Informationally Robust Implementation

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This Lecture

Some informationally robust analogues of first two lectures:

1. Full Implementation in General Environments
2. Allocation of a Single Good with Interdependent Values
Complete Information Environment

- society \( \{1, \ldots, n\} \)
- set of outcomes \( A \)
- states of the world \( \Theta \)
- preferences \( u_i (a, \theta) \)
- state \( \theta \) is common knowledge among players
Problem 1: Complete Information Implementation

- designer does not know $\theta$
- chooses mechanism:
  - player $i$ strategies $M_i$
  - game form $g : M_1 \times \ldots \times M_n \to A$
- wants to implement function $f : \Theta \to A$ in Nash equilibrium, i.e., find a mechanism such that, for each $\theta$, all Nash equilibria $(m_1, \ldots, m_n)$ satisfy

$$g(m_1, \ldots, m_n) = f(\theta)$$
**DEFINITION.** Social choice function $f$ is **Maskin monotonic** if $f(\theta') = f(\theta)$ whenever, for each $i$ and $y$

\[
u_i(f(\theta), \theta) \geq \nu_i(y, \theta) \Rightarrow \nu_i(f(\theta), \theta') \geq \nu_i(y, \theta')\]

**THEOREM.** Maskin monotonicity is necessary and "almost" sufficient for $f$ to be implementable in Nash equilibrium.
Checklist for Complete Information

1. Interpretable necessary condition
2. Proof of (almost) sufficiency using exotic mechanisms
3. Applications addressing multiple equilibria using simpler mechanisms
Incomplete Information

- society \( \{1, \ldots, n\} \)
- set of outcomes \( A \)
- player \( i \) types \( \Theta_i \)
- preferences \( u_i (a, (\theta_i, \theta_{-i})) \)
Incomplete Information

- designer still does not know $\theta = (\theta_1, \ldots, \theta_n)$
- the mechanism is still
  - player $i$ messages $M_i$
  - game form $g : M_1 \times \ldots \times M_n \rightarrow A$
Problem 2: Bayesian Implementation

- there is a common prior $\pi^* \in \Delta(\Theta)$
- strategy for player $i$ is now a mapping $s_i : \Theta_i \rightarrow M_i$
- a strategy profile $(s_1,\ldots,s_n)$ is a *Bayes Nash equilibrium* if

$$\sum_{\theta} \pi^* (\theta) u_i \left( g \left( s_i (\theta_i), s_{-i} (\theta_{-i}) \right), (\theta_i, \theta_{-i}) \right) \geq \sum_{\theta} \pi^* (\theta) u_i \left( g \left( s'_i (\theta_i), s_{-i} (\theta_{-i}) \right), (\theta_i, \theta_{-i}) \right)$$

for all $i$ and $s'_i : \Theta_i \rightarrow M_i$

- designer wants to implement social choice function $f : \Theta_1 \times \ldots \times \Theta_n \rightarrow A$ in Bayes Nash equilibrium: all Bayes Nash equilibria $(s_1,\ldots,s_n)$ satisfy

$$g \left( s_1 (\theta_1), \ldots, s_n (\theta_n) \right) = f \left( \theta_1, \ldots, \theta_n \right)$$
Bayesian Incentive Compatibility

**Definition**

Social choice function $f$ satisfies Bayesian incentive compatibility if

$$\sum_{\theta_i} \pi^* (\theta_{-i} | \theta_i) u_i(f(\theta_i, \theta_{-i}), (\theta_i, \theta_{-i}))$$

$$\geq \sum_{\theta_{-i}} \pi^* (\theta_{-i} | \theta_i) u_i(f(\theta'_i, \theta_{-i}), (\theta_i, \theta_{-i}))$$

for all $i$, $\theta_i$ and $\theta'_i$.

By the revelation principle, Bayesian incentive compatibility is already necessary for partial implementation.

**Definition**

Social choice function satisfies Bayesian monotonicity if......
Characterization

**Theorem**

*Bayesian incentive compatibility and Bayesian monotonicity are necessary and "almost" sufficient for \( f \) to be implementable in Bayes Nash equilibrium.*
Checklist for Bayesian Implementation

1. Interpretable necessary condition  NOT REALLY
2. Proof of (almost) sufficiency using exotic mechanisms  YES
3. Applications addressing multiple equilibria using simpler mechanisms  only exchange economies?
Robust Implementation

- as before but replace common prior $\pi^* \in \Delta (\Theta_1 \times ... \times \Theta_n)$
- instead assume type space:
  - player $i$ types $T_i$
  - mappings
    1. preferences $\hat{\theta}_i : T_i \rightarrow \Theta_i$
    2. beliefs $\hat{\pi}_i : T_i \rightarrow \Delta(T_{-i})$
- require Bayesian implementation on all type spaces ($\approx$ universal type space)
- yesterday’s lecture: equivalent to (belief-free) rationalizable implementation on all payoff type spaces
Rationalizable Implementation

