitional Dynamics III

- Stability result can be seen diagrammatically in the Figure:
    - Starting from initial capital stock $k(0) < k^*$, economy grows towards k^*, *capital deepening* and growth of per capita income.
    - If economy were to start with $k'(0) > k^*$, reach the steady state by decumulating capital and contracting.
- As a consequence:

Proposition Suppose that Assumptions 1 and 2 hold, and $k(0) < k^*$, then $\{w(t)\}_{t=0}^{\infty}$ is an increasing sequence and $\{R(t)\}_{t=0}^{\infty}$ is a decreasing sequence. If $k(0) > k^*$, the opposite results apply.

- Thus far Solow growth model has a number of nice properties, but no growth, except when the economy starts with $k(0) < k^*$.

Transitional Dynamics in Figure

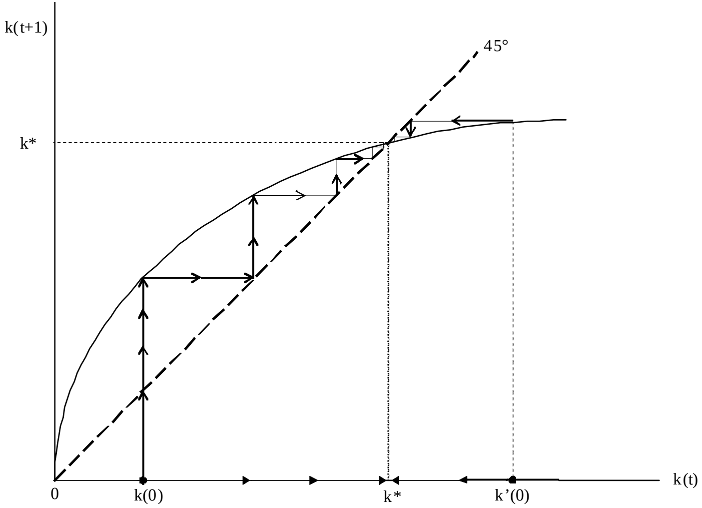


Figure: Transitional dynamics in the basic Solow model.

From Difference to Differential Equations I

- Start with a simple difference equation

$$x(t+1) - x(t) = g(x(t)) . \tag{22}$$

- Now consider the following approximation for any $\Delta t \in [0,1]$,

$$x(t+\Delta t) - x(t) \simeq \Delta t \cdot g(x(t)),$$

- When $\Delta t = 0$, this equation is just an identity. When $\Delta t = 1$, it gives (22).
- In-between it is a linear approximation, not too bad if $g(x) \simeq g(x(t))$ for all $x \in [x(t), x(t+1)]$

From Difference to Differential Equations II

- Divide both sides of this equation by Δt, and take limits

$$\lim_{\Delta t \to 0} \frac{x(t + \Delta t) - x(t)}{\Delta t} = \dot{x}(t) \simeq g(x(t)), \tag{23}$$

where

$$\dot{x}(t) \equiv \frac{dx(t)}{dt}$$

- Equation (23) is a differential equation representing (22) for the case in which t and $t + 1$ is "small".

The Fundamental Equation of the Solow Model in Continuous Time I

- Nothing has changed on the production side, so (11) still give the factor prices, now interpreted as instantaneous wage and rental rates.
- Savings are again

$$S(t) = sY(t),$$

- Consumption is given by (7) above.
- Introduce population growth,

$$L(t) = \exp(nt) L(0). \tag{24}$$

- Recall

$$k(t) \equiv \frac{K(t)}{L(t)},$$

The Fundamental Equation of the Solow Model in Continuous Time II

- Implies

$$\begin{aligned} \frac{\dot{k}(t)}{k(t)} &= \frac{\dot{K}(t)}{K(t)} - \frac{\dot{L}(t)}{L(t)}, \\ &= \frac{\dot{K}(t)}{K(t)} - n. \end{aligned}$$

- From the limiting argument leading to equation (23),

$$\dot{K}(t) = sF[K(t), L(t), A(t)] - \delta K(t).$$

- Using the definition of $k(t)$ and the constant returns to scale properties of the production function,

$$\frac{\dot{k}(t)}{k(t)} = s\frac{f(k(t))}{k(t)} - (n+\delta), \tag{25}$$

The Fundamental Equation of the Solow Model in Continuous Time III

Definition In the basic Solow model in continuous time with population growth at the rate n, no technological progress and an initial capital stock $K(0)$, an equilibrium path is a sequence of capital stocks, labor, output levels, consumption levels, wages and rental rates $[K(t), L(t), Y(t), C(t), w(t), R(t)]_{t=0}^{\infty}$ such that $L(t)$ satisfies (24), $k(t) \equiv K(t)/L(t)$ satisfies (25), $Y(t)$ is given by the aggregate production function, $C(t)$ is given by (7), and $w(t)$ and $R(t)$ are given by (11).

- As before, *steady-state* equilibrium involves $k(t)$ remaining constant at some level k^*.

Steady State With Population Growth

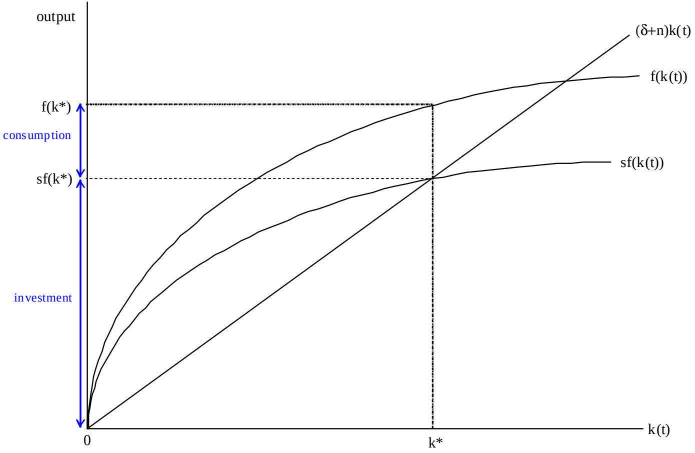


Figure: Investment and consumption in the steady-state equilibrium with population growth.

