14.452 Economic Growth: Lectures 6 and 7, Neoclassical Growth

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Introduction

- Ramsey or Cass-Koopmans model: differs from the Solow model only because it explicitly models the consumer side and endogenizes savings.
- Beyond its use as a basic growth model, also a workhorse for many areas of macroeconomics.

Preferences, Technology and Demographics I

- Infinite-horizon, continuous time.
- Representative household with instantaneous utility function

$$u\left(c\left(t\right)\right), \tag{1}$$

Assumption $u\left(c\right)$ is strictly increasing, concave, twice continuously differentiable with derivatives u' and u'', and satisfies the following Inada type assumptions:

$$\lim_{c \to 0} u'\left(c\right) = \infty \text{ and } \lim_{c \to \infty} u'\left(c\right) = 0.$$

- Suppose representative household represents set of identical households (normalized to 1).
- Each household has an instantaneous utility function given by (1).
- $L\left(0\right) = 1$ and

$$L\left(t\right) = \exp\left(nt\right). \tag{2}$$

Preferences, Technology and Demographics II

- All members of the household supply their labor inelastically.
- Objective function of each household at $t = 0$:

$$U(0) \equiv \int_0^\infty \exp(-(\rho - n)t)\, u(c(t))\, dt, \tag{3}$$

where
  - $c(t)$=consumption per capita at t,
  - ρ=subjective discount rate, and effective discount rate is $\rho - n$.
- Objective function (3) embeds:
  - Household is fully altruistic towards all of its future members, and makes allocations of consumption (among household members) cooperatively.
  - Strict concavity of $u(\cdot)$
- Thus each household member will have an equal consumption

$$c(t) \equiv \frac{C(t)}{L(t)}$$

Preferences, Technology and Demographics III

- Utility of $u(c(t))$ per household member at time t, total of $L(t)u(c(t)) = \exp(nt)u(c(t))$.
- With discount at rate of $\exp(-\rho t)$, obtain (3).

ASSUMPTION 4′.

$$\rho > n.$$

- Ensures that in the model without growth, discounted utility is finite. Will strengthen it in model with growth.
- Start model without any technological progress.
- Factor and product markets are competitive.
- Production possibilities set of the economy is represented by

$$Y(t) = F[K(t), L(t)],$$

- Standard constant returns to scale and Inada assumptions still hold.

Preferences, Technology and Demographics IV

- Per capita production function $f(\cdot)$

$$
\begin{aligned}
y(t) &\equiv \frac{Y(t)}{L(t)} \\
&= F\left[\frac{K(t)}{L(t)}, 1\right] \\
&\equiv f(k(t)),
\end{aligned}
$$

where, as before,

$$k(t) \equiv \frac{K(t)}{L(t)}. \tag{4}$$

- Competitive factor markets then imply:

$$R(t) = F_K[K(t), L(t)] = f'(k(t)). \tag{5}$$

and

$$w(t) = F_L[K(t), L(t)] = f(k(t)) - k(t) f'(k(t)). \tag{6}$$

Preferences, Technology and Demographics V

- Denote asset holdings of the representative household at time t by $\mathcal{A}(t)$. Then,

$$\dot{\mathcal{A}}(t) = r(t)\mathcal{A}(t) + w(t)L(t) - c(t)L(t)$$

- $r(t)$ is the risk-free market flow rate of return on assets, and $w(t)L(t)$ is the flow of labor income earnings of the household.
- Defining per capita assets as

$$a(t) \equiv \frac{\mathcal{A}(t)}{L(t)},$$

we obtain:

$$\dot{a}(t) = (r(t) - n)a(t) + w(t) - c(t). \tag{7}$$

- Household assets can consist of capital stock, $K(t)$, which they rent to firms and government bonds, $B(t)$.

Preferences, Technology and Demographics VI

- With uncertainty, households would have a portfolio choice between $K(t)$ and riskless bonds.
- With incomplete markets, bonds allow households to smooth idiosyncratic shocks. But for now no need.
- Thus, market clearing $\Rightarrow$

$$a(t) = k(t).$$

- No uncertainty depreciation rate of δ implies

$$r(t) = R(t) - \delta. \tag{8}$$

The Budget Constraint I

- The differential equation

$$\dot{a}(t) = (r(t) - n)\, a(t) + w(t) - c(t)$$

  is a flow constraint
- Not sufficient as a proper budget constraint unless we impose a lower bound on assets.
- Three options:
  1. Lower bound on assets such as $a(t) \geq 0$ for all t
  2. Natural debt limit (see notes).
  3. No-Ponzi Game Condition.
- The first two are not always applicable, so the third is most general.

The Budget Constraint II

- Write the single budget constraint of the form:

$$\int_0^T c(t) L(t) \exp\left(\int_t^T r(s)\,ds\right) dt + \mathcal{A}(T) \quad (9)$$
$$= \int_0^T w(t) L(t) \exp\left(\int_t^T r(s)\,ds\right) dt + \mathcal{A}(0) \exp\left(\int_0^T r(s)\,ds\right)$$

- Differentiating this expression with respect to T and dividing $L(t)$ gives (7).
- Now imagine that (9) applies to a finite-horizon economy ending at date T.
- Flow budget constraint (7) by itself does not guarantee that $\mathcal{A}(T) \geq 0$.
- Thus in finite-horizon we would simply impose (9) as a boundary condition.

