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Priority Design in Centralized Matching Markets

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In many centralized matching markets, agents' property rights over objects are derived from a coarse transformation of an underlying score. Prominent examples include the distance-based system employed by Boston Public Schools, where students who lived within a certain radius of each school were prioritized over all others, and the income-based system used in New York public housing allocation, where eligibility is determined by a sharp income cutoff. Motivated by this, we study how to optimally coarsen an underlying score. Our main result is that, for any continuous objective function and under stable matching mechanisms, the optimal design can be attained by splitting agents into at most three indifference classes for each object. We provide insights into this design problem in three applications: distance-based scores in Boston Public Schools, test-based scores for Chicago exam schools, and income-based scores in New York public housing allocation.

Key words: Matching theory, Market design, Priority design, Allocative efficiency.

JEL Codes: C78, D47, D61.

1. INTRODUCTION

In recent years, across countries such as the U.S., England, and Chile, ever more school districts, national university admissions boards and public authorities have adopted centralized matching mechanisms to allocate objects to agents. In many such markets, the property rights that agents receive over such objects are given by *priorities* derived from various criteria such as academic attainment, income or distance. Informed by the extensive academic literature on matching, these authorities have often introduced *stable* matching mechanisms.[1] Stable mechanisms respect these priorities in a natural sense by guaranteeing that no given agent strictly prefers the assigned object of another agent with lower priority.[2] This makes priorities (and the property rights they encode) critical to the realized distribution of outcomes in these markets across race, socioeconomic group, and space.

1. See, for example, Balinski and Sönmez (1999), Abdulkadiroğlu and Sönmez (2003), Roth *et al.* (2004), and Abdulkadiroğlu *et al.* (2005).

2. In the context of student assignment, stable mechanisms are both individually rational and eliminate justified envy.

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In this context, a large matching literature takes priorities as primitive and studies the design of the allocation mechanism (Roth, 2002). However, priorities often appear to be designed by the relevant authorities as a function of some other, underlying score.

Prominently, Boston Public Schools (BPS) wished to ensure that students were able to attend schools close to their own homes in order to both reduce transportation costs and improve community cohesion (Dur *et al.*, 2018). However, they did not introduce a priority that ranked students strictly according to their distance from each school. Instead, in the walk-zone assignment system employed by BPS until 2013, at each school students were partitioned into two groups: walk-zone students who lived within a certain radius of the school and the others who did not. Moreover, the New York City Housing Authority (NYCHA) has the goal of providing public housing to those who cannot afford adequate housing in the absence of assistance (Collinson *et al.*, 2015; Arnosti and Shi, 2020). However, agents' priorities are not strictly decreasing in their income. Instead, a household is only eligible for public housing if its income is less than a certain fraction of New York's Area Median Income (AMI).[3] Similar design concerns that trade off diversity and admitting the most academically qualified students are present in the Chicago Public Schools (CPS) system which uses scores derived from the academic merit of students.[4]

From these examples, it is clear that these authorities not only design priorities, but also choose coarse priority structures that do not reverse an underlying score. However, there exists little theoretical work that approaches the problem of optimal priority design.[5] Therefore, we study the problem of a mechanism designer that is faced with an underlying score (such as distance, income, or academic achievement in our running examples) and has the power to design priorities. Following the priority designs we see in our applications, we restrict attention to priority designs that do not reverse the given underlying score.[6] We call such designs *priority coarsenings*, as they result in a coarser ordering of students relative to the initial scores. Our main results show that, under any stable matching mechanism, the set of implementable allocations can be attained by designs that split agents into at most three object-specific indifference classes (Theorem 1). Moreover, when the mechanism designer has a continuous objective function, an optimal design exists (Theorem 2) and therefore requires only three indifference classes.

Concretely, we establish a general continuum matching market framework in the spirit of Abdulkadiroğlu *et al.* (2015) and Azevedo and Leshno (2016) for assessing the question of optimal priority design. In the *ex-ante stage* of the model, the mechanism designer knows the joint distribution of agents' preferences and rankings according to an underlying score and chooses a rule that coarsens the underlying score to maximize some arbitrary, continuous objective function. Types are then realized and in the *interim stage* agents are matched to objects according to a stable matching mechanism. Our framework features a unique stable matching, which makes it possible

3. In both BPS and NYCHA, the mechanism used during the matching process requires a strict priority order. In both markets, this strict order is obtained by assigning a random tie-breaker number for the agents and prioritizing according to the random tie-breaker number within each priority class.

4. To balance these competing objectives, CPS chose a policy that divided students into four socioeconomic tiers, reserved 70% of the capacity at exam schools for these tiers in equal proportion and left the remaining 30% open to students from all groups (Ellison and Pathak, 2021).

5. Exceptions to this include the work by Echenique and Yenmez (2015) and Erdil and Kumano (2019) who discuss issues related to substitutable priorities. Relatedly, Hafalir *et al.* (2013) and Ehlers *et al.* (2014) study the design of reserves as implemented through the use of differing priorities in different slots and affirmative action policies with upper and lower bounds, while Dur *et al.* (2020) study the design of precedence orders over these slots in the CPS context.

6. Beyond the relevance in our applications of restricting attention to priority coarsenings, in certain contexts such as CPS where the underlying score is derived from academic performance, reversing scores may give rise to incentive compatibility issues whereby students intentionally perform worse on exams. Sönmez (2013) documents a case in the US Military Academy and Reserve Officer Training Corps where having a lower score might be more advantageous for the graduates. As a result, cadets tried to intentionally lower their scores to get assigned to a more preferred army branch.

to express the type-contingent probability of assignment without reference to the specific stable mechanism under consideration.[7] This object functions as an *allocation* in the ex-ante stage, specifying the probability that each type of agent is assigned to each object as a function of the coarsening chosen by the designer.

Using this framework, we establish our main theoretical contribution that the optimal design can be attained by partitioning agents into at most three indifference classes at each object. The intuition for this result depends crucially on specific types of equivalence classes that emerge following the selection of a coarsening, *lottery classes*. Lottery classes have the property that some agents within them have positive probability of receiving that object and an object they rank lower in their preferences. Therefore, some agents in a lottery class have probability of being assigned to that object strictly between zero and one. We show that under a stable mechanism, there can be at most one lottery class of agents per object. This result holds for the following simple reason: if there is more than one lottery class, there will be at least two equivalence classes with interior probabilities of assignment and it must be that some agents in a lower equivalence class will be assigned the object while some agents in the higher class will not, violating stability (see the example below for a simple demonstration of this in the discrete context). Furthermore, all agents in higher priority classes are guaranteed to receive the object if they prefer it, and all agents in lower priority classes never receive the object. Thus, all classes above and below the lottery class can be merged without affecting the allocation, so the outcome of any coarsening can be replicated by a coarsening that splits agents into at most three indifference classes.[8]

We leverage the simplifying power of our theoretical results in our applications. First, we study the design of distance-based priorities in the BPS system. The following discrete example provides a simple illustration of the trinary optimality result and explores how the trade-off between diversity and distance travelled shapes the optimal policy in this context. Assume there are four students, i_1, i_2, i_3, i_4, and one school with two slots. Each student i_n lives n miles away from the school and i_3 belongs to an under-represented group of society. The mechanism designer gains $\alpha \in [0,1]$ utility from admitting the student from the underrepresented group, i_3, and loses utility equal to $1-\alpha$ times the total distance that admitted students travel. Mirroring the coarse walk-zone priorities in BPS, the mechanism designer decides how to coarsen students' distance to maximize this objective.

Owing to Theorem 1, it is without loss of optimality to consider priority coarsenings that divide students into at most three groups. In this particular example, it can be shown that an optimal policy is one of three such coarsenings: a two-zone priority with a cut-off at 2.5 miles, a two-zone priority with a cut-off at 3.5 miles and a three-zone priority with cut-offs at 1.5 and 3.5 miles. The first option guarantees that i_1 and i_2 are admitted for sure, the second option assigns i_1, i_2, and i_3 to the school with probability 2/3 each and the third option assigns i_1 with unit probability and i_2 and i_3 with probability half.

When are each of these optimal? A quick calculation shows that for $\alpha \leq 1/2$, the first option is optimal. This is intuitive as the first option guarantees i_1 and i_2 are admitted and for low values of α, minimizing distance travelled is much more important than diversity. For $\alpha \geq 3/4$, the second option is optimal as it maximizes the probability that the under-represented student i_3 is admitted. This is also intuitive as in this case the mechanism designer values diversity much more than distance. For $\alpha \in (1/2, 3/4)$, the third option is optimal. Here, the mechanism designer

7. We obtain a unique stable matching through the assumption that there is full support of all student types in our economy. We show that our results are robust to relaxation of this assumption in Supplementary Appendix F.

8. It is clear that this construction requires knowledge of the structure of the economy on the part of the planner. We discuss the validity of this assumption in our applications and robustness of our results to aggregate uncertainty in Section 3.4 and Supplementary Appendix C.

not only wants to have a diverse student body but also cares about distance travelled, and hence entails a strong preference for admitting i_1.

While we here provided a discrete example for simplicity and to exemplify our theoretical results, in our BPS application we employ the continuum framework to provide analytical results on the optimality of various walk-zone policies. In particular, we argue that the pursued policy of two zones is compatible with a planner who places a large weight on neighborhood assignment. However, were diversity concerns to dominate, there remains additional policy latitude for a planner to adopt a three-zone policy—which our trinary optimality result ensures is the only potential welfare-improving deviation from a simple two-zone policy.

Second, we study how priority design could be used in the design of exams in the CPS system. We show how coarser grading can increase the admissions of minority groups who score less well on exams and show how the trade-off between diversity and admitting the best scoring students shapes the structure of the optimal exam design. In particular, we show that when diversity concerns are sufficiently strong, pooling students' exam scores in up to three groups constitutes optimal policy.

Finally, we study the design of income-based priorities in public housing allocation by NYCHA. Relative to the other applications, this features complications as we must consider the dynamic nature of public housing allocation. Nevertheless, in the steady state of the dynamic matching model we develop, our Theorems 1 and 2 apply directly: the planner need only introduce two income cutoffs. We show how the trade-off between widening eligibility for public housing and targeting the allocation to the most needy shapes the optimal policy. In particular, when there is a sufficiently strong relationship between income and outside options, the optimal policy excludes the richer agents from eligibility for public housing—rationalizing the policy pursued by NYCHA. However, we also show a three-tiered system may improve welfare in the case of sufficient heterogeneity in outside options.

Related literature. The theory and design of matching markets was pioneered by Gale and Shapley (1962). Balinski and Sönmez (1999) and Abdulkadiroğlu and Sönmez (2003) introduced the problems of student assignment and school choice, respectively. Much of the literature following these seminal papers focused on allocation mechanisms that take property rights (encoded in priorities) as given. While there is a small literature that studies the effects of certain classes of priorities on certain mechanisms (Echenique and Yenmez, 2015; Erdil and Kumano, 2019),[9] this article develops the novel idea that priorities can be viewed as choice objects.

Our analysis is closely related to the rapidly growing literature on matching in the presence of distributional constraints and affirmative action. Kojima (2012) studies the widely used majority quotas and shows that such policies can harm all minority students. Hafalir *et al.* (2013) approach this question from a mechanism design perspective and introduce minority reserves to overcome the shortcomings of quota policies. Ehlers *et al.* (2014) generalize these reserves to incorporate multiple priority levels and accommodate further policies used in practice, such as floors and ceilings, and develop new mechanisms for hard or soft floors and ceilings. Reserve policies have also been generalized by Kominers and Sönmez (2016), who introduce and study

9. In particular, Erdil and Kumano (2019) study substitutable priorities with ties and propose an algorithm to improve the widely used Deferred Acceptance algorithm from an efficiency point of view. Echenique and Yenmez (2015) emphasize that substitutability is in conflict with schools' preferences for diversity and study different rules for incorporating such preferences into the assignment mechanism.

matching with slot-specific priorities.[10] In an alternative approach to the literature with quotas and reserves, Kamada and Kojima (2017, 2018) and Goto *et al.* (2017) study stability and efficiency in matching-with-constraints models. Finally, Çelebi and Flynn (2021) study the trade-offs between using minority quotas (and reserves) and score subsidies in affirmative action.

The common approach of papers studying priority structures, distributional constraints, and affirmative action in their respective contexts is to take the priority structure as given and analyze the properties of different mechanisms. The key difference between our paper and these literatures is our introduction of, and focus on, priority coarsenings that introduce indifferences into underlying scores as a tool to design priorities.[11] In this context, in contrast to the existing literature, we compare different priority structures for a fixed stable mechanism and investigate the relationship between the allocation, welfare and the priority structure.

We apply our general results regarding the design of priorities to three different settings, which have themselves been the subject of previous research that fixes priorities and compares different mechanisms in specific contexts. First, we study distance-based priorities and the design of walk-zones with reference to BPS, a setting that has been studied prominently by Dur *et al.* (2018). Second, motivated by the CPS assignment system outlined in Dur *et al.* (2020), we study how diversity considerations in an environment with priorities based upon student achievement affect optimal priority design. Finally, we study income-based priorities and the allocation of public housing in NYCHA, a context studied by Arnosti and Shi (2020) in their analysis of the design of lotteries and waitlists under fixed priorities.[12] Furthermore, while not explicitly featured as applications in this article, the design concerns we highlight are not unique to the U.S. context. For example, Sönmez and Yenmez (2019a,b,c) and Aygun and Bó (2021) study affirmative action policies in India and Brazil, respectively, and propose mechanisms for each context. In all of these contexts, priority coarsenings offer a new policy lever that could be useful in cases where current policies, such as quotas in affirmative action, are controversial.

Outline. The rest of the article proceeds as follows. Section 2 introduces the matching model and priority coarsening. Section 3 studies optimal priority design and provides our main results (Theorems 1 and 2). Motivated by the BPS context, Section 4 applies our results in the case of distance-based scores and considers optimal walk-zone design. Section 5 uses our results to study priority design with test-based scores, as in CPS. In Section 6, we augment our framework to analyze the design of income-based scores and the allocation of public housing in a dynamic matching model. Section 7 concludes.

2. MODEL

In this section, we develop a model of a matching market with a continuum of students as in Abdulkadiroğlu *et al.* (2015) and Azevedo and Leshno (2016) to study how and when priority

10. Further papers that analyze reserve-like policies include Doğan (2016), who proposes an assignment rule that never harms all minority students. Fragiadakis and Troyan (2017) propose a dynamic quota mechanism to improve allocations under hard bounds.

11. As the main policy tool available to the designer in our paper is the "coarseness" of the priorities to be used in the mechanism, our article is implicitly related to the literature that studies matching markets under indifferences. Following our applications, we require stability with respect to the tie-broken priorities and abstract away from the issues studied in that literature such as alternative stability criteria (Kesten and Ünver, 2015), computation of stable and efficient matchings (Erdil and Ergin, 2008, 2017), correlated lotteries (Ashlagi and Shi, 2014), random assignments under constraints (Budish *et al.*, 2013), and efficiency improving lottery mechanisms (Kesten *et al.*, 2017).

12. Relatedly, Leshno (2019) and Bloch and Cantala (2017) study models of dynamic waitlists and argue that randomized assignments will improve welfare. Geyer and Sieg (2013), Waldinger (2018), and Sieg and Yoon (2020) estimate empirical models of public housing allocations and compare different mechanisms.