Steady State of the Solow Model in Continuous Time

- Equilibrium path (25) has a unique *steady state* at k^*, which is given by a slight modification of (13) above:

$$\frac{f(k^*)}{k^*} = \frac{n+\delta}{s}. \tag{26}$$

Proposition Consider the basic Solow growth model in continuous time and suppose that Assumptions 1 and 2 hold. Then there exists a unique steady state equilibrium where the capital-labor ratio is equal to $k^* \in (0, \infty)$ and is given by (26), per capita output is given by

$$y^* = f(k^*)$$

and per capita consumption is given by

$$c^* = (1-s) f(k^*).$$

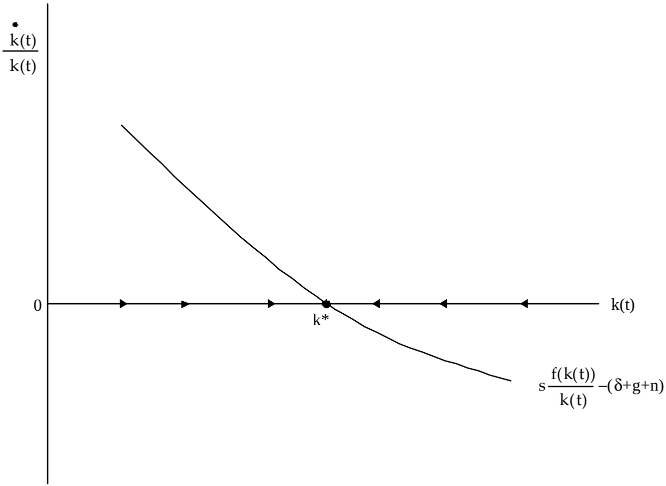
- Similar comparative statics to the discrete time model.

Transitional Dynamics in the Continuous Time Solow Model I

Simple Result about Stability In Continuous Time Model

- Let $g : \mathbb{R} \rightarrow \mathbb{R}$ be a differentiable function and suppose that there exists a unique x^* such that $g(x^*) = 0$. Moreover, suppose $g(x) < 0$ for all $x > x^*$ and $g(x) > 0$ for all $x < x^*$. Then the steady state of the nonlinear differential equation $\dot{x}(t) = g(x(t))$, x^*, is globally asymptotically stable, i.e., starting with any $x(0)$, $x(t) \rightarrow x^*$.

Simple Result in Figure



Transitional Dynamics in the Continuous Time Solow Model II

Proposition Suppose that Assumptions 1 and 2 hold, then the basic Solow growth model in continuous time with constant population growth and no technological change is globally asymptotically stable, and starting from any $k(0) > 0$, $k(t) \to k^*$.

- **Proof:** Follows immediately from the Theorem above by noting whenever $k < k^*$, $sf(k) - (n + \delta)k > 0$ and whenever $k > k^*$, $sf(k) - (n + \delta)k < 0$.
- Figure: plots the right-hand side of (25) and makes it clear that whenever $k < k^*$, $\dot{k} > 0$ and whenever $k > k^*$, $\dot{k} < 0$, so k monotonically converges to k^*.

A First Look at Sustained Growth I

- Cobb-Douglas already showed that when α is close to 1, adjustment to steady-state level can be very slow.
- Simplest model of sustained growth essentially takes $\alpha = 1$ in terms of the Cobb-Douglas production function above.
- Relax Assumptions 1 and 2 and suppose

$$F\left[K\left(t\right), L\left(t\right), A\left(t\right)\right] = AK\left(t\right), \tag{27}$$

  where $A > 0$ is a constant.
- So-called "AK" model, and in its simplest form output does not even depend on labor.
- Results we would like to highlight apply with more general constant returns to scale production functions,

$$F\left[K\left(t\right), L\left(t\right), A\left(t\right)\right] = AK\left(t\right) + BL\left(t\right), \tag{28}$$

A First Look at Sustained Growth II

- Assume population grows at n as before (cfr. equation (24)).
- Combining with the production function (27),

$$\frac{\dot{k}(t)}{k(t)} = sA - \delta - n.$$

- Therefore, if $sA - \delta - n > 0$, there will be sustained growth in the capital-labor ratio.
- From (27), this implies that there will be sustained growth in output per capita as well.

A First Look at Sustained Growth III

Proposition Consider the Solow growth model with the production function (27) and suppose that $sA - \delta - n > 0$. Then in equilibrium, there is sustained growth of output per capita at the rate $sA - \delta - n$. In particular, starting with a capital-labor ratio $k(0) > 0$, the economy has

$$k(t) = \exp((sA - \delta - n)t)\, k(0), \text{ and}$$

$$y(t) = \exp((sA - \delta - n)t)\, Ak(0).$$

- Note no transitional dynamics.
- Unattractive features:
  1. Knife-edge case, requires the production function to be ultimately linear in the capital stock.
  2. Implies that as time goes by the share of national income accruing to capital will reach 1.
  3. Technological progress seems to be a major (perhaps the most major) factor in long-run economic growth.

Sustained Growth in Figure

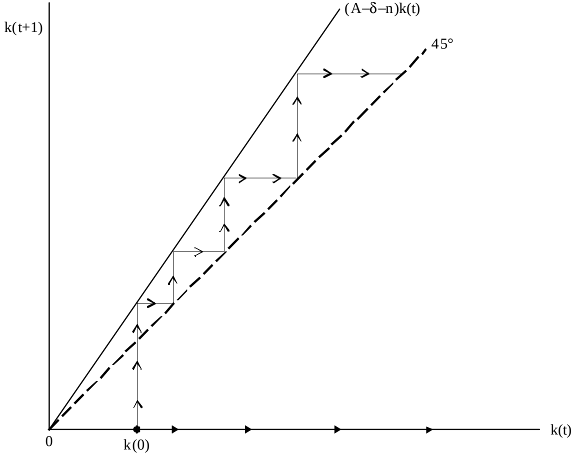


Figure: Sustained growth with the linear AK technology with $sA - \delta - n > 0$.

Balanced Growth I

- Production function $F[K(t), L(t), A(t)]$ is too general.
- May not have *balanced growth*, i.e. a path of the economy consistent with the *Kaldor facts* (Kaldor, 1963).
- Kaldor facts:
    - while output per capita increases, the capital-output ratio, the interest rate, and the distribution of income between capital and labor remain roughly constant.
- We know that the capital share of national income is not really constant, and has been increasing over the last 30 years or so. Nevertheless, its "relative constancy" for almost a century might be an argument for sticking to Kaldor facts.
- More importantly, balanced growth is a very simple starting point.

Balanced Growth II

- Note capital share in national income is about $1/3$, while the labor share is about $2/3$.
- Ignoring land, not a major factor of production.
- But in poor countries land is a major factor of production.
- This pattern often makes economists choose $AK^{1/3}L^{2/3}$.
- Main advantage from our point of view is that balanced growth is the same as a steady-state in transformed variables
  - i.e., we will again have $\dot{k} = 0$, but the definition of k will change.
- But important to bear in mind that growth has many non-balanced features.
  - e.g., the share of different sectors changes systematically.

