14.452 Economic Growth: Lectures 5 and 6, Neoclassical Growth

Daron Acemoglu

MIT

November 8 and 12, 2013.

Foundations of Neoclassical Growth

- Solow model: constant saving rate.
- More satisfactory to specify the *preference orderings* of individuals and derive their decisions from these preferences.
- Enables better understanding of the factors that affect savings decisions.
- Enables to discuss the "optimality" of equilibria
- Whether the (competitive) equilibria of growth models can be "improved upon".
- Notion of improvement: Pareto optimality.

Preliminaries

- Consider an economy consisting of a unit measure of infinitely-lived households.
- I.e., an uncountable number of households: e.g., the set of households $\mathcal{H}$ could be represented by the unit interval $[0, 1]$.
- Emphasize that each household is infinitesimal and will have no effect on aggregates.
- Can alternatively think of $\mathcal{H}$ as a countable set of the form $\mathcal{H} = \{1, 2, ..., M\}$ with $M = \infty$, without any loss of generality.
- Advantage of unit measure: averages and aggregates are the same
- Simpler to have $\mathcal{H}$ as a finite set in the form $\{1, 2, ..., M\}$ with M large but finite.
- Acceptable for many models, but with overlapping generations require the set of households to be infinite.

Time Separable Preferences

- Standard assumptions on preference orderings so that they can be represented by utility functions.
- In addition, **time separable preferences:** each household i has an *instantaneous (Bernoulli) utility function* (or felicity function):

$$u_i\left(c_i\left(t\right)\right),$$

- $u_i : \mathbb{R}_+ \rightarrow \mathbb{R}$ is increasing and concave and $c_i\left(t\right)$ is the consumption of household i.
- Note instantaneous utility function is *not* specifying a complete preference ordering over all commodities—here consumption levels in all dates.
- Instead, household i preferences at time $t = 0$ are obtained by combining this with *exponential* discounting.

Infinite Horizon and the Representative Household

- Thus given by the following von Neumann-Morgenstern expected utility function:

$$\mathbb{E}_0^i \sum_{t=0}^{T} \beta_i^t u_i\left(c_i\left(t\right)\right), \tag{1}$$

  where $\beta_i \in (0, 1)$ is the discount factor of household i, where $T < \infty$ or $T = \infty$ are the two cases to consider.

- To model households in infinite horizon, these two would then correspond to

  1. overlapping generations→finite planning horizon (generally...);
  2. "infinitely lived" or consisting of overlapping generations with full altruism linking generations→infinite planning horizon

- The second is often assumed because the standard approach in macroeconomics is to impose the existence of a *representative household*—costs of this to be discussed below.

Time Consistency

- Exponential discounting and time separability: ensure "time-consistent" behavior.
- A solution $\{x(t)\}_{t=0}^{T}$ (possibly with $T = \infty$) is *time consistent* if:
  - whenever $\{x(t)\}_{t=0}^{T}$ is an optimal solution starting at time $t = 0$, $\{x(t)\}_{t=t'}^{T}$ is an optimal solution to the continuation dynamic optimization problem starting from time $t = t' \in [0, T]$.

Challenges to the Representative Household

- An economy *admits a representative household* if preference side can be represented *as if* a single household made the aggregate consumption and saving decisions subject to a single budget constraint.
- This description concerning a representative household is purely positive
- Stronger notion of "normative" representative household: if we can also use the utility function of the representative household for welfare comparisons.
- Simplest case that will lead to the existence of a representative household: suppose each household is identical.

Representative Household II

- If instead households are not identical but assume can model *as if* demand side generated by the optimization decision of a representative household:
- More realistic, but:
  1. The representative household will have positive, but not always a normative meaning.
  2. Models with heterogeneity: often not lead to behavior that can be represented as if generated by a representative household.

Theorem **(Debreu-Mantel-Sonnenschein Theorem)** Let $\varepsilon > 0$ be a scalar and $N < \infty$ be a positive integer. Consider a set of prices $\mathbf{P}_\varepsilon = \left\{ p \in \mathbb{R}_+^N \colon\ p_j / p_{j'} \geq \varepsilon \text{ for all } j \text{ and } j' \right\}$ and any continuous function $\mathbf{x} : \mathbf{P}_\varepsilon \to \mathbb{R}_+^N$ that satisfies Walras' Law and is homogeneous of degree 0. Then there exists an exchange economy with N commodities and $H < \infty$ households, where the aggregate demand is given by $\mathbf{x}(p)$ over the set $\mathbf{P}_\varepsilon$.

Representative Household IV

- That excess demands come from optimizing behavior of households puts no restrictions on the form of these demands.
  - E.g., $\mathbf{x}(p)$ does not necessarily possess a negative-semi-definite Jacobian or satisfy the weak axiom of revealed preference (requirements of demands generated by individual households).
- Hence without imposing further structure, impossible to derive specific $\mathbf{x}(p)$'s from the maximization behavior of a single household.
- Severe warning against the use of the representative household assumption.
- Partly an outcome of very strong income effects:
  - special but approximately realistic preference functions, and restrictions on distribution of income rule out arbitrary aggregate excess demand functions.

Gorman Aggregation

- Recall an indirect utility function for household i, $v_i\left(p, y^i\right)$, specifies (ordinal) utility as a function of the price vector $p = \left(p_1, ..., p_N\right)$ and household's income y^i.
- $v_i\left(p, y^i\right)$: homogeneous of degree 0 in p and y.

Theorem **(Gorman's Aggregation Theorem)** Consider an economy with a finite number $N < \infty$ of commodities and a set $\mathcal{H}$ of households. Suppose that the preferences of household $i \in \mathcal{H}$ can be represented by an indirect utility function of the form

$$v^i\left(p, y^i\right) = a^i\left(p\right) + b\left(p\right) y^i, \qquad (2)$$

then these preferences can be aggregated and represented by those of a representative household, with indirect utility

$$v\left(p, y\right) = \int_{i \in \mathcal{H}} a^i\left(p\right) di + b\left(p\right) y,$$

where $y \equiv \int_{i \in \mathcal{H}} y^i di$ is aggregate income.

Linear Engel Curves

- Demand for good j (from Roy's identity):

$$x_j^i\left(p, y^i\right) = -\frac{1}{b\left(p\right)}\frac{\partial a^i\left(p\right)}{\partial p_j} - \frac{1}{b\left(p\right)}\frac{\partial b\left(p\right)}{\partial p_j}y^i.$$

- Thus linear Engel curves.
- "Indispensable" for the existence of a representative household.
- Let us say that there exists a *strong representative household* if redistribution of income or endowments across households does not affect the demand side.
- Gorman preferences are sufficient for a strong representative household.
- Moreover, they are also *necessary* (with the same $b\left(p\right)$ for all households) for the economy to admit a strong representative household.
  - The proof is easy by a simple variation argument.

Importance of Gorman Preferences

- Gorman Preferences limit the **extent of income effects** and enables the aggregation of individual behavior.
- Integral is "Lebesgue integral," so when $\mathcal{H}$ is a finite or countable set, $\int_{i\in\mathcal{H}} y^i di$ is indeed equivalent to the summation $\sum_{i\in\mathcal{H}} y^i$.
- Stated for an economy with a finite number of commodities, but can be generalized for infinite or even a continuum of commodities.
- Note all we require is there exists a monotonic transformation of the indirect utility function that takes the form in (2)—as long as no uncertainty.
- Contains some commonly-used preferences in macroeconomics.

Normative Representative Household

- Gorman preferences also imply the existence of a normative representative household.
- Recall an allocation is *Pareto optimal* if no household can be made strictly better-off without some other household being made worse-off.

Existence of Normative Representative Household

Theorem (Existence of a Normative Representative Household) Consider an economy with a finite number $N < \infty$ of commodities, a set $\mathcal{H}$ of households and a convex aggregate production possibilities set Y.. Suppose that the preferences of each household $i \in \mathcal{H}$ take the Gorman form, $v^i\left(p, y^i\right) = a^i\left(p\right) + b\left(p\right) y^i$.

1. Then any allocation that maximizes the utility of the representative household, $v\left(p, y\right) = \sum_{i \in \mathcal{H}} a^i\left(p\right) + b\left(p\right) y$, with $y \equiv \sum_{i \in \mathcal{H}} y^i$, is Pareto optimal.
2. Moreover, if $a^i\left(p\right) = a^i$ for all p and all $i \in \mathcal{H}$, then any Pareto optimal allocation maximizes the utility of the representative household.