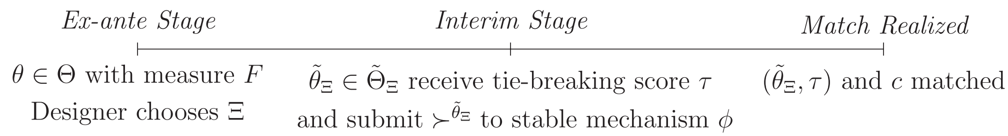

FIGURE 1

Model timeline

design can affect the allocation of objects and welfare. We proceed with the standard matching literature language of matching students and schools. However, as we later show, our analysis is of relevance beyond this context. In the ex-ante stage, the mechanism designer has a prior over the distribution of student types (that comprise preferences over schools, underlying scores at each school, and other identifying information) in the population and chooses a rule that coarsens the underlying scores of students into the priorities they will hold in the interim stage. In the interim stage, types are realized, and students submit their preferences to a stable matching mechanism that uses these priorities. Finally, the students are matched to the schools and payoffs are realized. The model timeline is shown in Figure 1.

2.1. *Ex-ante stage and priority coarsening*

There are a finite set of schools, denoted by $\mathcal{C}=\{c_0,c_1,\dots,c_n\}$ where c_0 is a dummy school that corresponds to a student going unmatched, and a unit measure of students. Let $\theta=(\{u_c^\theta,s_c^\theta\}_{c\in\mathcal{C}},\kappa)$ denote the type of a student whose utility from going to school c is u_c^θ, who has score $s_c^\theta\in S_c=[0,1]$ in school c, where S_c denotes the set of possible scores at school c. For example, S_c could contain possible distances from a school, or students' scores in an exam. Finally, κ denotes any other information about the socio-economic situation or minority status of the student.[13] We use u^θ, s^θ, and κ^θ to, respectively, denote the utility profile, score profile and additional information of a student with type θ. The set of student types is denoted by Θ, over which there is a probability measure F. $Q=(Q_0,\dots,Q_n)$ denotes the capacities of schools.[14] The economy can therefore be summarized by the triple $\Omega=(F,Q,\Theta)$.

In the ex-ante stage, the designer transforms the students' scores into the priorities that will be used in the matching mechanism. Formally, a priority design at school $c\in\mathcal{C}$ is a function $\Xi_c: S_c\to P_c$ that maps students' scores $s_c\in S_c$ into their priority $\Xi_c(s_c)\in P_c$, where $P_c\subseteq S_c=[0,1]$. A *priority design* is then a function that collects each school's design $\Xi(s)\equiv(\Xi_1(s_1),\dots,\Xi_n(s_n))$, with corresponding domain $S=\prod_{c=1}^n S_c$ and range $P=\prod_{c=1}^n P_c$. As we have motivated, we will restrict attention to coarsening rules: priority designs that coarsen, but do not reverse students' scores.

Definition 1. *A coarsening rule is a priority design $\Xi:S\to P$ such that for all $c\in\mathcal{C}$ and all $s,s'\in S$ such that $s_c\geq s'_c$, we have that $\Xi_c(s_c)\geq\Xi_c(s'_c)$. Moreover, for each $c\in\mathcal{C}$, P_c is either finite or Ξ_c is the identity function.*

The final condition that either the set of priorities at each school P_c is finite or the priority design leaves scores unchanged is a technical one that ensures ties resulting from coarse priorities can be broken while maintaining a well-defined economy. A natural example of coarsening that many will be familiar with is the conversion of fine numerical exam scores (ranging from 1 to

13. We will suppress κ until the applications sections, as it is irrelevant for allocations.

14. As school c_0 is a dummy school representing outside options, it is without loss of generality to set $Q_0=1$.

100) into letter grades (ranging from F to A). Example 1 demonstrates a coarsening from strict scores over $[0,1]$ to a priority structure with three indifference classes, while Example 2 shows the planner may choose to use the scores as priorities without transforming them.

Example 1. *There is one school,* $|\mathcal{C}|=1$*, scores lie in* $S=[0,1]$*, and we coarsen these strict scores into three indifference classes* $P=\{\frac{1}{3},\frac{2}{3},1\}$ *according to the rule:*

$$\Xi(s)=\begin{cases}\frac{1}{3}, & s\in[0,1/3),\\ \frac{2}{3}, & s\in[1/3,2/3),\\ 1, & s\in[2/3,1].\end{cases} \tag{1}$$

Specifically, Ξ *takes any student who had an initial score lower than* $1/3$ *and gives them a priority of* $\frac{1}{3}$*, students with initial score between* $1/3$ *and* $2/3$ *are given priority* $\frac{2}{3}$*, and students with initial score greater than* $2/3$ *are given priority* 1.

Example 2. *Let* $S=P$ *and* Ξ_c *be the identity function for all* $c\in\mathcal{C}$*. Then for any s,* $\Xi_c(s)=s$*, i.e. the priorities are identical to the scores.*

We argue that non-reversal is a relevant and natural property to demand from a priority design in our setting. From a practical perspective, such interventions seem feasible from a political economy point-of-view and have occurred in markets with an established score structure, such as distance-based priorities (as in BPS) and priorities that depend on a measurable statistic such as income (as in NYCHA).[15] Beyond the main applications of this paper, coarse priorities have also been advocated for in the 2018 U.S. Centers for Disease Control Vaccine Allocation Guideline, which divides the general population into four tiers based on their age (CDC, 2018). Furthermore, many states in the U.S. have recently adopted priority systems in the allocation of ventilators based on the Sequential Organ Failure Assessments (SOFA) score (Piscitello *et al.*, 2020; Pathak *et al.*, 2020), which maps continuous measures of patient health to a discrete set of values for six organ systems. Moreover, when scores are based on achievement (as in CPS) or can otherwise be gamed, a transformation that does not satisfy non-reversal may incentivize students to obtain lower scores, which is clearly undesirable.[16]

We argue that requiring stability with respect to the designed priorities is reasonable for three reasons. First, if students only know the coarsened scores, a student within a given priority class would not be able to block a match between a student in the same priority class and a school. This lack of knowledge seems natural, for instance, in the exam schools context where two students who receive the same grade do not have access the underlying raw score from which their grades were constructed. Moreover, school boards in centralized matching markets have the ability to enforce matches and prevent schools and students from matching outside of the system. Second, if we interpret stability as encoding procedural fairness and preventing legal challenge, then stability with respect to coarsened scores retains these properties. Finally, as our examples in the introduction from BPS, CPS, and NYCHA attest, authorities do in fact engage in the design of priorities and employ matching mechanisms that yield stable outcomes.

15. See Sections 4 and 6 and the references therein for more detail on these contexts.

16. As discussed in footnote 6, Sönmez (2013) provides a concrete example of such incentive compatibility issues in the context of the U.S. Military Academy.

2.2. *The interim stage: matching model*

We now study how to map the choice of coarsening by the designer Ξ to a matching in the interim stage. To do this, we first construct from our economy in the *ex-ante* stage an ordinal economy that transforms utility values of the agents to ordinal preferences for the matching stage. To this end, for any type $\theta=(u^\theta,s^\theta)$, we define the corresponding ordinal type $\tilde{\theta}=(\succ^{\tilde{\theta}},s^{\tilde{\theta}})$ by computing their induced ordinal preferences by ranking the schools in decreasing order according to u_c^θ and imposing that $s^\theta=s^{\tilde{\theta}}$.[17] Defining R as the set of ordinal preference relations over $\mathcal{C}$, we see that the set of induced types has product structure $\tilde{\Theta}=R\times S$. The distinction between a type and an induced type is subtle. As we consider an ex-ante stage for the purposes of performing welfare calculations in our later analysis, types refer to both a student's v.N-M preferences and their scores, while a student's induced type refers to the student's induced ordinal preferences and their scores.[18] An economy $\Omega=(F,Q,\Theta)$ thereby results in an ordinal economy $\tilde{\Omega}=(\tilde{F},Q,\tilde{\Theta})$, where $\tilde{F}$ is the probability measure over $\tilde{\Theta}$ induced by F.[19] We further make the following technical assumption that $\tilde{F}$ admits a density $\tilde{f}$ that has full support and has no mass points:[20]

Assumption 1. *The density of all ordinal types $\tilde{f}$ is well-defined and $\tilde{f}(\tilde{\theta})>0$ for all $\tilde{\theta}\in\tilde{\Theta}$.*

Second, we now show how the planner's choice of coarsening Ξ affects the ordinal economy in the interim stage. This transforms each ordinal type $\tilde{\theta}=(\succ^{\tilde{\theta}},s^{\tilde{\theta}})\in\tilde{\Theta}$ into a new ordinal type $\tilde{\theta}_\Xi=(\succ^{\tilde{\theta}},\Xi(s^{\tilde{\theta}}))\in\tilde{\Theta}_\Xi$ by replacing the score vector $s^{\tilde{\theta}}$ with the priority $s^{\tilde{\theta}_\Xi}\equiv\Xi(s^{\tilde{\theta}})$, and changes the set of students from $\tilde{\Theta}$ to $\tilde{\Theta}_\Xi$ and the probability measure from $\tilde{F}$ to $\tilde{F}_\Xi$.[21] Priority coarsening introduces indifferences, the existence of which necessitates tie-breaking to compute matchings. To this end, we augment the model with tie-breakers. Each student $\tilde{\theta}_\Xi$, in addition to her ordinal preferences and priority, receives a tie-breaker number $\tau\in[0,1]$, where $\tau\sim U[0,1]$. Thus, the distribution over types in the economy with tie-breakers $\tilde{F}_\Xi^\tau$ on $\tilde{\Theta}_\Xi^\tau=\tilde{\Theta}_\Xi\times[0,1]$ is almost surely such that $\tilde{f}_\Xi^\tau(\tilde{\theta}_\Xi,\tau)=\tilde{f}_\Xi(\tilde{\theta}_\Xi)$ for all $\tau\in[0,1]$.[22] This results in the coarsened ordinal economy with tie-breakers $\tilde{\Omega}_\Xi^\tau=(\tilde{F}_\Xi^\tau,Q,\tilde{\Theta}_\Xi^\tau)$, which lies in the set of all strict ordinal economies $\mathcal{O}$.

We are now ready to define the matching mechanism that applies in the coarsened ordinal economy with tie-breakers. A matching in this environment is a function $\mu:\mathcal{C}\cup\tilde{\Theta}_\Xi^\tau\rightarrow 2^{\tilde{\Theta}_\Xi^\tau}\cup\mathcal{C}$, where $\mu(\tilde{\theta}_\Xi,\tau)\in\mathcal{C}$ is the school that any ordinal type $\tilde{\theta}_\Xi$ with tie-breaker τ is assigned and $\mu(c)\subseteq\tilde{\Theta}_\Xi^\tau$ is the set of students assigned to school c.[23] Let $\mathcal{M}$ be the set of all matchings.

17. For example, if $u_{c_1}^\theta=1$, $u_{c_2}^\theta=2$, and $u_{c_3}^\theta=3$, the ordinal preferences are $\succ^\theta=c_3,c_2,c_1$. If $u_c^\theta=u_{c'}^\theta$ for $c\neq c'$, then students break the ties randomly.

18. Despite being irrelevant for allocations, the cardinal utility will later matter for the welfare analysis we perform in our applications.

19. To obtain $\tilde{F}$ from F, for each induced type $\tilde{\theta}=(\succ^{\tilde{\theta}},s^{\tilde{\theta}})\in\tilde{\Theta}$, simply compute the measure of types $\theta=(u^\theta,s^\theta)\in\Theta$ such that u^θ induces the ordinal preferences $\succ^{\tilde{\theta}}$.

20. See Footnote 19 to see how this condition can be easily translated to a primitive condition on F.

21. See that $\tilde{\Theta}_\Xi=R\times P$, where P is the range of Ξ. To construct $\tilde{F}_\Xi$ from $\tilde{F}$, for all types $\tilde{\theta}_\Xi=(\succ^{\tilde{\theta}_\Xi},s^{\tilde{\theta}_\Xi})\in\tilde{\Theta}_\Xi$, we compute the measure under $\tilde{F}$ of all types $\tilde{\theta}\in\tilde{\Theta}$ such that $\succ^{\tilde{\theta}}=\succ^{\tilde{\theta}_\Xi}$ and $\Xi(s^{\tilde{\theta}})=s^{\tilde{\theta}_\Xi}$.

22. See Lemma 1 in Supplementary Appendix B for a formal statement and proof. Also note that whenever a coarsening is not the identity, $\tilde{f}_\Xi$ is a probability mass function.

23. The mathematical definition of a matching for the strict continuum economy we study (with ordinal types $\tilde{\Theta}_\Xi^\tau$) follows Azevedo and Leshno (2016) and requires that μ satisfies the following four properties: (i) $\mu(\tilde{\theta}_\Xi,\tau)\in\mathcal{C},\forall(\tilde{\theta}_\Xi,\tau)\in\tilde{\Theta}_\Xi^\tau$; (ii) $\mu(c)\subseteq\tilde{\Theta}_\Xi^\tau$ is measurable, $\tilde{F}_\Xi^\tau(\mu(c))\leq Q_c,\forall c\in\mathcal{C}$; (iii) $c=\mu(\tilde{\theta}_\Xi,\tau)\iff(\tilde{\theta}_\Xi,\tau)\in\mu(c)$; (iv) $\{(\tilde{\theta}_\Xi,\tau)\in\tilde{\Theta}_\Xi^\tau:c\succ^{\tilde{\theta}_\Xi}\mu(\tilde{\theta}_\Xi,\tau)\}$ is open $\forall c\in\mathcal{C}\setminus\{c_0\}$. This last requirement imposes that the set of students that prefer their match to any given school (excluding the outside option) is open.

A matching mechanism ϕ is a function $\phi: \mathcal{O} \to \mathcal{M}$ that assigns a matching to each ordinal economy. Blocking and stability are defined as follows. A student-school pair $(\tilde{\theta}_\Xi, \tau, c)$ *blocks* a matching μ if the student prefers c to her match under μ and either the school c does not fill its capacity or the school c is matched to another student who has strictly lower score than $(\tilde{\theta}_\Xi, \tau)$. Formally, $(\tilde{\theta}_\Xi, \tau, c)$ blocks μ if $c \succ^{\tilde{\theta}_\Xi} \mu(\tilde{\theta}_\Xi, \tau)$ and either (i) $\tilde{F}^{\tau}_{\Xi}(\mu(c)) < Q_c$, or (ii) there exists $(\tilde{\theta}'_\Xi, \tau') \in \mu(c)$ with (a) $s_c^{\tilde{\theta}_\Xi} > s_c^{\tilde{\theta}'_\Xi}$, or (b) $s_c^{\tilde{\theta}_\Xi} = s_c^{\tilde{\theta}'_\Xi}$ and $\tau > \tau'$. A matching μ is *stable* if there are no blocking pairs. A mechanism ϕ is stable if it returns a stable matching for all economies.

