6.207/14.15: Networks
Lecture 9: Introduction to Game Theory–1

Daron Acemoglu and Asu Ozdaglar
MIT

October 13, 2009

Outline

- Decisions, Utility Maximization
- Games and Strategies
- Best Responses and Dominant Strategies
- Nash Equilibrium
- Applications
- Next Lecture: Mixed Strategies, Existence of Nash Equilibria, and Dynamic Games.

- Reading:
- Osborne, Chapters 1-2.
- EK, Chapter 6.

Motivation

- In the context of social networks, or even communication networks, agents make a variety of choices.
- For example:
  - What kind of information to share with others you are connected to.
  - How to evaluate information obtained from friends, neighbors, coworkers and media.
  - Whether to trust and form friendships.
  - Which of the sellers in your neighborhood to use.
  - Which websites to visit.
  - How to map your drive in the morning (or equivalently how to route your network traffic).
- In all of these cases, *interactions* with other agents you are connected to affect your payoff, well-being, utility.
- How to make decisions in such situations?
- $\rightarrow$ "multiagent decision theory" or game theory.

"Rational Decision-Making"

- Powerful working hypothesis in economics: individuals act *rationally* in the sense of choosing the option that gives them higher "payoff".
  - Payoff here need not be monetary payoff. Social and psychological factors influence payoffs and decisions.
  - Nevertheless, the rational decision-making paradigm is useful because it provides us with a (testable) theory of economic and social decisions.
- We often need only **ordinal** information; i.e., two options a and b, and we imagine a (real-valued) **utility function** $u(\cdot)$ that represents the ranking of different options, and we simply check whether $u(a) \geq u(b)$ or $u(a) \leq u(b)$.
  - In these cases, if a utility function $u(\cdot)$ represents preferences, so does any strictly monotonic transformation of $u(\cdot)$.
- But in game theory we often need **cardinal** information because decisions are made under natural or strategic uncertainty. The theory of decision-making under uncertainty was originally developed by John von Neumann and Oskar Morgenstern.

Decision-Making under Uncertainty

- von Neumann and Morgenstern posited a number of "reasonable" axioms that rational decision-making under uncertainty should satisfy. From these, they derived the expected utility theory.
- Under uncertainty, every choice induces *a lottery*, that is, a probability distribution over different outcomes.
  - E.g., one choice would be whether to accept a gamble which pays \$10 with probability 1/2 and makes you lose \$10 with probability 1/2.
- von Neumann and Morgenstern's expected utility theory shows that (under their axioms) there exists a utility function (also referred to as Bernoulli utility function) $u(c)$, which gives the utility of consequence (outcome) c.
- Then imagine that choice a induces a probability distribution $F^a(c)$ over consequences.

Decision-Making under Uncertainty (continued)

- Then the utility of this choice is given by **the expected utility** according to the probability distribution $F^a(c)$:

$$U(a) = \int u(c)\, dF^a(c).$$

  In other words, this is the expectation of the utility $u(c)$, evaluated according to the probability distribution $F^a(c)$.

- More simply, if $F^a(c)$ is a continuous distribution with density $f^a(c)$, then

$$U(a) = \int u(c)\, f^a(c)\, dc,$$

  or if it is a discrete distribution where outcome outcome c_i has probability p_i^a (naturally with $\sum_i p_i^a = 1$), then

$$U(a) = \sum_i p_i^a u(c_i).$$

Decision-Making under Uncertainty (continued)

- Given expected utility theory and our postulate of "rationality," single person decision problems are (at least conceptually) simple.
- If there are two actions, a and b, inducing probability distributions $F^a(c)$ and $F^b(c)$, then the individual should choose a over b only if

$$U(a) = \int u(c)\, dF^a(c) \geq U(b) = \int u(c)\, dF^b(c).$$

From Single Person to Multiperson Decision Problems

- But in a social situation (or more specifically, in a "social network" situation), the utility of an agent or probability distribution over outcomes depends on actions of others.
- A simple game of "partnership" represented as a **matrix game**:

Player 1 \ Player 2	work hard	shirk
work hard	$(2, 2)$	$(-1, 1)$
shirk	$(1, -1)$	$(0, 0)$

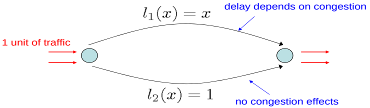
- Here the first number is the payoff to player (partner) 1 and the second number is the payoff to player 2. More formally, the cell indexed by row x and column y contains a pair, (a, b) where $a = u_1(x, y)$ and $b = u_2(x, y)$.
- These numbers could be just monetary payoffs, or it could be inclusive of "social preferences" (the fact that you may be altruistic towards your partner or angry at him or her).
- Should you play "work hard" or "shirk"?