The Budget Constraint III

- Infinite-horizon case: no-Ponzi-game condition,

$$\lim_{t\to\infty} a(t) \exp\left(-\int_0^t (r(s)-n)\,ds\right) \geq 0. \tag{10}$$

- Transversality condition ensures individual would never want to have positive wealth asymptotically, so no-Ponzi-game condition can be strengthened to (though not necessary in general):

$$\lim_{t\to\infty} a(t) \exp\left(-\int_0^t (r(s)-n)\,ds\right) = 0. \tag{11}$$

The Budget Constraint IV

- To understand no-Ponzi-game condition, multiply both sides of (9) by $\exp\left(-\int_0^T r(s)\,ds\right)$:

$$\exp\left(-\int_0^t r(s)\,ds\right)\left[\int_0^T c(t)\,L(t)dt+\mathcal{A}(T)\right]$$
$$=\int_0^T w(t)\,L(t)\exp\left(-\int_0^t r(s)\,ds\right)dt+\mathcal{A}(0),$$

- Divide everything by $L(0)$ and note that $L(t)$ grows at the rate n,

$$\int_0^T c(t)\exp\left(-\int_0^t (r(s)-n)\,ds\right)dt$$
$$+\exp\left(-\int_0^T (r(s)-n)\,ds\right)a(T)$$
$$=\int_0^T w(t)\exp\left(-\int_0^t (r(s)-n)\,ds\right)dt+a(0).$$

The Budget Constraint V

- Take the limit as $T \to \infty$ and use the no-Ponzi-game condition (11) to obtain

$$\begin{aligned} & \int_0^\infty c(t) \exp\left(-\int_0^t (r(s) - n)\, ds\right) dt \\ = \quad & a(0) + \int_0^\infty w(t) \exp\left(-\int_0^t (r(s) - n)\, ds\right) dt, \end{aligned}$$

- Thus no-Ponzi-game condition (11) essentially ensures that the individual's lifetime budget constraint holds in infinite horizon.

Definition of Equilibrium

Definition A competitive equilibrium of the Ramsey economy consists of paths $[C(t), K(t), w(t), R(t)]_{t=0}^{\infty}$, such that the representative household maximizes its utility given initial capital stock $K(0)$ and the time path of prices $[w(t), R(t)]_{t=0}^{\infty}$, and all markets clear.

- Notice refers to the entire path of quantities and prices, not just steady-state equilibrium.

Definition A competitive equilibrium of the Ramsey economy consists of paths $[c(t), k(t), w(t), R(t)]_{t=0}^{\infty}$, such that the representative household maximizes (3) subject to (7) and (10) given initial capital-labor ratio $k(0)$, factor prices $[w(t), R(t)]_{t=0}^{\infty}$ as in (5) and (6), and the rate of return on assets $r(t)$ given by (8).

Household Maximization I

- Maximize (3) subject to (7) and (11).
- First ignore (11) and set up the current-value Hamiltonian:

$$\hat{H}(a, c, \mu) = u(c(t)) + \mu(t)[w(t) + (r(t) - n)a(t) - c(t)],$$

- *Maximum Principle* $\Rightarrow$ "candidate solution"

$$\begin{aligned} \hat{H}_c(a, c, \mu) &= u'(c(t)) - \mu(t) = 0 \\ \hat{H}_a(a, c, \mu) &= \mu(t)(r(t) - n) \\ &= -\dot{\mu}(t) + (\rho - n)\mu(t) \\ \lim_{t\to\infty}[\exp(-(\rho - n)t)\mu(t)a(t)] &= 0. \end{aligned}$$

  and the transition equation (7).
- Notice transversality condition is written in terms of the current-value costate variable.

Household Maximization II

- For any $\mu(t) > 0$, $\hat{H}(a, c, \mu)$ is a concave function of (a, c) and strictly concave in c.
- The first necessary condition implies $\mu(t) > 0$ for all t.
- Therefore, *Sufficient Conditions* imply that the candidate solution is an optimum (is it unique?)
- Rearrange the second condition:

$$\frac{\dot{\mu}(t)}{\mu(t)} = -(r(t) - \rho), \tag{12}$$

- First necessary condition implies,

$$u'(c(t)) = \mu(t). \tag{13}$$

Household Maximization III

- Differentiate with respect to time and divide by $\mu(t)$,

$$\frac{u''(c(t))\,c(t)}{u'(c(t))}\frac{\dot{c}(t)}{c(t)} = \frac{\dot{\mu}(t)}{\mu(t)}.$$

- Substituting into (12), obtain another form of the consumer Euler equation:

$$\frac{\dot{c}(t)}{c(t)} = \frac{1}{\varepsilon_u(c(t))}(r(t) - \rho) \qquad (14)$$

  where

$$\varepsilon_u(c(t)) \equiv -\frac{u''(c(t))\,c(t)}{u'(c(t))} \qquad (15)$$

  is the elasticity of the marginal utility $u'(c(t))$.

- Consumption will grow over time when the discount rate is less than the rate of return on assets.

Household Maximization IV

- Speed at which consumption will grow is related to the elasticity of marginal utility of consumption, $\varepsilon_u(c(t))$.
- Even more importantly, $\varepsilon_u(c(t))$ is the inverse of the *intertemporal elasticity of substitution*:
  - regulates willingness to substitute consumption (or any other attribute that yields utility) over time.
  - Elasticity between dates t and $s > t$ is defined as

$$\sigma_u(t,s) = -\frac{d\log(c(s)/c(t))}{d\log(u'(c(s))/u'(c(t)))}.$$

  - As $s \downarrow t$,

$$\sigma_u(t,s) \to \sigma_u(t) = -\frac{u'(c(t))}{u''(c(t))\,c(t)} = \frac{1}{\varepsilon_u(c(t))}. \tag{16}$$

Household Maximization V

- Integrating (12),

$$\begin{aligned}\mu(t) &= \mu(0)\exp\left(-\int_0^t (r(s)-\rho)\,ds\right)\\ &= u'(c(0))\exp\left(-\int_0^t (r(s)-\rho)\,ds\right),\end{aligned}$$

- Substituting into the transversality condition,

$$\begin{aligned}0 &= \lim_{t\to\infty}\left[\exp\left(-(\rho-n)\,t\right)a(t)\,u'(c(0))\exp\left(-\int_0^t (r(s)-\rho)\,ds\right)\right.\\ 0 &= \lim_{t\to\infty}\left[a(t)\exp\left(-\int_0^t (r(s)-n)\,ds\right)\right].\end{aligned}$$

- Thus the "strong version" of the no-Ponzi condition, (11) has to hold.