- Iterated Deletion Solution Concept:
  - initialize $R^0_i(\theta_i) = M_i$
  - for $k \geq 0$, inductive define $R^{k+1}_i(\theta_i)$ to be the set of messages such that there exists $\nu_i \in \Delta(M_{-i} \times \Theta_{-i})$ such that
    
    $1) \nu_i(m_{-i}, \theta_{-i}) > 0 \Rightarrow m_j \in R^k_j(\theta_j)$ for all $j$
    
    $2) m_i \in \arg\max \sum_{m'_i} \nu_i(m_{-i}, \theta_{-i}) u_i(g(m'_i, m_{-i}), (\theta_i, \theta_{-i}))$

  - $R^\infty_i(\theta_i) = \bigcap_{k \geq 1} R^k_i(\theta_i)$

- PROPOSITION: social choice function $f$ is robustly implemented if $m \in R^\infty(\theta) \Rightarrow g(m) = f(\theta)$
Theorem

Robust monotonicity is necessary and "almost" sufficient for \( f \) to be robustly implementable.
Maskin Monotonicity and Whistle-Blowing

Social choice function $f$ is **Maskin monotonic** if $f (\theta') = f (\theta)$ whenever, for each $i$ and $y$,

$$u_i (f (\theta), \theta) \geq u_i (y, \theta) \Rightarrow u_i (f (\theta), \theta') \geq u_i (y, \theta')$$

- Equivalent definition: Social choice function $f$ is Maskin monotonic if whenever $f (\theta') \neq f (\theta)$, there exist $i$ and $y$ such that $u_i (y, \theta') > u_i (f (\theta), \theta')$ and $u_i (f (\theta), \theta) \geq u_i (y, \theta)$
- Breaking it down:
  - a deception is a mis-report $\theta \rightarrow \theta'$
  - a deception is acceptable if $f (\theta') = f (\theta)$
  - the reward set for player $i$ at $\theta$ is the set of outcomes $y$ that do not break the good equilibrium, i.e.,
    $$\{y | u_i (f (\theta), \theta) \geq u_i (y, \theta)\}$$
  - a deception $\theta \rightarrow \theta'$ is refutable if there exists a whistle-blower $i$ and an outcome $y$ in the state $\theta$ reward set, such that $u_i (y, \theta') > u_i (f (\theta), \theta')$
  - a social choice function satisfies Maskin monotonicity if every unacceptable deception is refutable
Robust Monotonicity Definition

- A deception is a profile $\beta = (\beta_i)_{i=1}^n$ with each $\beta_i : \Theta_i \to 2^{\Theta_i} / \emptyset_i$
- A deception is acceptable if $\theta' \in \beta(\theta) \Rightarrow f(\theta') = f(\theta)$
- The reward set for player $i$ at type profile $\theta'_{-i}$ is

$$Y_i(\theta'_{-i}) = \left\{ y \mid \text{for all } \theta''_i, \text{ either } f(\theta''_i, \theta'_{-i}) = y \text{ or } u_i(f(\theta''_i, \theta'_{-i}), (\theta''_i, \theta'_{-i})) > u_i(y, (\theta''_i, \theta'_{-i})) \right\}$$
Robust Monotonicity Definition

- A deception is *strictly refutable* if there exists a whistle blower $i$ and $\theta'_i \in \beta_i(\theta_i)$ such that for all $\theta'_{-i}$ and $\psi_i \in \Delta(\beta^{-1}_{-i}(\theta'_{-i}))$ there exists $y \in Y_i(\theta'_{-i})$ such that

  $$\sum_{\theta_{-i} \in \beta^{-1}_{-i}(\theta'_{-i})} \psi_i(\theta_{-i}) u_i(y, (\theta_i, \theta_{-i})) > \sum_{\theta_{-i} \in \beta^{-1}_{-i}(\theta'_{-i})} \psi_i(\theta_{-i}) u_i(f(\theta'_i, \theta'_{-i}), (\theta_i, \theta_{-i}))$$

- Social choice function satisfies *strict robust monotonicity* if every unacceptable deviation is strictly refutable.
Sketch of Proof

- Suppose that \( f \) is robustly implementable and fix a mechanism that robustly implements it.
- Suppose that deception \( \beta \) is not strictly refutable.
- Let
  \[
  S_i^\beta (\theta_i) = \bigcup_{\theta'_i \in \beta_i(\theta_i)} R_i^\infty (\theta'_i)
  \]
- Because \( \beta \) is not strictly refutable \( S_i^\beta (\theta_i) \subseteq R_i^\infty (\theta_i) \) for all \( i \) and \( \theta_i \).
- This implies that \( \beta \) is acceptable
Ex Post Incentive Compatibility

- social choice function $f$ satisfies strict ex post incentive compatibility (EPIC) if

$$u_i(f(\theta_i, \theta_{-i}), (\theta_i, \theta_{-i})) \geq u_i(f(\theta'_i, \theta_{-i}), (\theta_i, \theta_{-i}))$$

for all $i, \theta_{-i}, \theta_i$ and $\theta'_i$; with strictly inequality unless $f(\theta_i, \theta'_{-i}) = f(\theta'_i, \theta'_{-i})$ for all $\theta'_{-i}$.

