Heterogeneous Learning in Product Markets *

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November 2021

Abstract

We study how market structures and asymmetries in learning technologies affect trade in a product market. In this market, a new product of unknown quality is introduced to challenge an existing product of known quality. We show that market efficiency is achieved both under monopoly and competition if buyers are symmetric in the amount of information they generate when they consume the new product. If buyers are instead asymmetric, only the monopolistic market is efficient. We identify the inefficiencies and analyze their properties. Finally, we explore our results' robustness under different assumptions about the ability to price discriminate.

Keywords: Market structure, Market power, Trading, Learning.
JEL Classification: C7

*First version January 2020. This version November 2021. We thank Daron Acemoglu, Kerry Back, Pierpaolo Battigalli, Ricardo Caballero, Roberto Corrao, Gonzalo Cisternas, Glenn Ellison, Drew Fudenberg, Leonid Kogan, Kevin Li, Andrey Malenko, Stephen Morris, Tobias Salz, Philipp Strack, and Michael Whinston for great and very helpful comments. Giacomo Lanzani gratefully acknowledges the financial support of the Guido Cazzavillan scholarship.

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1 Introduction

E-commerce marketplaces provide easy access for sellers to compete and sell new products. These marketplaces are also platforms on which consumers share their reviews. These reviews are essential sources of information about the quality or adoption potential of new products, and they have implications for the pricing of old (existing) products.[1]

In these markets, many available reviews of individual experiences imply that the belief about the new product is approximately the same for all market participants. Still, a crucial form of heterogeneity across agents is present: the agents provide feedback and reviews in different forms (from a brief 1-to-5 scale review to detailed textual comments) and with different levels of accuracy. This heterogeneity of feedback for the new product is natural. Buyers differ significantly in terms of the depth of their experience with the product: even if all of them care about using the best product, some may adapt to it faster and use ancillary features more frequently, generating deeper knowledge of the product. The fact that those different experiences translate into a public belief is particularly realistic in current markets with frequent and detailed feedback about consumer experiences through surveys or posted reviews.

Given this heterogeneity, how do reviews affect price competition and the learning of Bayesian market participants? When does a buyer decide to purchase a new product? Does competition improve welfare? What type of distortions arise with competition? And which policies can help improve market efficiency? In this paper, we answer these questions, focusing on the role of buyers' heterogeneity in providing reviews.

To study these questions, we consider a market for indivisible experience goods, where a new product of unknown quality is introduced to challenge an existing (old) product of known quality. Two states correspond to the possible quality levels of the new product. The true state is initially unknown, and the market participants are Bayesian agents who can gradually learn the state through reviews of the buyers experiencing the new product. Importantly, we allow the buyers to be heterogeneous in the (expected)

[1] User reviews, particularly on sites like Amazon, mean a great deal to shoppers. "*A product that has just one review is 65% more likely to be purchased than a product that has none*", according to Matt Moog, CEO of Power Reviews. He added that one-third of online shoppers refuse to purchase products that have not received positive customer feedback.

information about the quality of the new product that they generate by consuming it. This is a reduced form representation of the differing (exogenous) accuracy of their feedback. Since their use of the product generates more learning about product quality, we refer to the agents who provide more accurate signals as better learners.[2]

To analyze the impact of market power, we consider the efficient consumption pattern and the decentralized market outcomes (i.e., the Markov perfect equilibria) under both a monopolistic firm providing both products and competition between two firms, each specializing in one of the products. Here efficiency is identified with the consumption patterns that maximize the expected total surplus of the market participants.

We first establish that both the efficient and the decentralized market outcomes feature a sequence of belief thresholds. When there is little confidence in the new product's quality, only the best learners consume it. Over time, if the reviews of best learners sufficiently improve the market's confidence in the new product, then the worse learners also start buying it. In other words, all solutions feature a **beta phase** in which only the best learners experience the new (unknown) product. We explicitly derive the beta phase and its expected length in terms of the endogenous model parameters.

We then consider the efficiency properties of different market structures, with particular attention to the comparison between monopoly and competition. The main finding is that the relative welfare performance of monopolistic and duopolistic market structures crucially relies on the learning technology across buyers. We show that both market structures lead to efficiency when buyers are homogeneous in their learning technologies. In contrast, when buyers are heterogeneous, competition is no longer efficient, whereas the monopolistic market remains so.