Infinite Planning Horizon I

- Most growth and macro models assume that individuals have an infinite-planning horizon. How could this be a good assumption?
- One possibility: intergenerational altruism or from the "bequest" motive.
- Imagine an individual who lives for one period and has a single offspring (who will also live for a single period and beget a single offspring etc.).
- Individual not only derives utility from his consumption but also from the bequest he leaves to his offspring.
- For example, utility of an individual living at time t is given by

$$u\left(c\left(t\right)\right) + U^{b}\left(b\left(t\right)\right),$$

- $c\left(t\right)$ is his consumption and $b\left(t\right)$ denotes the bequest left to his offspring.
- For concreteness, suppose that the individual has total income $y\left(t\right)$, so that his budget constraint is

Infinite Planning Horizon II

- $U^b(\cdot)$: how much the individual values bequests left to his offspring.
- Benchmark might be "purely altruistic:" cares about the utility of his offspring (with some discount factor).
- Let discount factor between generations be β.
- Assume offspring will have an income of w without the bequest.
- Then the utility of the individual can be written as

$$u(c(t)) + \beta V(b(t) + w),$$

- $V(\cdot)$: continuation value, the utility that the offspring will obtain from receiving a bequest of $b(t)$ (plus his own w).
- Value of the individual at time t can in turn be written as

$$V(y(t)) = \max_{c(t)+b(t)\leq y(t)} \{u(c(t)) + \beta V(b(t) + w(t+1))\},$$

Infinite Planning Horizon III

- Canonical form of a dynamic programming representation of an infinite-horizon maximization problem.
- Under some mild technical assumptions, this dynamic programming representation is equivalent to maximizing

$$\sum_{s=0}^{\infty} \beta^s u\left(c_{t+s}\right)$$

at time t.

- Each individual internalizes utility of all future members of the "dynasty".
- Fully altruistic behavior within a dynasty ("dynastic" preferences) will also lead to infinite planning horizon.

The Representative Firm I

- While not all economies would admit a representative household, standard assumptions (in particular no production externalities and competitive markets) are sufficient to ensure a representative firm.

Theorem (The Representative Firm Theorem) Consider a competitive production economy with $N \in \mathbb{N} \cup \{+\infty\}$ commodities and a countable set $\mathcal{F}$ of firms, each with a convex production possibilities set $Y^f \subset \mathbb{R}^N$. Let $p \in \mathbb{R}_+^N$ be the price vector in this economy and denote the set of profit maximizing net supplies of firm $f \in \mathcal{F}$ by $\hat{Y}^f(p) \subset Y^f$ (so that for any $\hat{y}^f \in \hat{Y}^f(p)$, we have $p \cdot \hat{y}^f \geq p \cdot y^f$ for all $y^f \in Y^f$). Then there exists a *representative firm* with production possibilities set $Y \subset \mathbb{R}^N$ and set of profit maximizing net supplies $\hat{Y}(p)$ such that for any $p \in \mathbb{R}_+^N$, $\hat{y} \in \hat{Y}(p)$ if and only if $\hat{y}(p) = \sum_{f \in \mathcal{F}} \hat{y}^f$ for some $\hat{y}^f \in \hat{Y}^f(p)$ for each $f \in \mathcal{F}$.

The Representative Firm II

- Why such a difference between representative household and representative firm assumptions? Income effects.
- Changes in prices create income effects, which affect different households differently.
- No income effects in producer theory, so the representative firm assumption is without loss of any generality.
- Does not mean that heterogeneity among firms is uninteresting or unimportant.
- Many models of endogenous technology feature productivity differences across firms, and firms' attempts to increase their productivity relative to others will often be an engine of economic growth.

Welfare Theorems I

- There should be a close connection between Pareto optima and competitive equilibria.
- Start with models that have a finite number of consumers, so $\mathcal{H}$ is finite.
- However, allow an infinite number of commodities.
- Results here have analogs for economies with a continuum of commodities, but focus on countable number of commodities.
- Let commodities be indexed by $j \in \mathbb{N}$ and $x^i \equiv \left\{x_j^i\right\}_{j=0}^{\infty}$ be the consumption bundle of household i, and $\omega^i \equiv \left\{\omega_j^i\right\}_{j=0}^{\infty}$ be its endowment bundle.
- Assume feasible x^i's must belong to some consumption set $X^i \subset \mathbb{R}_+^{\infty}$.
- Most relevant interpretation for us is that at each date $j = 0, 1, ...$, each individual consumes a finite dimensional vector of products.

Welfare Theorems II

- Thus $x_j^i \in X_j^i \subset \mathbb{R}_+^K$ for some integer K.
- Consumption set introduced to allow cases where individual may not have negative consumption of certain commodities.
- Let $\mathbf{X} \equiv \prod_{i \in \mathcal{H}} X^i$ be the Cartesian product of these consumption sets, the aggregate consumption set of the economy.
- Also use the notation $\mathbf{x} \equiv \left\{x^i\right\}_{i \in \mathcal{H}}$ and $\boldsymbol{\omega} \equiv \left\{\omega^i\right\}_{i \in \mathcal{H}}$ to describe the entire consumption allocation and endowments in the economy.
- Feasibility requires that $\mathbf{x} \in \mathbf{X}$.

Welfare Theorems III

- Each household in $\mathcal{H}$ has a well defined preference ordering over consumption bundles, given by some preference ordering $\succsim_i$and we assume that these can be represented by $u^i : X^i \to \mathbb{R}$, such that whenever $x' \succsim_i x$, we have $u^i(x') \geq u^i(x)$.
- Let $\mathbf{u} \equiv \left\{u^i\right\}_{i\in\mathcal{H}}$ be the set of utility functions.
- Production side: finite number of firms represented by $\mathcal{F}$
- Each firm $f \in \mathcal{F}$ is characterized by production set Y^f, specifies levels of output firm f can produce from specified levels of inputs.
- I.e., $y^f \equiv \left\{y_j^f\right\}_{j=0}^{\infty}$ is a feasible production plan for firm f if $y^f \in Y^f$.
- E.g., if there were only labor and a final good, Y^f would include pairs $(-l, y)$ such that with labor input l the firm can produce at most y.

Welfare Theorems IV

- Let $\mathbf{Y} \equiv \prod_{f \in \mathcal{F}} Y^f$ represent the aggregate production set and $\mathbf{y} \equiv \{y^f\}_{f \in \mathcal{F}}$ such that $y^f \in Y^f$ for all f, or equivalently, $\mathbf{y} \in \mathbf{Y}$.
- Ownership structure of firms: if firms make profits, they should be distributed to some agents
- Assume there exists a sequence of numbers (profit shares) $\boldsymbol{\theta} \equiv \left\{\theta_f^i\right\}_{f \in \mathcal{F}, i \in \mathcal{H}}$ such that $\theta_f^i \geq 0$ for all f and i, and $\sum_{i \in \mathcal{H}} \theta_f^i = 1$ for all $f \in \mathcal{F}$.
- θ_f^i is the share of profits of firm f that will accrue to household i.

Welfare Theorems V

- An economy $\mathcal{E}$ is described by $\mathcal{E} \equiv (\mathcal{H}, \mathcal{F}, \mathbf{u}, \boldsymbol{\omega}, \mathbf{Y}, \mathbf{X}, \boldsymbol{\theta})$.
- An allocation is $(\mathbf{x}, \mathbf{y})$ such that $\mathbf{x}$ and $\mathbf{y}$ are feasible, that is, $\mathbf{x} \in \mathbf{X}$, $\mathbf{y} \in \mathbf{Y}$, and $\sum_{i \in \mathcal{H}} x_j^i \leq \sum_{i \in \mathcal{H}} \omega_j^i + \sum_{f \in \mathcal{F}} y_j^f$ for all $j \in \mathbb{N}$.
- A price system is a sequence $p \equiv \{p_j\}_{j=0}^{\infty}$, such that $p_j \geq 0$ for all j.
- We can choose one of these prices as the numeraire and normalize it to 1.
- Also define $p \cdot x$ as the inner product of p and x, i.e., $p \cdot x \equiv \sum_{j=0}^{\infty} p_j x_j$.

Definition Household $i \in \mathcal{H}$ is *locally non-satiated* if at each x^i, $u^i(x^i)$ is strictly increasing in at least one of its arguments at x^i and $u^i(x^i) < \infty$.