In this environment, any coarsening Ξ and the matching mechanism ϕ together induce a probability distribution for each student type over the school that they are ultimately assigned. We call this distribution an *allocation* $g_{(\Xi,\phi)}: \Theta \times \mathcal{C} \to [0,1]$, with the probability that type θ is assigned to school c given by $g_{(\Xi,\phi)}(c|\theta)$.[24] We denote the set of potential allocations by $\mathcal{G}$.[25] We construct this allocation from the matching and distribution of tie-breakers by taking, for each tie-breaker realization of each student, the match the student receives and then integrating this over the uniform distribution of tie-breakers (see Lemma 2 in Supplementary Appendix B.1). These steps ensure that $g_{(\Xi,\phi)}$ is a well-defined distribution and respects the constraints imposed by the mechanism, including that no school is over capacity and no student has probability exceeding one of attending all potential schools.

Throughout the article, as we have motivated, we will assume the matching mechanism is stable.

Assumption 2. *The matching mechanism ϕ is stable.*

The importance of Assumptions 1 and 2 for our analysis is that Assumption 1 implies that there is a unique stable matching. Thus, after the planner fixes the coarsening in the *ex-ante* stage, Assumption 2 pins down the matching in the interim economy uniquely (see Lemma 1 in Supplementary Appendix B.1). Therefore, it is not important for us to specify which stable matching mechanism ϕ is. We will correspondingly suppress dependence on ϕ for the remainder of the analysis and write allocations as g_Ξ. A strong justification for these assumptions (which rule out multiple stable matchings) is that, empirically, the set of stable matchings has been found to be very small in large markets, including BPS.[26] Nevertheless, in Supplementary Appendix F, we relax this full-support assumption and allow multiple stable matchings to exist and we summarize the robustness of our main results to relaxing Assumption 1 in Section 3.5.

3. PRIORITY DESIGN

Having established a framework for the analysis and justified our restriction of the policy space of the designer to be that of priority coarsenings, we can now prove our main results (Theorems 1 and 2) and establish the existence of an optimal trinary coarsening. These results are stark as they reduce the complexity of finding the optimal coarsening from an infinite-dimensional problem to a $2|\mathcal{C}|$-dimensional problem and place a simple structure on optimal policies. We

24. Abdulkadiroğlu *et al.* (2017) use a similar representation to obtain the propensity score that any student is matched to a school to estimate treatment effects, albeit not as a function of any design tool of the policymaker.

25. Allocations will lie in (a subset of) the space of measurable functions with finite integral, so that $\mathcal{G} \subset L^1(\Theta \times \mathcal{C})$. See Supplementary Appendix B.2 for details on the measure space with respect to which we demand that g is measurable.

26. See Roth and Peranson (1999) for evidence from National Resident Matching Program and Pathak and Sönmez (2008) for evidence from BPS. In particular, in BPS for school years 2005–06 and 2006–07, there is only one stable matching for either year.

later leverage these results directly in applications to provide concrete insight into a number of important problems in market design.

3.1. *Trinary replication*

We now turn to proving the main result of the paper: any allocation achievable via a coarsening can be replicated with a trinary coarsening. Formally, we define a trinary coarsening as a coarsening such that the priority structure at each school features at most three equivalence classes of students.

Definition 2. *A coarsening $\Xi : S \to P$ is trinary if $|P_c| \leq 3$ for all schools $c \in \mathcal{C}$.*

Our main implementation result is stated formally as Theorem 1.

Theorem 1. *Suppose that a coarsening Ξ induces the allocation g_Ξ. There exists a trinary coarsening Ξ' that induces g_Ξ.*

Proof. See Appendix A.1. □

The basic intuition for this result and why it follows from stability is easily seen in the following example. Consider a model with an outside option and a school that all students prefer to the outside option. There is positive density of all scores between zero and one (in view of Assumption 1) at the school and the scores are coarsened into finitely many indifference classes. The matching mechanism is stable (by Assumption 2). Thus, if there is a positive probability that a student with lower priority is admitted to the school, then the higher priority student must not be allocated to the outside option, as that will violate stability. As a result, under any coarsening, there is at most one class of students (the lottery class) who have probability strictly between zero and one of being admitted. All students in higher priority classes are admitted with probability one while all students in lower priority classes are admitted with probability zero.[27] Thus, there can exist at most one lottery class, and the combination of stability and full support of ordinal types (Assumptions 2 and 1, respectively) pins down this uniqueness. The outcome of the coarsening can then be generated by an alternative coarsening that preserves the lottery class from the first coarsening, maps all students above this class into one class, and maps all students below the lottery class into another class. Hence, the outcome of any coarsening can be replicated by another coarsening with at most three indifferences classes at each school.

3.2. *The planner's objective*

To discuss optimal priority design, it is necessary to have an objective function for the planner. We assume that the planner has a complete and transitive preference over allocations g in the set

27. For concreteness, suppose instead that there were two equivalence classes A and B, where students in both have interior probabilities of assignment—so that there are two lottery classes—and students in A have higher priority than those in B. As these probabilities are interior, necessarily some students in A will not be allocated the school, while some students in B are. As there is full support of ordinal types, some students in A who are not admitted most prefer the given school. Moreover, they have higher priority than all assigned students from B. Thus, the outcome is unstable, as these rejected students in A and the school form a blocking pair. As the mechanism is assumed to be stable, this is a contradiction.

of all potential allocations $\mathcal{G}$ represented by a utility function $Z:\mathcal{G}\to\mathbb{R}$.[28] Moreover, we make the technical assumption that the utility function of the planner is continuous:

Assumption 3. *The social planner's objective function $Z:\mathcal{G}\to\mathbb{R}$ is continuous in g.*[29]

In the interests of clarity, we now discuss three natural specifications of planner utility that satisfy this assumption and will be used later in our applications: a utilitarian planner; a planner who cares about student utility with some penalty for deviating from the underlying score; and a planner who has affirmative action concerns.

1. A utilitarian social planner has utility function given by:

$$Z(g)=\int_{\Theta}\sum_{c\in\mathcal{C}}\lambda(\theta)u_c^{\theta}g(c|\theta)dF(\theta) \tag{2}$$

for some function yielding welfare weights $\lambda:\Theta\to\mathbb{R}_+$.
2. A priority-augmented social planner has utility function given by:

$$Z(g)=\int_{\Theta}\sum_{c\in\mathcal{C}}\left[u_c^{\theta}+\lambda(\theta)h(s_c^{\theta})\right]g(c|\theta)dF(\theta), \tag{3}$$

where $h:[0,1]\to\mathbb{R}$ is a monotonically increasing function that determines the base cost of the score not being met and $\lambda:\Theta\to\mathbb{R}$ is the weight of that loss for each underlying type θ.
3. An affirmative-action-concerned social planner has utility function given by:

$$Z(g)=\int_{\Theta}\sum_{c\in\mathcal{C}}u_c^{\theta}g(c|\theta)dF(\theta)+h\Big(\int_{\Theta}1\{\kappa^{\theta}\in D\}g(\hat{c}|\theta)dF(\theta)\Big) \tag{4}$$

where $\kappa^{\theta}\in D$ means that type θ is a student in the group which the planner wishes to ensure is more represented (recall that κ^{θ} corresponds to any non-preference or score information corresponding to a student), $\hat{c}$ is some given school, and h is a continuous and monotonically increasing function. This specification therefore rewards the planner for admitting more students in group D to the particular school $\hat{c}$. In practice, one might imagine that $\hat{c}$ is a high-quality school and D is an under-represented minority group.

Given this structure, the planner's problem is to choose a coarsening such that the induced allocation maximizes the planner's utility function over the set of potential coarsening rules. By Theorem 1, it is without loss of optimality for the planner to restrict attention to trinary coarsenings. That is, they can simply select two cutoff values for each school $v=\{\underline{P^c},\overline{P^c}\}_{c\in\mathcal{C}}$ where $v\in\mathcal{V}=\{v\in[0,1]^{2|\mathcal{C}|}|\overline{P^c}\geq\underline{P^c},\forall\ c\in\mathcal{C}\}$. In this representation, the $\overline{P^c}$ represent the score cutoffs for membership of the highest priority class and the $\underline{P^c}$ represent the score cutoffs for

28. Note that this rules out preferences that depend non-instrumentally on the coarsening itself. Namely, this rules out a preference for "simple" policies. We argue that this restriction is unimportant given the fact that optimal policies will be simple insofar as they are trinary in any case.

29. Recall from footnote 25 that $\mathcal{G}\subset L^1(\Theta\times\mathcal{C})$, so continuity is here meant with respect to the associated L^1-norm. See Supplementary Appendix B.2 for more details.

membership of the middle indifference class.[30] Hence, coarsening rules reduce simply to points in a closed subset of the unit hypercube, $\mathcal{V}$. Thus, the planner's problem can be stated as:

$$\mathcal{V}^* = \underset{v \in \mathcal{V}}{\arg\max} Z(g_v) \tag{5}$$

where g_v is the allocation induced by cutoff vector $v \in \mathcal{V}$.

3.3. *Optimal priority design*

Having established that any coarsening requires only three equivalence classes per school and set up the problem of the planner, we now show that there exists an optimal coarsening. First, we prove that g_v is continuous in v (Lemma 3 in Supplementary Appendix B). In view of the fact that the domain $\mathcal{V}$ is compact, and the objective function is continuous in g (Assumption 3), it follows that an optimal coarsening exists. This is formalized as Theorem 2 below.

Theorem 2. *$\mathcal{V}^*$ is non-empty. That is, there exists a trinary coarsening that is optimal.*

Proof. See Appendix A.2. □

The implications of Theorem 2 are significant. In particular, it reduces the dimensionality involved in finding the optimal coarsening from an infinite-dimensional problem to a $2|\mathcal{C}|$-dimensional problem, as now we only need to choose two numbers per school to attain any optimum. This is interesting as it not only implies that problems of priority design for school districts are substantially simpler than one might expect but also facilitates simple computation of the value of a given policy even in cases with a large number of schools. We later leverage this result to provide insights into the structure of the optimal priority design in each of our three leading applications: design of walk-zone policies under distance-based priorities in BPS; design of diversity policies under achievement-based priorities in CPS; and the allocation of affordable housing under income-based priorities by NYCHA.

3.4. *The impact of aggregate uncertainty*

In our analysis, we have assumed that the planner both knows the distribution of student types and that there is a continuum of students. In view of these assumptions, there is no aggregate uncertainty in the market and the planner knows that their choice of coarsening will lead to a particular, deterministic allocation. As most of the markets we study (BPS, CPS, and NYCHA) are large, have used centralized assignment mechanisms for a number of years, and are arguably likely to have similar distributions of types from year to year (they are stationary), we argue that this is a reasonable assumption.[31]

Nevertheless, to investigate the robustness of our results to aggregate uncertainty, in Supplementary Appendix C we study the same problem considered in the main text augmented

30. Following Azevedo and Leshno (2016), one can gain an interpretation of these thresholds in terms of the budget sets of students. If a student's score at a school exceeds $\overline{P}_c$ then that school is in their budget set with certainty. If a student's score lies between $\underline{P}_c$ and $\overline{P}_c$, then the school is in their budget set with some probability between zero and one. If a student's score lies below $\underline{P}_c$, then the school never lies in their budget set.

31. In particular, from 2010 to 2015 in CPS, we compute that the admissions cutoff for any school in the merit slots is within 3% of that school's average merit slot cutoff over this time period 96% of the time, providing strong evidence of approximate stationarity in this market. Cut-off score data are publicly available from CPS.

with uncertainty on the part of the planner regarding the distribution of student types in the population. Formally, we suppose that there is a finite set of probability measures $\mathcal{F}$ that the planner entertains as possible. In this context, Theorems 1 and 2 can be extended to show that an optimal coarsening still exists but that it may involve up to $2|\mathcal{F}|$ cutoffs at each school (Proposition 6 in Supplementary Appendix C). As a result, the presence of uncertainty can substantially complicate the problem of priority design and give rise to a less coarse priority structure (see Example 3 in Supplementary Appendix C for an explicit example of this). We can further characterize when uncertainty causes a welfare loss to the planner relative to the benchmark without aggregate uncertainty. In particular, uncertainty induces no welfare loss if and only if the *ex post* optimal lottery classes either coincide or never overlap across all states of the world (Proposition 7 in Supplementary Appendix C). Intuitively, it is exactly when aggregate uncertainty makes it impossible for the planner to target the same optimal lottery class across states of the world that this uncertainty has bite.

While we maintain that the assumption of no aggregate uncertainty is appropriate for our applications, these results suggest that our main results may have less bite in settings with appreciable aggregate uncertainty, as might be the case when markets change a great deal from year to year, or one is designing priorities in an unfamiliar market.

3.5. *Extensions: homogeneous coarsenings and multiple stable matchings*

In Supplementary Appendix D, we study an extension of the general analysis in this section where a planner is constrained to use the same coarsening at every school. We prove that suitably revised versions of Theorems 1 and 2 continue to hold in this setting, but now the designer needs to potentially specify up to $2|\mathcal{C}|$ cutoffs that are the same for each school (Proposition 8 in Supplementary Appendix D). We characterize when the imposition of homogeneity leads to a loss in welfare: there is no resulting loss in welfare when the cutoffs for the lottery classes of each school either coincide exactly or do not overlap at all (Proposition 9 in Supplementary Appendix D). Intuitively, as students who always or never gain admission to a school can receive the same allocation under homogeneity, the imposition of homogeneity leads to losses insofar as it makes it impossible to have the same regions of students (the lottery classes) who have fractional assignment probability to each school.

On the technical side, we proved our theoretical results under the condition that there is full support of ordinal types. When this assumption fails, there can be multiple stable matchings and one must address a number of technical details. To this end, in Supplementary Appendix F, we relax this condition and show that suitably modified versions of Theorems 1 and 2 continue to hold when the mechanism-designer optimal selection from the set of stable matchings is used (Theorems 3 and 4 in Supplementary Appendix F) and that Theorem 1 continues to hold under the student-optimal selection (Theorem 5 in Supplementary Appendix F). However, the student-optimal selection can cause the mechanism designer's objective to jump down, and so optima can fail to exist. Thus, Theorem 2 fails to hold under the student-optimal selection (see Example 5 in Supplementary Appendix F).

4. APPLICATION: DISTANCE-BASED PRIORITIES AND WALK-ZONE DESIGN

In many school districts, such as Boston, San Francisco, Denver, and much of the U.K., the distance between students and schools plays an important role in the assignment process. One widely studied example, which provides the concrete motivation for the theoretical exercise in this section, is the walk-zone assignment system Boston Public Schools (BPS) utilized until 2013. Under this policy, students were partitioned into two sets at all schools: the walk-zone students who live sufficiently close to the school and the others who do not.[32] In the language of our model, this corresponds closely to a situation where the underlying score is distance and the pursued policy is a coarsening that splits students into two groups.[33]

The main issue for BPS in designing its school admissions policy was the trade-off between two competing desires. On the one hand, it is desirable to have students attend schools that are closer to their homes on grounds of decreasing transportation costs for the school district and improving community cohesion. Indeed, Landsmark (2009) notes the costs of school transportation are very large for the district, at around \$70 million.[34] On the other hand, it is also desirable to ensure that schools have a diverse student body and for families to have greater choice over the schools they are able to attend. This concern is particularly relevant in communities that are socioeconomically segregated, such as those in Boston.[35] That this problem of conflicting objectives is at the heart of the design problem is attested to by Daley (1999), who notes that the walk-zone policy was created with the aim of "striking an uneasy compromise between neighborhood school advocates and those who want choice."