The lower welfare resulting from competition may seem counterintuitive: the intuition behind these results is that, in dynamic markets, bilateral contracting between two parties produces learning externalities on the other market actors, proportional to their value of information. In monopolistic markets, the monopolist's optimal pricing makes

[2]The assumption of an exogenous learning ability is approximately correct in many instances of online commerce, where the differences in the amount of feedback seem to mostly relate to individual attitudes, instead of strategic considerations. As a consequence, sellers can enhance information production only by targeting the best learners. We think that having sellers who endogenously increase the agent's feedback may be relevant for some applications. However, we do not follow this route in this paper, because we are interested in isolating the informational externalities due to learning asymmetries.

this value 0, even under asymmetric learning technologies. Crucially, we show that this result does not rely on the monopolist observing the consumer type. In contrast, under competition, part of these externalities accrue to the potential buyers not involved in the transaction, and therefore it is not internalized in prices. This situation is analogous to how a new retailer introduces its product on Amazon, offering discounts to a subset of consumers who can provide detailed reviews. Intuitively, the amount of the discount increases with the future market power of the entrant in case of success. Our results highlight that competition may induce objectively suboptimal discounting strategies.

Our model also yields novel implications in terms of the structure of the inefficiency induced by competition. Notably, as in the first-best, the equilibrium has a threshold structure. Only the best learners use the new product at a low level of confidence, while the worse learners move to the new product as the public confidence level increases. This equilibrium features efficiency for the top learners: the equilibrium threshold in beliefs to start serving the best learners is the same as the first-best one. This efficiency for the top learners implies that all the new products that are sufficiently promising (i.e., their prior market belief is high enough) are given a chance. However, competition distorts the threshold to move out from the beta phase and start serving the entire market.

We further investigate the comparative statics of the above inefficiency. Even if asymmetries in the learning technology are necessary to have an inefficient market outcome, the size of the distortion is not monotone in the amount of heterogeneity.

Finally, we consider a possible solution to the distortions induced by competition. Indeed, we show that the introduction of multilateral contracts leads to an efficient equilibrium outcome. Precisely, we increase the commitment power of the sellers by allowing them to make take-it-or-leave-it offers to multiple market participants. These offers pay good learners for having the product consumed by bad learners, to compensate good learners' information externality cost generated by bad learners. We prove that, if such contracts are feasible, the decentralized outcome is efficient, regardless of the heterogeneity in the learning technologies.

1.1 Related Literature

Dynamic pricing has a rich history. In general, time-varying prices may arise for a variety of reasons. For example, they might be due to learning about new experience goods (e.g., Bonatti, Cisternas and Toikka (2017)) or to product choice with social learning (e.g., Maglaras, Scarsini and Vaccari (2020)).[3,4] In contrast, in this paper, we consider dynamic pricing when consumers differ in the precision of the information they produce on the product quality. This is crucial and leads to rich predictions about how sellers discount and price discriminate between consumers based on their learning technologies. It also allows us to explore different questions, like the relative efficiency performance of monopoly and competition and how the inefficiency depends on the heterogeneity of the buyers. Several papers in this literature highlight that competition can harm welfare (e.g., Bergemann and Välimäki (1997, 2000), Fang, Noe and Strack (2020)), but they do not study the impact of heterogeneity in the learning technology of the agents.

Our paper also relates to the growing literature studying the role of online platforms in the sharing of information; see Bonatti and Cisternas (2020) and Vellodi (2021). For example, Acemoglu et al. (2019), like us, single out an externality induced by some consumers on others. However, they argue there is partial overlapping in the private information of the different consumers and the information provided by one consumer depresses the value of the information of the others. This paper also relates to the growing literature on innovation, strategic pricing, and externalities, including, in particular, papers on strategic information exchange (e.g., Sadler (Forthcoming)), and experimentation with technological innovation (e.g., Cerreia-Vioglio, Corrao and Lanzani (2020)).