Types of Neutral Technological Progress I

- For some constant returns to scale function $\tilde{F}$:
  - *Hicks-neutral* technological progress:

$$\tilde{F}\left[K\left(t\right),L\left(t\right),A\left(t\right)\right]=A\left(t\right)F\left[K\left(t\right),L\left(t\right)\right],$$

    - Relabeling of the isoquants (without any change in their shape) of the function $\tilde{F}\left[K\left(t\right),L\left(t\right),A\left(t\right)\right]$ in the L-K space.
  - *Solow-neutral* technological progress,

$$\tilde{F}\left[K\left(t\right),L\left(t\right),A\left(t\right)\right]=F\left[A\left(t\right)K\left(t\right),L\left(t\right)\right].$$

    - Capital-augmenting progress: isoquants shifting with technological progress in a way that they have constant slope at a given labor-output ratio.
  - *Harrod-neutral* technological progress,

$$\tilde{F}\left[K\left(t\right),L\left(t\right),A\left(t\right)\right]=F\left[K\left(t\right),A\left(t\right)L\left(t\right)\right].$$

    - Increases output as if the economy had more labor: slope of the isoquants are constant along rays with constant capital-output ratio.

Isoquants with Neutral Technological Progress

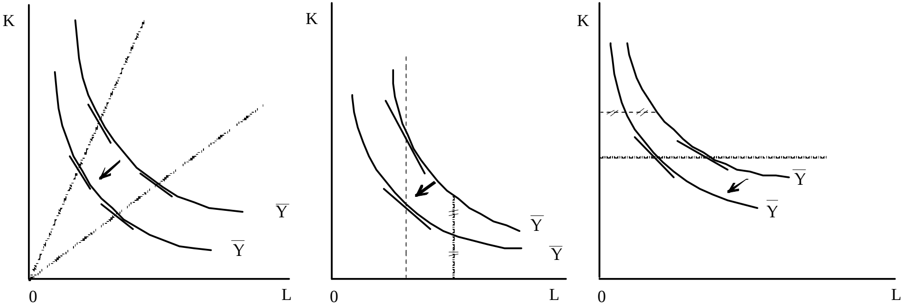


Figure: Hicks-neutral, Solow-neutral and Harrod-neutral shifts in isoquants.

Types of Neutral Technological Progress II

- Could also have a vector valued index of technology $\mathbf{A}(t) = (A_H(t), A_K(t), A_L(t))$ and a production function

$$\tilde{F}[K(t), L(t), \mathbf{A}(t)] = A_H(t) F[A_K(t) K(t), A_L(t) L(t)],$$

- Nests the constant elasticity of substitution production function introduced in the Example above.
- But even this is a restriction on the form of technological progress, $A(t)$ could modify the entire production function.
- Balanced growth necessitates that all technological progress be labor augmenting or Harrod-neutral.

Preliminaries

- Focus on continuous time models.
- Key elements of balanced growth: constancy of factor shares and of the capital-output ratio, $K(t)/Y(t)$.
- By factor shares, we mean

$$\alpha_L(t) \equiv \frac{w(t)L(t)}{Y(t)} \text{ and } \alpha_K(t) \equiv \frac{R(t)K(t)}{Y(t)}.$$

- By Assumption 1 and Euler Theorem $\alpha_L(t) + \alpha_K(t) = 1$.

Uzawa's Theorem

Theorem

(Uzawa I) *Suppose* $L(t) = \exp(nt) L(0)$,

$$Y(t) = \tilde{F}(K(t), L(t), \tilde{A}(t)),$$

$\dot{K}(t) = Y(t) - C(t) - \delta K(t)$, *and* $\tilde{F}$ *is CRS in* K *and* L. *Suppose for* $\tau < \infty$, $\dot{Y}(t)/Y(t) = g_Y > 0$, $\dot{K}(t)/K(t) = g_K > 0$ *and* $\dot{C}(t)/C(t) = g_C > 0$. *Then,*

1. $g_Y = g_K = g_C$; *and*
2. *for any* $t \geq \tau$, $\tilde{F}$ *can be represented as*

$$Y(t) = F(K(t), A(t) L(t)),$$

where $A(t) \in \mathbb{R}_+$, $F : \mathbb{R}_+^2 \to \mathbb{R}_+$ *is homogeneous of degree 1, and*

$$\dot{A}(t)/A(t) = g = g_Y - n.$$

Intuition

- From the aggregate resource constraint, $g_K > 0$ is only possible if output and capital grow at the same rate.
- Either this growth rate is equal to n and there is no technological change (i.e., proposition applies with $g = 0$), or the economy exhibits growth of per capita income and capital-labor ratio.
- The latter case creates an asymmetry between capital and labor: capital is accumulating faster than labor. Constancy of growth requires technological change to make up for this asymmetry.

Corollary Under the assumptions of Uzawa Theorem, after time τ technological progress can be represented as Harrod neutral (purely labor augmenting).

- Also, contrary to Uzawa's original theorem, not stated for equilibrium or a balanced growth path, but only for an asymptotic feasible path with constant rates of output, capital and consumption growth. **But**, the theorem gives only one representation.

Further Intuition

- Suppose the production function takes the special form $F\left[A_K(t)\,K(t)\,, A_L(t)\,L(t)\right]$.
- The stronger theorem implies that factor shares will be constant.
- Given constant returns to scale, this can only be the case when $A_K(t)\,K(t)$ and $A_L(t)\,L(t)$ grow at the same rate.
- The fact that the capital-output ratio is constant in steady state (or the fact that capital accumulates) implies that $K(t)$ must grow at the same rate as $A_L(t)\,L(t)$.
- Balanced growth possible only if $A_K(t)$ is asymptotically constant.
- Allows one important exception. If,

$$Y(t) = \left[A_K(t)\,K(t)\right]^{\alpha}\left[A_L(t)L(t)\right]^{1-\alpha},$$

then both $A_K(t)$ and $A_L(t)$ could grow asymptotically, while maintaining balanced growth. This is where the fact that Harrod-neutral technological change is just one representation is important.