Household Maximization VI

- Since $a(t) = k(t)$, transversality condition is also equivalent to

$$\lim_{t\to\infty}\left[\exp\left(-\int_0^t (r(s)-n)\,ds\right)k(t)\right] = 0$$

- Notice term $\exp\left(-\int_0^t r(s)\,ds\right)$ is a present-value factor: converts a unit of income at t to a unit of income at 0.
- When $r(s) = r$, factor would be $\exp(-rt)$. More generally, define an average interest rate between dates 0 and t

$$\bar{r}(t) = \frac{1}{t}\int_0^t r(s)\,ds. \tag{17}$$

- Thus conversion factor between dates 0 and t is

$$\exp(-\bar{r}(t)\,t),$$

Household Maximization VII

- And the transversality condition

$$\lim_{t\to\infty}\left[\exp\left(-\left(\bar{r}\left(t\right)-n\right)t\right)a\left(t\right)\right]=0. \tag{18}$$

- Recall solution to the differential equation

$$\dot{y}\left(t\right)=b\left(t\right)y\left(t\right)$$

  is

$$y\left(t\right)=y\left(0\right)\exp\left(\int_0^t b\left(s\right)ds\right),$$

- Integrate (14):

$$c\left(t\right)=c\left(0\right)\exp\left(\int_0^t \frac{r\left(s\right)-\rho}{\varepsilon_u\left(c\left(s\right)\right)}ds\right)$$

- Once we determine $c\left(0\right)$, path of consumption can be exactly solved out.

Household Maximization VIII

- Special case where $\varepsilon_u(c(s))$ is constant, $\varepsilon_u(c(s)) = \theta$:

$$c(t) = c(0)\exp\left(\left(\frac{\bar{r}(t) - \rho}{\theta}\right)t\right),$$

- Lifetime budget constraint simplifies to

$$\begin{aligned} & \int_0^\infty c(t)\exp\left(-(\bar{r}(t) - n)t\right)dt \\ = \quad & a(0) + \int_0^\infty w(t)\exp\left(-(\bar{r}(t) - n)t\right)dt, \end{aligned}$$

- Substituting for $c(t)$,

$$\begin{aligned} c(0) \quad = \quad & \int_0^\infty \exp\left(-\left(\frac{(1-\theta)\bar{r}(t)}{\theta} - \frac{\rho}{\theta} + n\right)t\right)dt \qquad (19) \\ & \times \left[a(0) + \int_0^\infty w(t)\exp\left(-(\bar{r}(t) - n)t\right)dt\right] \end{aligned}$$

Equilibrium Prices

- Equilibrium prices given by (5) and (6).
- Thus market rate of return for consumers, $r(t)$, is given by (8), i.e.,

$$r(t) = f'(k(t)) - \delta.$$

- Substituting this into the consumer's problem, we have

$$\frac{\dot{c}(t)}{c(t)} = \frac{1}{\varepsilon_u(c(t))}\left(f'(k(t)) - \delta - \rho\right) \tag{20}$$

- Equation (19) similarly generalizes for the case of iso-elastic utility function.

Optimal Growth I

- In an economy that admits a representative household, optimal growth involves maximization of utility of representative household subject to technology and feasibility constraints:

$$\max_{[k(t),c(t)]_{t=0}^{\infty}} \int_0^{\infty} \exp\left(-\left(\rho - n\right) t\right) u\left(c\left(t\right)\right) dt,$$

subject to

$$\dot{k}\left(t\right) = f\left(k\left(t\right)\right) - (n+\delta)k\left(t\right) - c\left(t\right),$$

and $k\left(0\right) > 0$.

- Versions of the First and Second Welfare Theorems for economies with a continuum of commodities: solution to this problem should be the same as the equilibrium growth problem.
- But straightforward to show the equivalence of the two problems.

Optimal Growth II

- Again set up the current-value Hamiltonian:

$$\hat{H}(k, c, \mu) = u(c(t)) + \mu(t)\left[f(k(t)) - (n+\delta)k(t) - c(t)\right],$$

- Candidate solution from the *Maximum Principle*:

$$\begin{aligned}
\hat{H}_c(k, c, \mu) &= 0 = u'(c(t)) - \mu(t), \\
\hat{H}_k(k, c, \mu) &= -\dot{\mu}(t) + (\rho - n)\mu(t) \\
&= \mu(t)\left(f'(k(t)) - \delta - n\right), \\
\lim_{t\to\infty}\left[\exp(-(\rho - n)t)\mu(t)k(t)\right] &= 0.
\end{aligned}$$

- *Sufficiency Theorem* $\Rightarrow$ unique solution ($\hat{H}$ and thus the maximized Hamiltonian strictly concave in k).

Optimal Growth III

- Repeating the same steps as before, these imply

$$\frac{\dot{c}(t)}{c(t)} = \frac{1}{\varepsilon_u(c(t))}\left(f'(k(t)) - \delta - \rho\right),$$

which is identical to (20), and the transversality condition

$$\lim_{t\to\infty}\left[k(t)\exp\left(-\int_0^t \left(f'(k(s)) - \delta - n\right) ds\right)\right] = 0,$$

which is, in turn, identical to (11).

- Thus the competitive equilibrium is a Pareto optimum and that the Pareto allocation can be decentralized as a competitive equilibrium.

Proposition In the neoclassical growth model described above, with standard assumptions on the production function (assumptions 1-4′), the equilibrium is Pareto optimal and coincides with the optimal growth path maximizing the utility of the representative household.

Steady-State Equilibrium I

- Steady-state equilibrium is defined as an equilibrium path in which capital-labor ratio, consumption and output are constant, thus:

$$\dot{c}(t) = 0.$$

- From (20), as long as $f(k^*) > 0$, *irrespective* of the exact utility function, we must have a capital-labor ratio k^* such that

$$f'(k^*) = \rho + \delta, \tag{21}$$

- Pins down the steady-state capital-labor ratio only as a function of the production function, the discount rate and the depreciation rate.
- *Modified golden rule*: level of the capital stock that does not maximize steady-state consumption, because earlier consumption is preferred to later consumption.