A Paradoxical Network Example

- Consider the following "non-atomic" game of traffic routing.
    - Non-atomic here refers to the fact that there is a "continuum" of players, so the effect of any given individual on "aggregates" is negligible.
- Each route has a cost (delay/latency) function $l_i(x_i)$ measuring costs of delay and congestion on link i as a function of the flow traffic x_i on this link. Suppose that motorists wish to minimize delay.
- Traditional Network Optimization Approach: choose the allocation of traffic so as to achieve some well-defined objective, such as minimizing aggregate delay.
- In practice, routing is "selfish," each motorist choosing the route that has the lowest delay.
- This problem was first studied by the famous economist Alfred Pigou.

A Paradoxical Network Example (continued)

- Consider the following example with a unit mass of traffic to be routed:



- System optimum (minimizing aggregate delay) can be found by solving
$$\min_{x_1+x_2\leq 1} C_{\text{system}}(x^S) = \sum_i l_i(x_i^S)x_i^S.$$

- First-order condition:

$$\begin{aligned} l_1(x_1) + x_1 l_1'(x_1) &= l_2(1-x_1) + (1-x_1)\, l_2'(1-x_1) \\ 2x_1 &= 1 \end{aligned}$$

- Hence, system optimum is to split traffic equally between two routes, giving:
$$\min_{x_1+x_2\leq 1} C_{\text{system}}(x^S) = \sum_i l_i(x_i^S)x_i^S = \frac{1}{4} + \frac{1}{2} = \frac{3}{4}.$$

A Paradoxical Network Example (continued)

- Suppose instead that there is selfish routing so that each motorist chooses the path with the lowest delay **taking aggregate traffic pattern as given** (is this reasonable?).
- This will give $x_1 = 1$ and $x_2 = 0$ (since for any $x_1 < 1$, $l_1(x_1) < 1 = l_2(1 - x_1)$). Aggregate delay is

$$C_{\text{eq}}(x^{WE}) = \sum_i l_i(x_i^{WE})x_i^{WE} = 1 + 0 = 1 > \frac{3}{4}.$$

- The outcome is "socially suboptimal"—a very common occurrence in game theory situations.
- Inefficiency sometimes quantified by the measure of **Price of anarchy**, defined as

$$\frac{C_{\text{system}}(x^S)}{C_{\text{eq}}(x^{WE})} = \frac{3}{4}.$$

Strategic Form Games

- Let us start with games in which all of the participants act simultaneously and without knowledge of other players' actions. Such games are referred to as strategic form games—or as normal form games or matrix games.
- For each game, we have to define
  1. The set of players.
  2. The strategies.
  3. The payoffs.
- More generally, we also have to define the *game form*, which captures the order of play (e.g., in chess) and information sets (e.g., in asymmetric information or incomplete information situations). But in strategic form games, play is simultaneous, so no need for this additional information.

Strategic Form Games (continued)

- More formally:

Definition

(Strategic Form Game) A strategic forms game is a triplet $\langle \mathcal{I}, (S_i)_{i\in\mathcal{I}}, (u_i)_{i\in\mathcal{I}} \rangle$ *such that*
$\mathcal{I}$ *is a finite set of players, i.e.,* $\mathcal{I} = \{1, \ldots, I\}$*;*
S_i *is the set of available actions for player* i*;*
$s_i \in S_i$ *is an action for player* i*;*
$u_i : S \to R$ *is the payoff (utility) function of player* i *where* $S = \prod_i S_i$ *is the set of all action profiles.*

- In addition, we use the notation

$s_{-i} = [s_j]_{j\neq i}$: vector of actions for all players except i.
$S_{-i} = \prod_{j\neq i} S_j$ is the set of all action profiles for all players except i
$(s_i, s_{-i}) \in S$ is a **strategy profile**, or outcome.