Thus motivated, we study the optimal distance-based priority design from the perspective of a mechanism designer who cares both about assigning students to schools they prefer and the distance students have to travel to their school. Our main results, Theorems 1 and 2, apply directly in this environment and imply that the optimal design can be attained via the use of at most three zones per school. We also show how the trade-off between distance and diversity shapes the structure of the desirable walk-zone policy and show when a simple walk-zone policy, corresponding to that pursued by BPS, is optimal.

4.1. *Model*

There is one school G with capacity $Q\in(0,1)$ and an outside option B. There is a unit measure of students who have bounded and positive Bernoulli utility $u\in\mathcal{U}$ from attending school G. The utility from attending B is normalized to zero. Students have underlying score $s\in[0,1]$ at school G. Students are indexed by their type $\theta=(u,s)$ and there is a joint distribution over the set of types $\Theta=\mathcal{U}\times[0,1]$ given by $f(\theta)$ such that there is a uniform distribution of underlying scores.[36] There is a continuous cost of students of score s attending G given by $C(s)$. This function can be interpreted as capturing transport costs, community cohesion, or fairness costs associated with a student of score s attending school G. Finally, for this section, we assume that the school board is utilitarian and has no distributional preferences:

$$Z(g)=\int_0^1\int_{\mathcal{U}}\big(u-C(s)\big)g(s)dF(u,s) \tag{6}$$

where g is the probability that a student with score s attends school G.

32. See Dur *et al.* (2018) for a more detailed account of this setting.

33. To account for additional dimensions such as sibling-based priorities, one need only construct a composite underlying score comprising distance and sibling status. All of our analysis then applies to the model with this composite score.

34. Moreover, Mayor Menino stated that (Goldstein, 2012): "Pick any street. A dozen children probably attend a dozen different schools... Parents might not know each other; children might not play together. They can't carpool, or study for the same tests."

35. Indeed, Levinson *et al.* (2012) note that increasing the priority of students who live closer to a school, as was the case under the walk-zone system, reduced the quality of schools certain socioeconomic groups could attend, making the assignment less equitable.

36. This is without loss in this environment as it simply redefines the scores over students.

We denote a priority coarsening as a vector of cutoffs $v = (v_1, \ldots, v_n)$, where $0 \equiv v_0 \leq v_1 \leq \cdots \leq v_n \leq v_{n+1} \equiv 1$. Students with $s \in [v_i, v_{i+1})$ for $i < n$ or $s \in [v_n, v_{n+1}]$ have the same priority at school G as all other students with scores within the same interval prior to tie-breaking. We label the set of such vectors for a given natural number n as $\mathcal{V}^n$. We further denote the probability that a student with $s \in [v_i, v_{i+1})$ goes to G under uniform tie-breaking by g_i^v. It is further useful to define the expected contribution to social welfare of a student with score s being assigned to school G:

$$W(s) = \mathbb{E}[u|s] - C(s) \equiv B(s) - C(s) \tag{7}$$

where $B(s)$ captures the benefit to social welfare of student with score s attending G which we assume to be continuous and $C(s)$ is the cost. Using this notation, the school district's value from a policy (n, v) is:

$$Z(n,v) = \sum_{i=1}^{n} g_i^v \int_{v_i}^{v_{i+1}} W(s)ds. \tag{8}$$

Thus, the school district faces the problem:

$$Z^* = \max_{n, v \in \mathcal{V}^n} Z(n,v) \tag{9}$$

It is important to note that this problem remains non-trivial as the choice object is of arbitrarily high-dimension. Our Theorems 1 and 2 makes this problem tractable:

Corollary 1. Under any stable mechanism, Z^* exists and there exists an optimal policy (n^*, v^*) such that $n^* = 2$.

Proof. See Appendix A.3. □

With Corollary 1 in hand, we can restrict attention to considering coarsening rules $v = (v_1, v_2) \in \mathcal{V}^2$. We now apply this result to solve the problem of the school district.

4.2. *Solving the school district's problem*

In view of Corollary 1, one notes that there are three types of regions that can arise depending on the priorities used. The first is an *acceptance region*, where students are assigned to the school with probability one ($s \geq v_2$). The second is a *lottery region*, where students are assigned to the school if they have a high enough lottery number ($s \in [v_1, v_2)$). The third is a *rejection region*, where students are never assigned to the school ($s < v_1$). Depending on the existence of such regions or not, there are five possible types of priorities that can be optimal:

1. A "double walk-zone": an interior case where all three regions exist
2. A "small walk-zone": a semi-interior case with an acceptance region and lottery region
3. A "large walk-zone": a semi-interior case with a lottery region and rejection region
4. "Full coarsening": a corner case with just a lottery region
5. "No coarsening": a corner case with an acceptance region and a rejection region

Moreover, in any of the (semi-)interior cases (*i.e.* any case excluding no coarsening or full coarsening), the optimality condition for the cutoffs is simple:

$$W(v_1) = W(v_2) = \frac{1}{v_2 - v_1} \int_{v_1}^{v_2} W(s)ds \tag{10}$$

That is to say, whenever a cutoff is on the interior, the cutoff is simply set to equalize the marginal contribution to social welfare of the student at the cutoff to the average contribution of all students in the lottery zone. As a result, under differentiability of W each optimal cutoff either satisfies equation 10 or it is on the boundary.[37]

Proposition 1. Any solution to the school district's problem (equation 9) $v^* = (v_1^*, v_2^*)$ must satisfy one of the following conditions for each v_i^*:

1. The cutoff is an interior optimum:

$$W(v_i^*) = \frac{1}{v_2^* - v_1^*} \int_{v_1^*}^{v_2^*} W(s) ds. \tag{11}$$

Moreover, the above equation is sufficient for v^* to be a local optimum whenever $W'(v_i^*) > 0$ for each cutoff to which this condition pertains.

2. The cutoff is on a boundary of the relevant constraint set:

$$v_1^* \in \{0, 1-Q\}, \quad v_2^* \in \{1-Q, 1\} \tag{12}$$

Proof. See Appendix A.4. □

This proposition shows which policies can be optimal and how to compute optima given a parametric environment, but it is otherwise silent on the forces that govern the structure of the optimum. In the next section, we study these questions in a more specialized environment.

4.3. *Optimal walk-zones in a parametric environment*

To study the key trade-off faced by BPS policymakers between assigning students to schools close to where they live and ensuring both choice and diversity, we now examine how the structure of a simple parametric environment with these features determines the optimal walk-zone structure. We further assume that there are two utility types of students $u \in \{u^R, u^P\}$. Moreover, we assume that conditional on these utility types, there is a higher density of u^R types with higher scores. Specifically, we assume that score and utility have following joint distribution:

$$f(u^R, s) = s \quad f(u^P, s) = 1 - s. \tag{13}$$

Typically, we will take $u^R < u^P$ and interpret this environment as one where poor students derive greater benefit from attending the good school relative to their outside option and the school is located in a neighbourhood primarily featuring rich students. The average marginal benefit to social welfare of students with score s is therefore given by:

$$B(s) = \mathbb{E}[u|s] = u^R s + u^P (1-s). \tag{14}$$

On the cost side, we specify a parametric cost function of admitting a student of score s given by:

$$C(s) = \frac{\alpha}{1 + \delta \exp\{\varepsilon(s - \bar{s})\}} + \mathbb{I}[s \le \bar{s}] \beta (\bar{s} - s)^\gamma, \tag{15}$$

37. Differentiability of W can be ensured by mild assumptions on primitives.

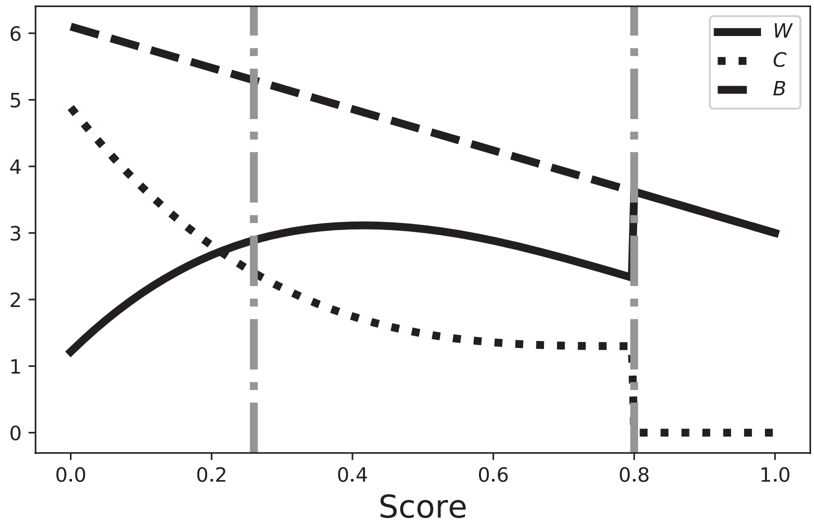

FIGURE 2
The W, B, and C functions with an optimal "double" Walk-zone

where $\alpha, \beta, \delta, \varepsilon \geq 0$, $\bar{s} \in (1-Q, 1)$ and $\gamma \geq 1$. See that this function accommodates the following two features. First, there is a score $\bar{s}$ below which there is a "sharp" increase in the cost of a student attending the school, whenever ε is large. This can be thought of as there being a distance $1-\bar{s}$ within which students can walk to school, beyond which walking becomes infeasible for students and transportation is required. Second, there are steadily increasing and perhaps convex costs of students having a low score, with β controlling the slope and γ the normalized convexity. This captures increasing costs of transportation and potentially fairness and community cohesion costs associated with admitting students who live far from the school.

As in the general analysis, the key object of interest for computing the optimal score cutoffs is the function:

$$W(s) = B(s) - C(s) \tag{16}$$

which captures the overall contribution to social welfare of a student who has underlying score of s. Moreover, given the parametric structure of B and C, W features the following trade-off: admitting poorer students increases the quality of the assignment but is more costly as those poorer students live further away.

To understand how these objects vary with the score in this parametric environment, Figure 2 plots the W, B, and C functions for a case where:

1. Students who live further away derive higher utility from attending the school $u^P > u^R$. This can be seen in the figure as the downward sloping benefit line.
2. Students who live beyond $\bar{s} = 0.8$ have rapidly increasing transport costs (large ε).
3. There is a convex cost of students beyond $\bar{s}$ attending the school, $\gamma > 1$.

Despite the parametric structure, this model is still sufficiently rich to demonstrate each of the five classes of walk-zone policy This is stated formally in Proposition 2:

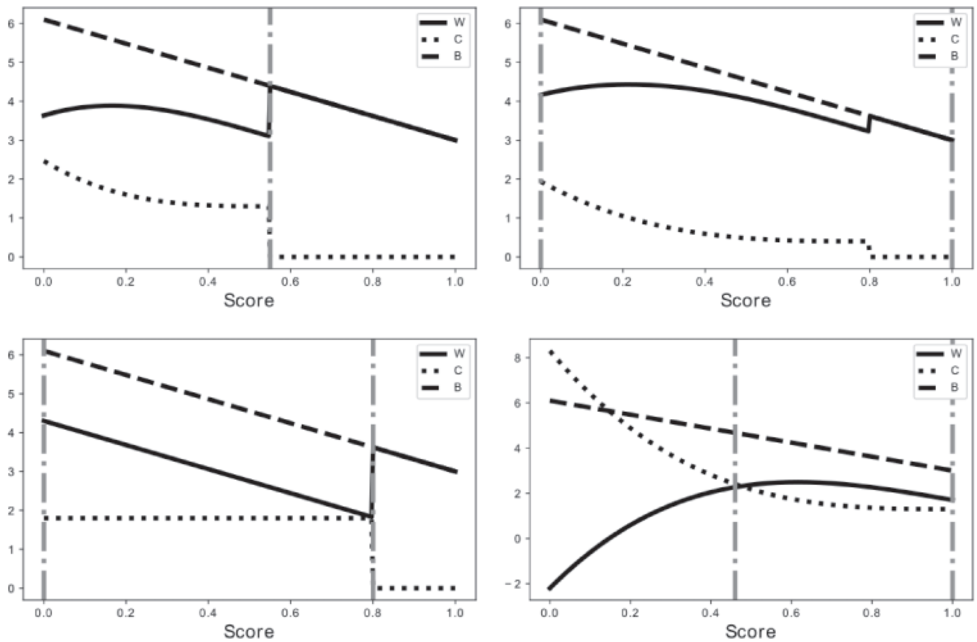

FIGURE 3

No coarsening (top-left pane), full coarsening (top-right pane), a “small” walk-zone (bottom-left pane) and a “large” walk-zone (bottom-right pane).

Proposition 2. For each of the five classes of policy, there exist open sets of $(\alpha, \beta, u_P, u_R)$ such that each class of policy is uniquely optimal for these parameters.

Proof. See Appendix A.5. □

Guided by this result, we now provide an intuitive discussion of when each of these classes of policy is optimal. First, we consider the double walk-zone case and refer the reader back to Figure 2, wherein the dashed grey lines in represent the optimal score cutoffs. See that in this example, the upper cutoff is close to $\bar{s} = 0.8$ and the lower cutoff is below the school's capacity. As a result, there is an acceptance region, a lottery region and a rejection region. Intuitively, the fixed cost being sufficiently large incentivizes the creation of the acceptance zone. Moreover, having a variable cost that is sufficiently large but not too large creates a region after $\bar{s}$ where the benefit increases more rapidly than cost, inducing coarsening. Finally, the convexity of cost eventually implies that cost exceeds benefit and it is optimal to create a rejection zone. Intuitively, the optimum simply balances the benefit of there being a higher average utility from students who are further from the school with the increasing cost of these students.

Second, we demonstrate the two corner cases that feature full coarsening and no coarsening. A simple case in which the designer will pursue full coarsening is a case where $u^P > u^R$ and there is no cost of students who live further away attending the school $\alpha = \beta = 0$. On the other hand, no coarsening will obtain in any case where $u^P < u^R$ or α and β are sufficiently large as it is optimal to simply admit all students who live as close to the school as is possible. The corner policies in these cases are shown in Figure 3.

Finally, the two simple walk-zone cases are also intuitive. When there is a very large fixed cost of students below $\bar{s}$, the mechanism designer wants to admit students who are sufficiently close with certainty. Moreover, when $u_P > u_R$ and β is sufficiently small, the mechanism designer wants to coarsen the remaining priority below $\bar{s}$ as much as possible. This is because the students who are furthest away contribute the most to social welfare as u^P types are relatively most dense. The case for a large walk-zone is similar. When the fixed cost is small but the cost of being further away is large, it is optimal to coarsen up to the point where the cost of being further away becomes too large. The simple walk-zone policies in these cases are shown in Figure 3.

Having shown how each policy can obtain depending on the strength of the trade-off between match quality and neighbourhood assignment, we supplement this analysis by deriving comparative statics in the optimal cutoffs as the relative strength of these motives changes. Concretely, we see how, in the case of an optimal double walk-zone, changes in transportation costs and relative utilities of students move the optimal cutoffs:

Proposition 3. Suppose that the solution to the planner's problem is interior and unique. The following comparative statics hold:

1. An increase in transportation costs enlarges the acceptance region and shrinks the lottery region:
$$\frac{\partial v_1^*}{\partial \beta} > 0, \quad \frac{\partial v_2^*}{\partial \beta} < 0. \tag{17}$$
2. An increase in relative utility of students who live farther away shrinks the acceptance region and enlarges the lottery region:
$$\frac{\partial v_1^*}{\partial u_P} < 0, \quad \frac{\partial v_2^*}{\partial u_P} > 0. \tag{18}$$

Proof. See Appendix A.6. □

The first of these results shows how increasing transportation costs for students who live furthest from the school will disincentivize the admission of poorer students. On the contrary, when the benefits of admitting poorer students increase, the reverse is true. Even though we do not explicitly model a preference for diversity here, this increase can be thought as a preference for such students from the perspective of the mechanism designer. If one interprets the mechanism designer as embodying the aggregate preferences of society, an increase in u_P may represent an improvement in the political organization of such parents or an increase in public pressure they exert. As one would expect, our comparative statics suggest that such an improvement will increase their representation in higher quality schools.