Implications for Factor Shares

- Suppose the labor-augmenting representation of the aggregate production function applies.
- Then note that with competitive factor markets, as $t \geq \tau$,

$$\begin{aligned} \alpha_K(t) &\equiv \frac{R(t)\,K(t)}{Y(t)} \\ &= \frac{K(t)}{Y(t)}\frac{\partial F[K(t), A(t)\,L(t)]}{\partial K(t)} \\ &= \alpha_K^*, \end{aligned}$$

- Second line uses the definition of the rental rate of capital in a competitive market
- Third line uses that $g_Y = g_K$ and $g_K = g + n$ from Uzawa Theorem and that F exhibits constant returns to scale so its derivative is homogeneous of degree 0.

Technological Progress in the Solow Model

- Uzawa Theorem's theorem is a distressing result.
- But it simplifies basic growth models considerably: production function must admit representation of the form

$$Y(t) = F[K(t), A(t)L(t)],$$

- Moreover, suppose

$$\frac{\dot{A}(t)}{A(t)} = g, \tag{29}$$

$$\frac{\dot{L}(t)}{L(t)} = n.$$

- Again using the constant saving rate

$$\dot{K}(t) = sF[K(t), A(t)L(t)] - \delta K(t). \tag{30}$$

The Solow Growth Model with Technological Progress: Continuous Time II

- Now define $k(t)$ as the *effective capital-labor* ratio, i.e.,

$$k(t) \equiv \frac{K(t)}{A(t)L(t)}. \tag{31}$$

- Slight but useful abuse of notation.
- Differentiating this expression with respect to time,

$$\frac{\dot{k}(t)}{k(t)} = \frac{\dot{K}(t)}{K(t)} - g - n. \tag{32}$$

- Output per unit of effective labor can be written as

$$\begin{aligned} \hat{y}(t) &\equiv \frac{Y(t)}{A(t)L(t)} = F\left[\frac{K(t)}{A(t)L(t)}, 1\right] \\ &\equiv f(k(t)). \end{aligned}$$

The Solow Growth Model with Technological Progress: Continuous Time III

- Income per capita is $y(t) \equiv Y(t)/L(t)$, i.e.,

$$\begin{aligned} y(t) &= A(t)\,\hat{y}(t) \qquad (33) \\ &= A(t)\,f(k(t)). \end{aligned}$$

- Clearly if $\hat{y}(t)$ is constant, income per capita, $y(t)$, will grow over time, since $A(t)$ is growing.
- Thus should not look for "steady states" where income per capita is constant, but for *balanced growth paths*, where income per capita grows at a constant rate.
- Some transformed variables such as $\hat{y}(t)$ or $k(t)$ in (32) remain constant.
- Thus balanced growth paths can be thought of as steady states of a transformed model.

The Solow Growth Model with Technological Progress: Continuous Time IV

- Hence use the terms "steady state" and balanced growth path interchangeably.
- Substituting for $\dot{K}(t)$ from (30) into (32):

$$\frac{\dot{k}(t)}{k(t)} = \frac{sF[K(t), A(t)L(t)]}{K(t)} - (\delta + g + n).$$

- Now using (31),

$$\frac{\dot{k}(t)}{k(t)} = \frac{sf(k(t))}{k(t)} - (\delta + g + n), \tag{34}$$

- Only difference is the presence of g: k is no longer the capital-labor ratio but the *effective* capital-labor ratio.

The Solow Growth Model with Technological Progress: Continuous Time V

Proposition Consider the basic Solow growth model in continuous time, with Harrod-neutral technological progress at the rate g and population growth at the rate n. Suppose that Assumptions 1 and 2 hold, and define the effective capital-labor ratio as in (31). Then there exists a unique steady state (balanced growth path) equilibrium where the effective capital-labor ratio is equal to $k^* \in (0, \infty)$ and is given by

$$\frac{f(k^*)}{k^*} = \frac{\delta + g + n}{s}. \tag{35}$$

Per capita output and consumption grow at the rate g.

The Solow Growth Model with Technological Progress: Continuous Time VI

- Equation (35), emphasizes that now total savings, $sf(k)$, are used for replenishing the capital stock for three distinct reasons:
  1. depreciation at the rate δ.
  2. population growth at the rate n, which reduces capital per worker.
  3. Harrod-neutral technological progress at the rate g.
- Now replenishment of effective capital-labor ratio requires investments to be equal to $(\delta + g + n)k$.

The Solow Growth Model with Technological Progress: Continuous Time VII

Proposition Suppose that Assumptions 1 and 2 hold, then the Solow growth model with Harrod-neutral technological progress and population growth in continuous time is asymptotically stable, i.e., starting from any $k(0) > 0$, the effective capital-labor ratio converges to a steady-state value k^* ($k(t) \to k^*$).

- Now model generates growth in output per capita, but entirely *exogenously*.

Comparative Dynamics I

- Comparative dynamics: dynamic response of an economy to a change in its parameters or to shocks.
- Different from comparative statics in Propositions above in that we are interested in the entire path of adjustment of the economy following the shock or changing parameter.
- For brevity we will focus on the continuous time economy.
- Recall

$$\dot{k}(t)/k(t) = sf(k(t))/k(t) - (\delta + g + n)$$

Comparative Dynamics in Figure

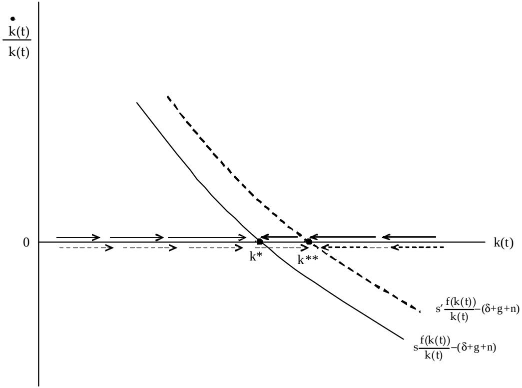


Figure: Dynamics following an increase in the savings rate from s to s'. The solid arrows show the dynamics for the initial steady state, while the dashed arrows show the dynamics for the new steady state.

Comparative Dynamics II

- One-time, unanticipated, permanent increase in the saving rate from s to s'.
    - Shifts curve to the right as shown by the dotted line, with a new intersection with the horizontal axis, k^{**}.
    - Arrows on the horizontal axis show how the effective capital-labor ratio adjusts gradually to k^{**}.
    - Immediately, the capital stock remains unchanged (since it is a *state* variable).
    - After this point, it follows the dashed arrows on the horizontal axis.
- s changes in unanticipated manner at $t = t'$, but will be reversed back to its original value at some known future date $t = t'' > t'$.
    - Starting at t', the economy follows the rightwards arrows until t'.
    - After t'', the original steady state of the differential equation applies and leftwards arrows become effective.
    - From t'' onwards, economy gradually returns back to its original balanced growth equilibrium, k^*.