- Latter requirement already implied by the fact that $u^i : X^i \to \mathbb{R}$. Let us impose this assumption.

Welfare Theorems VI

Definition A competitive equilibrium for the economy $\mathcal{E} \equiv (\mathcal{H}, \mathcal{F}, \mathbf{u}, \boldsymbol{\omega}, \mathbf{Y}, \mathbf{X}, \boldsymbol{\theta})$ is given by an allocation $\left(\mathbf{x}^* = \left\{x^{i*}\right\}_{i\in\mathcal{H}}, \mathbf{y}^* = \left\{y^{f*}\right\}_{f\in\mathcal{F}}\right)$ and a price system p^* such that

1. The allocation $(\mathbf{x}^*, \mathbf{y}^*)$ is feasible and market clearing, i.e., $x^{i*} \in X^i$ for all $i \in \mathcal{H}$, $y^{f*} \in Y^f$ for all $f \in \mathcal{F}$ and

$$\sum_{i\in\mathcal{H}} x_j^{i*} = \sum_{i\in\mathcal{H}} \omega_j^i + \sum_{f\in\mathcal{F}} y_j^{f*} \text{ for all } j \in \mathbb{N}.$$

2. For every firm $f \in \mathcal{F}$, y^{f*} maximizes profits, i.e.,

$$p^* \cdot y^{f*} \geq p^* \cdot y \text{ for all } y \in Y^f.$$

3. For every consumer $i \in \mathcal{H}$, x^{i*} maximizes utility, i.e.,

$$u^i\left(x^{i*}\right) \geq u^i\left(x\right) \text{ for all } x \text{ s.t. } x \in X^i \text{ and } p^* \cdot x \leq p^* \cdot x^{i*}.$$

Welfare Theorems VII

- Establish existence of competitive equilibrium with finite number of commodities and standard convexity assumptions is straightforward.
- With infinite number of commodities, somewhat more difficult and requires more sophisticated arguments.

Definition A feasible allocation $(\mathbf{x}, \mathbf{y})$ for economy $\mathcal{E} \equiv (\mathcal{H}, \mathcal{F}, \mathbf{u}, \boldsymbol{\omega}, \mathbf{Y}, \mathbf{X}, \boldsymbol{\theta})$ is *Pareto optimal* if there exists no other feasible allocation $(\hat{\mathbf{x}}, \hat{\mathbf{y}})$ such that $\hat{x}^i \in X^i$, $\hat{y}^f \in Y^f$ for all $f \in \mathcal{F}$,

$$\sum_{i \in \mathcal{H}} \hat{x}^i_j \leq \sum_{i \in \mathcal{H}} \omega^i_j + \sum_{f \in \mathcal{F}} \hat{y}^f_j \text{ for all } j \in \mathbb{N},$$

and

$$u^i\left(\hat{x}^i\right) \geq u^i\left(x^i\right) \text{ for all } i \in \mathcal{H}$$

with at least one strict inequality.

Welfare Theorems VIII

Theorem **(First Welfare Theorem I)** Suppose that $(\mathbf{x}^*, \mathbf{y}^*, p^*)$ is a competitive equilibrium of economy $\mathcal{E} \equiv (\mathcal{H}, \mathcal{F}, \mathbf{u}, \boldsymbol{\omega}, \mathbf{Y}, \mathbf{X}, \boldsymbol{\theta})$ with $\mathcal{H}$ finite. Assume that all households are locally non-satiated. Then $(\mathbf{x}^*, \mathbf{y}^*)$ is Pareto optimal.

Proof of First Welfare Theorem I

- To obtain a contradiction, suppose that there exists a feasible $(\hat{\mathbf{x}}, \hat{\mathbf{y}})$ such that $u^i\left(\hat{x}^i\right) \geq u^i\left(x^i\right)$ for all $i \in \mathcal{H}$ and $u^i\left(\hat{x}^i\right) > u^i\left(x^i\right)$ for all $i \in \mathcal{H}'$, where $\mathcal{H}'$ is a non-empty subset of $\mathcal{H}$.
- Since $\left(\mathbf{x}^*, \mathbf{y}^*, p^*\right)$ is a competitive equilibrium, it must be the case that for all $i \in \mathcal{H}$,

$$\begin{aligned} p^* \cdot \hat{x}^i &\geq p^* \cdot x^{i*} \\ &= p^* \cdot \left(\omega^i + \sum_{f \in \mathcal{F}} \theta_f^i y^{f*}\right) \end{aligned} \tag{3}$$

and for all $i \in \mathcal{H}'$,

$$p^* \cdot \hat{x}^i > p^* \cdot \left(\omega^i + \sum_{f \in \mathcal{F}} \theta_f^i y^{f*}\right). \tag{4}$$

Proof of First Welfare Theorem II

- Second inequality follows immediately in view of the fact that x^{i*} is the utility maximizing choice for household i, thus if $\hat{x}^i$ is strictly preferred, then it cannot be in the budget set.
- First inequality follows with a similar reasoning. Suppose that it did not hold.
- Then by the hypothesis of local-satiation, u^i must be strictly increasing in at least one of its arguments, let us say the j'th component of x.
- Then construct $\hat{x}^i(\varepsilon)$ such that $\hat{x}^i_j(\varepsilon) = \hat{x}^i_j$ and $\hat{x}^i_{j'}(\varepsilon) = \hat{x}^i_{j'} + \varepsilon$.
- For $\varepsilon \downarrow 0$, $\hat{x}^i(\varepsilon)$ is in household i's budget set and yields strictly greater utility than the original consumption bundle x^i, contradicting the hypothesis that household i was maximizing utility.
- Note local non-satiation implies that $u^i(x^i) < \infty$, and thus the right-hand sides of (3) and (4) are finite.

Proof of First Welfare Theorem III

- Now summing over (3) and (4), we have

$$\begin{aligned} p^{*} \cdot \sum_{i \in \mathcal{H}} \hat{x}^{i} &> p^{*} \cdot \sum_{i \in \mathcal{H}} \left(\omega^{i} + \sum_{f \in \mathcal{F}} \theta_{f}^{i} y^{f*} \right), \\ &= p^{*} \cdot \left(\sum_{i \in \mathcal{H}} \omega^{i} + \sum_{f \in \mathcal{F}} y^{f*} \right), \end{aligned} \tag{5}$$

- Second line uses the fact that the summations are finite, can change the order of summation, and that by definition of shares $\sum_{i \in \mathcal{H}} \theta_{f}^{i} = 1$ for all f.
- Finally, since $\mathbf{y}^{*}$ is profit-maximizing at prices p^{*}, we have that

$$p^{*} \cdot \sum_{f \in \mathcal{F}} y^{f*} \geq p^{*} \cdot \sum_{f \in \mathcal{F}} y^{f} \text{ for any } \left\{ y^{f} \right\}_{f \in \mathcal{F}} \text{ with } y^{f} \in Y^{f} \text{ for all } f \in \mathcal{F} \tag{6}$$

Proof of First Welfare Theorem IV

- However, by market clearing of $\hat{x}^i$ (Definition above, part 1), we have

$$\sum_{i\in\mathcal{H}} \hat{x}_j^i = \sum_{i\in\mathcal{H}} \omega_j^i + \sum_{f\in\mathcal{F}} \hat{y}_j^f,$$

- Therefore, by multiplying both sides by p^* and exploiting (6),

$$\begin{aligned} p^* \cdot \sum_{i\in\mathcal{H}} \hat{x}_j^i &\leq p^* \cdot \left(\sum_{i\in\mathcal{H}} \omega_j^i + \sum_{f\in\mathcal{F}} \hat{y}_j^f \right) \\ &\leq p^* \cdot \left(\sum_{i\in\mathcal{H}} \omega_j^i + \sum_{f\in\mathcal{F}} y_j^{f*} \right), \end{aligned}$$

- Contradicts (5), establishing that any competitive equilibrium allocation $(\mathbf{x}^*, \mathbf{y}^*)$ is Pareto optimal.

Welfare Theorems IX

- Proof of the First Welfare Theorem based on two intuitive ideas.
    1. If another allocation Pareto dominates the competitive equilibrium, then it must be non-affordable in the competitive equilibrium.
    2. Profit-maximization implies that any competitive equilibrium already contains the maximal set of affordable allocations.
- Note it makes no convexity assumption.
- Also highlights the importance of the feature that the relevant sums exist and are finite.
    - Otherwise, the last step would lead to the conclusion that "$\infty < \infty$".
- That these sums exist followed from two assumptions: finiteness of the number of individuals and non-satiation.

