Notes on "Financial Innovation in a General Equilibrium Model" by Wolfgang Pesendorfer (1995)

Juan Pablo Xandri

February 17th, 2010

Main idea: Given a set of nominal securities, intermediaries have a technology to create financial derivatives. An equilibrium concept is proposed and characterized. Moreover, the author investigates efficiency of the outcome, the effects on real indeterminacy and explores the possibility of redundant assets in equilibrium

1 Setup of the Model

- 2 period endowment economy: $t = 0$ and $t = 1$.
- At $t = 1$, a random state of nature, $s \in S$ is realized. Set of states is finite, $S = \{s_1, s_2, ..., s_S\}$. This random variable affects the endowment process of agents and the returns on available securities.
- One homogeneous consumption good, that can be consumed in both periods.
- Set of available core **nominal assets** $A = \{1, 2, ..., A\}$, with payoff matrix $R_{S\times A}$ with nonnegative entries. This returns are expressed in unit of account (i.e. money) and not on consumption goods. These assets are in **zero net supply**, and each agent in the economy (i.e. intermediaries and households) are endowed with zero amount of each. Moreover, both households and intermediaries have access to a market in which they can freely buy and sell this securities.
- 2 types of agents in this economy: **financial intermediaries** and **households**.
- Continuum of financial intermediaries, with a finite set of N types. These intermediaries can create assets backed on core assets and other securities issued by intermediaries. Intermediaries, when creating the securities they will issue decide both the payoffs each security will have in each state $s \in S$, and also how much of each asset they will produce. Each type has measure 1.
- Continuum of households, with a finite set of H types. Households are endowed with consumption goods in both periods, and the heterogeneity across types is both in preferences over consumption processes and on the endowment process. Each household type has measure 1.

- There are **constraints to accessing financial markets**. More specifically, a financial intermediary that wants to sell 1 unit of a new security she created to a household, must incur in a fixed cost of contacting that agent (i.e. call the agent by phone, going to her office, etc). If the contact is not made, then the agent will not know of the existence of the new security. Moreover, there are transaction costs in selling new securities both to households and other financial intermediaries, which are proportional to the amount of securities issued. Heterogeneity across intermediaries types comes from different cost functions.
- Because of these frictions, we will allow for nonlinear pricing functions for the financial innovations. This price functions are to be determined in equilibrium, and are taken as given both by financial intermediaries and households.

2 Financial Intermediaries (I)

There is a continuum of financial intermediaries in this economy. Each intermediary is characterized by a type $n \in N$, which is drawn from a finite set. Each type of intermediary has unit mass, and the heterogeneity in types will be expressed in differences in marketing and transaction costs, as will be showed shortly. Intermediaries in this model have to make decisions on **4 different levels**: the **set of securities** they will issue, **how much** of each security they will issue, **which portfolio** they will have to hold to collateralize the payments they have to make to the holders of the securities issued, and the **marketing strategy** they will follow when trying to sell the issued securities to households and other intermediaries. All decisions of financial intermediaries are made at $t = 0$.

So, to sum up, a type n **financial intermediary** (to be explained below) **has to choose 4 things:**

1. The set of securities she will create
2. How much of the issued securities she will sell both to households and other financial intermediaries
3. The portfolio she will hold on core nominal securities and of securities issued by other intermediaries, so she can collateralize all promised payments.
4. Which households and intermediaries will the intermediary "target" (which will be the marketing plan, to be explained later).

We will present each decision separately.

2.1 Financial Innovations

In the primitive economy, there exist a set of A nominal securities in zero net supply, which we will refer to as **core securities** or **core assets**. These securities are characterized by the matrix of returns R :

$$R = \left(\underbrace{R_1}_{S\times 1 \text{ return of core asset } 1} \quad \underbrace{R_2}_{S\times 1 \text{ return of core asset } 2} \quad \cdots \quad \underbrace{R_A}_{S\times 1 \text{ return of core asset } A} \right)$$

The first decision a financial intermediary has to make is what securities she will create. As we saw with the core securities, a security can be identified by a column vector of dimensions $S \times 1$, which states how much will it pay in units of account (i.e. how much money will it pay) in each state of the world $s \in S$.

Let J be the unit simplex in $\mathbb{R}^S : J = \left\{ d \in \mathbb{R}^s_+ : \sum_{s=s_1}^{s_S} d_s = 1 \right\}$. A **financial innovation** is a column vector Q_k of payoffs in $J \cup \left\{ 0 \in \mathbb{R}^S \right\}$. So, creating a security is equivalent to choosing the payoff vector it will have. When an intermediary issues a security Q_k, she promises to pay $Q_{k,s}$ "dollars" in state s to the holder of the security, per unit held. It is important to remember that this promises are made in unit of account, not in consumption goods. Moreover, implicitly we assume that intermediaries can only create assets that pay a positive amount in each state, presumably because of problems enforcing such contracts (that is, if say a household buys a security that in some states of the world gives a negative return, then in such a state the household should made a payment to the intermediary in such a state. This situation is excluded by the author).

The fact that an intermediary can only choose vectors from J is a normalization (since the units of the vector is not important). If it chooses a financial innovation given by $Q_k = 0$ it means that the innovation is trivial (that is, security k has not been issued). The reason for this will become apparent below.

For now, we will assume that any given intermediary can only issue at most K securities (that is, can only choose K vectors from $J \cup \{0\}$ as vectors of financial innovations). If chooses to issue a number less than K, then the rest of the financial innovations are taken to be 0 (that's why we included it as a possibility in the financial innovation set).

A **financial innovation plan for intermediary type** n, Q^n is defined simply as a matrix of dimensions $S \times K$ for which its column vectors are financial innovations:

$$Q^n = \left(\underbrace{Q^n_1}_{\substack{\text{column vector } S\times 1 \text{ of} \\ \text{returns of security } 1}} \quad \underbrace{Q^n_2}_{\substack{\text{column vector } S\times 1 \text{ of} \\ \text{returns of security } 2}} \quad \cdots \quad \underbrace{Q^n_K}_{\substack{\text{column vector } S\times 1 \text{ of} \\ \text{returns of security } K}} \right) \tag{1}$$

Clearly, since intermediaries can choose vectors that pay 0 in all states (that is, as if they did not issue that security) K is only an upper bound on the amount of securities they can issue. It will be shown below that if $K > \overline{K}$ for some $\overline{K}$, this upper bound restriction will not bind in any equilibrium, and the restriction will be immaterial (compare to Bisin (1995))

Based on this definition, define

$$Q \equiv \begin{pmatrix} Q^1 & Q^2 & \cdots & Q^N \end{pmatrix} \tag{2}$$

That is, matrix Q is a $S \times NK$ matrix that has the return vectors of all new securities created by all intermediary types. We will refer to this matrix as the **market payoff matrix**. Each of the components of Q are themselves matrices, each of dimension $S \times K$ as defined in (1).

Define also $M(Q)$ as the **set of securities actually created by financial intermediaries given the payoff matrix** Q (so, it gives the set of securities k such that $Q_k^n \neq 0$)

Three things limit the extent to which intermediaries can produce securities:

1. All securities have to be collateralized by other assets (to be defined shortly)
2. Financial intermediaries have to pay marketing costs associated with promoting the new financial products (i.e. getting customers and informing them about their financial products is costly). Again, heterogeneity across intermediary types comes from different marketing cost functions.
3. Marketing of assets is segmented into two markets: trading with other financial intermediaries, and trading with households (the "retail market").

In the next sections, we will study each of this aspects of financial innovation by intermediaries.

Important remark: Unlike Santos and Woodford (1997), both core assets and financially innovations are nominal assets, and not real assets. That is, they do not pay units of time $t = 1$ and state s consumption, but actually pay in units of account. The reason why we study this kind of securities is to address the issue of **real indeterminacy**, typical in economies with nominal securities and incomplete markets. This result can be illustrated by the following idea: since the securities pay in "dollars" then the real return of each security will be given by the nominal return at each state $s \in S$ divided by the price of the consumption good at that state. But then, if there is a source of randomness on the price level of the consumption good at $t = 2$ (for example, driven by a sunspot) then this will affect the real return of the securities at time $t = 1$, affecting in turn the resulting equilibrium allocation for each possible self-fulfilling variation in the price level. This is known as the **real indeterminacy of the price level,** since changes in the price level across state generates real differences in the resulting allocations. This phenomenon does not happen when securities pay off in real terms, since the returns are then independent of the price level.