- Robust monotonicity implies strict EPIC

- In private values environments, strict EPIC reduces to strict dominant strategies incentive compatibility

- To the extent robust monotonicity goes beyond strict EPIC, there must be restrictions on interdependence
Checklist for Robust Implementation

1. Interpretable necessary condition
   - better than Bayesian monotonicity?!  
   - $= \text{"NOT TOO MUCH INTERDEPENDENCE"}$

2. Proof of (almost) sufficiency using exotic mechanisms  YES

3. Applications addressing multiple equilibria using simpler mechanisms  YES, ALLOCATING A SINGLE GOOD WITH INTERDEPENDENT VALUES
Second Price Auction (Private Values)

- Allocating a good with private values $\theta_i \in [0, 1]$
- Only weakly dominated strategies: a problem for full implementation
- But can strengthen to strict second price auction to get full implementation:
  - second price auction with probability $1 - \varepsilon$,
    - allocate object to highest bidder
    - winner pays second highest bid
  - strict screening mechanism with probability $\varepsilon$: for each $i$, with probability $\frac{1}{n} b_i$
    - $i$ gets object
    - pays $\frac{1}{2} b_i$
- truth-telling is a strictly dominant strategy
- thus this ($\varepsilon$-efficient) allocation is robustly implemented
Adding Interdependence of Values

- Agent $i$ has payoff types $\theta_i \in [0, 1]$
- Agent $i$’s value is

$$v_i(\theta) = \theta_i + \gamma \sum_{j \neq i} \theta_j$$

- Assume that $\gamma < 1$: single crossing condition from lecture 2
Extended Second Price Auction

- strict extended second price auction:
  - extended second price auction with probability $1 - \varepsilon$,
    - allocate object to highest bidder $i$
    - winner pays $\max_{j \neq i} b_j + \gamma \sum_{j \neq i} b_j$
  - strict screening mechanism with probability $\varepsilon$: for each $i$ with probability $\frac{1}{n} b_i$,
    - $i$ gets object
    - pays $\frac{1}{2} b_i + \gamma \sum_{j \neq i} b_j$
- in extended second price auction, winner’s payment is independent of his bid and is his willingness to pay at the lowest bid at which he wins
Good News and Bad News

- truth-telling is a strict ex post equilibrium of this "direct" mechanism
- but existence of strict ex post equilibrium does not imply robust implementation
The direct mechanism robustly implements the (almost) efficient outcome if $|\gamma| < \frac{1}{n-1}$...

...because the direct mechanism has linear best response functions

$$a_i = \theta_i - \gamma \sum_{j \neq i} (a_j - \theta_j).$$

and we proved last lecture that there is a unique "truth-telling" rationalizable outcome in this game.
Failure of Full Implementation

- No mechanism implements the (almost) efficient outcome if $|\gamma| \geq \frac{1}{n-1} \ldots$
- We showed last lecture that every action was rationalizable in this game
- But no other mechanism would work either; two ways to see this:
  1. We can directly verify failure of robust monotonicity
  2. Also ad hoc argument (assuming $\gamma \geq 0$)
     - Consider a type space where whenever a player has payoff type $\theta_i$, he has dogmatic belief that each other player has payoff type
       $$\frac{1}{2} + \frac{1}{\gamma(n-1)} \left(\frac{1}{2} - \theta_i\right).$$
     - We know this is in the interval $[0, 1]$ only because $\gamma \geq \frac{1}{n-1}$
     - His valuation is
       $$\theta_i + \gamma(n-1) \left(\frac{1}{2} + \frac{1}{\gamma(n-1)} \left(\frac{1}{2} - \theta_i\right)\right) = \frac{1}{2} (1 + \gamma(n-1))$$
       independent of his type
     - always a pooling equilibrium
Consider an environment where players’ preferences depend on a statistic of players’ payoff types and are single crossing with respect to that statistic...

covers allocating a good with interdependent properties in lecture 2.

then robust implementation is possible if and only if it is possible in the payoff type direct mechanism

robust implementation is possible if and only if there is not too much interdependence
The Common Prior Assumption and Positive and Negative Interdependence

- We allowed for negative interdependence $\gamma \leq 0$. In this case, there are strategic complementarities and imposing the common prior does not make robust implementation easier.
- But if $\gamma \geq 0$....
  - there are strategic substitutes
  - arguments from last lecture establish robust implementation easier under the common prior
  - but arguments special