Welfare Theorems X

Theorem **(First Welfare Theorem II)** Suppose that $(\mathbf{x}^*, \mathbf{y}^*, p^*)$ is a competitive equilibrium of the economy $\mathcal{E} \equiv (\mathcal{H}, \mathcal{F}, \mathbf{u}, \boldsymbol{\omega}, \mathbf{Y}, \mathbf{X}, \boldsymbol{\theta})$ with $\mathcal{H}$ countably infinite. Assume that all households are locally non-satiated and that $p^* \cdot \omega^* = \sum_{i \in \mathcal{H}} \sum_{j=0}^{\infty} p_j^* \omega_j^i < \infty$. Then $(\mathbf{x}^*, \mathbf{y}^*, p^*)$ is Pareto optimal.

- **Proof:**
  - Same as before but now local non-satiation does not guarantee summations are finite (5), since we sum over an infinite number of households.
  - But since endowments are finite, the assumption that $\sum_{i \in \mathcal{H}} \sum_{j=0}^{\infty} p_j^* \omega_j^i < \infty$ ensures that the sums in (5) are indeed finite.

Welfare Theorems X

- Second Welfare Theorem (converse to First): whether or not $\mathcal{H}$ is finite is not as important as for the First Welfare Theorem.
- But requires assumptions such as the convexity of consumption and production sets and preferences, and additional requirements because it contains an "existence of equilibrium argument".
- Recall that the consumption set of each individual $i \in \mathcal{H}$ is $X^i \subset \mathbb{R}_+^\infty$.
- A typical element of X^i is $x^i = \left(x_1^i, x_2^i, ...\right)$, where x_t^i can be interpreted as the vector of consumption of individual i at time t.
- Similarly, a typical element of the production set of firm $f \in \mathcal{F}$, Y^f, is $y^f = \left(y_1^f, y_2^f, ...\right)$.
- Let us define $x^i\left[T\right] = \left(x_0^i, x_1^i, x_2^i, ..., x_T^i, 0, 0, ...\right)$ and $y^f\left[T\right] = \left(y_0^f, y_1^f, y_2^f, ..., y_T^f, 0, 0, ...\right)$.
- It can be verified that $\lim_{T\to\infty} x^i\left[T\right] = x^i$ and $\lim_{T\to\infty} y^f\left[T\right] = y^f$ in the product topology.

Second Welfare Theorem I

Theorem

Consider a Pareto optimal allocation $(\mathbf{x}^{**}, \mathbf{y}^{**})$ *in an economy described by* ω, $\{Y^f\}_{f\in\mathcal{F}}$, $\{X^i\}_{i\in\mathcal{H}}$, *and* $\{u^i(\cdot)\}_{i\in\mathcal{H}}$. *Suppose all production and consumption sets are convex, all production sets are cones, and all* $\{u^i(\cdot)\}_{i\in\mathcal{H}}$ *are continuous and quasi-concave and satisfy local non-satiation. Suppose also that* $0 \in X^i$, *that for each* $x, x' \in X^i$ *with* $u^i(x) > u^i(x')$ *for all* $i \in \mathcal{H}$, *there exists* $\bar{T}$ *such that* $u^i(x[T]) > u^i(x')$ *for all* $T \geq \bar{T}$ *and for all* $i \in \mathcal{H}$, *and that for each* $y \in Y^f$, *there exists* $\tilde{T}$ *such that* $y[T] \in Y^f$ *for all* $T \geq \tilde{T}$ *and for all* $f \in \mathcal{F}$.*Then this allocation can be decentralized as a competitive equilibrium.*

Second Welfare Theorem II

Theorem

(continued) *In particular, there exist* p^{**} *and* $(\boldsymbol{\omega}^{**}, \boldsymbol{\theta}^{**})$ *such that*

1. $\boldsymbol{\omega}^{**}$ *satisfies* $\omega = \sum_{i \in \mathcal{H}} \omega^{i**}$;
2. *for all* $f \in \mathcal{F}$,

$$p^{**} \cdot y^{f**} \leq p^{**} \cdot y \text{ for all } y \in Y^f;$$

3. *for all* $i \in \mathcal{H}$,

$$\text{if } x^i \in X^i \text{ involves } u^i\left(x^i\right) > u^i\left(x^{i**}\right), \text{ then } p^{**} \cdot x^i \geq p^{**} \cdot w^{i**},$$

where $w^{i**} \equiv \omega^{i**} + \sum_{f \in \mathcal{F}} \theta_f^{i**} y^{f**}$.

Moreover, if $p^{**} \cdot \mathbf{w}^{**} > 0$ *[i.e.,* $p^{**} \cdot w^{i**} > 0$ *for each* $i \in \mathcal{H}$*], then economy* $\mathcal{E}$ *has a competitive equilibrium* $(\mathbf{x}^{**}, \mathbf{y}^{**}, p^{**})$.

Welfare Theorems XII

- Notice:
  - if instead if we had a finite commodity space, say with K commodities, then the hypothesis that $0 \in X^i$ for each $i \in \mathcal{H}$ and $x, x' \in X^i$ with $u^i(x) > u^i(x')$, there exists $\bar{T}$ such that $u^i(x[T]) > u^i(x'[T])$ for all $T \geq \bar{T}$ and all $i \in \mathcal{H}$ (and also that there exists $\tilde{T}$ such that if $y \in Y^f$, then $y[T] \in Y^f$ for all $T \geq \tilde{T}$ and all $f \in \mathcal{F}$) would be satisfied automatically, by taking $\bar{T} = \tilde{T} = K$.
  - Condition not imposed in Second Welfare Theorem in economies with a finite number of commodities.
  - In dynamic economies, its role is changes in allocations at very far in the future should not have a large effect.
- The conditions for the Second Welfare Theorem are more difficult to satisfy than those for the First.
- Also the more important of the two theorems: stronger results that any Pareto optimal allocation can be *decentralized*.

Welfare Theorems XIII

- Immediate corollary is an existence result: a competitive equilibrium must exist.
- Motivates many to look for the set of Pareto optimal allocations instead of explicitly characterizing competitive equilibria.
- Real power of the Theorem in dynamic macro models comes when we combine it with models that admit a representative household.
- Enables us to characterize *the optimal growth allocation* that maximizes the utility of the representative household and assert that this will correspond to a competitive equilibrium.

Introduction

- Ramsey or Cass-Koopmans model: differs from the Solow model only because it explicitly models the consumer side and endogenizes savings.
- Beyond its use as a basic growth model, also a workhorse for many areas of macroeconomics.

Preferences, Technology and Demographics I

- Infinite-horizon, continuous time.
- Representative household with instantaneous utility function

$$u\left(c\left(t\right)\right), \tag{7}$$

Assumption $u\left(c\right)$ is strictly increasing, concave, twice continuously differentiable with derivatives u' and u'', and satisfies the following Inada type assumptions:

$$\lim_{c\to 0} u'\left(c\right) = \infty \text{ and } \lim_{c\to\infty} u'\left(c\right) = 0.$$

- Suppose representative household represents set of identical households (normalized to 1).
- Each household has an instantaneous utility function given by (7).
- $L\left(0\right) = 1$ and

$$L\left(t\right) = \exp\left(nt\right). \tag{8}$$

Preferences, Technology and Demographics II

- All members of the household supply their labor inelastically.
- Objective function of each household at $t = 0$:

$$U(0) \equiv \int_0^\infty \exp(-(\rho - n)t)\, u(c(t))\, dt, \qquad (9)$$

where $c(t)$=consumption per capita at t, and ρ=subjective discount rate, and effective discount rate is $\rho - n$.
- Continues time analogue of $\sum_{t=0}^\infty \beta_i^t u_i(c_i(t))$.
- Objective function (9) embeds:
  - Household is fully altruistic towards all of its future members, and makes allocations of consumption (among household members) cooperatively.
  - Strict concavity of $u(\cdot)$
- Thus each household member will have an equal consumption

$$c(t) \equiv \frac{C(t)}{L(t)}$$

Preferences, Technology and Demographics III

- Utility of $u(c(t))$ per household member at time t, total of $L(t)\,u(c(t)) = \exp(nt)\,u(c(t))$.
- With discount at rate of $\exp(-\rho t)$, obtain (9).