Strategies

- In game theory, a strategy is a complete description of how to play again.
- It requires full contingent planning. If instead of playing the game yourself, you had to delegate the play to a "computer" with no initiative, then you would have to spell out a full description of how the game would be played in every contingency.
- For example, in chess, this would be an impossible task (though in some simpler games, it can be done).
- Thinking in terms of strategies is important.
- But in strategic form games, there is no difference between an action and a pure strategy, and we will use them interchangeably.
- This will no longer be the case even for strategic form games when we turn to mixed strategies.

Finite Strategy Spaces

- When the strategy space is finite, and the number of players and actions is small, a game can be represented in matrix form.
- Recall that the cell indexed by row x and column y contains a pair, (a, b) where $a = u_1(x, y)$ and $b = u_2(x, y)$.

Example: Matching Pennies.

Player 1 \ Player 2	heads	tails
heads	$(-1, 1)$	$(1, -1)$
tails	$(1, -1)$	$(-1, 1)$

- This game represents pure conflict in the sense that one player's utility is the negative of the utility of the other player. Thus **zero sum game.**
  - More generally true for *strictly competitive games*, that is, games in which whenever one player wins the other one loses, though the sum of the payoffs need not be equal to 0.

Infinite Strategy Spaces

- Example: Cournot competition.
  - Two firms producing a homogeneous good for the same market
  - The action of a player i is a quantity, $s_i \in [0, \infty]$ (amount of good he produces).
  - The utility for each player is its total revenue minus its total cost,

$$u_i(s_1, s_2) = s_i p(s_1 + s_2) - cs_i$$

    where $p(q)$ is the price of the good (as a function of the total amount), and c is unit cost (same for both firms).
  - Assume for simplicity that $c = 1$ and $p(q) = \max\{0, 2 - q\}$
- Consider the best response correspondence for each of the firms, i.e., for each i, the mapping $B_i(s_{-i}) : S_{-i} \rightrightarrows S_i$ such that

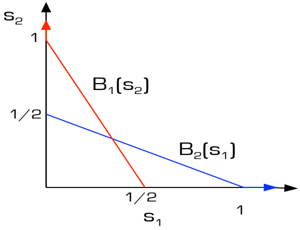
$$B_i(s_{-i}) \in \arg\max_{s_i \in S_i} u_i(s_i, s_{-i}).$$

- Why is this a "correspondence" not a function? When will it be a function?

Cournot Competition (continued)

- By using the first order optimality conditions, we have

$$B_i(s_{-i}) = \arg\max_{s_i \geq 0}(s_i(2 - s_i - s_{-i}) - s_i)$$
$$= \begin{cases} \frac{1-s_{-i}}{2} & \text{if } s_{-i} \leq 1, \\ 0 & \text{otherwise.} \end{cases}$$



- The figure illustrates the best response correspondences (which in this case are functions).
- Assuming that players are **rational and fully knowledgeable about the structure of the game and each other's rationality**, what should the outcome of the game be?

Dominant Strategies

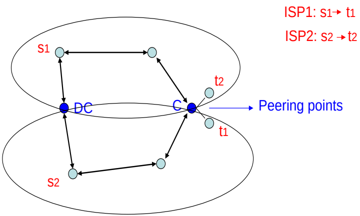
- Example: Prisoner's Dilemma.
  - Two people arrested for a crime, placed in separate rooms, and the authorities are trying to extract a confession.

prisoner 1 / prisoner 2	Confess	Don't confess
Confess	$(-2, -2)$	$(0, -3)$
Don't confess	$(-3, 0)$	$(0, 0)$

- What will the outcome of this game be?
- Regardless of what the other player does, playing "Confess" is better for each player.
- The action "Confess" **strictly dominates** the action "Don't confess"
- Prisoner's dilemma paradigmatic example of a self-interested, rational behavior not leading to jointly *(socially) optimal result*.

Prisoner's Dilemma and ISP Routing Game

- Consider two Internet service providers that need to send traffic to each other
- Assume that the unit cost along a link (edge) is 1



- This situation can be modeled by the "Prisoner's Dilemma" payoff matrix.

ISP 1 / ISP 2	Hot potato	Cooperate
Hot potato	$(-4, -4)$	$(-1, -5)$
Cooperate	$(-5, -1)$	$(-2, -2)$

Dominant Strategy Equilibrium

- Compelling notion of equilibrium in games would be dominant strategy equilibrium, where each player plays a dominant strategy.