Our paper is linked with works that study big data and the use-based evolution of beliefs about the quality of a product. Related questions to this type of belief dynamics have been addressed in different frameworks in several important papers (e.g., Bolton and Harris (1999), Mueller-Frank (2012), Papanastasiou, Bimpikis and Savva (2018), Koren and Mueller-Frank (2021)). For example, Park (2001) observed a possible linkage between learning asymmetries and efficiency, but he studies neither when it is possible nor the form of the inefficiency. The idea that platforms can aggregate information is linked

[3]Similar techniques have been applied to career concerns; see Cisternas (2018).

[4]Another cause suggested in the literature for varying prices over time is information diffusion due to the word of mouth effect (e.g., Ajorlou, Jadbabaie and Kakhbod (2018)).

to the literature on markets for big data (e.g. Begenau, Farboodi and Veldkamp (2018)), and mechanisms for pricing information (e.g., Eliaz, Eilat and Mu (2019)).[5] In contrast to these important works, we consider how the availability of information through heterogeneous sources affects welfare, market power (monopoly and competition), trading volume, the beta phase, and the nature of arising distortions in product markets. We further present policies that can reduce distortions. In this regard, this paper also relates to the body of work on heterogeneous learning in financial markets. However, the nature of the asymmetry differs, because most of the attention has been dedicated to heterogeneity in beliefs.[6] Finally, the effect of different ambiguity attitudes on learning has been studied by Battigalli et al. (2019).

The rest of the paper proceeds as follows. Section 2 introduces our formal model, and Section 3 studies the first-best consumption allocation. Section 4 moves to the analysis of the decentralized outcome and presents our main results. Section 5 proposes multilateral contracts for sellers and studies characteristics of the beta phase. Section 6 concludes. Most of the proofs are in the online appendix, except the ones of the main results that are collected in the appendix.

2 Model

We consider a product market where buyers (consumers) face two indivisible products. Product a is an established (old) good that creates a known flow of payoff for buyers. Product b is recently introduced, and its true (expected flow payoff) value is unknown to both sellers and buyers. That is, the consumption utility of product b depends on an unknown state $\theta \in \{\ell, h\}$, where the state is the expected flow utility of the new product.

There are $M \in \{1,2\}$ sellers and $n \geq 2$ possibly asymmetric buyers. When $M = 1$ a profit maximizing **monopolist** sells both of the products. In the **oligopoly** structure where $M = 2$, two different sellers compete strategically to sell the products; that is, one seller sells the new product, and the other sells the established one. In this case, we will

[5] See Bergemann and Bonatti (2019) for excellent surveys of this literature.

[6] Gennaioli and Shleifer (2018) study how investors and policymakers assign irrationally and inaccurately low probabilities to disaster outcomes leading to financial fragility. Veronesi (2019) considers general distributions of households' risk tolerance and beliefs about long-term growth.

label each of the sellers as the product he sells.

2.1 Buyers' asymmetry and flow payoffs

We assume that each product sold in time t will survive in $[t, t+dt)$ and generates the following flow of payoff for its buyer(s). Precisely, at $[t, t+dt)$, the old (established) product a creates

$$dC_{ai}(t) = \mu_a dt,$$

monetary value for buyer i, which represents the consumption utility the buyer i experiences from owning the old product. In state $\theta \in \{h, \ell\}$, the new product b creates

$$dC_{bi}(t) = \theta dt + \sigma_i dZ_{it},$$

monetary value for buyer i (if he owns it), where Z_{it}, $i = 1, 2, \cdots, n$, are independent standard Brownian motions (BMs) (Wiener processes). Therefore the expected experienced consumption utility of the new product is uncertain. We assume that the problem is not trivial, that is, $\ell < \mu_a < h$, and that the new product induces an expected nonnegative flow of utility in both states, that is, $\ell \geq 0$. Therefore the state θ determines the objectively preferable product, which is the same for every buyer. The state is initially unknown to all the market participants, and the buyers and the sellers share a common prior $\Pr\{\theta = h\} = \pi_0$ at time 0 when the new product starts to be offered.