2.2 Collateral Constraints and Production of Innovations

Once intermediaries decide which securities they will introduce (given by Q^n) each intermediary has to decide how much of each security she will produce to sell to households, and how much she will produce to sell to other intermediaries. Let us introduce some notation:

- $\theta^n_{m,k} \geq 0$: how many units of the $k-th$ asset originally created by intermediary m will be sold to households by intermediary n

- $\psi^n_{m,k}$: how many units of the $k-th$ asset originally created by intermediary m will be sold (or bought) to other intermediaries by intermediary n. If $\psi^n_{m,k} < 0$ means that this is a net purchase, and if $\psi^n_{m,k} > 0$ means its a net sale.

- $\theta^n \in \mathbb{R}^{NK}$: amount that intermediary n sells of each available new asset (created by all financial intermediaries) to households. Namely

$$\theta^n = \left(\theta^n_{1,1}, \cdots, \theta^n_{1,K}, \cdots, \theta^n_{N,1}, \cdots, \theta^n_{N,K}\right) \tag{3}$$

- $\psi^n \in \mathbb{R}^{NK}$: amount that intermediary n sells of each available new asset (created by all financial intermediaries) to other intermediaries. Namely

$$\psi^n = \left(\psi^n_{1,1}, \cdots, \psi^n_{1,K}, \cdots, \psi^n_{N,1}, \cdots, \psi^n_{N,K}\right) \tag{4}$$

Some restrictions are made over what an intermediary can sell and buy. Specifically:

1. Intermediary of type n can only sell to households assets created by herself (so $\theta^n_{m,k} = 0$ if $m \neq n$)
2. Intermediary of type n can only sell to other intermediaries assets created by herself (so $\psi^n_{m,k} \leq 0$ if $m \neq n$)

The most important but related restriction on the planned sales of the issued securities, however, is the so-called **collateral constraint**. Basically, in order to sell the created securities, the financial intermediary has to hold a portfolio of investments such that all promised payments in each state can actually be paid. In order to make this payments, financial intermediaries can invest in core assets and in assets created by other financial intermediaries.

Let $\zeta^n \in \mathbb{R}^A$ be the portfolio that intermediary n decides to purchase of core nominal assets. As we said before, both intermediaries and households are assumed to be **endowed with a zero quantity of these securities**. There exists a market in which both intermediaries and household freely buy and sell core securities, which are in zero net supply. By the construction of the vectors θ^n and ψ^n, the constraint on payments has to satisfy"

$$Q\theta^n + Q\psi^n \leq R\zeta^n \tag{5}$$

See that the first term of the LHS, $Q\theta^n = Q^n \begin{pmatrix} \theta^n_{n,1} \\ \theta^n_{n,2} \\ \vdots \\ \theta^n_{n,K} \end{pmatrix}$ is the $S \times 1$ vector of payments that the intermediary has to make at each state $s \in S$ to the households that hold the issued securities. The second term, $Q\psi^n$ is also a $S \times 1$ vector of net payments that the intermediary has to make at each state $s \in S$

to the intermediaries that hold the issued securities. However, unlike θ^n, $Q\psi^n$ also takes into account the purchases made of securities issued by other intermediaries (which were expressed as $\psi^n_{m,n} < 0$ if intermediary n bought securities issued by intermediary m) and the net payments she should thus receive at each state. On the RHS we have $R\zeta^n$, which is the $S \times 1$ vector of payoffs that the intermediary will receive by holding a portfolio ζ^n of core nominal assets.

So far, we introduced the first three decisions: which securities will be created, how much of each will be sold to households and other intermediaries, and the portfolio decisions that will allow them to actually make the promised payments. These can be summarized by a 4-tuple $(Q^n, \theta^n, \psi^n, \zeta^n)$, which we will refer to as a **production plan**. We summarize the restrictions on these production plans in the next definition

Definition 1 (Feasible production plan) *We say that a 4-tuple $(Q^n, \theta^n, \psi^n, \zeta^n)$is a **feasible production plan** if and only if:*

1.

$$\theta^n_{m,k} = 0 \text{ and } \psi^n_{m,k} \leq 0 \text{ for all } n, m \text{ such that } n \neq m. \tag{6}$$

2.

$$Q\left(\theta^n + \psi^n\right) \leq R\zeta^n \tag{7}$$

Let Y^n be the set of all feasible plans for type n, and note that **feasibility depends on the strategies of other intermediaries (through Q)**. But apart from this, there is no heterogeneity in the set of feasible plans across financial intermediary types, so $Y^n = Y$ for all n. The difference between different types of intermediaries will come from the marketing and transaction costs of trading the issued securities, which we will introduce now.

2.2.1 Marketing and Transaction costs

Suppose that the financial intermediary has so far chosen a feasible production plan $(Q^n, \theta^n, \psi^n, \zeta^n)$. If she were free access to capital markets and no transaction costs, no decisions should be made regarding to whom the financial intermediary should sell the financial innovations. However, two frictions limit the trading in financial innovations:

1. If an intermediary wants to sell a household a unit of financial innovation k, it has to pay a fixed cost $b \geq 0$. The idea of this feature, according to the author, is that to make the sale, the intermediary has to contact the household and explain the properties of this new innovation.
2. There is also a transaction cost per unit traded with both households and other financial intermediaries.

Let $p \in \mathbb{R}^S = (p_{s_1}, p_{s_2}, ..., p_{s_S})$ be the vector of commodity prices in units of account (i.e. dollars) at time $t = 1$ in each state of the world (we have normalized $p_0 = 1$).

Suppose intermediary n wants to sell $\theta^n_{n,k}$ units of security k (characterized as we saw before by its return vector, Q^n_k) to households, and $\psi^n_{n,k}$ units of security k to other financial intermediaries. For each unit marketed to intermediaries, intermediary type n must pay a (unit) cost of $c^n_{int}(Q^n_k, p)$ (*int* for "intermediaries") per asset sold, and likewise has a cost per unit sold to households, $c^n_{ret}(Q^n_k, p)$ (*ret* for "retail"). So, transaction costs depend on the payoff vector of the security issued. **All costs are stated in terms of period zero consumption** (so, they are costs in real terms). We can do this without affecting the real rates of return.

Moreover, in the retail market the intermediary has to pay a fixed cost $b \in \mathbb{R}_+$ for each new potential costumer reached, as we mentioned before. Then marketing costs are:

$$\text{Cost to target intermediaries} = \underbrace{c^n_{int}(Q^n_k, p)}_{\substack{\text{unit cost of trading security } k \\ \text{with intermediaries}}} * \underbrace{\psi^n_{n,k}}_{\substack{\text{how many units of security } k \\ \text{are sold to intermediaries}}}$$

$$\begin{aligned} \text{Cost to target households} \quad = & \quad \underbrace{b}_{\text{contacting cost per household}} * (\#\text{ households targeted}) \\ + & \quad \underbrace{c^n_{ret}(Q^n_k, p)}_{\substack{\text{unit cost of trading security } k \\ \text{with households}}} * \underbrace{\theta^n_{n,k}}_{\substack{\text{how many units of security } k \\ \text{are sold to households}}} \end{aligned}$$

Remarks:

(a) : **Why do costs depend on** p? : Since returns are in units of account, then it is natural that to assume that the real costs of selling a security could depend on the **real returns of the asset**, and not on the nominal returns. If a security k has a payoff vector in nominal, unit of account of $Q^n_k = \begin{pmatrix} Q^n_{k,1} \\ Q^n_{k,2} \\ \vdots \\ Q^n_{k,S} \end{pmatrix}$, then the real returns of this security in terms of time $t = 0$ consumption are

$$\widetilde{Q}^n_k \equiv \begin{pmatrix} \frac{1}{p_1} Q^n_{k,1} \\ \frac{1}{p_2} Q^n_{k,2} \\ \vdots \\ \frac{1}{p_S} Q^n_{k,S} \end{pmatrix} \tag{8}$$