Definition

(Dominant Strategy) *A strategy $s_i \in S_i$ is dominant for player i if*

$$u_i(s_i', s_{-i}) \geq u_i(s_i, s_{-i}) \qquad \text{for all } s_i' \in S_i \text{ and for all } s_{-i} \in S_{-i}.$$

Definition

(Dominant Strategy Equilibrium) *A strategy profile s^* is the dominant strategy equilibrium if for each player i, s_i^* is a dominant strategy.*

- These notions could be defined for strictly dominant strategies as well.

Dominant and Dominated Strategies

- Though compelling, dominant strategy equilibria do not always exist, for example, as illustrated by the partnership or the matching pennies games we have seen above.
- Nevertheless, in the prisoner's dilemma game, "confess, confess" is a dominant strategy equilibrium.
- We can also introduce the converse of the notion of dominant strategy, which will be useful next.

Definition

(Strictly Dominated Strategy) *A strategy $s_i \in S_i$ is strictly dominated for player i if there exists some $s_i' \in S_i$ such that*

$$u_i(s_i', s_{-i}) > u_i(s_i, s_{-i}) \qquad \text{for all } s_{-i} \in S_{-i}.$$

Dominated Strategies

Definition

(Weakly Dominated Strategy) *A strategy $s_i \in S_i$ is weakly dominated for player i if there exists some $s_i' \in S_i$ such that*

$$u_i(s_i', s_{-i}) \geq u_i(s_i, s_{-i}) \qquad \text{for all } s_{-i} \in S_{-i},$$

$$u_i(s_i', s_{-i}) > u_i(s_i, s_{-i}) \qquad \text{for some } s_{-i} \in S_{-i}.$$

- No player should play a strictly dominated strategy
- **Common knowledge** of payoffs and rationality results in **iterated elimination of strictly dominated strategies**

Iterated Elimination of Strictly Dominated Strategies

- Example: Iterated Elimination of Strictly Dominated Strategies.

prisoner 1 / prisoner 2	Confess	Don't confess	Suicide
Confess	$(-2, -2)$	$(0, -3)$	$(-2, -10)$
Don't confess	$(-3, 0)$	$(0, 0)$	$(0, -10)$
Suicide	$(-10, -2)$	$(-10, 0)$	$(-10, -10)$

- No dominant strategy equilibrium; because of the additional "suicide" strategy, which is a strictly dominated strategies for both players.
- No "rational" player would choose "suicide". Thus if prisoner 1 is certain that prisoner 2 is rational, then he can eliminate the latter's "suicide" strategy, and likewise for prisoner 2. Thus after one round of elimination of strictly dominated strategies, we are back to the prisoner's dilemma game, which has a *dominant strategy equilibrium*.
- Thus iterated elimination of strictly dominated strategies leads to a unique outcome, "confess, confess"—thus the game is **dominance solvable**.

Iterated Elimination of Strictly Dominated Strategies (continued)

More formally, we can follow the following iterative procedure:

- **Step 0:** Define, for each i, $S_i^0 = S_i$.
- **Step 1:** Define, for each i,

$$S_i^1 = \left\{ s_i \in S_i^0 \mid \nexists s_i' \in S_i^0 \text{ s.t. } u_i\left(s_i', s_{-i}\right) > u_i\left(s_i, s_{-i}\right) \ \forall\ s_{-i} \in S_{-i}^0 \right\}.$$

...

- **Step k:** Define, for each i,

$$S_i^k = \left\{ s_i \in S_i^{k-1} \mid \nexists s_i' \in S_i^{k-1} \text{ s.t. } u_i\left(s_i', s_{-i}\right) > u_i\left(s_i, s_{-i}\right) \ \forall\ s_{-i} \in S_{-i}^{k-1} \right\}.$$

- **Step ∞:** Define, for each i,

$$S_i^\infty = \cap_{k=0}^{\infty} S_i^k.$$

Iterated Elimination of Strictly Dominated Strategies (continued)

Theorem

Suppose that either (1) each S_i is finite, or (2) each $u_i(s_i, s_{-i})$ is continuous and each S_i is compact. Then S_i^∞ (for each i) is nonempty.