Growth Accounting I

- Aggregate production function in its general form:

$$Y(t) = F[K(t), L(t), A(t)].$$

- Combined with competitive factor markets, gives Solow (1957) *growth accounting framework.*
- Continuous-time economy and differentiate the aggregate production function with respect to time.
- Dropping time dependence,

$$\frac{\dot{Y}}{Y} = \frac{F_A A}{Y}\frac{\dot{A}}{A} + \frac{F_K K}{Y}\frac{\dot{K}}{K} + \frac{F_L L}{Y}\frac{\dot{L}}{L}. \quad (36)$$

Growth Accounting II

- Denote growth rates of output, capital stock and labor by $g \equiv \dot{Y}/Y$, $g_K \equiv \dot{K}/K$ and $g_L \equiv \dot{L}/L$.
- Define the contribution of technology to growth as

$$x \equiv \frac{F_A A}{Y}\frac{\dot{A}}{A}$$

- Recall with competitive factor markets, $w = F_L$ and $R = F_K$.
- Define factor shares as $\alpha_K \equiv RK/Y$ and $\alpha_L \equiv wL/Y$.
- Putting all these together, (36) the *fundamental growth accounting* equation

$$x = g - \alpha_K g_K - \alpha_L g_L. \tag{37}$$

- Gives estimate of contribution of technological progress, *Total Factor Productivity* (TFP) or Multi Factor Productivity as

$$\hat{x}(t) = g(t) - \alpha_K(t)\, g_K(t) - \alpha_L(t)\, g_L(t). \tag{38}$$

- All terms on right-hand side are "estimates" obtained with a range of assumptions from national accounts and other data sources.

Growth Accounting III

- In continuous time, equation (38) is exact.
- With discrete time, potential problem in using (38): over the time horizon factor shares can change.
- Use beginning-of-period or end-of-period values of α_K and α_L?
  - Either might lead to seriously biased estimates.
  - Best way of avoiding such biases is to use as high-frequency data as possible.
  - Typically use factor shares calculated as the average of the beginning and end of period values.
- In discrete time, the analog of equation (38) becomes

$$\hat{x}_{t,t+1} = g_{t,t+1} - \bar{\alpha}_{K,t,t+1} g_{K,t,t+1} - \bar{\alpha}_{L,t,t+1} g_{L,t,t+1}, \tag{39}$$

- $g_{t,t+1}$ is the growth rate of output between t and $t+1$; other growth rates defined analogously.

Growth Accounting IV

- Moreover,

$$\bar{\alpha}_{K,t,t+1} \equiv \frac{\alpha_K(t) + \alpha_K(t+1)}{2}$$

$$\text{and } \bar{\alpha}_{L,t,t+1} \equiv \frac{\alpha_L(t) + \alpha_L(t+1)}{2}$$

- Equation (39) would be a fairly good approximation to (38) when the difference between t and $t+1$ is small and the capital-labor ratio does not change much during this time interval.
- Solow's (1957) applied this framework to US data: a large part of the growth was due to technological progress.
- From early days, however, a number of pitfalls were recognized.
    - Moses Abramovitz (1956): dubbed the $\hat{x}$ term "the measure of our ignorance".
    - If we mismeasure g_L and g_K we will arrive at inflated estimates of $\hat{x}$.

Growth Accounting Results

- Example from Barro and Sala-i-Martin's textbook

Table 10.1
Growth Accounting for a Sample of Countries

Country	(1) Growth Rate of GDP	(2) Contribution from Capital	(3) Contribution from Labor	(4) TFP Growth Rate
Panel A: OECD Countries, 1947–73				
Canada ($\alpha = 0.44$)	0.0517	0.0254 (49%)	0.0088 (17%)	0.0175 (34%)
France[a] ($\alpha = 0.40$)	0.0542	0.0225 (42%)	0.0021 (4%)	0.0296 (54%)
Germany[b] ($\alpha = 0.39$)	0.0661	0.0269 (41%)	0.0018 (3%)	0.0374 (56%)
Italy[b] ($\alpha = 0.39$)	0.0527	0.0180 (34%)	0.0011 (2%)	0.0337 (64%)
Japan[b] ($\alpha = 0.39$)	0.0951	0.0328 (35%)	0.0221 (23%)	0.0402 (42%)
Netherlands[c] ($\alpha = 0.45$)	0.0536	0.0247 (46%)	0.0042 (8%)	0.0248 (46%)
U.K.[d] ($\alpha = 0.38$)	0.0373	0.0176 (47%)	0.0003 (1%)	0.0193 (52%)
U.S. ($\alpha = 0.40$)	0.0402	0.0171 (43%)	0.0095 (24%)	0.0135 (34%)
Panel B: OECD Countries, 1960–95				
Canada ($\alpha = 0.42$)	0.0369	0.0186 (51%)	0.0123 (33%)	0.0057 (16%)
France ($\alpha = 0.41$)	0.0358	0.0180 (53%)	0.0033 (10%)	0.0130 (38%)
Germany ($\alpha = 0.39$)	0.0312	0.0177 (56%)	0.0014 (4%)	0.0132 (42%)
Italy ($\alpha = 0.34$)	0.0357	0.0182 (51%)	0.0035 (9%)	0.0153 (42%)
Japan ($\alpha = 0.43$)	0.0566	0.0178 (31%)	0.0125 (22%)	0.0265 (47%)
U.K. ($\alpha = 0.37$)	0.0221	0.0124 (56%)	0.0017 (8%)	0.0080 (36%)
U.S. ($\alpha = 0.39$)	0.0318	0.0117 (37%)	0.0127 (40%)	0.0076 (24%)

Table continued

Growth Accounting Results (continued)

Table 10.1
(Continued)