Then, if for example the cost of selling a unit of an asset with real returns $\widetilde{Q}_k^n$ for type n intermediary in the intermediaries market is given by a function $\mathcal{C}_{int}^n\left(\widetilde{Q}_k^n\right)$, then we can write the cost function as:

$$c_{int}^n\left(Q_k^n, p\right) \equiv \mathcal{C}_{int}^n\left(\widetilde{Q}_k^n\right) \equiv \mathcal{C}_{int}^n\left(\frac{1}{p_1}Q_{k,1}^n, \frac{1}{p_2}Q_{k,2}^n, ..., \frac{1}{p_S}Q_{k,S}^n\right) \tag{9}$$

(b) : Why are not fixed costs in the intermediary market? The author assumes that the contacting cost b is a fixed cost to learning about new financial innovations. Basically, to target a client, the intermediary has to pay a fixed cost b per client (i.e. calling by phone, sending mail, etc). Moreover, the author assumes that financial intermediaries are "better informed", which is modeled here as sales among intermediaries being costless (that is, an intermediary does not have to incur in a marketing cost when selling to another intermediary, since they know the products better). However, this does not preclude the existence of transaction costs when selling to other financial intermediaries.

We assume the following properties regarding the cost functions:

1. For all $n \in N$, cost functions c_{int}^n and c_{ret}^n are continuous
2. $c_{int}^n(0, p) = c_{ret}^n(0, p) = 0$ for all $p \in \mathbb{R}_{++}^S$, all $n \in N$
3. $c_j^n(Q_k^n, \lambda p) = \frac{1}{\lambda}c_j^n(Q_k^n, p)$ for all p, λ and $j \in \{int, ret\}$
4. $c_{ret}^n(Q_k^n, p) > c_{int}^n(Q_k^n, p) > 0$ for all $c_{ret}^n(Q_k^n, p)$ and all $n \in N$

The first two assumptions are technical and simplify the analysis. Condition 3 implies that changes in the value of money across all states in the second period do not affect "the proportional innovation cost, so that if all payoffs were to be measured in cents instead of dollars, then the costs per unit would decrease by a factor of one hundred, leaving real costs unchanged". Condition 4 simply implies that targeting intermediaries in the marketing plan is cheaper than targeting households (since they are "better informed")

Remark: The introduction of the cost per household b gives incentives to financial intermediaries to "tailor" securities specific for certain types of households, in order to reduce the set of innovations traded with households (since each new security which might be introduced to a household costs $b \geq 0$ in terms of $t = 0$ consumption). This may cause the existence of **redundant assets** (i.e. assets with return vectors that are linear combinations of existing assets), as we will investigate in some examples.

The existence of this marketing and transaction costs makes financial intermediaries not only decide which assets to issue and how much of each they will produce, but also to whom the intermediaries will sell their innovations. This is the fourth aspect of the decision making process, the **marketing plan**. However, in order to set up this problem properly, we need first to define the household maximization problem, which we present in the next section. After presenting the household problem, we will return to set up the intermediaries problem, putting all the decisions we have been talking about together.

3 Households

There are a continuum of households distributed uniformly on $I = [0, H]$. There are also a finite set of types $\mathcal{H} = \{h_1, h_2, ..., h_H\}$ with $h_i = i$, and the set of agents that are of type h is $[h-1, h) \subseteq I$.

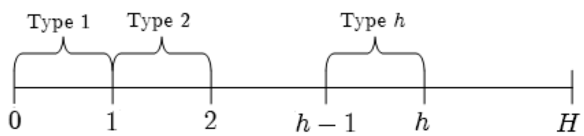

Figure 1: Household types

Each type has a differentiable utility function $u^h : \mathbb{R}^{S+1}_{++} \to \mathbb{R}$ over consumption at $t = 0$ and consumption over $t = 1$. Specifically, $u^h(x_0, x_1)$ with $x_0 \in \mathbb{R}_{++}$ consumption in period 0 and $x_1 \in \mathbb{R}^S_{++}$ the consumption vector of period 1 in each state of nature. The utility function is not required to satisfy the expected utility axioms.

Also, let $\omega^h \equiv \left(\omega^h_0, \omega^h_1\right) \in \mathbb{R}^{S+1}_{++}$ be the endowment of household of type h (which is deterministic at $t = 0$ and stochastic at $t = 1$). We will use notation $\omega(i) = (\omega_0(i), \omega_1(i))$ to allow in principle for different treatment of identical agents (regarding their type), so that we do not require all households of the same type to necessarily behave identically in equilibrium. The only extra assumption on preferences is that **the indifference curves do not touch the axes** (as in Cobb-Douglas utility).

Because of the transaction and marketing costs associated with sales of new securities created by financial intermediaries, we want to allow for the possibility of **nonlinear prices**. That is, if some household wants to buy $y^n_k > 0$ units of security k issued by intermediary type n, it will have to pay $r^n_k(y^n_k)$, a possible non-linear function of the amount bought. This pricing functions (for which there exist one per intermediary and per innovation) are to be defined in equilibrium, and both households and financial intermediaries are assumed to take this pricing functions as given. We need to allow for these non-linear pricing functions precisely because of the fixed cost of entry to the retail market (that is, the cost b). Later, we will actually show that in equilibrium, this is exactly the shape of the non-lnear pricing functions: a fixed fee plus a constant price per unit.

Let us introduce some notation:

- $r^n_k : \mathbb{R}_+ \to \mathbb{R}$, $r^n_k(y^n_k)$ is (as we mentioned above) the price function in units of account at period $t = 0$ (which because of the normalization, is in terms of time $t = 0$ consumption goods) in the retail market for y^n_k units of asset k, issued by intermediary n.
- $r : \mathbb{R}^{NK}_+ \to \mathbb{R}$ defined as

$$r(y) = \sum_{n \in N} \sum_{k=1}^{K} r^n_k(y^n_k) \tag{10}$$

is the total price to be paid for a demand of $y = \left(y_1^1, y_2^1, ..., y_K^1, ..., y_1^N, ..., y_K^N\right) \equiv \left(y^1, y^2, ..., y^N\right)$ units of securities issued by intermediaries. In this notation, y^i is the $K \times 1$ vector of holdings of intermediated assets created by intermediary n.

So, if a household decides to buy a portfolio $y \in \mathbb{R}_+^{NK}$ of securities issued by financial intermediaries at $t = 0$, it will have to pay for it $r(y)$ dollars.

- $P = \begin{pmatrix} p_{s_1} & 0 & \dots & 0 \\ 0 & p_{s_2} & \dots & 0 \\ \vdots & \dots & \ddots & \vdots \\ 0 & 0 & \dots & p_{s_S} \end{pmatrix}$ is the diagonal matrix of the equilibrium $t = 1$ prices for the consumption good, in terms of units of money, or equivalently in time $t = 0$ consumption.

- $q \in \mathbb{R}_{++}^A$ is the price vector of core securities at $t = 0$.

The problem a household must solve is to choose the consumption levels at $t = 0$ and at $t = 1$ at any contingency, together with the portfolio decisions at $t = 0$ of how much of each core nominal asset they will hold (given by $z \in \mathbb{R}^A$) and also of how many new securities issued by financial intermediaries the will buy (given by $y \in \mathbb{R}^{NK}$). Note that for a portfolio choice of z, the real returns that the household will have in state s (in terms of time $t = 1$ consumption) are:

$$\text{real return}(s) = \frac{1}{p_s} \underbrace{\sum_{a=1}^{A} R_{a,s} z_a}_{\substack{\text{nominal return of} \\ \text{portfolio } z \text{ at state } s}} \tag{11}$$

and therefore, the column vector $P^{-1}Rz$ gives the $S \times 1$ vector of real returns (in terms of consumption good) of portfolio z. Following the same reasoning, the column vector $P^{-1}Qy$ gives the $S \times 1$ vector of real returns of portfolio of financial innovations y. Households take the asset structure Q from the securities created by intermediaries as given, and also take as given the price function $r(y)$.