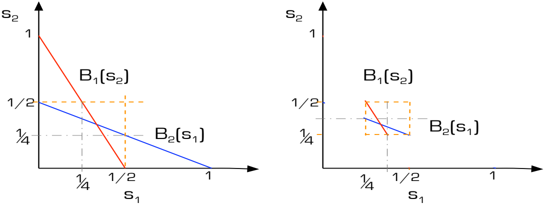
- Proof for part (1) is trivial.
- Proof for part (2) in homework.
- But note that S_i^∞ need not be a singleton.

How Reasonable is Dominance Solvability

- At some level, it seems very compelling. But consider the following game, often called the k- beauty game.
- Each of you will pick an integer between 0 and 100.
- The person who was closest to k times the average of the group will win a prize.
- How will you play this game? And why?

Revisiting Cournot Competition

- Apply iterated strict dominance to Cournot model to predict the outcome



- One round of elimination yields $S_1^1 = [0, 1/2]$, $S_2^1 = [0, 1/2]$
- Second round of elimination yields $S_1^1 = [1/4, 1/2]$, $S_2^1 = [1/4, 1/2]$
- It can be shown that the endpoints of the intervals converge to the intersection
- Most games not solvable by iterated strict dominance, need a stronger **equilibrium notion**

Pure Strategy Nash Equilibrium

Definition

(Nash equilibrium) *A (pure strategy) Nash Equilibrium of a strategic game* $\langle \mathcal{I}, (S_i)_{i \in \mathcal{I}}, (u_i)_{i \in \mathcal{I}} \rangle$ *is a strategy profile* $s^* \in S$ *such that for all* $i \in \mathcal{I}$

$$u_i(s_i^*, s_{-i}^*) \geq u_i(s_i, s_{-i}^*) \qquad \text{for all } s_i \in S_i.$$

- Why is this a "reasonable" notion?
- No player can profitably deviate given the strategies of the other players. Thus in Nash equilibrium, "best response correspondences intersect".
- Put differently, the conjectures of the players are *consistent*: each player i chooses s_i^* expecting all other players to choose s_{-i}^*, and each player's conjecture is verified in a Nash equilibrium.

Reasoning about Nash Equilibrium

- This has a "steady state" type flavor. In fact, two ways of justifying Nash equilibrium rely on this flavor:
    1. Introspection: what I do must be consistent with what you will do given your beliefs about me, which should be consistent with my beliefs about you,...
    2. Steady state of a learning or evolutionary process.
- An alternative justification: Nash equilibrium is *self-reinforcing*
    - If player 1 is told about player 2's strategy, in a Nash equilibrium she would have no incentive to change her strategy.

Role of Conjectures

- To illustrate the role of conjectures, let us revisit matching pennies

Player 1 \ Player 2	heads	tails
heads	$(-1, 1)$	$(1, -1)$
tails	$(1, -1)$	$(-1, 1)$

- Here, player 1 can play heads expecting player 2 to play tails. Player 2 can play tails expecting player 1 to play tails.
- But these conjectures are not consistent with each other.

Intersection of Best Responses

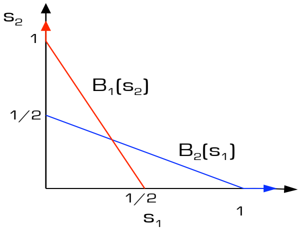
- Recall the best-response correspondence $B_i(s_{-i})$ of player i,

$$B_i(s_{-i}) \in \arg\max_{s_i \in S_i} u_i(s_i, s_{-i}).$$

- **Equivalent characterization:** an action profile s^* is a Nash equilibrium if and only if

$$s_i^* \in B_i(s_{-i}^*) \qquad \text{for all } i \in \mathcal{I}.$$

- Therefore, in Cournot as formulated above, unique Nash equilibrium.



Example: The Partnership Game

- Let us return to the partnership game we started with.

Player 1 \ Player 2	work hard	shirk
work hard	$(2, 2)$	$(-1, 1)$
shirk	$(1, -1)$	$(0, 0)$

- There are no dominant or dominated strategies.
- Work hard is a best response to work hard and shirk is a best response shirk for each player.
- Therefore, there are two pure strategy Nash equilibria (work hard, work hard) and (shirk, shirk).
- Depending on your conjectures ("expectations") about your partner, you can end up in a good or bad outcome.