5. APPLICATION: DESIGNING EXAMS FOR DIVERSITY

In our second application, we study how to optimally design exams in an environment where a planner cares about both admitting higher quality students and the diversity of the student body in a competitive exam school. Concretely, our analysis is motivated by exam schools in the Chicago Public Schools (CPS) system, where the district uses student achievement in entrance exams as the basis for its priorities but also has a long history of controversial diversity-based affirmative action policies.[38]

38. For more detail on the institutional setting of exam schools in the CPS system, we refer the reader to Dur *et al.* (2020) and Ellison and Pathak (2021).

The key trade-off faced by CPS is admitting the students with the highest academic qualifications while having racially and socioeconomically diverse student bodies.[39] Presently, CPS uses a strict ranking based on a composite academic score, which we take as the underlying score. In our model, the mechanism designer can also construct a coarser ranking that partitions students into different achievement levels, without releasing the strict ranking they obtained in the exam. A coarser ranking may help students with lower scores, who are potentially from less-advantaged socioeconomic backgrounds. However, this comes at the cost of admitting students with lower levels of achievement in the entrance exam. In this section, we employ Theorem 1 to characterize when such a coarse grading policy can be used to improve the allocation. We therefore explore the possibility of using priority coarsening to increase diversity without any other explicit affirmative action policies, thereby adding an additional policy lever for bodies such as CPS to consider.

5.1. *Model*

There is one school with capacity Q, to be interpreted as a desirable exam school. The exam school gives all students common utility u, which exceeds the utility they receive at their outside option. Students have exam scores $s\in[0,1]$. There are two different socioeconomic groups of equal size, κ_1 (the under-represented group) and κ_2, where the vector (s,κ) summarizes the type of any given student. The score densities of each type of students are given by $f_{\kappa_1}(s)$ and $f_{\kappa_2}(s)$, where $f_{\kappa_1}(s)+f_{\kappa_2}(s)=1$. Motivated by affirmative action concerns of CPS, we assume the mechanism designer has the following utility function:

$$Z(g)=\sum_{\kappa_i}\int_0^1 sg(c|s,\kappa_i)dF_{\kappa_i}(s)+h\left(\int_0^1 g(c|s,\kappa_1)dF_{\kappa_1}(s)\right), \tag{19}$$

where h is a strictly increasing function. The first term represents the benefit to the mechanism designer from assigning students with higher scores to the school. The second term represents the benefit of assigning students from the under-represented socioeconomic group.

5.2. *Optimal exam design*

We now study when designing the exam leads to welfare improvements. In this context, an exam design takes the form of score cutoffs $0\equiv v_0\leq v_1\leq\cdots\leq v_n\leq v_{n+1}\equiv 1$ such that students with $s\in[v_i,v_{i+1})$ for $i<n$ or $s\in[v_n,v_{n+1}]$ have the same priority as all other students with scores within the same interval prior to tie-breaking. It is natural to interpret such a design as providing a coarse grading of an entrance exam rather than simply ranking the students. Owing to Theorem 1, we can restrict attention to trinary coarsenings. Hence, it is without loss of generality to assume the mechanism designer picks two numbers $v_1<r$ and $v_2\geq r$ as score cut-offs where $r\equiv 1-Q$. Under a grading policy (v_1,v_2), we have that the allocation is given by:

$$g(c|s,\kappa_i)=\begin{cases}1 & \text{if } s\geq v_2\\ p_L(v_1,v_2) & \text{if } s\in[v_1,v_2)\\ 0 & \text{if } s<v_1\end{cases}, \tag{20}$$

39. This is attested to by the Blue Ribbon Commission (BRC), which was appointed to review CPS's policy remarks (Dur *et al.*, 2020): "The BRC wants these programs, [exam schools], to maintain their academic strength and excellent record of achievement, but also believes that diversity is an important part of the historical success of these programs."

where $p_L(v_1,v_2)=\frac{v_2-r}{v_2-v_1}$. The utility of the mechanism designer under any coarsening (v_1,v_2) is then given by:

$$Z(v_1,v_2)=\sum_{\kappa_i}\left(\int_{v_2}^{1} sdF_{\kappa_i}(s)+p_L(v_1,v_2)\int_{v_1}^{v_2} sdF_{\kappa_i}(s)\right)$$
$$+h\left(\int_{v_2}^{1} dF_{\kappa_1}(s)+p_L(v_1,v_2)\int_{v_1}^{v_2} dF_{\kappa_1}(s)\right). \tag{21}$$

We can now ask when exam design leads to welfare improvements. Formally, we say that exam design leads to welfare improvements if there exists a pair (v_1,v_2) such that $Z(v_1,v_2)>Z^{NC}$, where Z^{NC} is the utility the planner receives from not coarsening exam scores.[40] Using the structure of payoffs, one can achieve the following characterization of when exam design leads to welfare improvements:

Proposition 4. Exam design leads to welfare improvements if and only if there exists a pair (v_1,v_2) where $v_2>r$ and $v_1<r$ such that:

$$\frac{1}{2}[(v_2-r)(r-v_1)]<h\Big(\frac{1}{2}-F_{\kappa_1}(v_2)+(F_{\kappa_1}(v_2)-F_{\kappa_1}(v_1))p_L(v_1,v_2)\Big)-h\Big(\frac{1}{2}-F_{\kappa_1}(r)\Big) \tag{22}$$

Moreover, if h is linear with slope $\alpha>0$, this inequality reduces to:

$$(r-v_1)<2\alpha\Big(\frac{F_{\kappa_1}(v_2)-F_{\kappa_1}(v_1)}{v_2-v_1}-\frac{F_{\kappa_1}(v_2)-F_{\kappa_1}(r)}{v_2-r}\Big) \tag{23}$$

Proof. See Appendix A.7. □

This condition simply compares the loss from having students with lower scores to the benefit of (potentially) increasing the diversity of the student body through admitting a different composition of students. A simple sufficient condition for a priority coarsening to increase diversity is that the majority score distribution dominates the minority score distribution in the sense of first-order stochastic dominance. The trade-off between student exam scores and affirmative action is clear in the case with linear h. As v_1 decreases, students with lower scores gain admission, reducing the overall student quality. However, there is a benefit from increasing diversity if the ratio of under-represented students with scores in $[v_1,v_2)$ is larger than the ratio in $[r,v_2)$. The total benefit from diversity then depends the difference of these ratios and the preference for diversity, which is measured by α. In particular, the mechanism designer is more likely to improve the allocation via exam design if there are more under-represented students that are close to the no-affirmative-action cut-off r or she has a stronger preference for diversity.

6. APPLICATION: INCOME-BASED PRIORITIES AND THE ALLOCATION OF PUBLIC HOUSING

With more than 1 million housing units, and 1.6 million households on waiting lists, public housing programs in U.S. are very important for the welfare of low income households

40. The allocation in this case is such that students are allocated the school if their score exceeds r, and not otherwise. Plugging this allocation into equation 19 yields Z^{NC}.

(Collinson *et al.*, 2015). Because of this, many housing authorities employ various restrictions on the eligibility of applicants and then allocate the units via lotteries to those deemed eligible. Concretely, to be eligible for public housing provided by the New York City Housing Authority (NYCHA), a household must have income less than a certain fraction of New York's Area Median Income (AMI).[41]

As in our previous applications, determining eligibility requirements for housing assistance is a contentious topic. The main trade-off for policymakers appears to be between making a larger part of society eligible and targeting households that will gain most from the assistance—those with the worst outside housing options. That it is desirable to ensure wide eligibility in this context is reflected in the words of NYC Mayor Bill de Blasio:[42]

> "Affordable housing initiatives cannot just be for the lowest income folks, ... There has to be a place for work force housing and middle-class housing as well."

However, the targeting of those with low outside options that would cause them to have inadequate housing in the absence of intervention is also extremely important. To this end, NYC officials have recently been developing laws that would reserve 15% of affordable-housing projects for the homeless (Stewart *et al.*, 2019).

Thus motivated, we study the problem of a designer who determines the priority of households as a function of their income. To model this market requires a significant departure from the previous analysis: we must consider the dynamic nature of public housing allocation. To this end, we develop a dynamic matching model to study how these trade-offs affect priority design in the allocation of public housing. We leverage our general results (Theorems 1 and 2) to place a simple three-income-tier structure on the optimal policy and deliver insights regarding the optimal design.

6.1. *Model*

There is a continuum of agents $i \in [0,1]$ that differ in their income $s \in [0,1]$, where $s=1$ is the lowest level of income and $s=0$ is the highest level of income, and their outside options $\kappa \in \mathcal{K}$. Income and outside options have joint distribution $F(\kappa, s)$ such that the marginal distribution of incomes is uniform. Time is discrete and infinite $t \in \mathbb{N}$. Agents have discount factors β and die at rate $(1-\delta)$. Each period $(1-\delta)$ new agents are born from distribution F so that the population of agents always has unit measure.

A priority design is a vector $v=(v_1, v_2, \dots, v_n)$ such that $0 \equiv v_0 \leq v_1 \leq \cdots \leq v_n \leq v_{n+1} \equiv 1$. Agents with $s \in [v_i, v_{i+1})$ for $i<n$ or $s \in [v_n, v_{n+1}]$ have the same priority as all other agents with incomes within the same interval prior to tie-breaking. A stock of measure Q of *ex-ante* identical houses are available. Agents are allocated to houses via a stable mechanism with tie-breaking within any tier to determine the order. When an agent is allocated a house, they receive $\tilde{v} \sim \Lambda$ per period they inhabit that house, where $\tilde{v} \in [v_{\min}, v_{\max}]$. Once an agent accepts a house, they inhabit that house until their death. Each period an agent goes unmatched, they receive their outside option.

41. This fraction differs from development to development. Moreover, as Waldinger (2018) and Arnosti and Shi (2020) note, in some markets, there is also a minimum income level that applicants must clear. The reason for this is to make sure the applicant can pay the rent, which is a feasibility constraint rather than a policy with distributional motives. We abstract away from such issues in the present analysis.

42. Stewart *et al.* (2019).

Upon birth, agents are not allocated to public housing. Hence, their expected lifetime utility at birth is given by their value function when unmatched $V(\kappa,s,\nu)$. We suppose that the social planner has inequality-averse preferences given by the function:

$$Z(\nu)=\int_{\mathcal{K}}\int_0^1\left(\frac{V(\kappa,s,\nu)}{1-\gamma}\right)^{1-\gamma}dF(\kappa,s), \tag{24}$$

where γ indexes the degree of inequality aversion. Of particular interest is the Rawlsian limit $\gamma\to\infty$ where the planner seeks to maximize the welfare of worst-off agent.

In Supplementary Appendix E, we characterize the steady state of this dynamic matching model and derive a simple expression for the welfare of any priority design in terms of the steady state reservation values of each type (Proposition 10). Importantly, we can apply Theorems 1 and 2 directly in this steady state to show the following Corollary:

Corollary 2. In the steady state of the dynamic matching model, there exists an optimal priority design with two cutoffs $\nu^*=(\nu_1^*,\nu_2^*)$.

Proof. See Appendix A.8. □

This result greatly simplifies the analysis as we know that we need to specify at most two income cutoffs to find the optimal design.

6.2. *Optimal priority design*

Having understood the structure of the model, we now explore the problem of priority design from the perspective of the social planner. Given the nature of the fixed point equations for the equilibrium density of unmatched agents, finding an analytical characterization of the optimal cutoffs is challenging. Nevertheless, one can still establish interesting properties of the optimum and the trade-offs involved in the construction of optimal policy.

Given the system employed by NYCHA and elsewhere, it is of particular interest to study when we can rationalize a policy with the following feature: an income threshold below which agents have some probability of being allocated a house and above which they are ineligible. To this end, we show that when outside options are sufficiently increasing in income and the planner is sufficiently inequality averse that the richer agents are optimally excluded entirely from public housing (as in NYCHA). Conversely, if outside options are sufficiently similar across income groups, a sufficiently inequality averse social planner would like to give all agents positive probability of receiving public housing. Proposition 5 formalizes these statements:

Proposition 5. Suppose that agents have outside options given by a decreasing and differentiable function of their underlying score, *i.e.* an strictly increasing function of their income, $\kappa=h(s)$. In the limit of $\gamma\to\infty$:[43]

43. We say that a statement is true if h is sufficiently steep if there exists $\alpha<0$ such that the statement is true whenever:

$$h'(s)<\alpha \quad \forall s\in(0,1) \tag{25}$$

Likewise, we say that a statement is true if h is sufficiently flat if there exists $\alpha\leq 0$ such that the statement is true whenever:

$$h'(s)\geq\alpha \quad \forall s\in(0,1) \tag{26}$$

1. If h is sufficiently steep, then an optimal policy features a threshold of income above which agents have zero probability of receiving public housing
2. If h is sufficiently flat, then an optimal policy gives all agents a positive probability of receiving public housing

Proof. See Appendix A.9. □

This result highlights the key trade-off facing the planner between effective targeting and eligibility and shows how the strength of the relationship between outside options and incomes governs this trade-off. In particular, when h is very steep, those agents with the highest incomes also have relatively relatively high outside options. In this case, the poorest agents, even if they were to receive public housing in each period with certainty would have lower welfare than the richer agents. As a result, the targeting motive dominates the eligibility motive and the richer agents are excluded entirely from public housing. On the other hand, when h is very flat, all agents have very similar outside options and excluding any agent from receiving public housing will give rise to them being worse off than all agents who have some chance at public housing. To a sufficiently inequality-averse planner, this is unacceptable, and so the eligibility motive dominates the targeting motive.

The model moreover suggests that such an eligibility cutoff in income is optimal for similar reasons to those given by advocates in the New York public housing debate: the richest are sufficiently well off that we should reserve housing only for those who are needy. However, as Theorems 1 and 2 showed, the planner has additional latitude to introduce a tiered system with three tiers: one with unit probability of assignment, one with interior probability of assignment and one with zero probability of assignment. In Example 4 in Supplementary Appendix E.1, we construct an explicit example of when this is desirable with three groups of agents: rich, middle class, and poor agents. If poor agents have sufficiently low outside options relative to middle class agents who have sufficiently low outside options relative to rich agents, three priority tiers are strictly optimal. Intuitively, it is optimal to assign poor agents as soon as they are unmatched as their outside option is so bad (they may be homeless), while we wish to exclude rich agents from assignment altogether as in Proposition 5. Thus, when there are enough homes relative to poor agents, it is optimal to allocate the remaining homes via lottery to middle class agents.

7. CONCLUSION

Motivated by the clear design of priorities in centralized matching markets, we introduce and study the problem of optimal priority design subject to a constraint that an underlying score cannot be reversed. In our main results, we show that it is without loss of optimality for a mechanism designer to split agents into at most three indifference classes for each object (Theorem 1) and that an optimal policy exists (Theorem 2).