Take a household $i \in [h-1, h)$. Then, the problem the household solves is:

$$\max_{x \in \mathbb{R}_+^{S+1}, y \in \mathbb{R}_+^{NK}, z \in \mathbb{R}^A} u^h(x_0, x_1) \tag{12}$$

$$x_0 \leq \omega_0^h - \underbrace{q'z}_{\text{cost of portfolio } z} - \underbrace{r(y)}_{\text{cost of portfolio } y \in \mathbb{R}_+^{NK}}$$

$$x_1 \leq \underbrace{\omega_1^h}_{\text{endowment}} + \underbrace{P^{-1}Rz}_{\substack{\text{real returns on} \\ \text{core securities}}} + \underbrace{P^{-1}Qy}_{\substack{\text{real returns on} \\ \text{intermediated securities}}}$$

$$(x_0, x_1) \geq 0, y \geq 0$$

For this problem to make sense, we need a further equilibrium condition that must hold:

$$\begin{pmatrix} -qz \\ Rz \end{pmatrix} + \begin{pmatrix} -r(y) \\ Qy \end{pmatrix} \geq 0 \tag{13}$$

for all (z, y), so there is no arbitrage opportunities. See that this condition entails that whenever $Rz + Qy \geq 0$ (that is, there is a portfolio (z, y) that gives positive returns) then $qz + r(y) \leq 0$ (the portfolio must "cost" something, so there are no arbitrage opportunities). Compare this condition to Santos and Woodford (1997) no arbitrage condition.

Based on this problem, let $V^i(r, p, q, Q, \omega(i))$ be the value function for household i associated with problem (12), where $\omega(i) = \left(\omega_0^h, \omega_1^h\right)$ with $h : i \in [h-1, h)$

4 Financial Intermediaries (II)

As noted in the description of the intermediation technology, intermediaries need to choose a financial innovation and production plan 4-tuple $(Q^n, \theta^n, \psi^n, \zeta^n)$ (from now on, referred to as a "plan"). Also, since intermediaries need to target specific households and these are heterogeneous, they also need to specify the distribution of the output for each household. This will be given by a **marketing plan.** In this section we will define what we mean by marketing plans, and we will define what are the objectives of each financial intermediary

4.1 Marketing Plans

Take intermediary of type n, and suppose that she has already chosen a feasible production plan $(Q^n, \theta^n, \psi^n, \zeta^n)$. Then, she has decided to sell $\theta_{n,k}^n$ units of security k to households. However, since financial intermediaries have to incur in a cost when contacting each household to make the sale, she must also decide **how much of each security will she sell to each household**. Of course, such a consideration wouldn't be needed if there was a perfect competitive market in which households could go and buy any security they wish. However, the existing frictions in the financial markets make financial intermediaries also worry about specific targeting of different securities to different households. Formally, such a plan is given by a **marketing plan:**

Definition 2 (Marketing plans) *A **marketing plan** for an intermediary of type n , given the production plan $(Q^n, \theta^n, \psi^n, \zeta^n)$ is a measure η^n on $\left(\mathbb{R}^K, \mathcal{B}\left(\mathbb{R}^K\right)\right)$, $\left(\text{with } \mathcal{B}\left(\mathbb{R}^K\right) \text{ the Borel } \sigma\text{-algebra on } \mathbb{R}^K\right)$ such that*

$$\int_{\mathbb{R}^K} y\eta^n(dy) = \left(\theta_{n,1}^n, \theta_{n,2}^n, ..., \theta_{n,K}^n\right) \equiv \theta_n^n \tag{14}$$

While a bit technical, the intuition of the definition is straightforward: in the case for which there is a finite set of bundles of assets, then η^n would be simply a matrix that would state how many units of security k is sold to household h. Here, since we have a continuum of households, we need to think of this as a borel measure over the set of households. A thing to have in mind is that η^n **is not a probability distribution.**

Define $\chi : \mathbb{R}_+ \to \{0,1\}$, $\chi(y_k)$ as the characteristic function of $y_k > 0$. Then, since each different y is provided to a different household, the fixed cost of selling to households is

$$b \int_{\mathbb{R}^K} \left(\sum_{k=1}^{K} \chi(y_k) \right) \eta^n(dy) \tag{15}$$

4.2 Intermediaries Objectives

4.2.1 Reservation prices

So far, we described the endogenous objects that a financial intermediary has to choose: both a feasible production plan $(Q^n, \theta^n, \psi^n, \zeta^n)$ and a marketing plan to sell to households η^n. It only remains to study how intermediaries take their decisions.

First, we have to study their **sources of income** . Define the function $v : J \to \mathbb{R}_+$ as the **equilibrium price function of an innovation Q_k^n in the intermediaries market**. That is, if intermediary n sells ψ units of an intermediated security with payoff vector $Q_k^n \in J$ to another intermediary, then she will receive $v(Q_k^n) * \psi$ units of account for that sale. See that this function is defined for all potential innovations, not only the ones that will be chosen by intermediaries in equilibrium (which will be defined later). An important aspect to consider is that the price function v is defined in equilibrium, which means that it is taken as given by all intermediaries (i.e. intermediaries do not decide at which price they will sell their financial innovations).

Consider the set of all intermediated securities of all financial intermediaries, given by the payoff matrix Q and given prices $(r(.), p, q, v(.))$, (that is, prices of the consumption good, core nominal assets, and the price functions $r(.)$ and $v(.)$, all in terms of units of account) and imagine that intermediary n is thinking about the financial innovation plans she will be introducing. In order to do so, she takes as given the financial innovations taken by all intermediaries, including other intermediaries of her same type: i.e. takes both the market payoff matrix Q and the financial innovation plan taken by all other intermediaries of the same type, Q^n.

Then, the problem of the intermediary is to choose an optimal financial innovation plan $\widehat{Q}^n$, which could in principle differ from the financial innovation plans made by intermediaries of the same type, Q^n. We will say that an intermediary of type n has no incentives to deviate from Q^n when she optimally chooses $\widehat{Q}^n = Q^n$. This will be an equilibrium requirement when we define our equilibrium concept for this economy.

The assumption of a continuum of intermediaries of each type plays a role, since each intermediary is infinitesimal in this economy. Then, a new security issued by this intermediary is assumed **not to affect prices of all other financial innovations** created by the other financial innovators. At what prices will intermediary n be able to sell this plan? First we need to find **what is the price households would be willing to pay for this plan**. This will be given by the concept of a reservation price function.

Definition 3 (Reservation Price Function) *A function $\rho : J^K \times \mathbb{R}^K \to \mathbb{R}_+$ is a* ***reservation price function given*** *(Q, r, p, q) if*

$$\rho\left(\widehat{Q}^n, y^n \mid Q, r, p, q\right) = \max\left\{t : \text{ exists } i \in I \text{ such that } V^i\left(r, p, q, Q, \left(\omega_0(i) - t, \omega_1(i) + P^{-1}\widehat{Q}^n y^n\right)\right) \geq V^i(r, p, q, Q, \omega(i))\right\} \tag{16}$$

So, the reservation price function is the ***maximum price some household would be willing to pay to get the possibility of investing also in an innovation plan*** *$\widehat{Q}^n$*, ***given a portfolio holding of*** *y^n* ***of the assets*** *n* ***produces.***

See that this is exactly the definition of reservation price: the value t will be actually the value that makes the most willing household (that's why we are actually maximizing over willingness to pay over $i \in I$) indifferent between buying a portfolio $y^n \in \mathbb{R}^K_+$ from intermediary n, and then solving the problem (12) to get the demands of intermediated assets from every other intermediary.