Each buyer expresses the experienced consumption utility online. The volatility σ_i indicates how noisy the experience of buyer i reflects product's true consumption utility. Importantly, we let $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_n$. This means that buyers are possibly **asymmetric** in their accuracy in experiencing the new product. In this regard, we order buyers from the noisiest buyer 1 to the least noisy buyer n. We say that a buyer is a better learner the lower σ is. Clearly, when $\sigma_i = \sigma$ for all $i = 1, 2, \cdots, n$, then buyers are **symmetric**. To isolate the effect of heterogeneous learning technology (i.e., heterogenous σ_i), we assume that the buyers are otherwise identical and, in particular, that they share the same valuation for the product of unknown quality in both situations, i.e., h and ℓ are the same across buyers.[7]

[7]Our main results would continue to hold as long as the valuations of the different buyers are corre-

2.2 Trading volume and payoffs

At each period $[t, t+dt)$, a buyer at most uses (experiences) one product. Hence, at the beginning of period t, the order of buyer i is in $\{a, b, \emptyset\}$, where $\emptyset$ means that i does not hold any product in $[t, t+dt)$. We denote by $\xi_{ik}(t)$ the allocation process such that $\xi_{ik}(t) = 1$ if buyer i purchases product k in period t and $\xi_{ik}(t) = 0$ otherwise. Therefore the trading volume of product $k \in \{a, b\}$ at time t is

$$\mathrm{Vol}_k(t) = \sum_{i=1}^{n} \xi_{ik}(t).$$

Sellers have all the bargaining power, that is, offers are in take-it-or-leave-it forms. At the beginning of period t, the price of product k for buyer i posted by its seller is $p_{k,i}(t)$, for $k \in \{a, b\}$.

Both buyers and sellers are risk-neutral and forward-looking. They discount payoffs exponentially at a shared rate $\rho > 0$. Therefore the payoff of buyer i is given by

$$U_i^B = \mathbf{E}\Big[\int_0^\infty \rho e^{-\rho t} \sum_{k \in \{a,b\}} \underbrace{\xi_{ik}(t)}_{\text{order}} \Big(\underbrace{dC_{ki}(t)}_{\text{flow gain}} - \underbrace{p_{k,i}(t)}_{\text{payment}} \, dt \Big)\Big], \tag{1}$$

where $\mathbf{E}[\cdot]$ denotes the expectation operator. Without loss of generality, we normalize the production cost to 0 so that the payoff of the sellers equals the total revenues they obtain from the products they sell.

Importantly, how we compute these revenues depends on the market structure (monopoly versus oligopoly). Below we present the expected discounted payoffs in the two cases considered in the paper.

Monopoly. When there is a unique seller of both products, the seller's payoff is given by

$$U_m = \mathbf{E}\Big[\int_0^\infty \rho e^{-\rho t} \underbrace{\Big(\sum_{i=1}^{n} \xi_{ia}(t) p_{a,i}(t) + \sum_{i=1}^{n} \xi_{ib}(t) p_{b,i}(t) \Big)}_{\text{overall time } t \text{ monopoly profit (sale)}} dt\Big]. \tag{2}$$

lated. However, if they were independent, the information externality we single out below would disappear, and the equilibrium under competition would be efficient.

Under competition, the objective function of the two competing sellers is analogous, but it takes into account that each benefits only from his or her sales.

Oligopoly. Under oligopoly, the payoff of seller $k \in \{a, b\}$ is given by

$$U_k^S = \mathbf{E}\Big[\int_0^\infty \rho e^{-\rho t} \underbrace{\sum_{i=1}^n \xi_{ik}(t) p_{k,i}(t)\, dt}_{\substack{\text{time } t \text{ seller } k \\ \text{profit (sale)}}}\Big]. \tag{3}$$

2.3 Belief dynamics

At each time t, all the data about the buyers' flow of payoffs are public information. Therefore, even when the amount of information produced by buyers differs, there is a unique market belief about the type of the unknown product. Formally, let $\mathcal{F} = \{\mathcal{F}_t\}_{t\geq 0}$ be the filtration generated by the public information available up to time t, that is, the filtration generated by the public signal $(\mathbf{X}(t))_{t\geq 0}$, where $\mathbf{X}(t) = (X_1(t), ..., X_n(t))$ and

$$X_i(t) = \int_0^t \xi_{ib}(\tau) dC_{bi}(\tau).$$

With this, the public belief is denoted as

$$\pi_t := \Pr\big\{\theta = h | \mathcal{F}_t\big\}.$$

The following lemma characterizes the dynamics of the market belief in terms of the (endogenous) trading volume and learning technologies (i.e., σ_i) of buyers.[8]