Country	(1) Growth Rate of GDP	(2) Contribution from Capital	(3) Contribution from Labor	(4) TFP Growth Rate
Panel C: Latin American Countries, 1940–90				
Argentina	0.0279	0.0128	0.0097	0.0054
($\alpha = 0.54$)		(46%)	(35%)	(19%)
Brazil	0.0558	0.0294	0.0150	0.0114
($\alpha = 0.45$)		(53%)	(27%)	(20%)
Chile	0.0362	0.0120	0.0103	0.0138
($\alpha = 0.52$)		(33%)	(28%)	(38%)
Colombia	0.0454	0.0219	0.0152	0.0084
($\alpha = 0.63$)		(48%)	(33%)	(19%)
Mexico	0.0522	0.0259	0.0150	0.0113
($\alpha = 0.69$)		(50%)	(29%)	(22%)
Peru	0.0323	0.0252	0.0134	−0.0062
($\alpha = 0.66$)		(78%)	(41%)	(−19%)
Venezuela	0.0443	0.0254	0.0179	0.0011
($\alpha = 0.55$)		(57%)	(40%)	(2%)
Panel D: East Asian Countries, 1966–90				
Hong Kong[c]	0.073	0.030	0.020	0.023
($\alpha = 0.37$)		(41%)	(28%)	(32%)
Singapore	0.087	0.056	0.029	0.002
($\alpha = 0.49$)		(65%)	(33%)	(2%)
South Korea	0.103	0.041	0.045	0.017
($\alpha = 0.30$)		(40%)	(44%)	(16%)
Taiwan	0.094	0.032	0.036	0.026
($\alpha = 0.26$)		(34%)	(39%)	(28%)

Interpreting the Results

- Reasons for mismeasurement:
  - what matters is not labor hours, but effective labor hours
    - important—though difficult—to make adjustments for changes in the *human capital* of workers.
  - measurement of capital inputs:
    - in the theoretical model, capital corresponds to the final good used as input to produce more goods.
    - in practice, capital is machinery, need assumptions about how relative prices of machinery change over time.
    - typical assumption was to use capital expenditures but if machines become cheaper would severely underestimate g_K

A World of Augmented Solow Economies I

- Mankiw, Romer and Weil (1992) used regression analysis to take the augmented Solow model, with human capital, to data.
- Use the Cobb-Douglas model and envisage a world consisting of $j = 1, ..., N$ countries.
- "Each country is an island": countries do not interact (perhaps except for sharing some common technology growth).
- Country $j = 1, ..., N$ has the aggregate production function:

$$Y_j(t) = K_j(t)^{\alpha} H_j(t)^{\beta} (A_j(t) L_j(t))^{1-\alpha-\beta}.$$

- Nests the basic Solow model without human capital when $\alpha = 0$.
- Countries differ in terms of their saving rates, $s_{k,j}$ and $s_{h,j}$, population growth rates, n_j, and technology growth rates $\dot{A}_j(t) / A_j(t) = g_j$.
- Define $k_j \equiv K_j / A_j L_j$ and $h_j \equiv H_j / A_j L_j$.

A World of Augmented Solow Economies II

- Focus on a world in which each country is in their steady state
- Assuming that human capital also has depreciation, at the rate δ_h, and it is accumulated with the saving rate s_h, steady state values for country j would be (to be derived in recitation):

$$
\begin{aligned}
k_j^* &= \left(\left(\frac{s_{k,j}}{n_j + g_j + \delta_k} \right)^{1-\beta} \left(\frac{s_{h,j}}{n_j + g_j + \delta_h} \right)^{\beta} \right)^{\frac{1}{1-\alpha-\beta}} \\
h_j^* &= \left(\left(\frac{s_{k,j}}{n_j + g_j + \delta_k} \right)^{\alpha} \left(\frac{s_{h,j}}{n_j + g_j + \delta_h} \right)^{1-\alpha} \right)^{\frac{1}{1-\alpha-\beta}} .
\end{aligned}
$$

- Consequently:

$$
\begin{aligned}
y_j^*(t) &\equiv \frac{Y(t)}{L(t)} \qquad (40) \\
&= A_j(t) \left(\frac{s_{k,j}}{n_j + g_j + \delta_k} \right)^{\frac{\alpha}{1-\alpha-\beta}} \left(\frac{s_{h,j}}{n_j + g_j + \delta_h} \right)^{\frac{\beta}{1-\alpha-\beta}} .
\end{aligned}
$$

A World of Augmented Solow Economies II

- Here $y_j^*(t)$ stands for output per capita of country j along the balanced growth path.
- Note if g_j's are not equal across countries, income per capita will diverge.
- Mankiw, Romer and Weil (1992) make the following assumption:

$$A_j(t) = \bar{A}_j \exp(gt).$$

- Countries differ according to technology *level*, (initial level $\bar{A}_j$) but they share the same common technology growth rate, g.

A World of Augmented Solow Economies III

- Using this together with (40) and taking logs, equation for the balanced growth path of income for country $j = 1, ..., N$:

$$\ln y_j^*(t) = \ln \bar{A}_j + gt + \frac{\alpha}{1-\alpha-\beta} \ln \left(\frac{s_{k,j}}{n_j + g + \delta_k} \right) + \frac{\beta}{1-\alpha-\beta} \ln \left(\frac{s_{h,j}}{n_j + g + \delta_h} \right). \tag{41}$$

- Mankiw, Romer and Weil (1992) take:
  - $\delta_k = \delta_h = \delta$ and $\delta + g = 0.05$.
  - $s_{k,j}$=average investment rates (investments/GDP).
  - $s_{h,j}$=fraction of the school-age population that is enrolled in secondary school.

A World of Augmented Solow Economies IV

- Even with all of these assumptions, (41) can still not be estimated consistently.
- $\ln \bar{A}_j$ is unobserved (at least to the econometrician) and thus will be captured by the error term.
- Most reasonable models would suggest $\ln \bar{A}_j$'s should be correlated with investment rates.
- Thus an estimation of (41) would lead to omitted variable bias and inconsistent estimates.
- Implicitly, MRW make another *crucial* assumption, the **orthogonal technology assumption**:

$$\bar{A}_j = \varepsilon_j A, \text{ with } \varepsilon_j \text{ orthogonal to all other variables.}$$

Cross-Country Income Differences: Regressions I

- MRW first estimate equation (41) without the human capital term for the cross-sectional sample of non-oil producing countries

$$\ln y_j^* = \text{constant} + \frac{\alpha}{1-\alpha}\ln\left(s_{k,j}\right) - \frac{\alpha}{1-\alpha}\ln\left(n_j + g + \delta_k\right) + \varepsilon_j .$$

Cross-Country Income Differences: Regressions II

Estimates of the Basic Solow Model			
	MRW	Updated data	
	1985	1985	2000
$\ln(s_k)$	1.42	1.01	1.22
	(.14)	(.11)	(.13)
$\ln(n+g+\delta)$	-1.97	-1.12	-1.31
	(.56)	(.55)	(.36)
Adj R^2	.59	.49	.49
Implied α	.59	.50	.55
No. of observations	98	98	107