4.2.2 Relationship with Makowski (1980)

In order to have a better understanding on how intermediaries decide to introduce new financial innovations, it is useful to compare this model with the one developed in Makowski. See that in this model, because of the assumption that there is a continuum of intermediaries of each type, we can actually show that in any Walrasian equilibrium that we could consider holding the set of financial innovation fixed, all firms are **perfect competitors**. In Makowski's definition, a perfect competitor firm has two characteristics:

1. A perfect competitor cannot affect equilibrium prices by changing the set of traded commodities
2. A perfect competitor is inessential in terms of welfare: for any allocation in which the firm trades a non-zero amount of commodities, there exist another allocation in which that firm does not trade, and such that all households have the same welfare.

The first condition is clearly satisfied in this model, because of the continuum assumption. The second assumption comes from the fact that there is a continuum of measure 1 of each type of financial intermediary: this in turn implies that for any allocation in which some financial intermediary sells some financial innovations to a household, there always exist a continuum of identical intermediaries which can offer the exact same innovations and leave households indifferent.

As Makowski points out in his paper, a firm being a perfect competitor implies that in any quasi-equilibrium the firm could move the economy to, it can only charge the reservation price of new commodity to some household. This is because prices of all other commodities remain fixed, and there's always another firm which is willing to offer the exact same allocation as the deviating firm, so in the margin the household that buys from the deviating firm must be indifferent between buying and not buying. The quasi-equilibrium price in Makowski is the equivalent in this setting to the reservation price function concept defined in (16). Also as in Makowski, innovation from the set of marketed securities given by the payoff structure Q is an off-equilibrium phenomenon. That is, the set of securities marketed is such that no intermediary wants to "block it" (that is, change it by introducing new securities), so there's no innovation from the equilibrium Q.

However, in Makowski only linear pricing is allowed, whereas in this model, because of the fixed cost of contacting households, we allow for nonlinear pricing rules, which are also determined in equilibrium. But if we think of each potential portfolio holding $y \in \mathbb{R}^{NK}$ as a different good altogether and we discretize the set of portfolios that a household can buy, then we are actually back at Makowski's framework. This idea is actually used later to prove the existence of equilibria (as we will show later).

This is not the only connection with Makowski (1980): later in this notes we will see that the conditions that we need on the fundamentals of this model to get efficiency of equilibria are actually conceptually the same as in his paper.

4.2.3 Profit Function

Let us make some more definitions:

- $\widetilde{K}(Q^n) \subset M(Q)$ is the set of assets produced by intermediary n, if she chooses a financial innovation plan Q^n
- $L(Q^n) \equiv \left\{ y \in \mathbb{R}^K : y_k > 0 \text{ for } k \in \widetilde{K}(Q^n) \right\}$is the set of all portfolios that had a positive position already being offered, in an asset marketed at the financial innovation plan Q^n.

As we said before, an intermediary of type n takes as given the market payoff matrix Q, which includes also the financial innovation plan Q^n of all other intermediaries of the same type, and is thinking of choosing a financial innovation plan $\widehat{Q}^n$ which is potentially different from Q^n. Clearly, for portfolios that have a zero position on new securities introduced by the financial innovation plan $\widehat{Q}^n$, the deviating innovator must charge the same price as before for such portfolios, that is $r(y)$. For $y \notin L(Q^n)$ (that is, portfolios that has positive positions on assets not yet produced), the intermediary will charge the reservation price (which is equivalent to the concept of quasi-equilibrium price in Makowski, as we analyzed above). Then, total revenue for intermediary n from households is

$$\text{Revenue}_{ret} = \underbrace{\int_{L(Q^n)} r(y)\,\eta^n(dy)}_{\substack{\text{revenue from selling} \\ \text{portfolios on existing innovations}}} + \underbrace{\int_{\mathbb{R}^K - \{L(Q^n)\}} \rho\left(\widehat{Q}^n, y \mid Q, r, p, q\right)\eta^n(dy)}_{\substack{\text{revenue from selling} \\ \text{portfolios on new innovations}}} \tag{17}$$

Then, profits of choosing a production plan $\left(\widehat{Q}^n, \theta^n, \psi^n, \zeta^n\right)$ and a marketing plan η^n given prices $r(.), p, q$ and the market payoff matrix Q are defined as:

$$\pi^n\left(\widehat{Q}^n, \theta^n, \psi^n, \zeta^n, \eta^n \mid Q, r, p, q\right) = \text{Revenue}_{ret} + \underbrace{\sum_{n',k} \psi^n_{n',k} v\left(\widehat{Q}^{n'}_k\right)}_{\text{revenue from selling to intermediaries}} - \underbrace{q\zeta^n}_{\text{cost of core securities portfolio}}$$

$$-\left(\underbrace{b\int_{\mathbb{R}^K}\left(\sum_{k=1}^K \chi(y_k)\right)\eta^n(dy)}_{\text{cost of targeting households}} + \underbrace{\sum_{k=1}^K\left(c_2^n\left(\widehat{Q}^n_k, p\right)\theta^n_{n,k} + c_1^n\left(\widehat{Q}^n_k, p\right)\psi^n_{n,k}\right)}_{\text{transaction costs}}\right)$$

Then, the objective of intermediary of type n is to choose production plans $\left(\widehat{Q}^n, \theta^n, \psi^n, \zeta^n\right)$ together with a marketing plan η^n in order to maximize profits given prices and the set of securities traded by all intermediaries (including the securities that intermediary of type n is supposed to choose at Q)

5 Equilibrium

Once we studied the agents in this economy and their objectives, we are able to define what we mean by equilibrium in this model. The equilibrium concept proposed is a **competitive equilibrium with financial innovation (CEFI)** defined as follows

Definition 4 (Competitive Equilibrium with Financial Innovation) *A* ***competitive equilibrium with financial innovation (CEFI)*** *is a return matrix Q with set of actively traded securities $M(Q)$, a set of intermediaries production and marketing plans $\{(\theta^n, \psi^n, \zeta^n, \eta^n)\}_{n=1}^N$, consumers decision functions $\left\{x : I \to \mathbb{R}^{S+1}_{++}, z : I \to \mathbb{R}^A, y : I \to \mathbb{R}^{NK}_+\right\}$ and prices $p \in \mathbb{R}^S_{++}, q \in \mathbb{R}^A_{++}, r : \mathbb{R}^{NK} \to \mathbb{R}_+$ and $v : \mathbb{R}^{NK} \to \mathbb{R}_+$ such that:*

(i) : *x, y and z are measurable functions, such that for all i, $(x(i), y(i), z(i))$ solve (12) given prices and the market payoff matrix Q*

(ii) : *Choosing Q^n implicit in the matrix return Q together with $(\theta^n, \psi^n, \zeta^n, \eta^n)$ maximize profits for each intermediary of type n, for all $n = 1, 2, ..., N$ given prices*

(iii) : *Aggregate market clearing:*

$$\int_I z(i)\,di + \sum_{n=1}^{N} \zeta^n = 0 \tag{18}$$

$$\int_I y(i)\,di = \sum_{n=1}^{N} \theta^n \tag{19}$$

$$\sum_{n=1}^{N} \psi^n = 0 \tag{20}$$

$$\int_{i\in I} x_0(i)\,di + \sum_{n.k} \left(c_2^n\left(Q_k^n, p\right)\theta_{n,k}^n + c_1^n\left(Q_k^n, p\right)\psi_{n,k}^n \right) + \sum_{n=1}^{n=N} b \int_{\mathbb{R}^K} \left(\sum_{k=1}^{K} \chi\left(y_k\right) \right) \eta^n(dy) = \int_{i\in I} \omega_0(i)\,di \tag{21}$$

$$\int_{i\in I} x_{1,s}(i)\,di = \int_{i\in I} \omega_{1,s}(i)\,di \text{ for all } s \in S \tag{22}$$

(iv) : *Let γ^n be the derived measure for the demanded portfolios by intermediary n. That is, for a measurable set $E \subseteq \mathbb{R}_+^K$, we have*

$$\gamma^n(E) = \int_{i:y^n(i)\in E} di$$

Then $\gamma^n = \eta^n$ for all $n = 1, 2, ..., N$

Let's analyze this definition. Part (**i**) of the definition simply states that households maximize utility given prices and the set of securities introduced by financial intermediaries. Part (**ii**) is the equivalent in Makowski to the "no profitable deviations" condition of a Full Walrasian Equilibrium, together with profit maximization choices by financial intermediaries. Equations (18) to (22) are aggregate resources constraints on core nominal assets, intermediated assets sold to households, intermediated assets sold to intermediaries, time $t = 0$ consumption goods and time $t = 1$ consumption in each state $s \in S$, respectively. Part (**iv**) simply states that the demand of each household of the intermediated assets coincides with the marketing plan.