We apply these results and our framework to provide concrete insight into a number of important and widely studied centralized matching markets: BPS, CPS, and NYCHA. In each case, we study the trade-offs highlighted by the relevant policymakers and provide normative insights as to the nature of optimal priority structures and positive rationalizations of the policies pursued in practice.

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Supplementary Data

Supplementary data are available at Review of Economic Studies online.

A. OMITTED PROOFS

A.1. *Proof of Theorem 1*

Proof. Fix an arbitrary coarsening Ξ. Divide the schools to two subsets: $c\in\tilde{\mathcal{C}}$ if Ξ_c is finite and $c\in\hat{\mathcal{C}}$ if Ξ_c is the identity function. In what follows, we will construct an alternative coarsening Ξ' that has three indifference classes in some school c and induces the same allocation. Since c was arbitrary, replicating this for all c will yield a trinary coarsening Ξ'.

For any school $c\in\tilde{\mathcal{C}}$, the coarsening Ξ takes any student with score $s_c^{\tilde{\theta}}$ to an equivalence class, hence $\Xi_c(s_c^{\tilde{\theta}})\in\{P_1^c,P_2^c,\dots,P_N^c\}$ with $P_1^c<P_2^c<\dots<P_N^c$ for some N. By Lemma 1 in Supplementary Appendix B, there exists a matching $\tilde{\mu}$ in the coarsened ordinal economy with tie-breakers $\tilde{\Omega}_\Xi^\tau$ such that for all $(\tilde{\theta}_\Xi,\tau)\in\tilde{\Theta}_\Xi^\tau$, its matching is uniquely given by $\tilde{\mu}(\tilde{\theta}_\Xi,\tau)$. Moreover, by Lemma 2 in Supplementary Appendix B, any type $\theta\in\Theta$ whose coarsened ordinal type is given by $\tilde{\theta}_\Xi$ has assignment probability at each school c, $g_\Xi(c|\theta)$, that is obtained by integrating $\tilde{\mu}(\tilde{\theta}_\Xi,\tau)$ over τ.

We will construct an alternative coarsening Ξ' that has only three indifference classes for school c and induces the same allocation as Ξ, *i.e.* $g_\Xi(c|\theta)=g_{\Xi'}(c|\theta)$ for all $\theta\in\Theta$ and $c\in\mathcal{C}$. Let P_x^c be the lowest indifference class that has a student placed in school c, i.e. $\tilde{\mu}(\tilde{\theta}_\Xi,\tau)=c$ for some $(\tilde{\theta}_\Xi,\tau)$ with $s_c^{\tilde{\theta}_\Xi}=P_x^c$ and $\tilde{\mu}(\tilde{\theta}_\Xi,\tau)\neq c$ for all $(\tilde{\theta}_\Xi,\tau)$ with $s_c^{\tilde{\theta}_\Xi}=P_y^c$ where $y<x$.[44] Now, define Ξ' by merging all indifference classes above and below x for school c:

$$\Xi'_c(s_c^{\tilde{\theta}})=\begin{cases}P_1^c & \text{, if}\quad \Xi_c(s_c^{\tilde{\theta}})=P_z^c, z<x,\\ P_x^c & \text{, if}\quad \Xi_c(s_c^{\tilde{\theta}})=P_x^c,\\ P_N^c & \text{, if}\quad \Xi_c(s_c^{\tilde{\theta}})=P_z^c, z>x.\end{cases}\tag{A.1}$$

And $\Xi'_{c'}=\Xi_{c'}$ for all $c'\neq c$. To see that $\tilde{\mu}$ is stable under Ξ', assume (ordinal) student pair i,j (with scores $s_{c'}^i$ and $s_{c'}^j$ and tie-breakers τ_i and τ_j) and school c' blocks $\tilde{\mu}$. Then, $c'\succ_i\tilde{\mu}(i)$, $\tilde{\mu}(j)=c'$ and either $\Xi'_{c'}(s_{c'}^i)>\Xi'_{c'}(s_{c'}^j)$ or $\Xi'_{c'}(s_{c'}^i)=\Xi'_{c'}(s_{c'}^j)$ and $\tau_i>\tau_j$.[45] First, if $c'\neq c$, as the priority for school c' is same under both Ξ and Ξ', (i,j,c') block $\tilde{\mu}$ under Ξ, which is a contradiction. If $c=c'$, $\Xi'_{c'}(s_{c'}^i)>\Xi'_{c'}(s_{c'}^j)$ implies that $\Xi_{c'}(s_{c'}^i)>\Xi_{c'}(s_{c'}^j)$ and (i,j,c') again block $\tilde{\mu}$ under Ξ, which is a contradiction. Next, suppose $\Xi'_{c'}(s_{c'}^i)=\Xi'_{c'}(s_{c'}^j)$ and $\tau_i>\tau_j$. As $\tilde{\mu}(j)=c'$ and $\Xi'_{c'}(s_{c'}^i)=\Xi'_{c'}(s_{c'}^j)$, we have that $\Xi_{c'}(s_{c'}^i)\geq P_x^{c'}$. There are two cases, either $\Xi_{c'}(s_{c'}^i)>P_x^{c'}$ or $\Xi_{c'}(s_{c'}^i)=P_x^{c'}$. In the first case, from the definition of $P_x^{c'}$, there exists k such that $\tilde{\mu}(k)=c'$, $\Xi_{c'}(s_{c'}^k)=P_x^{c'}$. Then, $\Xi_{c'}(s_{c'}^i)>\Xi_{c'}(s_{c'}^k)$, $c'\succ_i\tilde{\mu}(i)$ and $\tilde{\mu}(k)=c'$, which means that (i,k,c') block $\tilde{\mu}$ under Ξ, a contradiction. In the second case, $\Xi_{c'}(s_{c'}^i)=P_x^{c'}$ and $\Xi'_{c'}(s_{c'}^i)=\Xi'_{c'}(s_{c'}^j)$ imply $\Xi_{c'}(s_{c'}^j)=P_x^{c'}$. However, this violates the stability of $\tilde{\mu}$ under Ξ as $\tau_i>\tau_j$, which is a contradiction. Hence, $\tilde{\mu}$ is stable under both Ξ and Ξ'. Moreover, the economy under Ξ' with tie-breakers retains the full support property, so there still is a unique stable matching for both economies (see Lemma 1 in Supplementary Appendix B.1). As the stable matching is unique in both economies, we use the same matching in the construction of g_Ξ and $g_{\Xi'}$, so $g_\Xi=g_{\Xi'}$ (applying Lemma 2 in Supplementary Appendix B.1).

Next, take $c\in\hat{\mathcal{C}}$. Let $\tilde{\mu}$ denote the unique stable matching under Ξ. For any school c, let t_c denote the threshold that the students must clear in order to gain admission to that school. Formally,

$$t_c=\inf\{s_c^\theta:\tilde{\mu}(\theta)=c\}.\tag{A.2}$$

Next, define Ξ' in the following way:

$$\Xi'_c(s_c^\theta)=\begin{cases}0 & \text{, if}\quad s_c^\theta<t_c,\\ 1 & \text{, if}\quad s_c^\theta\geq t_c.\end{cases}\tag{A.3}$$

Note that from the stability of $\tilde{\mu}$ and the definition of matching (in particular, property (iv) from footnote 23), there cannot be a student k such that $c'\succ_k\tilde{\mu}(k)$ and $s_{c'}^k\geq t_{c'}$. To see that $\tilde{\mu}$ is stable under Ξ', assume that student pair i,j (with scores $s_{c'}^i$ and $s_{c'}^j$) blocks it in school c'. Then, we have $c'\succ_i\tilde{\mu}(i)$, $\tilde{\mu}(j)=c'$ and $\Xi'_{c'}(i)\geq\Xi'_{c'}(j)$. As $\tilde{\mu}(j)=c'$, we have

44. Note that this allows $x=1$ and $x=N$, *i.e.,* this class can be the lowest indifference class or the highest indifference class.

45. We abuse notation slightly by evaluating $\tilde{\mu}$ under Ξ' as the set of types changes under Ξ'. However, this is not an issue as we explicitly refer to the uncoarsened ordinal types of students i and j.

$\Xi'_{c'}(i) = \Xi'_{c'}(j) = 1$. But then this implies that $s^i_{c'} \geq t_{c'}$, which contradicts the stability of $\tilde{\mu}$ under Ξ as $c' \succ_i \tilde{\mu}(i)$. By the same argument as in the first case, it follows that $g_\Xi = g_{\Xi'}$.

Applying this argument for all $c \in \mathcal{C}$ then yields a trinary coarsening Ξ' that replicates Ξ in the sense that $g_\Xi = g_{\Xi'}$. □

A.2. *Proof of Theorem 2*

Proof. First, by Theorem 1, we parameterize coarsenings Ξ by the outcome equivalent $v \in \mathcal{V}$, where we note that $\mathcal{V}$ is compact. Second, define $\tilde{g}: \mathcal{V} \to \mathcal{G}$ such that $\tilde{g}(v) = g_v$. By Lemma 3 in Supplementary Appendix B.2, we have that $\tilde{g}$ is continuous under the appropriate L^1-norm on $\mathcal{G}$. Third, define $\tilde{Z}: \mathcal{V} \to \mathbb{R}$ as $\tilde{Z} = Z \circ \tilde{g}$. By Assumption 3 that Z is continuous under the L^1-norm and the fact that $\tilde{g}$ is continuous, it then follows that $\tilde{Z}$ is continuous. Fourth, observe that we can write equation 5 as:

$$\max_{v \in \mathcal{V}} \tilde{Z}(v) \tag{A.4}$$

Finally, by the extreme value theorem as $\mathcal{V}$ is compact and $\tilde{Z}$ is continuous, it follows that $\mathcal{V}^*$ is non-empty. Thus, an optimal trinary coarsening exists. □

A.3. *Proof of Corollary 1*

Proof. This result follows from Theorem 1 specialized to an environment with two schools. However, in this case there is an alternative, simpler proof that we provide below for completeness.

In the first part of the proof, we show that under any stable mechanism, the allocation takes the following form:

$$g_i^v = \begin{cases} 1, v \geq \overline{v}, \\ p_L, v \in [\underline{v}, \overline{v}), \\ 0, v < \underline{v}. \end{cases} \tag{A.5}$$

for $0 \leq \underline{v} \leq \overline{v} \leq 1$ and $p_L \in [0,1]$. To see this, consider priority classes defined by: $i \in P_j \iff s_i \in [v_{j-1}, v_j)$ for $j \leq n$ and $i \in P_{n+1} \iff s_i \in [v_n, v_{n+1}]$. Now suppose that $\exists i \in P_j$ and $k \in P_l$ for $l < j$ such that $g_k^v > 0$. If $g_j^v < 1$, then a positive measure of students in P_j will not be assigned to G and a positive measure of students in P_k will be assigned to G. This violates stability. Hence, $g_j^v = 1$. Now suppose that $\exists i \in P_j$ and $k \in P_l$ for $l < j$ such that $g_j^v < 1$. By an identical argument, it must be that $g_k^v = 0$. By the above two conclusions, it must be that if there is any l such that $g_l^v \in (0,1)$ there is a unique P_j such that $g_j^v \in (0,1)$ and that $g_k^v = 1$ for $k > j$ and $g_k^v = 0$ for $k < j$. Taking $\overline{v} = v_j$ and $\underline{v} = v_{j-1}$ thereby proves that the allocation takes the form given in equation A.5.

Given this claim, we can take a (v,n) that induces g^v and construct a $(v',2)$ that also induces g^v. If there is no j such that $g_j^v \in (0,1)$, then we can simply take the lowest class k for which $g_k^v = 1$ and set $v'_1 = 0$ and $v'_2 = v_{k-1}$. If there is a j such that $g_j^v \in (0,1)$, then we can take $v'_1 = v_{j-1}$ and $v'_2 = v_j$. See that v' induces the same allocation as v in both cases.

Having now established that we can replicate any (v,n) with $(v',2)$, it remains to show that there exists an optimum to establish the result. See that the optimization problem by the replication result can be rewritten as:[46]

$$\begin{aligned} &\max_{v_1, v_2} \frac{Q-(1-v_2)}{v_2 - v_1} \int_{v_1}^{v_2} W(s)ds + \int_{v_2}^{1} W(s)ds \\ &\text{s.t.} \quad 0 \leq v_1 \leq 1 - Q \leq v_2 \leq 1 \end{aligned} \tag{A.6}$$

If the function $W(s)$ is continuous, then there must exist a solution by the Weierstrass Extreme Value Theorem as we are simply maximizing a continuous function over a compact domain. As B and C are continuous, then so too is W, so a solution exists. This completes the proof. □

A.4. *Proof of Proposition 1*

Proof. The school district's problem is given by:

$$\max_{n, v \in \mathcal{V}^n} \sum_{i=1}^{n} g_i^v \int_{v_i}^{v_{i+1}} W(s)ds \tag{A.7}$$

Applying the replication argument in Corollary 1, it is without loss of optimality to impose $n = 2$ and to have one class with a zero probability of assignment $[0, v_1)$, one class $[v_1, v_2)$ which faces a lottery of being assigned with probability

46. See Proposition 1 for the full argument.

$0 \le p_L(v_1, v_2) \le 1$ and one class $[v_2, 1]$ with a unit probability of assignment. As the planner has measure Q seats to assign, it must be that:

$$(1-v_2)+(v_2-v_1)p_L(v_1,v_2)=Q. \tag{A.8}$$

Or:

$$p_L(v_1,v_2)=\frac{Q-(1-v_2)}{v_2-v_1}. \tag{A.9}$$

Thus, the objective becomes:

$$V(v_1,v_2)=\frac{Q-(1-v_2)}{v_2-v_1}\int_{v_1}^{v_2} W(s)ds+\int_{v_2}^{1} W(s)ds. \tag{A.10}$$

We require that $v_1 \ge 0, v_2 \le 1, p_L(v_1,v_2) \in [0,1]$. These requirements reduce to:

$$0 \le v_1 \le 1-Q \le v_2 \le 1. \tag{A.11}$$

Thus the planner's problem is:

$$\begin{aligned} &\max_{v_1,v_2} \frac{Q-(1-v_2)}{v_2-v_1}\int_{v_1}^{v_2} W(s)ds+\int_{v_2}^{1} W(s)ds \\ &\text{s.t.} \quad 0 \le v_1 \le 1-Q \le v_2 \le 1. \end{aligned} \tag{A.12}$$

From the form of the problem, the Lagrangian can be stated as:

$$\begin{aligned} \mathcal{L}(v_1,v_2,\lambda_1,\lambda_2,\mu_1,\mu_2)=&\frac{Q-(1-v_2)}{v_2-v_1}\int_{v_1}^{v_2} W(s)ds+\int_{v_2}^{1} W(s)ds-\lambda_1\big((v_1-(1-Q)\big) \\ &-\lambda_2\big((1-Q)-v_2\big)+\mu_1 v_1+\mu_2(1-v_2), \end{aligned} \tag{A.13}$$

where the λ_i are the Lagrange multipliers on the constraints that the probability in the lottery zone does not exceed unity or become negative and the μ_i are Lagrange multipliers on the constraints that the cutoffs remain in the unit interval. See that there are five cases of interest.