Lemma 1. *[Belief Evolution] We have*

$$d\pi_t = \pi_t(1 - \pi_t)(h - \ell)\sqrt{\sum_{i=1}^n \frac{\xi_{ib}(t)}{\sigma_i^2}} dZ_t$$

where Z_t is a standard Wiener process with respect to the filtration $\mathcal{F}_t$. In particular, in the case

[8] The proof of this lemma resembles the proof of Lemma 1 of Bolton and Harris (1999), with some minor differences. Here the heterogeneous learning technologies play the role of the intensity of experimentation in that paper.

of symmetric buyers, we have

$$d\pi_t = \frac{\pi_t(1-\pi_t)(h-\ell)}{\sigma}\sqrt{Vol_b(t)}dZ_t.$$

The dynamics are intuitive: beliefs change more when the information provided by the consumers is better and when the market participants are less sure to start with (i.e., π is closer to $\frac{1}{2}$).

2.4 Learning Progression

How does the market belief improve over time? Which factors matter? In this environment, starting from confidence π_0, we can characterize the expected improvement in the market belief about unknown state θ to a target belief β obtained when only an arbitrary subset of buyers $M = \{m_1,...,m_j\}$ uses product b for the fixed span of time T. More formally,

$$MO\left(\pi_0,\beta,T,m_1,...,m_j\right) = \mathbf{E}_{\pi_0}\left[\max\left\{\pi_T-\beta,0\right\}|\forall t\in[0,T],\forall i\in\{1,n\},\xi_{ib}(t)=1_{\{i\in M\}}\right],$$

where $1_{\{A\}}$ is the indicator function on A.

The above expression introduces a natural way to measure the expected improvement in market optimism (MO) in comparison to a target level β, due to the experiences of buyers $\{m_1,...,m_j\}$ of the new product up to time T. To obtain intuition about the market optimism, we consider a special case with $\beta = \pi_0$ in the following result. A general result with arbitrary β is presented in Proposition 13 in the online appendix.

Proposition 1. *Let buyers $\{m_1,...,m_j\}$ use the new product b in the time interval $[0,T]$. Then the expected progression in the market belief π_t from the initial market belief π_0 is:*

$$MO(\pi_0,\pi_0,T,m_1,...,m_j) = \pi_0(1-\pi_0)\left(2\Phi\left(\frac{(h-\ell)}{2}\sqrt{\sum_{i=1}^{j}\frac{T}{\sigma_{m_i}^2}}\right)-1\right). \tag{4}$$

Moreover, $MO(\pi_0,\pi_0,T,m_1,...,m_j)$ is increasing in the horizon time T, the learning quality $\frac{1}{\sigma_{m_k}}$ of each buyer m_k, and the number of buyers experiencing the product b.

This corollary immediately shows intuitive comparative statics on this measure. Somewhat expectedly, MO increases with the horizon time T, the learning quality of buyer m_k, and the number of buyers experiencing product b.