Focal Points

- What do we do when there are multiple Nash equilibria?
  - Our models would not be making a unique prediction.
- Two different lines of attack:
  - Think of set valued predictions—i.e., certain outcomes are possible, and Nash equilibrium rules out a lot of other outcomes.
  - Think of **equilibrium selection**.
- Equilibrium selection is hard.
- Most important idea, Schelling's focal point.
- Some equilibria are more natural and will be expected.
- Schelling's example: ask the people to meet in New York, without specifying the place. Most people will go to Grand Central. Meeting at Grand Central, as opposed to meeting at any one of thousands of similar places, is a "focal point".

Examples: Battle of the Sexes and Matching Pennies

- Example: Battle of the Sexes (players wish to coordinate but have conflicting interests)

Player 1 \ Player 2	ballet	football
ballet	$(1, 4)$	$(0, 0)$
football	$(0, 0)$	$(4, 1)$

- Two Nash equilibria, (Ballet, Ballet) and (Soccer, Soccer).
- Example: Matching Pennies.

Player 1 \ Player 2	heads	tails
heads	$(-1, 1)$	$(1, -1)$
tails	$(1, -1)$	$(-1, 1)$

- No pure Nash equilibrium (but we will see in the next lecture that there exists a unique mixed strategy equilibrium).

Examples: Cournot Competition

- We now provide an explicit characterization of the Nash equilibrium of Cournot for a specific demand function.
- Suppose that both firms have marginal cost c and the inverse demand function is given by $P(Q) = \alpha - \beta Q$, where $Q = q_1 + q_2$, where $\alpha > c$. Then player i will maximize:

$$\begin{aligned} \max_{q_i \geq 0} \pi_i(q_1, q_2) &= [P(Q) - c]\, q_i \\ &= [\alpha - \beta(q_1 + q_2) - c]\, q_i. \end{aligned}$$

- To find the best response of firm i we just maximize this with respect to q_i, which gives first-order condition

$$[\alpha - c - \beta(q_1 + q_2)] - \beta q_i = 0.$$

- Therefore, the best response correspondence (function) or firm i can be written as

$$q_i = \frac{\alpha - c - \beta q_{-i}}{2\beta}.$$

Cournot Competition (continued)

- Now combining the two best response functions, we find the unique Cournot equilibrium as

$$q_1^* = q_2^* = \frac{\alpha - c}{3\beta}.$$

- Total quantity is $2(\alpha - c)/3\beta$, and thus the equilibrium price is

$$P^* = \frac{\alpha + 2c}{3}.$$

- It can be verified that if the two firms colluded, then they could increase joint profits by reducing total quantity to $(\alpha - c)/2\beta$ and increasing price to $(\alpha + c)/2$.

Examples: Bertrand Competition

- An alternative to the Cournot model is the Bertrand model of oligopoly competition.
- In the Cournot model, firms choose quantities. In practice, choosing prices may be more reasonable.
- What happens if two producers of a homogeneous good charge different prices? *Reasonable answer:* everybody will purchase from the lower price firm.
- In this light, suppose that the demand function of the industry is given by $Q(p)$ (so that at price p, consumers will purchase a total of $Q(p)$ units).
- Suppose that two firms compete in this industry and they both have marginal cost equal to $c > 0$ (and can produce as many units as they wish at that marginal costs).

Bertrand Competition (continued)

- Then the profit function of firm i can be written as

$$\pi\left(p_i, p_{-i}\right) = \begin{cases} Q\left(p_i\right)\left(p_i - c\right) & \text{if } p_{-i} > p_i \\ \frac{1}{2}Q\left(p_i\right)\left(p_i - c\right) & \text{if } p_{-i} = p_i \\ 0 & \text{if } p_{-i} < p_i \end{cases}$$

- Actually, the middle row is arbitrary, given by some ad hoc "tiebreaking" rule. Imposing such tie-breaking rules is often not "kosher" as the homework will show.

Proposition

In the two-player Bertrand game there exists a unique Nash equilibrium given by $p_1 = p_2 = c$.

Bertrand Competition (continued)

Proof: Method of "finding a profitable deviation".