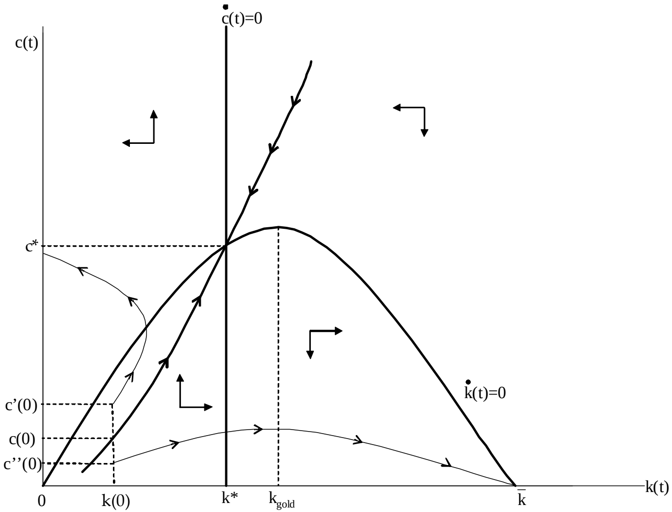

Figure: Steady state in the baseline neoclassical growth model

Steady-State Equilibrium II

- Given k^*, steady-state consumption level:

$$c^* = f(k^*) - (n+\delta)k^*, \tag{22}$$

- Given Assumption 4′, a steady state where the capital-labor ratio and thus output are constant necessarily satisfies the transversality condition.

Proposition In the neoclassical growth model described above, with Assumptions 1, 2, assumptions on utility above and Assumption 4′, the steady-state equilibrium capital-labor ratio, k^*, is uniquely determined by (21) and is independent of the utility function. The steady-state consumption per capita, c^*, is given by (22).

- Parameterize the production function as follows

$$f(k) = A\tilde{f}(k),$$

Steady-State Equilibrium III

- Since $f(k)$ satisfies the regularity conditions imposed above, so does $\tilde{f}(k)$.

Proposition Consider the neoclassical growth model described above, with Assumptions 1, 2, assumptions on utility above and Assumption 4′, and suppose that $f(k) = A\tilde{f}(k)$. Denote the steady-state level of the capital-labor ratio by $k^*(A, \rho, n, \delta)$ and the steady-state level of consumption per capita by $c^*(A, \rho, n, \delta)$ when the underlying parameters are A, ρ, n and δ. Then we have

$$
\begin{aligned}
\frac{\partial k^*(\cdot)}{\partial A} &> 0, \ \frac{\partial k^*(\cdot)}{\partial \rho} < 0, \ \frac{\partial k^*(\cdot)}{\partial n} = 0 \text{ and } \frac{\partial k^*(\cdot)}{\partial \delta} < 0 \\
\frac{\partial c^*(\cdot)}{\partial A} &> 0, \ \frac{\partial c^*(\cdot)}{\partial \rho} < 0, \ \frac{\partial c^*(\cdot)}{\partial n} < 0 \text{ and } \frac{\partial c^*(\cdot)}{\partial \delta} < 0.
\end{aligned}
$$

Steady-State Equilibrium IV

- Instead of the saving rate, it is now the discount factor that affects the rate of capital accumulation.
- Loosely, lower discount rate implies greater patience and thus greater savings.
- Without technological progress, the steady-state saving rate can be computed as

$$s^* = \frac{\delta k^*}{f(k^*)}. \tag{23}$$

- Rate of population growth has no impact on the steady state capital-labor ratio, which contrasts with the basic Solow model.
    - result depends on the way in which intertemporal discounting takes place.
- k^* and thus c^* do *not* depend on the instantaneous utility function $u(\cdot)$.
    - form of the utility function only affects the transitional dynamics
    - not true when there is technological change,.

Transitional Dynamics I

- Equilibrium is determined by two differential equations:

$$\dot{k}(t) = f(k(t)) - (n+\delta)k(t) - c(t) \tag{24}$$

and

$$\frac{\dot{c}(t)}{c(t)} = \frac{1}{\varepsilon_u(c(t))}\left(f'(k(t)) - \delta - \rho\right). \tag{25}$$

- Moreover, we have an initial condition $k(0) > 0$, also a boundary condition at infinity,

$$\lim_{t\to\infty}\left[k(t)\exp\left(-\int_0^t \left(f'(k(s)) - \delta - n\right) ds\right)\right] = 0.$$

Transitional Dynamics II

- Appropriate notion of *saddle-path stability*:
  - consumption level (or equivalently μ) is the control variable, and $c(0)$ (or $\mu(0)$) is free: has to adjust to satisfy transversality condition
  - since $c(0)$ or $\mu(0)$ can jump to any value, need that there exists a one-dimensional manifold tending to the steady state (*stable arm*).
  - If there were more than one path equilibrium would be indeterminate.
- Economic forces are such that indeed there will be a one-dimensional manifold of stable solutions tending to the unique steady state.
- See Figure.

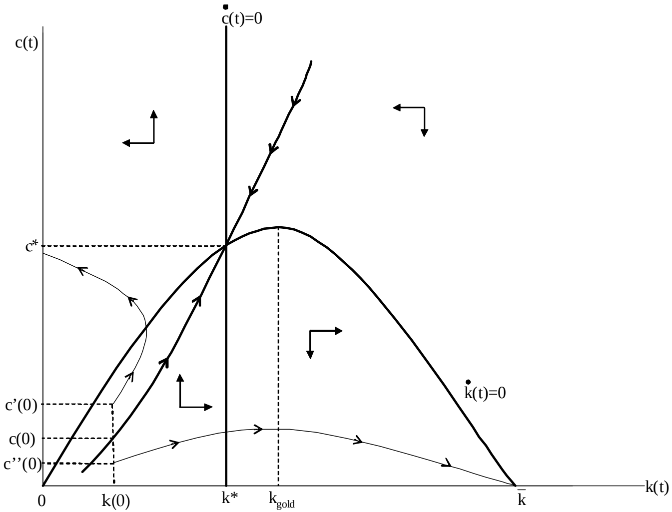

Figure: Transitional dynamics in the baseline neoclassical growth model

Transitional Dynamics: Sufficiency

- Why is the stable arm unique?
- Three different (complementary) lines of analysis
  1. Sufficiency Theorem
  2. Global Stability Analysis
  3. Local Stability Analysis
- *Sufficiency Theorem*: solution starting in $c(0)$ and limiting to the steady state satisfies the necessary and sufficient conditions, and thus unique solution to household problem and unique equilibrium.