ASSUMPTION 4′.

$$\rho > n.$$

- Ensures that in the model without growth, discounted utility is finite. Will strengthen it in model with growth.
- Start model without any technological progress.
- Factor and product markets are competitive.
- Production possibilities set of the economy is represented by

$$Y(t) = F[K(t), L(t)],$$

- Standard constant returns to scale and Inada assumptions still hold.

Preferences, Technology and Demographics IV

- Per capita production function $f(\cdot)$

$$\begin{aligned} y(t) &\equiv \frac{Y(t)}{L(t)} \\ &= F\left[\frac{K(t)}{L(t)}, 1\right] \\ &\equiv f(k(t)), \end{aligned}$$

where, as before,

$$k(t) \equiv \frac{K(t)}{L(t)}. \tag{10}$$

- Competitive factor markets then imply:

$$R(t) = F_K[K(t), L(t)] = f'(k(t)). \tag{11}$$

and

$$w(t) = F_L[K(t), L(t)] = f(k(t)) - k(t) f'(k(t)). \tag{12}$$

Preferences, Technology and Demographics V

- Denote asset holdings of the representative household at time t by $\mathcal{A}(t)$. Then,

$$\dot{\mathcal{A}}(t) = r(t)\mathcal{A}(t) + w(t)L(t) - c(t)L(t)$$

- $r(t)$ is the risk-free market flow rate of return on assets, and $w(t)L(t)$ is the flow of labor income earnings of the household.
- Defining per capita assets as

$$a(t) \equiv \frac{\mathcal{A}(t)}{L(t)},$$

we obtain:

$$\dot{a}(t) = (r(t) - n)a(t) + w(t) - c(t). \tag{13}$$

- Household assets can consist of capital stock, $K(t)$, which they rent to firms and government bonds, $B(t)$.

Preferences, Technology and Demographics VI

- With uncertainty, households would have a portfolio choice between $K(t)$ and riskless bonds.
- With incomplete markets, bonds allow households to smooth idiosyncratic shocks. But for now no need.
- Thus, market clearing $\Rightarrow$

$$a(t) = k(t).$$

- No uncertainty depreciation rate of δ implies

$$r(t) = R(t) - \delta. \tag{14}$$

The Budget Constraint

- The differential equation

$$\dot{a}(t) = (r(t) - n)\, a(t) + w(t) - c(t)$$

  is a flow constraint
- Not sufficient as a proper budget constraint unless we impose a lower bound on assets.
- Three options:
  1. Lower bound on assets such as $a(t) \geq 0$ for all t
  2. Natural debt limit.
  3. No Ponzi Game Condition.

The No Ponzi Game Condition

- Infinite-horizon no Ponzi game condition is:

$$\lim_{t \to \infty} a(t) \exp\left(-\int_0^t (r(s) - n)\, ds\right) \geq 0. \tag{15}$$

- Transversality condition ensures individual would never want to have positive wealth asymptotically, so no Ponzi game condition can be strengthened to (though not necessary in general):

$$\lim_{t \to \infty} a(t) \exp\left(-\int_0^t (r(s) - n)\, ds\right) = 0. \tag{16}$$

Understanding the No Ponzi Game Condition

- Why?
- Write the single budget constraint of the form:

$$\begin{aligned} & \int_0^T c(t) L(t) \exp\left(\int_t^T r(s)\, ds\right) dt + \mathcal{A}(T) \\ = & \int_0^T w(t) L(t) \exp\left(\int_t^T r(s)\, ds\right) dt + \mathcal{A}(0) \exp\left(\int_0^T r(s)\, ds\right). \end{aligned} \tag{17}$$

- Differentiating with respect to T and dividing $L(t)$ gives (13).
- Now imagine that (17) applies to a finite-horizon economy .
- Flow budget constraint (13) by itself does not guarantee that $\mathcal{A}(T) \geq 0$.
- Thus in finite-horizon we would simply impose (17) as a boundary condition.
- The no Ponzi game condition is the infinite horizon equivalent of this (obtained by dividing by $L(t)$ and multiplying both sides by $\exp\left(-\int_0^T r(s)\, ds\right)$ and taking the limit as $T \to \infty$).

Definition of Equilibrium

Definition A competitive equilibrium of the Ramsey economy consists of paths $[C(t), K(t), w(t), R(t)]_{t=0}^{\infty}$, such that the representative household maximizes its utility given initial capital stock $K(0)$ and the time path of prices $[w(t), R(t)]_{t=0}^{\infty}$, and all markets clear.

- Notice refers to the entire path of quantities and prices, not just steady-state equilibrium.

Definition A competitive equilibrium of the Ramsey economy consists of paths $[c(t), k(t), w(t), R(t)]_{t=0}^{\infty}$, such that the representative household maximizes (9) subject to (13) and (15) given initial capital-labor ratio $k(0)$, factor prices $[w(t), R(t)]_{t=0}^{\infty}$ as in (11) and (12), and the rate of return on assets $r(t)$ given by (14).

Household Maximization I

- Maximize (9) subject to (13) and (16).
- First ignore (16) and set up the current-value Hamiltonian:

$$\hat{H}(a, c, \mu) = u(c(t)) + \mu(t)[w(t) + (r(t) - n)a(t) - c(t)],$$

- *Maximum Principle* $\Rightarrow$ "candidate solution"

$$\begin{aligned} \hat{H}_c(a, c, \mu) &= u'(c(t)) - \mu(t) = 0 \\ \hat{H}_a(a, c, \mu) &= \mu(t)(r(t) - n) \\ &= -\dot{\mu}(t) + (\rho - n)\mu(t) \\ \lim_{t\to\infty}[\exp(-(\rho - n)t)\mu(t)a(t)] &= 0. \end{aligned}$$

  and the transition equation (13).
- Notice transversality condition is written in terms of the current-value costate variable.

Household Maximization II

- For any $\mu(t) > 0$, $\hat{H}(a, c, \mu)$ is a concave function of (a, c) and strictly concave in c.
- The first necessary condition implies $\mu(t) > 0$ for all t.
- Therefore, *Sufficient Conditions* imply that the candidate solution is an optimum (is it unique?)
- Rearrange the second condition:

$$\frac{\dot{\mu}(t)}{\mu(t)} = -(r(t) - \rho), \tag{18}$$

- First necessary condition implies,

$$u'(c(t)) = \mu(t). \tag{19}$$

Household Maximization III

- Differentiate with respect to time and divide by $\mu(t)$,

$$\frac{u''(c(t))\,c(t)}{u'(c(t))}\frac{\dot{c}(t)}{c(t)} = \frac{\dot{\mu}(t)}{\mu(t)}.$$

- Substituting into (18) gives

$$\frac{\dot{c}(t)}{c(t)} = \frac{1}{\varepsilon_u(c(t))}(r(t) - \rho) \tag{20}$$

where

$$\varepsilon_u(c(t)) \equiv -\frac{u''(c(t))\,c(t)}{u'(c(t))} \tag{21}$$

is the elasticity of the marginal utility $u'(c(t))$ or the inverse of the *intertemporal elasticity of substitution*.

- Consumption will grow over time when the discount rate is less than the rate of return on assets.

Household Maximization IV

- Integrating (18),

$$\mu(t) = \mu(0)\exp\left(-\int_0^t (r(s)-\rho)\,ds\right)$$
$$= u'(c(0))\exp\left(-\int_0^t (r(s)-\rho)\,ds\right),$$

- Substituting into the transversality condition,

$$0 = \lim_{t\to\infty}\left[\exp(-(\rho-n)t)\,a(t)\,u'(c(0))\exp\left(-\int_0^t (r(s)-\rho)\,ds\right)\right.$$
$$0 = \lim_{t\to\infty}\left[a(t)\exp\left(-\int_0^t (r(s)-n)\,ds\right)\right].$$

- Thus the "strong version" of the no-Ponzi condition, (16) has to hold.

Household Maximization V

- Since $a(t) = k(t)$, transversality condition is also equivalent to

$$\lim_{t\to\infty}\left[\exp\left(-\int_0^t (r(s)-n)\,ds\right)k(t)\right]=0$$

- Notice term $\exp\left(-\int_0^t r(s)\,ds\right)$ is a present-value factor: converts a unit of income at t to a unit of income at 0.
- When $r(s) = r$, factor would be $\exp(-rt)$. More generally, define an average interest rate between dates 0 and t given by $\frac{1}{t}\int_0^t r(s)\,ds$.