Cross-Country Income Differences: Regressions III

- Their estimates for $\alpha / (1-\alpha)$, implies that α must be around $2/3$, but should be around $1/3$.
- The most natural reason for the high implied values of α is that ε_j is correlated with $\ln(s_{k,j})$, either because:
  1. the orthogonal technology assumption is not a good approximation to reality or
  2. there are also human capital differences correlated with $\ln(s_{k,j})$.
- Mankiw, Romer and Weil favor the second interpretation and estimate the augmented model,

$$\begin{aligned} \ln y_j^* &= \text{cst} + \frac{\alpha}{1-\alpha-\beta} \ln(s_{k,j}) - \frac{\alpha}{1-\alpha-\beta} \ln(n_j + g + \delta_k) \\ &\quad + \frac{\beta}{1-\alpha-\beta} \ln(s_{h,j}) - \frac{\beta}{1-\alpha-\beta} \ln(n_j + g + \delta_h) + \varepsilon_j. \end{aligned} \tag{42}$$

Estimates of the Augmented Solow Model

	MRW	Updated data	
	1985	1985	2000
$\ln(s_k)$	.69	.65	.96
	(.13)	(.11)	(.13)
$\ln(n+g+\delta)$	-1.73	-1.02	-1.06
	(.41)	(.45)	(.33)
$\ln(s_h)$	.66	.47	.70
	(.07)	(.07)	(.13)
Adj R^2	.78	.65	.60
Implied α	.30	.31	.36
Implied β	.28	.22	.26
No. of observations	98	98	107

Cross-Country Income Differences: Regressions IV

- If these regression results are reliable, they give a big boost to the augmented Solow model.
  - Adjusted R^2 suggests that three quarters of income per capita differences across countries can be explained by differences in their physical and human capital investment.
- Immediate implication is technology (TFP) differences have a somewhat limited role.
- But this conclusion should not be accepted without further investigation.

Challenges to Regression Analyses I

- **Technology differences across countries are not orthogonal to all other variables.**
- $\bar{A}_j$ is correlated with measures of s_j^h and s_j^k for two reasons.
  1. *omitted variable bias*: societies with high $\bar{A}_j$ will be those that have invested more in technology for various reasons; same reasons likely to induce greater investment in physical and human capital as well.
  2. *reverse causality*: complementarity between technology and physical or human capital imply that countries with high $\bar{A}_j$ will find it more beneficial to increase their stock of human and physical capital.
- In terms of (42), implies that key right-hand side variables are correlated with the error term, ε_j.
- OLS estimates of α and β and R^2 are biased upwards.

Challenges to Regression Analyses II

- **β is too large relative to what we should expect on the basis of microeconometric evidence.**
- The working age population enrolled in school ranges from 0.4% to over 12% in the sample of countries.
- Predicted log difference in incomes between these two countries is

$$\frac{\beta}{1-\alpha-\beta}\left(\ln 12 - \ln(0.4)\right) = 0.66 \times \left(\ln 12 - \ln(0.4)\right) \approx 2.24.$$

- Thus a country with schooling investment of over 12 should be about $\exp(2.24) - 1 \approx 8.5$ times richer than one with investment of around 0.4.

Challenges to Regression Analyses III

- Take Mincer regressions of the form:

$$\ln w_i = \mathbf{X}_i'\gamma + \phi S_i, \tag{43}$$

- Microeconometrics literature suggests that ϕ is between 0.06 and 0.10.
- Can deduce how much richer a country with 12 if we assume:
  1. That the micro-level relationship as captured by (43) applies identically to all countries.
  2. That there are no *human capital externalities*.
- Then: a country with 12 more years of average schooling should have between $\exp(0.10 \times 12) \simeq 3.3$ and $\exp(0.06 \times 12) \simeq 2.05$ times the stock of human capital of a county with fewer years of schooling.

Challenges to Regression Analyses IV

- Thus holding other factors constant, this country should be about 2-3 times as rich as the country with zero years of average schooling.
- Much less than the 8.5 fold difference implied by the Mankiw-Romer-Weil analysis.
- Thus β in MRW is too high relative to the estimates implied by the microeconometric evidence and thus likely upwardly biased.
- Overestimation of β is, in turn, most likely related to correlation between the error term ε_j and the key right-hand side regressors in (42).
- We have so far discussed cross-country "levels" regressions, similar issues apply to "growth regressions" but we have also seen in the first lecture how one might make partial progress here.

Calibrating Productivity Differences I

- The problems with regression analysis with cross-country data have motivated some macroeconomists to turn to "calibration"-type exercises.
- Suppose each country has access to the Cobb-Douglas aggregate production function:

$$Y_j = K_j^{\alpha} \left(A_j H_j\right)^{1-\alpha}, \tag{44}$$

- Each worker in country j has S_j years of schooling.
- Then using the Mincer equation (43) ignoring the other covariates and taking exponents, H_j can be estimated as

$$H_j = \exp\left(\phi S_j\right) L_j,$$

- Does not take into account differences in other "human capital" factors, such as experience.

Calibrating Productivity Differences II

- Let the rate of return to acquiring the Sth year of schooling be $\phi(S)$.
- A better estimate of the stock of human capital can be constructed as

$$H_j = \sum_S \exp\{\phi(S)\,S\}\,L_j(S)$$

- $L_j(S)$ now refers to the total employment of workers with S years of schooling in country j.
- Series for K_j can be constructed from Summers-Heston dataset using investment data and the perpetual inventory method.

$$K_j(t+1) = (1-\delta)\,K_j(t) + I_j(t)\,,$$

- Assume, following Hall and Jones that $\delta = 0.06$.
- With same arguments as before, choose a value of $1/3$ for α.