Proposition In the neoclassical growth model described above, with Assumptions 1, 2, assumptions on utility above and Assumption $4'$, there exists a unique equilibrium path starting from any $k(0) > 0$ and converging to the unique steady-state (k^*, c^*) with k^* given by (21). Moreover, if $k(0) < k^*$, then $k(t) \uparrow k^*$ and $c(t) \uparrow c^*$, whereas if $k(0) > k^*$, then $k(t) \downarrow k^*$ and $c(t) \downarrow c^*$.

Global Stability Analysis

- Alternative argument:
  - if $c(0)$ started below it, say $c''(0)$, consumption would reach zero, thus capital would accumulate continuously until the maximum level of capital (reached with zero consumption) $\bar{k} > k_{gold}$. This would violate the transversality condition. Can be established that transversality condition necessary in this case, thus such paths can be ruled out.
  - if $c(0)$ started above this stable arm, say at $c'(0)$, the capital stock would reach 0 in finite time, while consumption would remain positive. But this would violate feasibility (a little care is necessary with this argument, since necessary conditions do not apply at the boundary).

Local Stability Analysis I

- Linearize the set of differential equations, and looking at their eigenvalues.
- Recall the two differential equations:

$$\dot{k}(t) = f(k(t)) - (n+\delta)k(t) - c(t)$$

and

$$\frac{\dot{c}(t)}{c(t)} = \frac{1}{\varepsilon_u(c(t))}\left(f'(k(t)) - \delta - \rho\right).$$

- Linearizing these equations around the steady state (k^*, c^*), we have (suppressing time dependence)

$$\begin{aligned}\dot{k} &= \text{constant} + \left(f'(k^*) - n - \delta\right)(k - k^*) - c \\ \dot{c} &= \text{constant} + \frac{c^* f''(k^*)}{\varepsilon_u(c^*)}(k - k^*).\end{aligned}$$

Local Stability Analysis II

- From (21), $f'(k^*) - \delta = \rho$, so the eigenvalues of this two-equation system are given by the values of ξ that solve the following quadratic form:

$$\det \begin{pmatrix} \rho - n - \xi & -1 \\ \frac{c^* f''(k^*)}{\varepsilon_u(c^*)} & 0 - \xi \end{pmatrix} = 0.$$

- Since $c^* f''(k^*) / \varepsilon_u(c^*) < 0$, there are two real eigenvalues, one negative and one positive.
- Thus local analysis also leads to the same conclusion, but can only establish local stability.

Technological Change and the Neoclassical Model

- Extend the production function to:

$$Y(t) = F[K(t), A(t)L(t)], \tag{26}$$

where

$$A(t) = \exp(gt)A(0).$$

- A consequence of Uzawa Theorem.: (26) imposes purely labor-augmenting—Harrod-neutral—technological change.
- Continue to adopt all usual assumptions, and Assumption 4′ will be strengthened further in order to ensure finite discounted utility in the presence of sustained economic growth.

Technological Change II

- Define

$$\hat{y}(t) \equiv \frac{Y(t)}{A(t)L(t)} = F\left[\frac{K(t)}{A(t)L(t)}, 1\right] \equiv f(k(t)),$$

where

$$k(t) \equiv \frac{K(t)}{A(t)L(t)}. \tag{27}$$

- Also need to impose a further assumption on preferences in order to ensure balanced growth.

Technological Change III

- Define balanced growth as a pattern of growth consistent with the *Kaldor facts* of constant capital-output ratio and capital share in national income.
- These two observations together also imply that the rental rate of return on capital, $R(t)$, has to be constant, which, from (8), implies that $r(t)$ has to be constant.
- Again refer to an equilibrium path that satisfies these conditions as a balanced growth path (BGP).
- Balanced growth also requires that consumption and output grow at a constant rate. Euler equation

$$\frac{\dot{c}(t)}{c(t)} = \frac{1}{\varepsilon_u(c(t))}(r(t) - \rho).$$

Technological Change IV

- If $r(t) \to r^*$, then $\dot{c}(t)/c(t) \to g_c$ is only possible if $\varepsilon_u(c(t)) \to \varepsilon_u$, i.e., if the elasticity of marginal utility of consumption is asymptotically constant.
- Thus balanced growth is only consistent with utility functions that have asymptotically constant elasticity of marginal utility of consumption.

Proposition Balanced growth in the neoclassical model requires that asymptotically (as $t \to \infty$) all technological change is purely labor augmenting and the elasticity of intertemporal substitution, $\varepsilon_u(c(t))$, tends to a constant ε_u.

Example: CRRA Utility I

- Recall the Arrow-Pratt coefficient of relative risk aversion for a twice-continuously differentiable concave utility function $U(c)$ is

$$\mathcal{R} = -\frac{U''(c)\,c}{U'(c)}.$$

- Constant relative risk aversion (CRRA) utility function satisfies the property that $\mathcal{R}$ is constant.
- Integrating both sides of the previous equation, setting $\mathcal{R}$ to a constant, implies that the family of CRRA utility functions is given by

$$U(c) = \begin{cases} \frac{c^{1-\theta}-1}{1-\theta} & \text{if } \theta \neq 1 \text{ and } \theta \geq 0 \\ \ln c & \text{if } \theta = 1 \end{cases},$$

with the coefficient of relative risk aversion given by θ.

Example: CRRA Utility II

- With time separable utility functions, the inverse of the elasticity of intertemporal substitution (defined in equation (16)) and the coefficient of relative risk aversion are identical.
- Thus the family of CRRA utility functions are also those with constant elasticity of intertemporal substitution.
- Link this utility function to the Gorman preferences: consider a slightly different problem in which an individual has preferences defined over the consumption of N commodities $\{c_1, ..., c_N\}$ given by

$$U(\{c_1, ..., c_N\}) = \begin{cases} \sum_{j=1}^{N} \frac{c_j^{1-\theta}}{1-\theta} & \text{if } \theta \neq 1 \text{ and } \theta \geq 0 \\ \sum_{j=1}^{N} \ln c_j & \text{if } \theta = 1 \end{cases}. \tag{28}$$