1. Both v_1 and v_2 are unconstrained: $\lambda_1=\lambda_2=\mu_1=\mu_2=0$. In this case, the Lagrangian becomes:

$$\mathcal{L}(v_1,v_2)=\frac{Q-(1-v_2)}{v_2-v_1}\int_{v_1}^{v_2} W(s)ds+\int_{v_2}^{1} W(s)ds. \tag{A.14}$$

Taking FOCs:

$$\begin{aligned} p_L(v_1,v_2)W(v_1)&=p_{Lv_1}(v_1,v_2)\int_{v_1}^{v_2} W(s)ds, \\ (1-p_L(v_1,v_2))W(v_2)&=p_{Lv_2}(v_1,v_2)\int_{v_1}^{v_2} W(s)ds. \end{aligned} \tag{A.15}$$

Noting that:

$$\begin{aligned} p_{Lv_1}&=\frac{p_L(v_1,v_2)}{v_2-v_1} \\ p_{Lv_2}&=\frac{1-p_L(v_1,v_2)}{v_2-v_1}. \end{aligned} \tag{A.16}$$

Plugging these relations into the FOCs yields:

$$\begin{aligned} W(v_1)&=\frac{1}{v_2-v_1}\int_{v_1}^{v_2} W(s)ds \\ W(v_2)&=\frac{1}{v_2-v_1}\int_{v_1}^{v_2} W(s)ds. \end{aligned} \tag{A.17}$$

Thus, when either FOC holds, it must be the case that the marginal contribution to social welfare is equal to average utility within the lottery zone. It further follows that when both FOCs hold that:

$$W(v_1)=W(v_2). \tag{A.18}$$

This reveals that an interior optimum, it must be that the marginal lottery zone student on average contributes as much to welfare as the marginal student in the zone that gets in with probability one.

However, for the above to hold, we must ensure that the SOCs hold. To this end, it is sufficient to show that the Hessian of the Lagrangian is negative definite. Taking second derivatives and re-arranging:

$$\begin{aligned}
\mathcal{L}_{v_1v_1}(v_1,v_2) &= \frac{2p_L(v_1,v_2)}{v_2-v_1}\Big[\frac{1}{v_2-v_1}\int_{v_1}^{v_2} W(s)ds - W(v_1)\Big] - p_L(v_1,v_2)W'(v_1)\\
\mathcal{L}_{v_1v_2}(v_1,v_2) &= \frac{1-p_L(v_1,v_2)}{v_2-v_1}\Big[\frac{1}{v_2-v_1}\int_{v_1}^{v_2} W(s)ds - W(v_1)\Big]\\
&\quad -\frac{p_L(v_1,v_2)}{v_2-v_1}\Big[\frac{1}{v_2-v_1}\int_{v_1}^{v_2} W(s)ds - W(v_2)\Big]\\
\mathcal{L}_{v_2v_2}(v_1,v_2) &= \frac{2(1-p_L(v_1,v_2))}{v_2-v_1}\Big[W(v_2) - \frac{\int_{v_1}^{v_2} W(s)ds}{v_2-v_1}\Big] - (1-p_L(v_1,v_2))W'(v_2).
\end{aligned} \tag{A.19}$$

To show that the Hessian of the Lagrangian is negative (semi-)definite, it suffices to show that $\mathcal{L}_{v_1v_1}(v_1,v_2), \mathcal{L}_{v_2v_2}(v_1,v_2) \leq 0$ with one equality strict at a point satisfying the FOCs. See that if the FOC for v_1 holds that:

$$\mathcal{L}_{v_1v_1}(v_1,v_2) = -p_L(v_1,v_2)W'(v_1). \tag{A.20}$$

And if the FOC for v_2 holds:

$$\mathcal{L}_{v_2v_2}(v_1,v_2) = -(1-p_L(v_1,v_2))W'(v_2). \tag{A.21}$$

Thus, if both FOCs are satisfied at (v_1,v_2), it suffices that $W'(v_1), W'(v_2) \geq 0$ (with one strict) for (v_1,v_2) to be a local optimum.

2. The lottery probability is equal to unity or zero: $\mu_1 = \mu_2 = \lambda_i = 0$ for one and only one i. In this case, there is an acceptance zone and a rejection zone. As a result, the objective takes the value:

$$V = \int_{1-Q}^{1} W(s)ds \tag{A.22}$$

which is simply the total value of all students living closest to the school.

3. Only the lower cutoff is zero: $\mu_2 = \lambda_1 = \lambda_2 = 0$ and $\mu_1 > 0$. In this case, $v_2 > 1-Q$ and $v_1 = 0$. Thus, the Lagrangian is:

$$\mathcal{L}(v_2) = \frac{Q-(1-v_2)}{v_2}\int_0^{v_2} W(s)ds + \int_{v_2}^{1} W(s)ds. \tag{A.23}$$

Taking the FOC with respect to v_2 as in the first case yields:

$$W(v_2) = \frac{1}{v_2}\int_0^{v_2} W(s)ds \tag{A.24}$$

which reveals that the marginal student who gets in for sure should contribute as much to social welfare as the average student in the lottery zone. The SOC for this to be an optimum is:

$$-(1-p_L(0,v_2))W'(v_2) \leq 0 \tag{A.25}$$

Or simply:

$$W'(v_2) \geq 0 \tag{A.26}$$

4. Only the upper cutoff is zero: $\mu_1 = \lambda_1 = \lambda_2 = 0$ and $\mu_2 > 0$. Following the same approach as for the lower cutoff being zero. The Lagrangian is:

$$\mathcal{L}(v_1) = \frac{Q}{1-v_1}\int_{v_1}^{1} W(s)ds. \tag{A.27}$$

Taking the FOC with respect to v_1 yields:

$$W(v_1) = \frac{1}{1-v_1}\int_{v_1}^{1} W(s)ds \tag{A.28}$$

again yielding the insight that the marginal student in the lottery zone should contribute as much to social welfare as the average student in the walk-zone.

The SOC for this to be an optimum is given by:

$$-p_L(v_1,1)W'(v_1) \leq 0. \tag{A.29}$$

Or simply:

$$W'(v_1) \geq 0. \tag{A.30}$$

5. Both the upper and lower cutoff are one and zero, respectively: $\lambda_1 = \lambda_2 = 0$ and $\mu_1, \mu_2 > 0$. In this case, the value is simply given by:

$$V = Q \int_0^1 W(s) ds \tag{A.31}$$

so value is simply the total utility of all students, weighted by the probability that they are admitted to the school.

From the above analysis, it follows that whenever a cutoff is interior and W is differentiable, the following two conditions are necessary and sufficient for a local optimum:

$$\begin{aligned} W(v_i^*) &= \frac{1}{v_2^* - v_1^*} \int_{v_1^*}^{v_2^*} W(s) ds \\ W'(v_i^*) &> 0. \end{aligned} \tag{A.32}$$

As a result, an optimal cutoff either satisfies the above conditions, lies on the boundary of the constraint set, or lies at a point of non-differentiability of W. This concludes the proof. □

A.5. *Proof of Proposition 2*

Proof. We explicitly provide sufficient conditions on the parameters such that each class of policy is optimal:

1. Full coarsening: if $u^P > u^R$, then there exist $\bar{\alpha}, \bar{\beta}$ such that when $\alpha < \bar{\alpha}$ and $\beta < \bar{\beta}$, $v_1^* = 0$ and $v_2^* = 1$. To show this, we want to prove in the case that $u^p > u^R$ that there exist $\bar{\alpha}, \bar{\beta} > 0$ such that when $\alpha < \bar{\alpha}$ and $\beta < \bar{\beta}$ the optimum is $v_1^* = 0$ and $v_2^* = 1$. To this end, it is sufficient to show that for $\alpha < \bar{\alpha}$ and $\beta < \bar{\beta}$, $W(s)$ is strictly decreasing on $[0, 1]$. See that $W(s)$ is given by:

$$W(s) = u^P + s(u^R - u^P) - \frac{\alpha}{1 + \delta \exp\{\varepsilon(s - \bar{s})\}} - \mathbb{I}[s \leq \bar{s}] \beta (\bar{s} - s)^\gamma. \tag{A.33}$$

As $u^R > u^P$, this is clearly the case so long as α, β are sufficiently small.

2. No coarsening: if $u^P < u^R$ then $v_1^* \leq 1 - Q, v_2^* = 1 - Q$. Alternatively, if $u^P > u^R$ there exists $\underline{\alpha}, \underline{\beta}$ such that when $\underline{\alpha} < \alpha$ and $\underline{\beta} < \beta$, $v_1^* \leq 1 - Q$, and $v_2^* = 1 - Q$. In this case, it is sufficient to show that $W(s)$ is strictly increasing. This is transparently the case when $u^P < u^R$. Moreover, whenever $\bar{s} > 1 - Q$, when $u^P > u^R$ and α, β are sufficiently large, then for $s < \bar{s}$, W(s) is strictly increasing and it is optimal to set $v_2^* = 1 - Q$ and $v_1^* < 1 - Q$.

3. "Small" walk-zone: if $u^P > u^R$, $\gamma > 1$ and $\varepsilon \to \infty$ then there exist $\bar{\beta}$ and $\underline{\alpha}$ such that when $\beta < \bar{\beta}$ and $\alpha > \underline{\alpha}$, $v_1^* = 0$ and $v_2^* \in (1 - Q, 1)$. In this case, it is sufficient to show that W(s) is strictly decreasing for $s < \bar{s}$ and $W(s) < W(s')$ for all $s < \bar{s} < s'$. This is clearly the case for β sufficiently small and α sufficiently large.

4. "Large" walk-zone: it is sufficient to show that there is a $\tilde{s} < 1 - Q$ such that for $s < \tilde{s}$, W(s) is increasing and for $s > \tilde{s}$, W(s) is decreasing and that $W(0) < W(1)$. In this case, it follows that $v_1^* \in (0, 1 - Q)$ and $v_2^* = 1$. If $u^P > u^R$ and $\gamma > 1$ then there exist $\underline{\beta}, \bar{\beta}$ and $\bar{\alpha}$ such that when $\underline{\beta} < \beta < \bar{\beta}$ and $\alpha < \bar{\alpha}$, the above is true.

5. "Double" walk-zone: if $u^P > u^R$ and $\gamma > 1$ and $\varepsilon \to \infty$ then there exist $\underline{\beta}, \bar{\beta}, \underline{\alpha} > 0$ such that when $\underline{\beta} < \beta < \bar{\beta}$ and $\underline{\alpha} < \alpha$, $v_1^* \in (0, 1 - Q)$ and $v_2^* \in (1 - Q, 1)$. In this case, it is sufficient to show that for $W(s) < W(s')$ for all $s < \bar{s} < s'$ and there is a $\tilde{s} < 1 - Q$ such that $W(s)$ is increasing for $s < \tilde{s}$ and decreasing over $\bar{s} > s > \tilde{s}$ and $W(0) < W(\bar{s})$. The first condition is satisfied for α sufficiently large. The second is satisfied so long as β is neither too small nor too large.

□

A.6. *Proof of Proposition 3*

Proof. We first show that for any parameter λ in which W is differentiable that:

$$\frac{\partial v_i^*}{\partial \lambda} \lesseqgtr 0 \Longleftrightarrow \int_{v_1^*}^{v_2^*} \left(W_\lambda(s) - W_\lambda(v_i^*)\right) ds \lesseqgtr 0 \tag{A.34}$$

for $i \in \{1, 2\}$. This result allows one to find comparative statics in any parametric environment and is perhaps of independent interest.

First, note that from optimality and interiority of (v_1^*, v_2^*) and from Proposition 1, we have that

$$W(v_2^*) = W(v_1^*) = \frac{1}{v_2^* - v_1^*} \int_{v_1^*}^{v_2^*} W(s) ds. \tag{A.35}$$

Differentiating $(v_2^*-v_1^*)W(v_i^*)=\int_{v_1^*}^{v_2^*}W(s)ds$ implicitly with respect to λ, we obtain

$$\begin{aligned}\left(\frac{\partial v_2^*}{\partial\lambda}-\frac{\partial v_1^*}{\partial\lambda}\right)W(v_i^*)+(v_2^*-v_1^*)\left(W_\lambda(v_i^*)+\frac{\partial v_i^*}{\partial\lambda}W'(v_i^*)\right)=\\ \frac{\partial v_2^*}{\partial\lambda}W(v_2^*)-\frac{\partial v_1^*}{\partial\lambda}W(v_1^*)+\int_{v_1^*}^{v_2^*}W_\lambda(s)ds.\end{aligned}\tag{A.36}$$

Plugging in $W(v_2^*)=W(v_1^*)=W(v_i^*)$ to the RHS of this equation, we obtain

$$\begin{aligned}\left(\frac{\partial v_2^*}{\partial\lambda}-\frac{\partial v_1^*}{\partial\lambda}\right)W(v_i^*)+(v_2^*-v_1^*)\left(W_\lambda(v_i^*)+\frac{\partial v_i^*}{\partial\lambda}W'(v_i^*)\right)=\\ \left(\frac{\partial v_2^*}{\partial\lambda}-\frac{\partial v_1^*}{\partial\lambda}\right)W(v_i^*)+\int_{v_1^*}^{v_2^*}W_\lambda(s)ds.\end{aligned}\tag{A.37}$$

Cancelling the terms and re-arranging, we obtain

$$\begin{aligned}(v_2^*-v_1^*)W'(v_i^*)\frac{\partial v_i^*}{\partial\lambda}&=\int_{v_1^*}^{v_2^*}W_\lambda(s)ds-(v_2^*-v_1^*)W_\lambda(v_i^*)\\ &=\int_{v_1^*}^{v_2^*}\big(W_\lambda(s)-W_\lambda(v_i^*)\big)ds.\end{aligned}\tag{A.38}$$

As the solution is interior, we have $1>v_2^*>1-Q>v_1^*>0$ and from Proposition 1, $W'(v_i^*)>0$, which proves that $\frac{\partial v_i^*}{\partial\lambda}$ and $\int_{v_1^*}^{v_2^*}\big(W_\lambda(s)-W_\lambda(v_i^*)\big)ds$ have the same sign for $i\in\{1,2\}$.

To prove the first part of the proposition, note that $W_\beta(s)=-\mathbb{I}[s\leq\bar{s}](\bar{s}-s)^\gamma$, which is increasing in s. Moreover, as $\bar{s}>1-Q$, in an interior solution, $v_1^*<\bar{s}$. Then for any $s\in(v_1^*,v_2^*)$, $W_\beta(v_1^*)<W_\beta(s)\leq W_\beta(v_2^*)$. Thus:

$$\int_{v_1^*}^{v_2^*}\big(W_\beta(s)-W_\beta(v_1^*)\big)ds>0>\int_{v_1^*}^{v_2^*}\big(W_\beta(s)-W_\beta(v_2^*)\big)ds\tag{A.39}$$

So $\frac{\partial v_1^*}{\partial\beta}>0$ and $\frac{\partial v_2^*}{\partial\beta}<0$ obtains by equation A.34.