Calibrating Productivity Differences III

- Given series for H_j and K_j and a value for α, construct "predicted" incomes at a point in time using

$$\hat{Y}_j = K_j^{1/3} \left(A_{US} H_j \right)^{2/3}$$

- A_{US} is computed so that $Y_{US} = K_{US}^{1/3} \left(A_{US} H_{US} \right)^{2/3}$.
- Once a series for $\hat{Y}_j$ has been constructed, it can be compared to the actual output series.
- Gap between the two series represents the contribution of technology.
- Alternatively, could back out country-specific technology terms (relative to the United States) as

$$\frac{A_j}{A_{US}} = \left(\frac{Y_j}{Y_{US}} \right)^{3/2} \left(\frac{K_{US}}{K_j} \right)^{1/2} \left(\frac{H_{US}}{H_j} \right).$$

Calibrating Productivity Differences IV

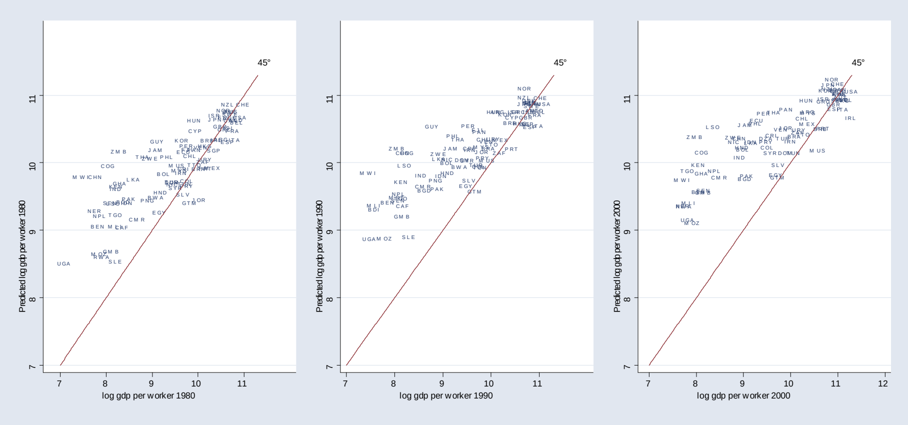


Figure: Calibrated technology levels relative to the US technology (from the Solow growth model with human capital) versus log GDP per worker, 1980, 1990 and 2000.

Calibrating Productivity Differences V

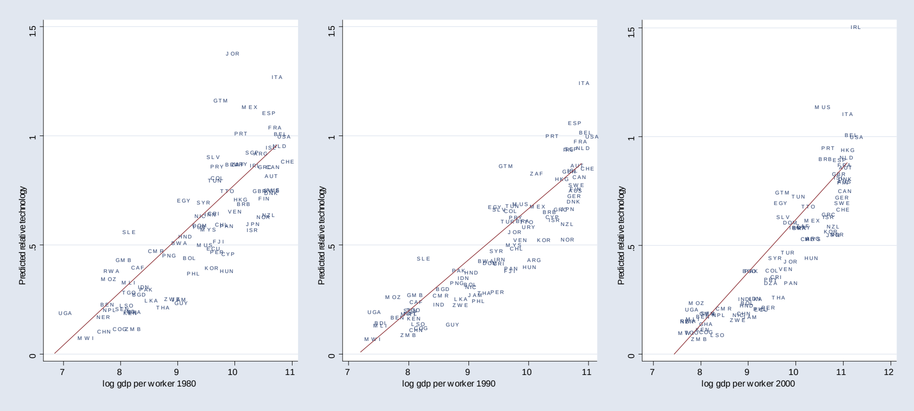


Figure: Calibrated technology levels relative to the US technology (from the Solow growth model with human capital) versus log GDP per worker, 1980, 1990 and 2000.

Calibrating Productivity Differences VI

The following features are noteworthy:

1. Differences in physical and human capital still matter a lot.
2. However, differently from the regression analysis, this exercise also shows significant *technology (productivity) differences*.
3. Same pattern visible in the next three figures for the estimates of the technology differences, A_j / A_{US}, against log GDP per capita in the corresponding year.
4. Also interesting is the pattern that the empirical fit of the neoclassical growth model seems to deteriorate over time.

Challenges to Callibration I

- In addition to the standard assumptions of competitive factor markets, we had to assume :
  - no human capital externalities, a Cobb-Douglas production function, and a range of approximations to measure cross-country differences in the stocks of physical and human capital.
- The calibration approach is in fact a close cousin of the growth-accounting exercise (sometimes referred to as "levels accounting").
- Imagine that the production function that applies to all countries in the world is

$$F\left(K_j, H_j, A_j\right),$$

- Assume countries differ according to their physical and human capital as well as technology—but not according to F.

Challenges to Callibration II

- Rank countries in descending order according to their physical capital to human capital ratios, K_j/H_j Then

$$\hat{x}_{j,j+1} = g_{j,j+1} - \bar{\alpha}_{K,j,j+1} g_{K,j,j+1} - \bar{\alpha}_{Lj,j+1} g_{H,j,j+1}, \tag{45}$$

- where:
  - $g_{j,j+1}$: proportional difference in output between countries j and $j+1$,
  - $g_{K,j,j+1}$: proportional difference in capital stock between these countries and
  - $g_{H,j,j+1}$: proportional difference in human capital stocks.
  - $\bar{\alpha}_{K,j,j+1}$ and $\bar{\alpha}_{Lj,j+1}$: average capital and labor shares between the two countries.
- The estimate $\hat{x}_{j,j+1}$ is then the proportional TFP difference between the two countries.

Challenges to Callibration III

- Levels-accounting faces two challenges.
  1. Data on capital and labor shares across countries are not widely available. Almost all exercises use the Cobb-Douglas approach (i.e., a constant value of α_K equal to $1/3$).
  2. The differences in factor proportions, e.g., differences in K_j / H_j, across countries are large. An equation like (45) is a good approximation when we consider small (infinitesimal) changes.

What Does All This Mean?

- There is also a sense perhaps that these are all "weak tests".
- They impose the structure of the Solow model on the data or exploit the quasi-balanced growth properties.
- These tests do not shed much light on any of the following questions:
  1. To what extent the equilibrium is efficient or "inside the production possibilities frontier"?
  2. Is technology driven by market and other incentives or mostly evolving exogenously?
  3. Is the way that these neoclassical models frame the effects of technology appropriate?
  4. What about recent tectonic shifts?
  5. And what is a proximate cause and what is a fundamental cause?

From Correlates to Fundamental Causes

- In this lecture, the focus has been on proximate causes— importance of human capital, physical capital and technology.
- Let us now return to the list of potential fundamental causes discussed in the first lecture:
  1. luck (or multiple equilibria)
  2. geographic differences
  3. institutional differences
  4. cultural differences
- Do we need to worry about the relationship between these fundamental causes and the correlates of growth? In what way? Where is theory useful?

Conclusions

- Message is somewhat mixed.
    - On the positive side, despite its simplicity, the Solow model has enough substance that we can take it to data in various different forms, including TFP accounting, regression analysis and calibration.
    - On the negative side, however, no single approach is entirely convincing.
- Complete agreement is not possible, but safe to say that consensus favors the interpretation that cross-country differences in income per capita cannot be understood solely on the basis of differences in physical and human capital
- Differences in TFP are not necessarily due to technology in the narrow sense.
- It is also useful and important to think about *fundamental causes*, what lies behind the factors taken as given either Solow model.