Example: CRRA Utility III

- Suppose this individual faces a price vector $\mathbf{p} = (p_1, ..., p_N)$ and has income y, so that his budget constraint is

$$\sum_{j=1}^{N} p_j c_j \leq y. \tag{29}$$

- Maximizing utility subject to this budget constraint leads to the indirect utility function

$$v(p,y) = \frac{y^{\frac{\sigma-1}{\sigma}}}{\left[\sum_{j=1}^{N} p_j^{1-\sigma}\right]^{1/\sigma}}$$

- A monotonic transformation (raise it to the power $\sigma / (\sigma - 1)$) leads to Gorman class: CRRA utility functions are within the Gorman class

Example: CRRA Utility IV

- If all individuals have CRRA utility functions, then we can aggregate their preferences and represent them as if it belonged to a single individual.
- Now consider a dynamic version of these preferences (defined over infinite horizon):

$$U = \begin{cases} \sum_{t=0}^{\infty} \beta^t \frac{c(t)^{1-\theta}-1}{1-\theta} & \text{if } \theta \neq 1 \text{ and } \theta \geq 0 \\ \sum_{t=0}^{\infty} \beta^t \ln c(t) & \text{if } \theta = 1 \end{cases}.$$

- The important feature here is not that the coefficient of relative risk aversion constant, but that the intertemporal elasticity of substitution is constant.

Technological Change V

- Given the restriction that balanced growth is only possible with a constant elasticity of intertemporal substitution, start with

$$u\left(c\left(t\right)\right)=\begin{cases}\frac{c(t)^{1-\theta}-1}{1-\theta} & \text{if } \theta\neq 1 \text{ and } \theta\geq 0\\ \ln c(t) & \text{if } \theta=1\end{cases},$$

- Elasticity of marginal utility of consumption, ε_u, is given by θ.
- When $\theta = 0$, these represent linear preferences, when $\theta = 1$, we have log preferences, and as $\theta \to \infty$, infinitely risk-averse, and infinitely unwilling to substitute consumption over time.
- Assume that the economy admits a representative household with CRRA preferences

$$\int_0^\infty \exp\left(-(\rho-n)t\right)\frac{\tilde{c}\left(t\right)^{1-\theta}-1}{1-\theta}dt, \tag{30}$$

Technological Change VI

- $\tilde{c}(t) \equiv C(t)/L(t)$ is per capita consumption.
- Refer to this model, with labor-augmenting technological change and CRRA preference as given by (30) as the *canonical model*
- Euler equation takes the simpler form:

$$\frac{\dot{\tilde{c}}(t)}{\tilde{c}(t)} = \frac{1}{\theta}(r(t) - \rho). \tag{31}$$

- Steady-state equilibrium first: since with technological progress there will be growth in per capita income, $\tilde{c}(t)$ will grow.

Technological Change VII

- Instead define

$$c(t) \equiv \frac{C(t)}{A(t)L(t)} \equiv \frac{\tilde{c}(t)}{A(t)}.$$

- This normalized consumption level will remain constant along the BGP:

$$\frac{\dot{c}(t)}{c(t)} \equiv \frac{\dot{\tilde{c}}(t)}{\tilde{c}(t)} - g = \frac{1}{\theta}\left(r(t) - \rho - \theta g\right).$$

Technological Change VIII

- For the accumulation of capital stock:

$$\dot{k}(t) = f(k(t)) - c(t) - (n + g + \delta) k(t),$$

where $k(t) \equiv K(t) / A(t) L(t)$.

- Transversality condition, in turn, can be expressed as

$$\lim_{t \to \infty} \left\{ k(t) \exp\left(- \int_0^t \left[f'(k(s)) - g - \delta - n \right] ds \right) \right\} = 0. \qquad (32)$$

- In addition, equilibrium $r(t)$ is still given by (8), so

$$r(t) = f'(k(t)) - \delta$$

Technological Change IX

- Since in steady state $c(t)$ must remain constant:

$$r(t) = \rho + \theta g$$

or

$$f'(k^*) = \rho + \delta + \theta g, \tag{33}$$

- Pins down the steady-state value of the normalized capital ratio k^* uniquely.
- Normalized consumption level is then given by

$$c^* = f(k^*) - (n + g + \delta) k^*, \tag{34}$$

- Per capita consumption grows at the rate g.

Technological Change X

- Because there is growth, to make sure that the transversality condition is in fact satisfied substitute (33) into (32):

$$\lim_{t \to \infty} \left\{ k(t) \exp\left(-\int_0^t [\rho - (1-\theta) g - n] \, ds \right) \right\} = 0,$$

- Can only hold if $\rho - (1-\theta) g - n > 0$, or alternatively :

ASSUMPTION 4:

$$\rho - n > (1-\theta) g.$$

- Remarks:
  - Strengthens Assumption $4'$ when $\theta < 1$.
  - Alternatively, recall in steady state $r = \rho + \theta g$ and the growth rate of output is $g + n$.
  - Therefore, equivalent to requiring that $r > g + n$.

Technological Change XI

Proposition Consider the neoclassical growth model with labor augmenting technological progress at the rate g and preferences given by (30). Suppose that Assumptions 1, 2, assumptions on utility above hold and $\rho - n > (1-\theta) g$. Then there exists a unique balanced growth path with a normalized capital to effective labor ratio of k^*, given by (33), and output per capita and consumption per capita grow at the rate g.

- Steady-state capital-labor ratio no longer independent of preferences, depends on θ.
  - Positive growth in output per capita, and thus in consumption per capita.
  - With upward-sloping consumption profile, willingness to substitute consumption today for consumption tomorrow determines accumulation and thus equilibrium effective capital-labor ratio.

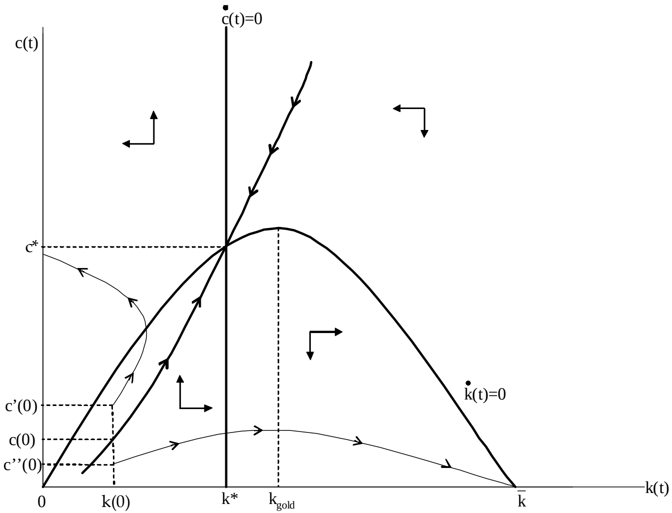

Figure: Transitional dynamics in the neoclassical growth model with technological change.