Equilibrium Prices

- Equilibrium prices given by (11) and (12).
- Thus market rate of return for consumers, $r(t)$, is given by (14), i.e.,

$$r(t) = f'(k(t)) - \delta.$$

- Substituting this into the consumer's problem, we have

$$\frac{\dot{c}(t)}{c(t)} = \frac{1}{\varepsilon_u(c(t))}\left(f'(k(t)) - \delta - \rho\right) \tag{22}$$

Optimal Growth I

- In an economy that admits a representative household, optimal growth involves maximization of utility of representative household subject to technology and feasibility constraints:

$$\max_{[k(t),c(t)]_{t=0}^{\infty}} \int_0^{\infty} \exp\left(-\left(\rho - n\right) t\right) u\left(c\left(t\right)\right) dt,$$

subject to

$$\dot{k}\left(t\right) = f\left(k\left(t\right)\right) - (n+\delta)k\left(t\right) - c\left(t\right),$$

and $k(0) > 0$.

- Versions of the First and Second Welfare Theorems for economies with a continuum of commodities: solution to this problem should be the same as the equilibrium growth problem.
- But straightforward to show the equivalence of the two problems.

Optimal Growth II

- Again set up the current-value Hamiltonian:

$$\hat{H}(k, c, \mu) = u(c(t)) + \mu(t)\left[f(k(t)) - (n+\delta)k(t) - c(t)\right],$$

- Candidate solution from the *Maximum Principle*:

$$\begin{aligned} \hat{H}_c(k, c, \mu) &= 0 = u'(c(t)) - \mu(t), \\ \hat{H}_k(k, c, \mu) &= -\dot{\mu}(t) + (\rho - n)\mu(t) \\ &= \mu(t)\left(f'(k(t)) - \delta - n\right), \\ \lim_{t\to\infty}\left[\exp(-(\rho - n)t)\mu(t)k(t)\right] &= 0. \end{aligned}$$

- *Sufficiency Theorem* $\Rightarrow$ unique solution ($\hat{H}$ and thus the maximized Hamiltonian strictly concave in k).

Optimal Growth III

- Repeating the same steps as before, these imply

$$\frac{\dot{c}(t)}{c(t)} = \frac{1}{\varepsilon_u(c(t))}\left(f'(k(t)) - \delta - \rho\right),$$

which is identical to (22), and the transversality condition

$$\lim_{t\to\infty}\left[k(t)\exp\left(-\int_0^t \left(f'(k(s)) - \delta - n\right) ds\right)\right] = 0,$$

which is, in turn, identical to (16).

- Thus the competitive equilibrium is a Pareto optimum and that the Pareto allocation can be decentralized as a competitive equilibrium.

Proposition In the neoclassical growth model described above, with standard assumptions on the production function (assumptions 1-4′), the equilibrium is Pareto optimal and coincides with the optimal growth path maximizing the utility of the representative household.

Steady-State Equilibrium I

- Steady-state equilibrium is defined as an equilibrium path in which capital-labor ratio, consumption and output are constant, thus:

$$\dot{c}(t) = 0.$$

- From (22), as long as $f(k^*) > 0$, *irrespective* of the exact utility function, we must have a capital-labor ratio k^* such that

$$f'(k^*) = \rho + \delta. \tag{23}$$

- Pins down the steady-state capital-labor ratio only as a function of the production function, the discount rate and the depreciation rate.
- *Modified golden rule*: level of the capital stock that does not maximize steady-state consumption, because earlier consumption is preferred to later consumption.

Steady-State Equilibrium II

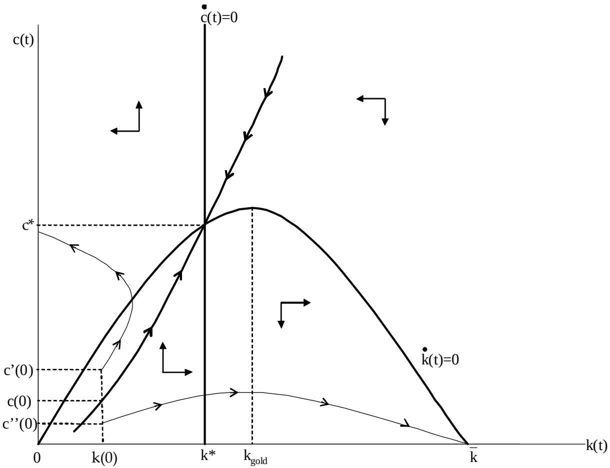

Figure: Steady state in the baseline neoclassical growth model

Steady-State Equilibrium III

- Given k^*, steady-state consumption level:

$$c^* = f(k^*) - (n+\delta)k^*, \tag{24}$$

- Given Assumption $4'$, a steady state where the capital-labor ratio and thus output are constant necessarily satisfies the transversality condition.

Proposition In the neoclassical growth model described above, with Assumptions 1, 2, assumptions on utility above and Assumption $4'$, the steady-state equilibrium capital-labor ratio, k^*, is uniquely determined by (23) and is independent of the utility function. The steady-state consumption per capita, c^*, is given by (24).

- Comparative statics again straightforward.

Steady-State Equilibrium IV

- Instead of the saving rate, it is now the discount factor that affects the rate of capital accumulation.
- Loosely, lower discount rate implies greater patience and thus greater savings.
- Without technological progress, the steady-state saving rate can be computed as

$$s^* = \frac{\delta k^*}{f(k^*)}. \tag{25}$$

- Rate of population growth has no impact on the steady state capital-labor ratio, which contrasts with the basic Solow model.
  - result depends on the way in which intertemporal discounting takes place.
- k^* and thus c^* do *not* depend on the instantaneous utility function $u(\cdot)$.
  - form of the utility function only affects the transitional dynamics
  - not true when there is technological change,.

Transitional Dynamics I

- Equilibrium is determined by two differential equations:

$$\dot{k}(t) = f(k(t)) - (n+\delta)k(t) - c(t) \tag{26}$$

and

$$\frac{\dot{c}(t)}{c(t)} = \frac{1}{\varepsilon_u(c(t))}\left(f'(k(t)) - \delta - \rho\right). \tag{27}$$

- Moreover, we have an initial condition $k(0) > 0$, also a boundary condition at infinity,

$$\lim_{t\to\infty}\left[k(t)\exp\left(-\int_0^t \left(f'(k(s)) - \delta - n\right) ds\right)\right] = 0.$$

Transitional Dynamics II

- Appropriate notion of *saddle-path stability*:
  - consumption level (or equivalently μ) is the control variable, and $c(0)$ (or $\mu(0)$) is free: has to adjust to satisfy transversality condition
  - since $c(0)$ or $\mu(0)$ can jump to any value, need that there exists a one-dimensional manifold tending to the steady state (*stable arm*).
  - If there were more than one path equilibrium would be indeterminate.
- Economic forces are such that indeed there will be a one-dimensional manifold of stable solutions tending to the unique steady state.
- See Figure.

Transitional Dynamics III

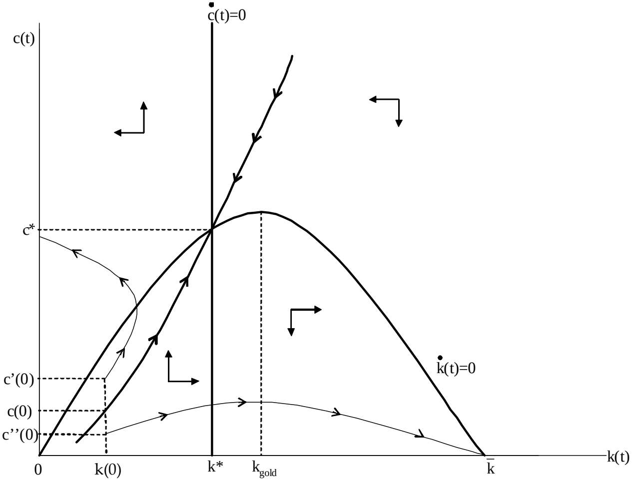

Figure: Transitional dynamics in the baseline neoclassical growth model

Transitional Dynamics: Sufficiency

- Why is the stable arm unique?
- Three different (complementary) lines of analysis
  1. Sufficiency Theorem
  2. Global Stability Analysis
  3. Local Stability Analysis
- *Sufficiency Theorem*: solution starting in $c(0)$ and limiting to the steady state satisfies the necessary and sufficient conditions, and thus unique solution to household problem and unique equilibrium.