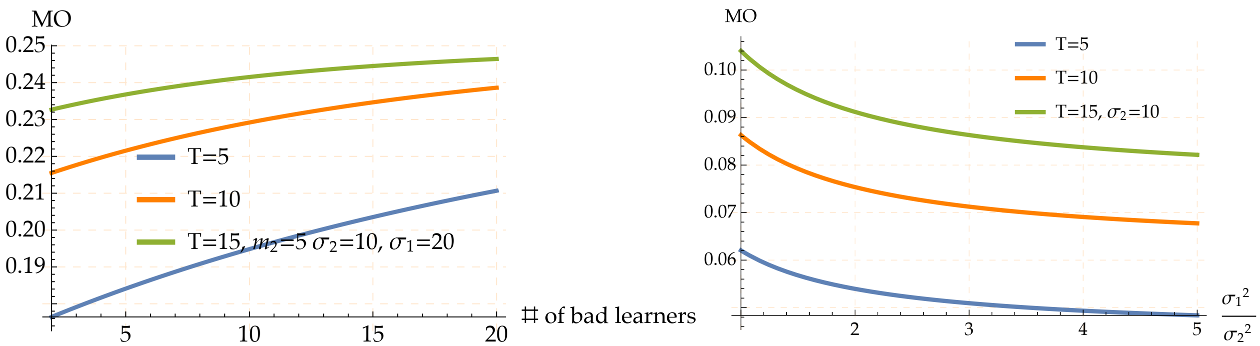


Figure 1: The left panel $MO(\pi_0, \pi_0, T, m_1, ..., m_j)$ increases with T and the number of buyers experiencing the product b. The right panel shows $MO(\pi_0, \pi_0, T, m_1, m_2)$ increases with the quality of learning.

Next, we leverage the previous results on the public belief dynamics to study the optimal choice of buyers and the dynamic pricing of sellers under different market structures. First, we study the optimal consumption pattern for a planner who wants to maximize the sum of the utilities of the market participants. Then we consider the decentralized equilibrium that arises when each market participant best replies to the strategy of the opponents, and we explore the difference between these two situations.

3 The first-best— efficient strategies

The first-best formulation. In this section, we consider the social welfare-maximizing strategies; that is, we specify strategies that maximize the sum of the utilities of all market participants. To be feasible, the strategies must only reflect the available information. This requirement is formalized by letting $\xi_{ik} \in \Xi_{ik}$ for all $i \in \{1, ..., n\}$ and $k \in \{a, b\}$, where Ξ_{ik} is the set of consumption allocations for buyer i and product k that are progressively measurable with respect to the filtration $\mathcal{F}$.

Given buyers' and sellers' payoffs (see (1)-(3)), the payments cancel each other out in the welfare-maximization problem. As a result, the objective function is the discounted sum of the consumption utility of the buyers:

$$W(\pi) = \max_{\xi_{ik}\in\Xi_{ik}} \mathbf{E}\left[\sum_{i=1}^{n}\sum_{k\in\{a,b\}}\int_0^{\infty}\rho e^{-\rho t}\xi_{ik}(t)dC_{ki}(t)\right].$$

Therefore efficiency only depends on the consumption of each agent, regardless of the transfers. Given that the system is time-invariant, the optimal ξ_{ik} only depends on the public belief π, and the maximization can be mapped into an optimal stopping problem (e.g., Karatzas (1984)). With this, the Hamilton-Jacobi-Bellman (HJB) equation for this problem is given by:

$$W(\pi) = \max_{\xi_{ik}\in\Xi_{ik}}\left\{\sum_{i=1}^{n}\left(\xi_{ia}\mu_a + \xi_{ib}\mathbf{E}_{\pi}[\theta]\right) + W''(\pi)\sum_{i=1}^{n}\xi_{ib}\frac{g(\pi,h,\ell)}{2\rho\sigma_i^2}\right\}$$

where $g(\pi,h,\ell) = \left((h-\ell)\pi(1-\pi)\right)^2$. Since the planner's instantaneous gain from allocating consumers to the risky good is linear in π_t, the efficient allocation is pinned down by a simplea sequence of cutoffs on the public belief $(\pi_{\text{fb},i})_{i=1}^{n}$. The consumer i buys product b at time t if and only if $\pi_t > \pi_{\text{fb,i}}$, i.e.,

$$\xi_{ib}(t) = 1_{\{\pi_t > \pi_{\text{fb,i}}\}} \quad \forall\, i = 1,2\cdots,n. \tag{5}$$

3.1 Symmetric buyers

In the case of symmetric buyers, there is only one cutoff π_{fb}. In this case, the first-best cutoff and welfare have closed-form expressions summarized by the next two propositions.

Proposition 2. *The first-best (social welfare) maximizing cutoff is given by*

$$\pi_{\text{fb}} = \frac{(\mu_a-\ell)\left(\sqrt{1+8\frac{\sigma^2\rho}{n(h-\ell)^2}}-1\right)}{(\ell+h)-2\mu_a+(h-\ell)\sqrt{1+8\frac{\sigma^2\rho}{n(h-\ell)^2}}}. \tag{6}$$

Moreover, π_{fb} is increasing in μ_a, σ^2, and ρ. It is decreasing in h and n.