Technological Change XII

- Steady-state effective capital-labor ratio, k^*, is determined endogenously, but steady-state growth rate of the economy is given exogenously and equal to g.

Proposition Consider the neoclassical growth model with labor augmenting technological progress at the rate g and preferences given by (30). Suppose that Assumptions 1, 2, assumptions on utility above hold and $\rho - n > (1 - \theta) g$. Then there exists a unique equilibrium path of normalized capital and consumption, $(k(t), c(t))$ converging to the unique steady-state (k^*, c^*) with k^* given by (33). Moreover, if $k(0) < k^*$, then $k(t) \uparrow k^*$ and $c(t) \uparrow c^*$, whereas if $k(0) > k^*$, then $c(t) \downarrow k^*$ and $c(t) \downarrow c^*$.

Example: CRRA and Cobb-Douglas

- Production function is given by $F(K, AL) = K^{\alpha}(AL)^{1-\alpha}$, so that

$$f(k) = k^{\alpha},$$

- Thus $r = \alpha k^{\alpha-1} - \delta$.
- Suppressing time dependence, Euler equation:

$$\frac{\dot{c}}{c} = \frac{1}{\theta}\left(\alpha k^{\alpha-1} - \delta - \rho - \theta g\right),$$

- Accumulation equation:

$$\frac{\dot{k}}{k} = k^{\alpha-1} - \delta - g - n - \frac{c}{k}.$$

- Define $z \equiv c/k$ and $x \equiv k^{\alpha-1}$, which implies that $\dot{x}/x = (\alpha - 1)\,\dot{k}/k$.

Example II

- Therefore,

$$\frac{\dot{x}}{x} = -(1-\alpha)(x-\delta-g-n-z) \tag{35}$$

$$\frac{\dot{z}}{z} = \frac{\dot{c}}{c} - \frac{\dot{k}}{k},$$

- Thus

$$\begin{aligned} \frac{\dot{z}}{z} &= \frac{1}{\theta}(\alpha x-\delta-\rho-\theta g) - x+\delta+g+n+z \\ &= \frac{1}{\theta}((\alpha-\theta)x-(1-\theta)\delta+\theta n) - \frac{\rho}{\theta} + z. \end{aligned} \tag{36}$$

- Differential equations (35) and (36) together with the initial condition $x(0)$ and the transversality condition completely determine the dynamics of the system.

Comparative Dynamics I

- Comparative statics: changes in steady state in response to changes in parameters.
- Comparative dynamics look at how the entire equilibrium path of variables changes in response to a change in policy or parameters.
- Look at the effect of a change in tax on capital (or discount rate ρ)
- Consider the neoclassical growth in steady state (k^*, c^*).
- Tax declines to $\tau' < \tau$.
- From Propositions above, after the change there exists a unique steady state equilibrium that is saddle path stable.
- Let this steady state be denoted by (k^{**}, c^{**}).
- Since $\tau' < \tau$, $k^{**} > k^*$ while the equilibrium growth rate will remain unchanged.

Comparative Dynamics II

- Figure: drawn assuming change is unanticipated and occurs at some date T.
- At T, curve corresponding to $\dot{c}/c = 0$ shifts to the right and laws of motion represented by the phase diagram change.
- Following the decline c^* is above the stable arm of the new dynamical system: consumption must drop immediately
- Then consumption slowly increases along the stable arm
- Overall level of normalized consumption will necessarily increase, since the intersection between the curve for $\dot{c}/c = 0$ and for $\dot{k}/k = 0$ will necessarily be to the left side of k_{gold}.

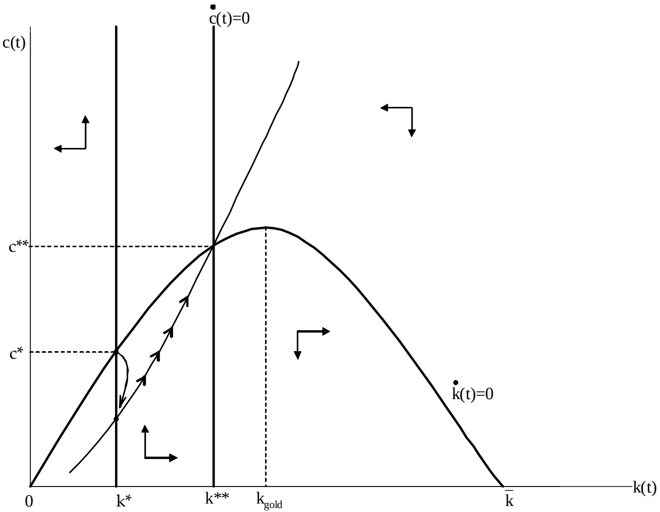

Figure: The dynamic response of capital and consumption to a decline in capital taxation from τ to $\tau' < \tau$.

The Role of Policy I

- Growth of per capita consumption and output per worker (per capita) are determined exogenously.
- But level of income, depends on $1/\theta$, ρ, δ, n, and naturally the form of $f(\cdot)$.
- Proximate causes of differences in income per capita: here explain those differences only in terms of preference and technology parameters.
- Link between proximate and potential fundamental causes:
  - e.g. intertemporal elasticity of substitution and the discount rate can be as related to cultural or geographic factors.
- But an explanation for cross-country and over-time differences in economic growth based on differences or changes in preferences is unlikely to be satisfactory.
- More appealing: link incentives to accumulate physical capital (and human capital and technology) to the institutional environment.