Proposition In the neoclassical growth model described above, with Assumptions 1, 2, assumptions on utility above and Assumption 4′, there exists a unique equilibrium path starting from any $k(0) > 0$ and converging to the unique steady-state (k^*, c^*) with k^* given by (23). Moreover, if $k(0) < k^*$, then $k(t) \uparrow k^*$ and $c(t) \uparrow c^*$, whereas if $k(0) > k^*$, then $k(t) \downarrow k^*$ and $c(t) \downarrow c^*$.

Global Stability Analysis

- Alternative argument:
  - if $c(0)$ started below it, say $c''(0)$, consumption would reach zero, thus capital would accumulate continuously until the maximum level of capital (reached with zero consumption) $\bar{k} > k_{gold}$. This would violate the transversality condition. Can be established that transversality condition necessary in this case, thus such paths can be ruled out.
  - if $c(0)$ started above this stable arm, say at $c'(0)$, the capital stock would reach 0 in finite time, while consumption would remain positive. But this would violate feasibility (a little care is necessary with this argument, since necessary conditions do not apply at the boundary).

Local Stability Analysis I

- Linearize the set of differential equations, and looking at their eigenvalues.
- Recall the two differential equations:

$$\dot{k}(t) = f(k(t)) - (n+\delta)k(t) - c(t)$$

and

$$\frac{\dot{c}(t)}{c(t)} = \frac{1}{\varepsilon_u(c(t))}\left(f'(k(t)) - \delta - \rho\right).$$

- Linearizing these equations around the steady state (k^*, c^*), we have (suppressing time dependence)

$$\begin{aligned}\dot{k} &= \text{constant} + \left(f'(k^*) - n - \delta\right)(k - k^*) - c\\ \dot{c} &= \text{constant} + \frac{c^* f''(k^*)}{\varepsilon_u(c^*)}(k - k^*).\end{aligned}$$

Local Stability Analysis II

- From (23), $f'(k^*) - \delta = \rho$, so the eigenvalues of this two-equation system are given by the values of ξ that solve the following quadratic form:

$$\det \begin{pmatrix} \rho - n - \xi & -1 \\ \frac{c^* f''(k^*)}{\varepsilon_u(c^*)} & 0 - \xi \end{pmatrix} = 0.$$

- Since $c^* f''(k^*) / \varepsilon_u(c^*) < 0$, there are two real eigenvalues, one negative and one positive.
- Thus local analysis also leads to the same conclusion, but can only establish local stability.

Neoclassical Growth Model in Discrete Time

- Economically, nothing is different in discrete time.
- Mathematically, a few details need to be sorted out.
- Sometimes discrete time will be more convenient to work with, and sometimes continuous time.
- See recitation for details of the discrete time model.

Technological Change and the Neoclassical Model

- Extend the production function to:

$$Y(t) = F[K(t), A(t)L(t)], \tag{28}$$

  where

$$A(t) = \exp(gt)A(0).$$

- A consequence of Uzawa Theorem.: (28) imposes purely labor-augmenting—Harrod-neutral—technological change.
- Continue to adopt all usual assumptions, and Assumption 4′ will be strengthened further in order to ensure finite discounted utility in the presence of sustained economic growth.

Technological Change II

- Define

$$\begin{aligned}\hat{y}(t) &\equiv \frac{Y(t)}{A(t)L(t)} \\ &= F\left[\frac{K(t)}{A(t)L(t)}, 1\right] \\ &\equiv f(k(t)),\end{aligned}$$

where

$$k(t) \equiv \frac{K(t)}{A(t)L(t)}. \tag{29}$$

- Also need to impose a further assumption on preferences in order to ensure balanced growth.

Technological Change III

- Define balanced growth as a pattern of growth consistent with the *Kaldor facts* of constant capital-output ratio and capital share in national income.
- These two observations together also imply that the rental rate of return on capital, $R(t)$, has to be constant, which, from (14), implies that $r(t)$ has to be constant.
- Again refer to an equilibrium path that satisfies these conditions as a balanced growth path (BGP).
- Balanced growth also requires that consumption and output grow at a constant rate. Euler equation

$$\frac{\dot{c}(t)}{c(t)} = \frac{1}{\varepsilon_u(c(t))}(r(t) - \rho).$$

Technological Change IV

- If $r(t) \to r^*$, then $\dot{c}(t)/c(t) \to g_c$ is only possible if $\varepsilon_u(c(t)) \to \varepsilon_u$, i.e., if the elasticity of marginal utility of consumption is asymptotically constant.
- Thus balanced growth is only consistent with utility functions that have asymptotically constant elasticity of marginal utility of consumption.

Proposition Balanced growth in the neoclassical model requires that asymptotically (as $t \to \infty$) all technological change is purely labor augmenting and the elasticity of intertemporal substitution, $\varepsilon_u(c(t))$, tends to a constant ε_u.

Example: CRRA Utility I

- Recall the Arrow-Pratt coefficient of relative risk aversion for a twice-continuously differentiable concave utility function $U(c)$ is

$$\mathcal{R} = -\frac{U''(c)\, c}{U'(c)}.$$

- Constant relative risk aversion (CRRA) utility function satisfies the property that $\mathcal{R}$ is constant.
- Integrating both sides of the previous equation, setting $\mathcal{R}$ to a constant, implies that the family of CRRA utility functions is given by

$$U(c) = \begin{cases} \frac{c^{1-\theta}-1}{1-\theta} & \text{if } \theta \neq 1 \text{ and } \theta \geq 0 \\ \ln c & \text{if } \theta = 1 \end{cases},$$

  with the coefficient of relative risk aversion given by θ.
- Details: see recitation.

Technological Change V

- Given the restriction that balanced growth is only possible with a constant elasticity of intertemporal substitution, start with

$$u(c(t)) = \begin{cases} \frac{c(t)^{1-\theta}-1}{1-\theta} & \text{if } \theta \neq 1 \text{ and } \theta \geq 0 \\ \ln c(t) & \text{if } \theta = 1 \end{cases},$$

- Elasticity of marginal utility of consumption, ε_u, is given by θ.
- When $\theta = 0$, these represent linear preferences, when $\theta = 1$, we have log preferences, and as $\theta \to \infty$, infinitely risk-averse, and infinitely unwilling to substitute consumption over time.
- Assume that the economy admits a representative household with CRRA preferences

$$\int_0^\infty \exp(-(\rho - n)t) \frac{\tilde{c}(t)^{1-\theta} - 1}{1-\theta} dt, \tag{30}$$

Technological Change VI

- $\tilde{c}(t) \equiv C(t) / L(t)$ is per capita consumption.
- Refer to this model, with labor-augmenting technological change and CRRA preference as given by (30) as the *canonical model*
- Euler equation takes the simpler form:

$$\frac{\dot{\tilde{c}}(t)}{\tilde{c}(t)} = \frac{1}{\theta}(r(t) - \rho). \tag{31}$$

- Steady-state equilibrium first: since with technological progress there will be growth in per capita income, $\tilde{c}(t)$ will grow.

Technological Change VII

- Instead define

$$\begin{aligned} c(t) &\equiv \frac{C(t)}{A(t)L(t)} \\ &\equiv \frac{\tilde{c}(t)}{A(t)}. \end{aligned}$$

- This normalized consumption level will remain constant along the BGP:

$$\begin{aligned} \frac{\dot{c}(t)}{c(t)} &\equiv \frac{\dot{\tilde{c}}(t)}{\tilde{c}(t)} - g \\ &= \frac{1}{\theta}\left(r(t) - \rho - \theta g\right). \end{aligned}$$

Technological Change VIII

- For the accumulation of capital stock:

$$\dot{k}(t) = f(k(t)) - c(t) - (n + g + \delta) k(t),$$

where $k(t) \equiv K(t) / A(t) L(t)$.

- Transversality condition, in turn, can be expressed as

$$\lim_{t \to \infty} \left\{ k(t) \exp\left(- \int_0^t \left[f'(k(s)) - g - \delta - n \right] ds \right) \right\} = 0. \qquad (32)$$

- In addition, equilibrium $r(t)$ is still given by (14), so

$$r(t) = f'(k(t)) - \delta$$

Technological Change IX

- Since in steady state $c(t)$ must remain constant:

$$r(t) = \rho + \theta g$$

or

$$f'(k^*) = \rho + \delta + \theta g, \tag{33}$$

- Pins down the steady-state value of the normalized capital ratio k^* uniquely.
- Normalized consumption level is then given by

$$c^* = f(k^*) - (n + g + \delta) k^*, \tag{34}$$

- Per capita consumption grows at the rate g.