Some notes on this definition:

- The definition is somewhat incomplete in the original paper. However, the proof of the main theorems use this notion of equilibrium
- This definition is basically identical to the Full Walrasian Equilibrium of Makowski (1980), with the difference that here we have non-linear equilibrium price rules and marketing plans (which are not needed in the later paper).

One of the main theorems of this paper consists of proving that equilibria like this always exist, and that equilibrium asset prices are in the shape of two-part tariffs. The proof is omitted (highly technical)

Theorem 5 (Existence of CEFI) *In the described economy, if* $K \geq \overline{K} \equiv N(S+1)[N+(A+1+(S+1)N)H]$ *then there exist a competitive equilibrium with financial innovation* ***(CEFI)****. Moreover, in every equilibrium the function* $r_k^n(y)$ *for intermediary type* n *and security* k *can be chosen to be of the form* $r_k^n = b + \widetilde{r}_k^n * y$ *for all* (k,n)*, with* $\widetilde{r}_k^n \in \mathbb{R}_+$

The idea of the proof is first to prove existence in a finite good economy approximation, in which pairs (Q_k^n, t) with t the price and Q_k^n the asset are identified as one commodities, with a discretization of both the set of returns and the set of prices that intermediaries can charge for them. After that, the equilibrium is proven to be the limit as the grids are taken to be finer and finer. The restriction $K \geq \overline{K}$ is to guarantee that intermediaries are not restricted in their issues of assets, since each type of household will not hold positive amounts of intermediated assets for more than $N(S+A)+1$ securities. See that a single household needs no more than A core securities and S Arrow-Debreu securities (which are by construction linearly independent in their returns) which could in principle be completely personalized to her. Since there are N types, the most securities there might be is $NS + NA = N(S+A)$. Note that we have a result similar to Bisin (1998) in the sense that the restriction for assets bounded from above is just a technical issue.

The shape of the price function comes from the fact that in equilibrium, given the homogeneity of the profit function for intermediaries, they will be making zero profits in equilibrium, so the cost of trading per household must be paid by the household themselves. Then, although we allow for very general nonlinearities in the pricing rules, it is basically a two part tariff: a household must pay a fixed fee to buy security k from intermediary type n, and then pay a constant price per unit. Moreover, the fixed fee coincides with the marketing cost of contacting that particular household.

6 Redundant Securities - An example

As we mentioned before, the fixed fee to trade in intermediated assets give rise to a gain for using fewer securities to transfer resources across states of the world. Then, "personalized" or "tailored" assets might be useful, and this would make the equilibrium set of issued securities to be redundant (in the sense that the columns of Q form a linearly dependent set). Authors propose the following example:

- 6 types of households, and 3 state of the world in period $t = 1$, each of them that happens with probability $\frac{1}{3}$
- Identical utility, given by $u(x_0, (x_{1,1}, x_{1,2}, x_{1,3})) = \frac{1}{3}x_0 + \frac{1}{3}\left(x_{1,1} - \frac{1}{3}x_{1,1}^2\right) + \frac{1}{3}\left(x_{1,2} - \frac{1}{3}x_{1,2}^2\right) + \frac{1}{3}\left(x_{1,3} - \frac{1}{3}x_{1,3}^2\right)$ (so that in particular, it satisfies the axioms of the expected utility theory)
- All households have an endowment of 2 units of the consumption good in the first period, and each

type of household has endowments in the second period of

$$\begin{aligned}
\omega_1 &= \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \omega_2 = \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix}, \omega_3 = \begin{pmatrix} 3 \\ 1 \\ 2 \end{pmatrix} \\
\omega_4 &= \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix}, \omega_5 = \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix}, \omega_6 = \begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix}
\end{aligned}$$

(all combinations of $\{1, 2, 3\}$)

- There is only one standard core security, that pays 1 unit per unit bought. That is, in terms of the notation above:

$$R = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

- Only one type of intermediary, with $b = 0.1$ and $c_{int}(Q_k, p) = c^n_{ret}(Q_k, p) = c'Q_k$ with $c \simeq 0$. That is, we assume that the transaction costs of selling the intermediated assets is basically zero, and the only real cost is the targeting cost to households.

Clearly, this example violates some of the assumptions made on the general setup. More specifically, $c = 0$, and the shape of the utility functions violate the assumptions. It can be shown that $c = 0$ gives further an indeterminacy in prices. Then, we will take the equilibrium with $p_s = 1$ for all s.

Note that to achieve full insurance each household only needs 1 asset more, and not two: for example, for household of type 1, if she has an asset that pays 2 if state s_1 is realized, 1 if state s_2 is realized and 0 otherwise, then the agent can buy one unit of this asset and go short on the riskless core asset, giving a sure income of 2 in period 1 . This kind of scheme is analogous for all households, and therefore since $c = 0$, the intermediary would be willing in equilibrium to produce each of these securities for each households: in equilibrium it will hold that

$$\begin{aligned}
Q_1 &= \begin{pmatrix} \frac{2}{3} \\ \frac{1}{3} \\ 0 \end{pmatrix}, Q_2 = \begin{pmatrix} \frac{1}{3} \\ 0 \\ \frac{2}{3} \end{pmatrix}, Q_3 = \begin{pmatrix} 0 \\ \frac{2}{3} \\ \frac{1}{3} \end{pmatrix} \\
Q_4 &= \begin{pmatrix} 0 \\ \frac{1}{3} \\ \frac{2}{3} \end{pmatrix}, Q_5 = \begin{pmatrix} \frac{1}{3} \\ \frac{2}{3} \\ 0 \end{pmatrix}, Q_6 = \begin{pmatrix} \frac{2}{3} \\ 0 \\ \frac{1}{3} \end{pmatrix}
\end{aligned}$$

so

$$Q = \begin{pmatrix} \frac{2}{3} & \frac{1}{3} & 0 & 0 & \frac{1}{3} & \frac{2}{3} \\ \frac{1}{3} & 0 & \frac{2}{3} & \frac{1}{3} & \frac{2}{3} & 0 \\ 0 & \frac{2}{3} & \frac{1}{3} & \frac{2}{3} & 0 & \frac{1}{3} \end{pmatrix}$$

Based on the previous analysis, we must have that each household buys 1 unit of the intermediated asset tailored specifically to it (if we normalized the prices of each good to 1) so $\theta = (1,1,1,1,1,1)$ (that is, produces one of each). See that if the financial intermediary buys 2 units of the core nominal asset from households (which as we said, hold negative positions in it, so this happens in equilibrium), then we have that:

$$Q\theta = \begin{pmatrix} \frac{2}{3} & \frac{1}{3} & 0 & 0 & \frac{1}{3} & \frac{2}{3} \\ \frac{1}{3} & 0 & \frac{2}{3} & \frac{1}{3} & \frac{2}{3} & 0 \\ 0 & \frac{2}{3} & \frac{1}{3} & \frac{2}{3} & 0 & \frac{1}{3} \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \\ 2 \end{pmatrix} = R * 2$$

So the collateral constraint (7) holds. Moreover, one can show that the aggregate demand of the core nominal asset by household is actually -2, and also that buying these assets (and paying the fixed fee b) is optimal for households, so this is in fact, an equilibrium.

Note that if for example asset Q_3 is not available, household 3 could still get full insurance, but it would be more costly (she has to use 2 of the intermediated securities instead of just one). Then, there would be a profitable opportunity for some intermediary to provide this security and get some positive profit, which cannot happen in equilibrium.

As we will see in the following section, although there are redundant assets in the sense given before, in this case this outcome is constrained efficient!