The Role of Policy II

- Simple way: through differences in policies.
- Introduce linear tax policy: returns on capital net of depreciation are taxed at the rate τ and the proceeds of this are redistributed back to the consumers.
- Capital accumulation equation remains as above:

$$\dot{k}(t) = f(k(t)) - c(t) - (n + g + \delta) k(t),$$

- But interest rate faced by households changes to:

$$r(t) = (1 - \tau)\left(f'(k(t)) - \delta\right),$$

The Role of Policy III

- Growth rate of normalized consumption is then obtained from the consumer Euler equation, (31):

$$\begin{aligned}\frac{\dot{c}(t)}{c(t)} &= \frac{1}{\theta}\left(r(t)-\rho-\theta g\right). \\ &= \frac{1}{\theta}\left((1-\tau)\left(f'(k(t))-\delta\right)-\rho-\theta g\right).\end{aligned}$$

- Identical argument to that before implies

$$f'(k^*) = \delta + \frac{\rho+\theta g}{1-\tau}. \tag{37}$$

- Higher τ, since $f'(\cdot)$ is decreasing, reduces k^*.
- Higher taxes on capital have the effect of depressing capital accumulation and reducing income per capita.
- But have not so far offered a reason why some countries may tax capital at a higher rate than others.

A Quantitative Evaluation I

- Consider a world consisting of a collection $\mathcal{J}$ of closed neoclassical economies (with the caveats of ignoring technological, trade and financial linkages across countries
- Each country $j \in \mathcal{J}$ admits a representative household with identical preferences,

$$\int_0^\infty \exp\left(-\rho t\right) \frac{C_j^{1-\theta}-1}{1-\theta} dt. \tag{38}$$

- There is no population growth, so c_j is both total or per capita consumption.
- Equation (38) imposes that all countries have the same discount rate ρ.
- All countries also have access to the same production technology given by the Cobb-Douglas production function

$$Y_j = K_j^{1-\alpha}\left(AH_j\right)^{\alpha}, \tag{39}$$

- H_j is the exogenously given stock of effective labor (human capital).

A Quantitative Evaluation II

- The accumulation equation is

$$\dot{K}_j = I_j - \delta K_j.$$

- The only difference across countries is in the budget constraint for the representative household,

$$(1+\tau_j)\, I_j + C_j \leq Y_j, \tag{40}$$

- τ_j is the tax on investment: varies across countries because of policies or differences in institutions/property rights enforcement.
- $1+\tau_j$ is also the relative price of investment goods (relative to consumption goods): one unit of consumption goods can only be transformed into $1/(1+\tau_j)$ units of investment goods.
- The right-hand side variable of (40) is still Y_j: assumes that $\tau_j I_j$ is wasted, rather than simply redistributed to some other agents.

A Quantitative Evaluation III

- Without major consequence since CRRA preferences (38) can be exactly aggregated across individuals.
- Competitive equilibrium: solution to maximization of (38) subject to (40) and the capital accumulation equation.
- Euler equation of the representative household

$$\frac{\dot{C}_j}{C_j} = \frac{1}{\theta}\left(\frac{(1-\alpha)}{(1+\tau_j)}\left(\frac{AH_j}{K_j}\right)^{\alpha} - \delta - \rho\right).$$

- Steady state: because A is assumed to be constant, the steady state corresponds to $\dot{C}_j / C_j = 0$. Thus,

$$K_j = \frac{(1-\alpha)^{1/\alpha} AH_j}{\left[(1+\tau_j)(\rho+\delta)\right]^{1/\alpha}}$$

A Quantitative Evaluation IV

- Thus countries with higher taxes on investment will have a lower capital stock, lower capital per worker, and lower capital output ratio (using (39) the capital output ratio is simply $K/Y = (K/AH)^{\alpha}$) in steady state.
- Substituting into (39), and comparing two countries with different taxes (but the same human capital):

$$\frac{Y(\tau)}{Y(\tau')} = \left(\frac{1+\tau'}{1+\tau}\right)^{\frac{1-\alpha}{\alpha}} \tag{41}$$

- So countries that tax investment at a higher rate will be poorer.
- Advantage relative to Solow growth model: extent to which different types of distortions will affect income and capital accumulation is determined endogenously.
- A plausible value for α is $2/3$, since this is the share of labor income in national product.

A Quantitative Evaluation V

- For differences in τ's across countries there is no obvious answer:
  - popular approach: obtain estimates of τ from the relative price of investment goods (as compared to consumption goods)
  - data from the Penn World tables suggest there is a large amount of variation in the relative price of investment goods.
- E.g., countries with the highest relative price of investment goods have relative prices almost eight times as high as countries with the lowest relative price.
- Using $\alpha = 2/3$, equation (41) implies:

$$\frac{Y(\tau)}{Y(\tau')} \approx 8^{1/2} \approx 3.$$

- Thus, even very large differences in taxes or distortions are unlikely to account for the large differences in income per capita that we observe.

A Quantitative Evaluation VI

- Parallels discussion of the Mankiw-Romer-Weil approach:
  - differences in income per capita unlikely to be accounted for by differences in capital per worker alone.
  - need sizable differences in the efficiency with which these factors are used, absent in this model.
- But many economists have tried (and still try) to use versions of the neoclassical model to go further.
- Motivation is simple: if instead of using $\alpha = 2/3$, we take $\alpha = 1/3$

$$\frac{Y(\tau)}{Y(\tau')} \approx 8^2 \approx 64.$$

- Thus if there is a way of increasing the responsiveness of capital or other factors to distortions, predicted differences across countries can be made much larger.

A Quantitative Evaluation VII

- To have a model in which $\alpha = 1/3$, must have additional accumulated factors, while still keeping the share of labor income in national product roughly around $2/3$.
- E.g., include human capital, but human capital differences appear to be insufficient to explain much of the income per capita differences across countries.
- Or introduce other types of capital or perhaps technology that responds to distortions in the same way as capital.

Conclusions

- Major contribution: open the black box of capital accumulation by specifying the preferences of consumers.
- Also by specifying individual preferences we can explicitly compare equilibrium and optimal growth.
- Paves the way for further analysis of capital accumulation, human capital and endogenous technological progress.
- Did our study of the neoclassical growth model generate new insights about the sources of cross-country income differences and economic growth relative to the Solow growth model? Largely no.
- This model, by itself, does not enable us to answer questions about the fundamental causes of economic growth.
- But it clarifies the nature of the economic decisions so that we are in a better position to ask such questions.