To prove the second part of the proposition, note that $W_{u_p}(s)=1-s$ is decreasing in s. Then for any $s\in(v_1^*,v_2^*)$, $W_{u_p}(v_1^*)>W_{u_p}(s)>W_{u_p}(v_2^*)$. Thus

$$\int_{v_1^*}^{v_2^*}\big(W_{u_p}(s)-W_{u_p}(v_1^*)\big)ds<0<\int_{v_1^*}^{v_2^*}\big(W_{u_p}(s)-W_{u_p}(v_2^*)\big)ds\tag{A.40}$$

As a result, from equation A.34, we have $\frac{\partial v_1^*}{\partial up}<0$ and $\frac{\partial v_2^*}{\partial up}>0$. Other comparative statics can be derived in a similar fashion. □

A.7. *Proof of Proposition 4*

Proof. The utility with cut-offs v_1,v_2 is given by:

$$\begin{aligned}Z(v_1,v_2)=\sum_{\kappa_i}\int_{v_2}^1 sdF_{\kappa_i}(s)+\frac{v_2-r}{v_2-v_1}\sum_{\kappa_i}\int_{v_1}^{v_2}sdF_{\kappa_i}(s)\\ +h\Big(\int_{v_2}^1 dF_{\kappa_1}(s)+\frac{v_2-r}{v_2-v_1}\int_{v_1}^{v_2}dF_{\kappa_1}(s)\Big)\end{aligned}\tag{A.41}$$

As $f_{\kappa_1}(s)+f_{\kappa_2}(s)=1$, by computing the score integrals and re-arranging, we have that:

$$Z(v_1,v_2)=\frac{1}{2}\Big[1-v_2^2+(v_2+v_1)(v_2-r)\Big]+h\Big(\int_{v_2}^1 dF_{\kappa_1}(s)+\frac{v_2-r}{v_2-v_1}\int_{v_1}^{v_2}dF_{\kappa_1}(s)\Big)\tag{A.42}$$

Similarly, utility without coarsening is given by:

$$Z^{NC}=\sum_{\kappa_i}\int_r^1 sdF_{\kappa_i}(s)+h\Big(\int_r^1 dF_{\kappa_1}(s)\Big)=\frac{1}{2}(1-r^2)+h\Big(\int_r^1 dF_{\kappa_1}(s)\Big)\tag{A.43}$$

Thus:

$$Z(v_1,v_2)-Z^{NC}=\frac{1}{2}[(v_2-r)(v_1-r)] \\ +h\Big(\int_{v_2}^{1}dF_{\kappa_1}(s)+\frac{v_2-r}{v_2-v_1}\int_{v_1}^{v_2}dF_{\kappa_1}(s)\Big)-h\Big(\int_{r}^{1}dF_{\kappa_1}(s)\Big) \tag{A.44}$$

For $Z(v_1,v_2)>Z^{NC}$ to hold, it must be that:

$$\frac{1}{2}[(v_2-r)(r-v_1)]<h\Big(\frac{1}{2}-F_{\kappa_1}(v_2)+(F_{\kappa_1}(v_2)-F_{\kappa_1}(v_1))\frac{v_2-r}{v_2-v_1}\Big)-h\Big(\frac{1}{2}-F_{\kappa_1}(r)\Big) \tag{A.45}$$

Which proves the first part of the result. Plugging in $h(x)=\alpha x$, we have:

$$[(v_2-r)(r-v_1)]<2\alpha\Big(F_{\kappa_1}(r)-F_{\kappa_1}(v_2)+(F_{\kappa_1}(v_2)-F_{\kappa_1}(v_1))\frac{v_2-r}{v_2-v_1}\Big) \tag{A.46}$$

Dividing both sides by (v_2-r):

$$(r-v_1)<2\alpha\Big(\frac{F_{\kappa_1}(v_2)-F_{\kappa_1}(v_1)}{v_2-v_1}-\frac{F_{\kappa_1}(v_2)-F_{\kappa_1}(r)}{v_2-r}\Big) \tag{A.47}$$

Yielding the second part of the result. □

A.8. *Proof of Corollary 2*

Proof. This is an immediate consequence of Proposition 10 in Supplementary Appendix E. □

A.9. *Proof of Proposition 5*

Proof. First, let us compute social welfare function under the induced trinary coarsening of any policy, v. In the $\gamma\to\infty$ limit, we have that social welfare is given by (up to constant of proportionality that we omit):

$$Z(v)=\min_{\kappa\in\mathcal{K},s\in[0,1]}\bar{v}(\kappa,P(s,v)). \tag{A.48}$$

Note that agents have outside options given by $\kappa=h(s)$, where h is decreasing. Thus, within any equivalence class, the agent with the lowest expected utility is the agent at the upper threshold for each equivalence class. Moreover, we know that the allocation probabilities for these agents are given by 0, $p_L(v)$ and 1, respectively. Thus:

$$Z(v)=\min\{\bar{v}(h(v_1),0),\bar{v}(h(v_2),p_L(v)),\bar{v}(h(1),1)\}. \tag{A.49}$$

We now prove both parts of the proposition. In the first case, suppose that all agents have positive assignment probability. It follows that the coarsening is given by a single number $v=v_2$. Under this policy, welfare is given by:[47]

$$Z(v)=\min\{\bar{v}(h(v_2),p_L(v)),\bar{v}(h(1),1)\} \tag{A.50}$$

Now consider an alternative policy $v'=(\varepsilon,v_2)$ for $0<\varepsilon<v_2$. Under this policy, welfare is given by:

$$Z(v')=\min\big\{\bar{v}(h(\varepsilon),0),\bar{v}(h(v_2),p_L(v')),\bar{v}(h(1),1)\big\} \tag{A.51}$$

If h is sufficiently steep, then $\bar{v}(h(\varepsilon),0)>\bar{v}(h(v_2),1)$. Moreover, $p_L(v')>p_L(v)$. Thus, $Z(v')\geq Z(v)$. Thus an optimal coarsening features a lower cutoff and consequently some agents who have zero assignment probability.

For the second part of the proposition, simply take h to be the constant function. The optimal policy is $v=(0,1)$ and all agents have interior assignment probability. This completes the proof. □

REFERENCES

ABDULKADIROĞLU, A., ANGRIST, J. D., NARITA, Y., et al. (2017), "Research Design Meets Market Design: Using Centralized Assignment for Impact Evaluation", *Econometrica*, **85**, 1373–1432.

ABDULKADIROĞLU, A., CHE, Y.-K. and YASUDA, Y. (2015), "Expanding 'Choice' in School Choice", *American Economic Journal: Microeconomics*, **7**, 1–42.

47. In the knife-edge case with full coarsening, $Z(v)=\bar{v}(h(1),p_L(v))$. The steps below still follow.

ABDULKADIROĞLU, A., PATHAK, P. A. and ROTH, A. E. (2005), "The New York City High School Match", *American Economic Review*, **95**, 364–367.
ABDULKADİROĞLU, A. and SÖNMEZ, T. (2003), "School Choice: A Mechanism Design Approach", *American Economic Review*, **93**, 729–747.
ARNOSTI, N. and SHI, P. (2020), "Design of Lotteries and Wait-lists for Affordable Housing Allocation", *Management Science*, **66**, 2291–2307.
ASHLAGI, I. and SHI, P. (2014), "Improving Community Cohesion in School Choice via Correlated-lottery Implementation", *Operations Research*, **62**, 1247–1264.
AYGUN, O. and BÓ, I. (2021), "College Admission with Multidimensional Privileges: The Brazilian Affirmative Action Case", *American Economic Journal: Microeconomics*, **13**, 1–28.
AZEVEDO, E. M. and LESHNO, J. D. (2016), "A Supply and Demand Framework for Two-sided Matching Markets", *Journal of Political Economy*, **124**, 1235–1268.
BALINSKI, M. and SÖNMEZ, T. (1999), "A Tale of Two Mechanisms: Student Placement", *Journal of Economic Theory*, **84**, 73–94.
BLOCH, F. and CANTALA, D. (2017), "Dynamic Assignment of Objects to Queuing Agents", *American Economic Journal: Microeconomics*, **9**, 88–122.
BUDISH, E., CHE, Y.-K., KOJIMA, F., et al. (2013), "Designing Random Allocation Mechanisms: Theory and Applications", *American Economic Review,* **103**, 585–623.
ÇELEBI, O. and FLYNN, J. P. (2021), "Priorities *vs.* quotas" Available at SSRN 3562665.
CDC. (2018), "Interim Updated Planning Guidance on Allocating and Targeting Pandemic Influenza Vaccine during an Influenza Pandemic" https://www.cdc.gov/flu/pandemic-resources/pdf/2018-Influenza-Guidance.pdf.
COLLINSON, R., ELLEN, I. G. and LUDWIG, J. (2015), "Low-income Housing Policy", in Moffitt, R. A. (ed) *Economics of Means-Tested Transfer Programs in the United States, Vol.* **2** (University of Chicago Press) 59–126.
DALEY, B. (1999), "Plan Drops Race Role in Enrollment, Compromise Misses Point, Critics Say" (Boston Globe).
DOĞAN, B. (2016), "Responsive Affirmative Action in School Choice", *Journal of Economic Theory*, **165**, 69–105.
DUR, U., KOMINERS, S. D., PATHAK, P. A. et al. (2018), "Reserve Design: Unintended Consequences and the Demise of Boston's Walk Zones", *Journal of Political Economy*, **126**, 2457–2479.
DUR, U., PATHAK, P. A. and SÖNMEZ, T. (2020), "Explicit vs. Statistical Targeting in Affirmative Action: Theory and Evidence from Chicago's Exam Schools", *Journal of Economic Theory*, **187**, 104996.
ECHENIQUE, F. and YENMEZ, M. B., (2015), "How to Control Controlled School Choice", *American Economic Review*, **105**, 2679–2694.
EHLERS, L., HAFALIR, I. E., YENMEZ, M. B. et al. (2014), "School Choice with Controlled Choice Constraints: Hard Bounds versus Soft Bounds", *Journal of Economic Theory*, **153**, 648–683.
ELLISON, G. and PATHAK, P. (2021), "The Efficiency of Race-neutral Alternatives to Race-based Affirmative Action: Evidence from Chicago's Exam Schools", *American Economic Review*, **111**, 943–975.
ERDIL, A. and ERGIN, H. (2008), "What's the Matter with Tie-breaking? Improving Efficiency in School Choice", *American Economic Review*, **98**, 669–689.
ERDIL, A. and ERGIN, H. (2017), "Two-sided Matching with Indifferences", *Journal of Economic Theory*, **171**, 268–292.
ERDIL, A. and KUMANO, T. (2019), "Efficiency and Stability under Substitutable Priorities with ties", *Journal of Economic Theory*, **184**, 104950.
FRAGIADAKIS, D. and TROYAN, P. (2017), "Improving Matching under Hard Distributional Constraints", *Theoretical Economics*, **12**, 863–908.
GALE, D. and SHAPLEY, L. S. (1962), "College Admissions and the Stability of Marriage", *American Mathematical Monthly*, **69**, 9–15.
GEYER, J. and SIEG, H. (2013), "Estimating a Model of Excess Demand for Public Housing", *Quantitative Economics*, **4**, 483–513.
GOLDSTEIN, D. (2012), "Bostonians Committed to School Diversity Haven't Given Up on Busing", *The Atlantic*, https://www.theatlantic.com/politics/archive/2012/10/bostonians-committed-to-school-diversity-havent-given-up-on-busing/429060/.
GOTO, M., KOJIMA, F., KURATA, R., et al. (2017), "Designing Matching Mechanisms under General Distributional Constraints", *American Economic Journal: Microeconomics*, **9**, 226–262.
HAFALIR, I. E., YENMEZ, M. B. and YILDIRIM, M. A. (2013), "Effective Affirmative Action in School Choice", *Theoretical Economics*, **8**, 325–363.
KAMADA, Y. and KOJIMA, F. (2017), "Stability Concepts in Matching under Distributional Constraints", *Journal of Economic Theory*, **168**, 107–142.
KAMADA, Y. and KOJIMA, F. (2018), "Stability and Strategy-Proofness for Matching with Constraints: A Necessary and Sufficient Condition", *Theoretical Economics*, **13**, 761–793.
KESTEN, O., KURINO, M. and NESTEROV, A. S. (2017), "Efficient Lottery Design", *Social Choice and Welfare*, **48**, 31–57.
KESTEN, O. and ÜNVER, M. U. (2015), "A Theory of School-Choice Lotteries", *Theoretical Economics*, **10**, 543–595.
KOJIMA, F. (2012), School Choice: Impossibilities for Affirmative Action", *Games and Economic Behavior*, **75**, 685–693.
KOMINERS, S. D. and SÖNMEZ, T., (2016), "Matching with Slot-Specific Priorities: Theory", *Theoretical Economics*, **11**, 683–710.
LANDSMARK, T. (2009), "It's Time to End Busing in Boston" (Boston Globe, Opinion-Editorial).
LESHNO, J. (2019), "Dynamic Matching in Overloaded Waiting Lists" (SSRN 2967011).

LEVINSON, M. L., NOONAN, J. M., FAY, J. Z. et al. (2012), "First-Round Analysis of BPS Proposed 6-Zone, 9-Zone, 11-Zone, and 23-Zone School Assignment Plans" (Techincal Report, HGSE).

PATHAK, P. A. and SÖNMEZ, T. (2008), "Leveling the Playing Field: Sincere and Sophisticated Players in the Boston Mechanism", *American Economic Review*, **98**, 1636–1652.

PATHAK, P. A., SÖNMEZ, T., UNVER, M. U. et al. (2020), "Leaving No Ethical Value Behind: Triage Protocol Design for Pandemic Rationing" (NBER Working Paper).

PISCITELLO, G. M., KAPANIA, E. M., MILLER, W. D. et al. (2020), "Variation in Ventilator Allocation Guidelines by US State during the Coronavirus Disease 2019 Pandemic: A Systematic Review", *JAMA Network Open*, **3**, e2012606.

ROTH, A. E. (2002), "The Economist as Engineer: Game Theory, Experimentation, and Computation as Tools for Design Economics", *Econometrica*, **70**, 1341–1378.

ROTH, A. E. and PERANSON, E., (1999), "The Redesign of the Matching Market for american physicians: Some Engineering Aspects of Economic Design", *American Economic Review*, **89**, 748–780.

ROTH, A. E., SÖNMEZ, T. and ÜNVER, M. U. (2004), "Kidney Exchange", *The Quarterly Journal of Economics,* **119**, 457–488.

SIEG, H. and YOON, C. (2020), "Waiting for Affordable Housing in New York City", *Quantitative Economics*, **11**, 277–313.

SÖNMEZ, T. (2013), "Bidding for Army Career Specialties: Improving the ROTC Branching Mechanism", *Journal of Political Economy*, **121**, 186–219.

SÖNMEZ, T. and YENMEZ, M. B. (2019a), "Affirmative Action in India via Vertical and Horizontal Reservations" (Unpublished mimeo).

SÖNMEZ, T. and YENMEZ, M. B. (2019b), "Affirmative Action with Overlapping Reserves" (Technical Report, Boston College Department of Economics).

SÖNMEZ, T. and YENMEZ, M. B. (2019c), "Constitutional Implementation of Vertical and Horizontal Reservations in India: A Unified Mechanism for Civil Service Allocation and College Admissions" (Technical Report, Boston College Department of Economics).

STEWART, N., MAYS, J. C. and HAAG, M. (2019), "Facing Homeless Crisis, New York Aims for 1,000 New Apartments a Year" (New York Times).

WALDINGER, D. (2018), "Targeting In-kind Transfers through Market Design: A Revealed Preference Analysis of Public Housing Allocation", *American Economic Review*, **111**, 2660–2696.