7 Constrained Efficiency

Since this economy is prone to the problem of real indeterminacy (which we commented at the beginning), the author proposes to study efficiency given a price vector $p \in \mathbb{R}^S_{++}$, which is referred to as constrained efficiency. This concept is defined formally as follows:

Definition 6 (Constrained Innovation Efficiency) *An allocation $x : I \to \mathbb{R}^{S+1}_+$ is **constrained innovation efficient with respect to** p if there does not exist innovations $\widehat{Q}$, trading plans $\left(\left\{\widehat{\theta}^n, \widehat{\psi}^n, \widehat{\zeta}^n\right\}, \widehat{y}, \widehat{z}\right)$ and a period 0 consumption function $\widehat{x}_0 : I \to \mathbb{R}_+$ such that:*

(i) : *Trading plans are feasible for intermediaries and allocations are resource feasible: i.e. they satisfy the resource constraints of market clearing (18) to (22)*

(ii) : $u^h(\widehat{x}_0(i), \widehat{x}_1(i)) > u^h(x(i))$ *for all* $i \in I$, $h : i \in [h-1, h)$, *with* $\widehat{x}_1(i) \equiv \omega_1(i) + P^{-1}R\widehat{z}(i) + P^{-1}\widehat{Q}\widehat{y}(i)$

A problem that might arise is that there are complementarities in security creation, that might not be internalized by intermediaries. This is also analogous to the source of potential inefficiencies of the Full Walrasian equilibrium in makowski, and is illustrated in this setting by the following very simple example:

7.1 Example:

- 2 types of households: 1 and 2 and 6 states of nature in $t = 1$
- Preferences are given by

$$
\begin{aligned}
u^1(x_0, x_1) &= x_0 + \sum_{s=1}^{6}\left(x_{1s} - \frac{1}{4}x_{1s}^2\right) \\
u^2(x_0, x_1) &= x_0 + \frac{1}{4}\sum_{s=1}^{6} x_{1s}
\end{aligned}
$$

- Endowments are

$$
\begin{aligned}
\omega_1 &= (1, 2, 1, 1, 2, 1) \\
\omega_2 &= (2, 2, 2, 2, 2, 2)
\end{aligned}
$$

- There are 3 core nominal assets, with returns

$$
\begin{aligned}
R^1 &= (1, 1, 0, 0, 0, 0) \\
R^2 &= (0, 1, 1, 1, 1, 0) \\
R^3 &= (0, 0, 0, 0, 0, 1)
\end{aligned}
$$

- 2 intermediaries. Cost functions are

$$
\begin{aligned}
c^1(Q) &= K\sum_{s=3}^{s=6} Q_s \\
c^2(Q) &= K\sum_{s=1}^{s=4} Q_s
\end{aligned}
$$

From the technology, we can see that producing assets that payoff only in states $s \in \{1, 2\}$ can be costlessly produced by intermediary 1, while the analogous is true for $s \in \{5, 6\}$ and intermediary 2. Also, since the costs are proportionally to the returns they have to pay to agents, intermediary 1 will only produce assets that cost zero in equilibrium, and the same is true for intermediary 2 (This is equivalent of actually assuming that intermediary 1 has an infinite cost of producing any security which pays a positive amount in states 3 to 6, and likewise for intermediary type 2 for securities that pay off in states 1 to 4.To make sure this happens in equilibrium, we can take K to be large enough as to prohibit type 1 to produce any security that pays off in states 3 to 6, and analogously for intermediary type 2. Finally, since these assets cost zero, we can without loss of generality assume that 1 produces Arrow securities for states $\{1,2\}$ and 2 produces Arrow securities for states $\{5,6\}$ .That is:

$$Q^1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}, Q^2 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}$$

It can be showed that with no innovation (incomplete market setting) there is actually a unique equilibrium, with $p_{1s} = 1$ for all s, $q_3 = q_1 = \frac{1}{2}$, $q_2 = 1$ and there is no trade among households.

Conjecture first that an equilibrium exists in which $p = 1$ and both intermediaries produce these securities. Clearly, agent 2 being risk neutral does not care about financial innovation. However, 1 could have full insurance if both intermediaries produce these securities. It can be shown that the increase in utility from the availability of full insurance possibilities is 0.375. Then, since 4 new securities are available, if $4b < 0.375$ then the agent would be willing to buy them (and it would be socially efficient since securities are costlessly produced) and there would be an equilibrium in which both types of intermediaries produce securities.

Now, imagine that one of the intermediaries does not produce any security. What will do the other one? If only one of the intermediaries produce and the other does not, then the increment in utility for agent 1 will be 0.15, and then if $2b < 0.15$, the remaining intermediary will be willing to produce these securities. However, if $b \in \left(\frac{0.15}{2}, \frac{0.375}{4}\right) = (0.075\,, 0.094)$ it will be the case that there is two equilibria: one in which both intermediaries produce securities, and one in which they will not, and the latter is inefficient. Notice the nature of the coordination game played here among the intermediaries.

In this example we can see the strategic complementarity of the intermediaries: by creating more securities, agents are more willing to incur in the fixed costs of financial intermediation. In a competitive equilibrium, this pecuniary externalities are not internalized.

Note: Another possibility that is not mentioned here of strategic complementarity is that when some intermediary produces some asset, it can be used as collateral for other intermediaries, affecting her strategy space.

7.2 Conditions for Constrained Efficiency

The following theorem provides some very restrictive sufficient conditions for competitive equilibria with financial innovation to be constrained innovation efficient

Theorem 7 (Suficient Conditions for Efficiency) *A competitive equilibrium with financial innovation is constrained innovation efficient if either of these conditions hold:*

$(\mathbf{i}):\ N=1$

$(\mathbf{ii}):\ b=0$

Proof. Let's prove first the sufficiency of $(\mathbf{i})$: Given (p,q), assume that this result does not hold, so there exist consumption allocations and trading plans as in the definition of constrained efficiency provided before. Let

$$t\left(i\right)=\omega_{0}\left(i\right)-\left(\widehat{x}_{0}\left(i\right)+q\widehat{z}\left(i\right)\right)$$

to be a price that the innovator would charge for the potential innovation $\widehat{Q}^{n}$ implicit in (ii) of the definition of constrained efficiency. Then the revenue that the intermediary would collect is

$$\int_{I}\left(\omega_{0}\left(i\right)-\widehat{x}_{0}\left(i\right)-q\widehat{z}\left(i\right)\right)di=\left(\int_{I}\omega_{0}\left(i\right)di-\int_{I}\widehat{x}_{0}\left(i\right)di\right)-q\int_{I}\widehat{z}\left(i\right)di$$

From the resource constraint (21) we get that the excedent $\left(\int_{I}\omega_{0}\left(i\right)di-\int_{I}\widehat{x}_{0}\left(i\right)di\right)$ is equal to the total of innovation costs, and $-q\int_{I}\widehat{z}\left(i\right)di$ cover the cost of buying a portfolios $\widehat{z}\left(i\right)$ of nominal core assets, by (18). Therefore, choosing this prices would make the intermediary able to provide this innovation at zero cost. However, since all households prefer strictly this plan, she could charge the positive reservation price for all households, and therefore make a profitable deviation, which is a contradiction

For $(\mathbf{ii})$: also given (p,q) if $b=0$ this implies that all intermediaries charge linear prices $r_{k}^{n}\left(y\right)=r_{k}^{n}*y$. Define, for any innovation Q

$$r\left(Q\right)=\max_{h}\sum_{s}MRS_{s,0}\left(h\right)\frac{Q_{s}}{P_{s}}$$

with $MRS_{s,0}\left(h\right)$ the equilibrium marginal rate of substitution between consumption in state s and consumption at $t=0$, and from the assumptions made about the indifference curves (that they do not touch the axis) they are well defined. We want to show first that $r\left(Q\right)$ is an equilibrium price function. First, note that households would demand the same amount as before at this prices (it will still satisfy their FOCs) and if intermediaries supply assets to households that value it the most, then the price they would charge in equilibrium would be actually r. Since price are linear in this setting, and we are holding the production side fixed (which is CRS), then we can apply the first welfare theorem, and attain efficiency (for given prices) ■

The authors do not provide necessary conditions for this result. In any case, the restrictions we need to impose on the economy to achieve efficiency are quite restrictive.