- Can $p_1 \geq c > p_2$ be a Nash equilibrium? No because firm 2 is losing money and can increase profits by raising its price.
- Can $p_1 = p_2 > c$ be a Nash equilibrium? No because either firm would have a profitable deviation, which would be to reduce their price by some small amount (from p_1 to $p_1 - \varepsilon$).
- Can $p_1 > p_2 > c$ be a Nash equilibrium? No because firm 1 would have a profitable deviation, to reduce its price to $p_2 - \varepsilon$.
- Can $p_1 > p_2 = c$ be a Nash equilibrium? No because firm 2 would have a profitable deviation, to increase its price to $p_1 - \varepsilon$.
- Can $p_1 = p_2 = c$ be a Nash equilibrium? Yes, because no profitable deviations. Both firms are making zero profits, and any deviation would lead to negative or zero profits.

Examples: Second Price Auction

- Second Price Auction (with Complete Information) The second price auction game is specified as follows:
- An object to be assigned to a player in $\{1, .., n\}$.
- Each player has her own valuation of the object. Player i's valuation of the object is denoted v_i. We further assume that $v_1 > v_2 > ... > 0$.
- Note that for now, we assume that everybody knows all the valuations $v_1, \ldots, v_n$, i.e., this is a complete information game. We will analyze the incomplete information version of this game in later lectures.
  - The assignment process is described as follows:
  - The players simultaneously submit bids, $b_1, .., b_n$.
  - The object is given to the player with the highest bid (or to a random player among the ones bidding the highest value).
  - The winner pays the second highest bid.
  - The utility function for each of the players is as follows: the winner receives her valuation of the object minus the price she pays, i.e., $v_i - b_j$; everyone else receives 0.

Second Price Auction (continued)

Proposition

In the second price auction, truthful bidding, i.e., $b_i = v_i$ for all i, is a Nash equilibrium.

Proof: We want to show that the strategy profile $(b_1, .., b_n) = (v_1, .., v_n)$ is a Nash Equilibrium—**a truthful equilibrium**.

- First note that if indeed everyone plays according to that strategy, then player 1 receives the object and pays a price v_2.
- This means that her payoff will be $v_1 - v_2 > 0$, and all other payoffs will be 0. Now, player 1 has no incentive to deviate, since her utility can only decrease.
- Likewise, for all other players $v_i \neq v_1$, it is the case that in order for v_i to change her payoff from 0 she needs to bid more than v_1, in which case her payoff will be $v_i - v_1 < 0$.
- Thus no incentive to deviate from for any player.

Second Price Auction (continued)

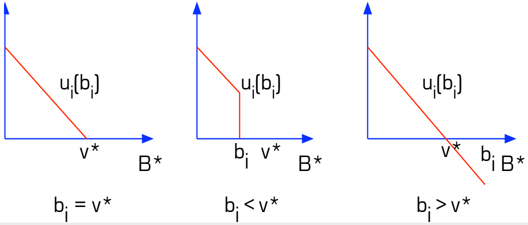
- Are There Other Nash Equilibria? In fact, there are also unreasonable Nash equilibria in second price auctions.
- We show that the strategy $(v_1, 0, 0, ..., 0)$ is also a Nash Equilibrium.
- As before, player 1 will receive the object, and will have a payoff of $v_1 - 0 = v_1$. Using the same argument as before we conclude that none of the players have an incentive to deviate, and the strategy is thus a Nash Equilibrium.
- It can be verified the strategy $(v_2, v_1, 0, 0, ..., 0)$ is also a Nash Equilibrium.
- Why?

Second Price Auction (continued)

- Nevertheless, the truthful equilibrium, where , $b_i = v_i$, is the **Weakly Dominant Nash Equilibrium**
- In particular, truthful bidding, $b_i = v_i$, weakly dominates all other strategies.
- Consider the following picture proof where B^* represents the maximum of all bids excluding player i's bid, i.e.

$$B^* = \max_{j \neq i} b_j,$$

and v^* is player i's valuation and the vertical axis is utility.



Second Price Auction (continued)

- The first graph shows the payoff for bidding one's valuation. In the second graph, which represents the case when a player bids lower than their valuation, notice that whenever $b_i \leq B^* \leq v^*$, player i receives utility 0 because she loses the auction to whoever bid B^*.
- If she would have bid her valuation, she would have positive utility in this region (as depicted in the first graph).
- Similar analysis is made for the case when a player bids more than their valuation.
- An immediate implication of this analysis is that other equilibria involve the play of weakly dominated strategies.