Technological Change X

- Because there is growth, to make sure that the transversality condition is in fact satisfied substitute (33) into (32):

$$\lim_{t\to\infty}\left\{k\left(t\right)\exp\left(-\int_0^t\left[\rho-\left(1-\theta\right)g-n\right]ds\right)\right\}=0,$$

- Can only hold if $\rho-\left(1-\theta\right)g-n>0$, or alternatively :

ASSUMPTION 4:

$$\rho-n>\left(1-\theta\right)g.$$

- Remarks:
  - Strengthens Assumption $4'$ when $\theta<1$.
  - Alternatively, recall in steady state $r=\rho+\theta g$ and the growth rate of output is $g+n$.
  - Therefore, equivalent to requiring that $r>g+n$.

Technological Change XI

Proposition Consider the neoclassical growth model with labor augmenting technological progress at the rate g and preferences given by (30). Suppose that Assumptions 1, 2, assumptions on utility above hold and $\rho - n > (1 - \theta) g$. Then there exists a unique balanced growth path with a normalized capital to effective labor ratio of k^*, given by (33), and output per capita and consumption per capita grow at the rate g.

- Steady-state capital-labor ratio no longer independent of preferences, depends on θ.
  - Positive growth in output per capita, and thus in consumption per capita.
  - With upward-sloping consumption profile, willingness to substitute consumption today for consumption tomorrow determines accumulation and thus equilibrium effective capital-labor ratio.

Transitional Dynamics with Technological Change

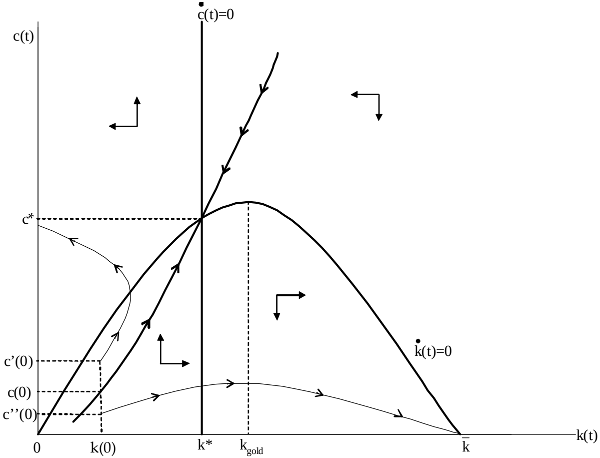

Figure: Transitional dynamics in the neoclassical growth model with technological change.

Technological Change XII

- Steady-state effective capital-labor ratio, k^*, is determined endogenously, but steady-state growth rate of the economy is given exogenously and equal to g.

Proposition Consider the neoclassical growth model with labor augmenting technological progress at the rate g and preferences given by (30). Suppose that Assumptions 1, 2, assumptions on utility above hold and $\rho - n > (1-\theta) g$. Then there exists a unique equilibrium path of normalized capital and consumption, $(k(t), c(t))$ converging to the unique steady-state (k^*, c^*) with k^* given by (33). Moreover, if $k(0) < k^*$, then $k(t) \uparrow k^*$ and $c(t) \uparrow c^*$, whereas if $k(0) > k^*$, then $c(t) \downarrow k^*$ and $c(t) \downarrow c^*$.

Example: CRRA and Cobb-Douglas

- One solvable case: CRRA (or even better log) preferences and Cobb-Douglas production function, given by $F(K, AL) = K^{\alpha}(AL)^{1-\alpha}$, so that
$$f(k) = k^{\alpha}.$$
- See recitation.

Comparative Dynamics I

- Comparative statics: changes in steady state in response to changes in parameters.
- Comparative dynamics look at how the entire equilibrium path of variables changes in response to a change in policy or parameters.
- Look at the effect of a change in tax on capital (or discount rate ρ)
- Consider the neoclassical growth in steady state (k^*, c^*).
- Tax declines to $\tau' < \tau$.
- From Propositions above, after the change there exists a unique steady state equilibrium that is saddle path stable.
- Let this steady state be denoted by (k^{**}, c^{**}).
- Since $\tau' < \tau$, $k^{**} > k^*$ while the equilibrium growth rate will remain unchanged.

Comparative Dynamics II

- Figure: drawn assuming change is unanticipated and occurs at some date T.
- At T, curve corresponding to $\dot{c}/c = 0$ shifts to the right and laws of motion represented by the phase diagram change.
- Following the decline c^* is above the stable arm of the new dynamical system: consumption must drop immediately
- Then consumption slowly increases along the stable arm
- Overall level of normalized consumption will necessarily increase, since the intersection between the curve for $\dot{c}/c = 0$ and for $\dot{k}/k = 0$ will necessarily be to the left side of k_{gold}.

Comparative Dynamics III

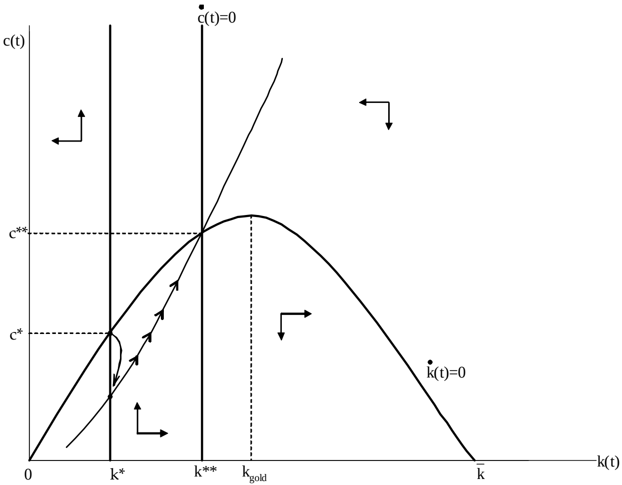

Figure: The dynamic response of capital and consumption to a decline in capital

The Role of Policy I

- Growth of per capita consumption and output per worker (per capita) are determined exogenously.
- But level of income, depends on $1/\theta$, ρ, δ, n, and naturally the form of $f(\cdot)$.
- Proximate causes of differences in income per capita: here explain those differences only in terms of preference and technology parameters.
- Link between proximate and potential fundamental causes:
  - e.g. intertemporal elasticity of substitution and the discount rate can be as related to cultural or geographic factors.
- But an explanation for cross-country and over-time differences in economic growth based on differences or changes in preferences is unlikely to be satisfactory.
- More appealing: link incentives to accumulate physical capital (and human capital and technology) to the institutional environment.

The Role of Policy II

- Simple way: through differences in policies.
- Introduce linear tax policy: returns on capital net of depreciation are taxed at the rate τ and the proceeds of this are redistributed back to the consumers.
- Capital accumulation equation remains as above:

$$\dot{k}(t) = f(k(t)) - c(t) - (n + g + \delta)\, k(t),$$

- But interest rate faced by households changes to:

$$r(t) = (1 - \tau)\left(f'(k(t)) - \delta\right),$$

The Role of Policy III

- Growth rate of normalized consumption is then obtained from the consumer Euler equation, (31):

$$\begin{aligned} \frac{\dot{c}(t)}{c(t)} &= \frac{1}{\theta}\left(r(t) - \rho - \theta g\right). \\ &= \frac{1}{\theta}\left((1-\tau)\left(f'(k(t)) - \delta\right) - \rho - \theta g\right). \end{aligned}$$

- Identical argument to that before implies

$$f'(k^*) = \delta + \frac{\rho + \theta g}{1-\tau}. \tag{35}$$

- Higher τ, since $f'(\cdot)$ is decreasing, reduces k^*.
- Higher taxes on capital have the effect of depressing capital accumulation and reducing income per capita.
- But have not so far offered a reason why some countries may tax capital at a higher rate than others.

Conclusions

- Major contribution: open the black box of capital accumulation by specifying the preferences of consumers.
- Also by specifying individual preferences we can explicitly compare equilibrium and optimal growth.
- Paves the way for further analysis of capital accumulation, human capital and endogenous technological progress.
- Did our study of the neoclassical growth model generate new insights about the sources of cross-country income differences and economic growth relative to the Solow growth model? Largely no.
- This model, by itself, does not enable us to answer questions about the fundamental causes of economic growth.
- But it clarifies the nature of the economic decisions so that we are in a better position to ask such questions.