See that this is analogous to the conditions on efficiency of FWE seen in Makowski. The first condition of $N=1$ means that all intermediaries must have the same technology, which is identical to the condition of similar technologies we saw in the notes on Makowski. The reason for this condition follows the same logic: the source of potential inefficiencies comes from the fact that if the set of commodities that if introduced together would increase welfare are divided among different types of intermediaries. Therefore, if one of the intermediaries does not introduce a certain commodity, the other intermediary might not find it profitable to do it either. However, if all intermediaries (firms in Makowski) have identical technologies, then any single firm can exploit this profitable innovations, and so any equilibrium must be constrained efficient.

The other condition is $b = 0$, which is analogous to the convexity and smoothness conditions in Makowski. The existence of a fixed fee acts as a fixed cost on introducing new commodities, since a household must pay a fixed cost per new commodity traded. Then, as in the example we saw above, the existence of only one of two complementary securities may not be very useful for a household by itself, and although it increases welfare marginally, it is not worth the fixed cost the household has to pay. However, if the two assets are traded, then the complementarities between them makes it worthy for the households to trade in both. Now, when $b = 0$ this effect is not present, and the value of each new commodity will cost exactly the marginal value it gives to the household, recovering then the efficiency of the equilibrium. And is exactly when $b = 0$ that the production technologies of the financial intermediaries (that are both the production of new securities and the marketing activities they have to engage in order to sell them) are convex and smooth, as in Makowski's paper.

8 Real Indeterminacy and Small intermediary costs

If costs are too high, then innovation will not occur in equilibrium, and we will be back to the setting of incomplete markets. If we apply the results of Geneakopolos and Mas-Collel (1979) to this setting with no financial innovation, then the set of equilibrium consumption allocations will be a $S-1$ manifold, so there is huge indeterminacy. However, if costs were close to zero, we would expect assets to span the whole $\mathbb{R}^S$ space for each agent, and reduce the indeterminacy. This is exactly what the author proves in this section.

Let

$$P_\xi^S = \left\{p \in \mathbb{R}_+^S : \sum p_s = 1, p_s \geq \xi > 0\right\}$$

to be the set of prices bounded away from $\xi > 0$. Also, define a **cost structure** as a collection $\left(\{(c_{int}^n(.), c_{ret}^n(.))\}_{n=1}^N, b\right)$ that represent the set of costs for all intermediaries. We will measure "smallness" of the costs by premultiplying all costs by $\delta > 0$: $\left(\{(\delta c_{int}^n(.), \delta c_{ret}^n(.))\}_{n=1}^N, \delta b\right)$. From the literature on equilibria with incomplete markets, we know that the set of equilibrium allocations is a $S+1-$dimensional manifold (Geneakopolos and Mas-Collel (1989)), which is known as real intederminancy, since teh set of equilibrium allocations is huge. The following theorem states that taken some price as given, as $\delta \to 0$ the set of equilibrium allocation shrinkens, limiting a finite set of points (which happens generically in the complete markets case). This idea can be illustrated in the following figure, where δ and $\delta' < \delta$ are two possible parameters on cost structures:

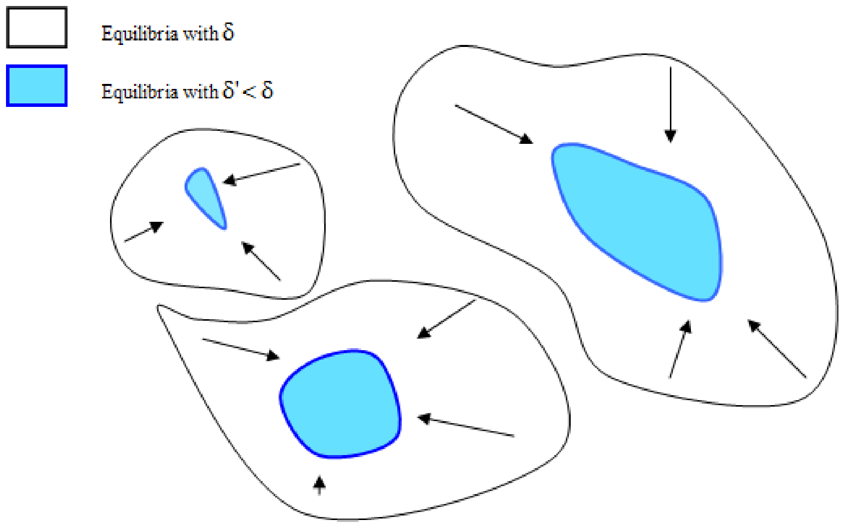

Figure 2: Set of equilibrium allocations, as $\delta \to 0$

Theorem 8 *For all* $\varepsilon > 0$, *there exist some* $\overline{\delta} > 0$ *such that if* $\delta < \overline{\delta}$ *and cost structure is* $\left(\{(\delta c^n_{int}(.), \delta c^n_{ret}(.))\}^N_{n=1}, \delta b\right)$ *, then for an open, dense subset of endowments there is a finite collection of neighborhoods* $\{B_1, B_2, ..., B_\kappa\}$ *with each having a diameter less than* ε*, such that all equilibrium consumption allocations* x*, satisfy* $x \in \bigcup_j B_j$

Proof is omitted. Is technical, although not as much as the one in theorem 1. The following figure illustrates why this statement is equivalent to showing that the set of equilibrium allocations is close to finite:

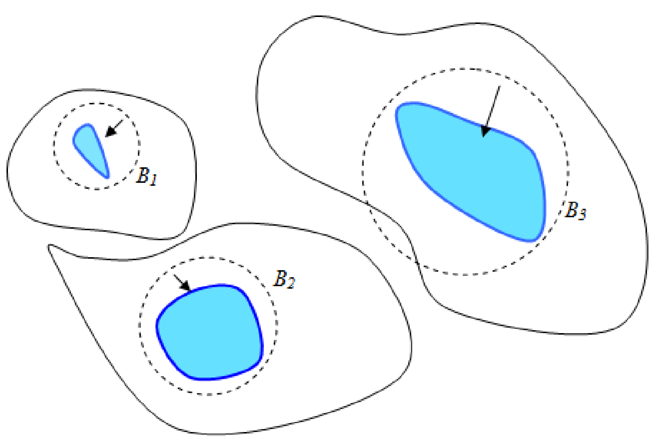

Figure 3: Idea of Theorem 8

An interesting special case is noted in the paper: suppose that for all $s \in S$, there exist some core security $a(s)$ such that $R_s^{a(s)} > 0$. Also, define x^* as an allocation satisfying:

1. There exist $p^* \in \mathbb{R}_{++}^{S+1}$ such that $p^* x^*(i) \leq p^* \omega(i)$ for all $i \in I$
2. $u^i(x^*(i)) \geq u^i(\widehat{x})$ for all $\widehat{x} : p^* \widehat{x} \leq p^* \omega(i)$ for all $i \in I$
3. $\int_I x^*(i)\, di = \int_I \omega(i)\, di$

That is, x^* is a complete market equilibrium allocation (which we know to be Pareto optimal). Then the following proposition can be showed as a corollary of the previous theorem

Corollary 9 (Closeness to Complete Markets) *If for all $s \in S$ there exist some $a(s) \in A : R_s^{a(s)} > 0$, then for every $\varepsilon > 0$ and every competitive equilibrium with financial innovation allocation x, there exist $\overline{\delta}$ such that if $\delta < \overline{\delta}$ and the cost structure is $\left(\{(\delta c_{int}^n(.), \delta c_{ret}^n(.))\}_{n=1}^N, \delta b\right)$, we have that $\inf_{x^* \in CME} \|x - x^*\|_\infty < \varepsilon$ (with CME the set of complete markets equilibrium allocations)*

This means that for all competitive equilibrium with financial allocation x, there will exist some complete markets equilibrium allocation arbitrarily close if the costs are low enough, which makes this allocation close to efficient.