It is interesting how the first-best cutoff changes with the fundamentals. First, since the known product acts as an outside option, a higher μ_a is easily seen to induce a higher π_{fb}. On the other hand, a larger h increases the value of choosing alternative b through two channels. First, it increases the instantaneous value given a particular belief. Second, it increases the learning value by making $(h - \ell)$ larger. Therefore it unambiguously induces a lower π_{fb}. The effect of a larger ℓ is instead ambiguous: it reduces the value of experimentation, but it makes the instantaneous reward of choosing b larger. The effects of the information processing technology, the discount factor, and the number of buyers are unambiguous and intuitive. The larger σ^2 (or ρ), the less attractive experimentation and the higher π_{fb}. Conversely, more patient buyers stop experimenting at a lower π_{fb}. Finally, notice that the existence of the public platform makes the benefit of learning linearly increase with the number of buyers, and therefore a larger n induces a lower π_{fb}. These comparative statics are presented in Figure 6 in the Online Appendix.

We can use the derived optimal strategy and cutoff value to compute the total welfare of the agents. The following proposition provides an explicit formula.

Proposition 3. *The first-best social welfare equals $nW_{avg}(\pi)$, where*

$$W_{avg}(\pi) = \begin{cases} \mu_a & \text{if } 0 \leq \pi \leq \pi_{\text{fb}}; \\ \mathbf{E}_\pi[\theta] + \varphi\Big[\pi^{\frac{1}{2}\left(1-\sqrt{1+8\frac{\sigma^2\rho}{n(h-\ell)^2}}\right)}(1-\pi)^{\frac{1}{2}\left(\sqrt{1+8\frac{\sigma^2\rho}{n(h-\ell)^2}}+1\right)}\Big] & \text{if } \pi_{\text{fb}} < \pi \leq 1; \end{cases}$$

and

$$\varphi := 2\frac{h-\mu_a}{\sqrt{1+8\frac{\sigma^2\rho}{n(h-\ell)^2}}-1}\left(\frac{\pi_{\text{fb}}}{1-\pi_{\text{fb}}}\right)^{\frac{1}{2}\left(\sqrt{1+8\frac{\sigma^2\rho}{n(h-\ell)^2}}+1\right)}.$$

Moreover, W_{avg} is strictly convex in $\pi_{\text{fb}} < \pi \leq 1$.

Intuitively, the welfare function is flat before the optimal cutoff. The per-consumer value there equals the flow of payoff guaranteed by the known product bought by all the consumers. After the cutoff, the value increases with the probability assigned to the high quality of the unknown product. There the convexity of the welfare is due to the known product that acts as an outside option whenever the market participants are more confident that the unknown product is low quality.

Next we move to the more interesting case of different learning technologies for the buyers. There the optimal cutoffs do not admit an explicit expression, but we obtain some qualitative properties of the welfare-maximizing consumption pattern that we can later use to draw comparisons with the decentralized market outcome.

3.2 Asymmetric buyers

We consider the case of 2 buyers. The extension to n buyers is straightforward and does not provide additional insights. Recall that we have $\sigma_1 > \sigma_2$. It is useful to consider the average (per consumer) first-best welfare. The dynamic programming implies that W_{avg} satisfies the following HJB:

$$W_{avg}(\pi) = \max_{\xi_{1b}\in\Xi_{1b},\xi_{2b}\in\Xi_{2b}} \left\{ \mu_a + \sum_{i=1}^{2} \frac{\xi_{ib}}{2} \left(\mathbf{E}_\pi[\theta] - \mu_a + \frac{g(\pi,h,\ell)W''_{avg}(\pi)}{\rho\sigma_i^2} \right) \right\}. \tag{7}$$

Denote ξ^*_{1b},ξ^*_{2b} the optimizer which maximizes the average utility above. The optimality condition in each ξ_{ib} yields

$$\xi^*_{ib} = \begin{cases} 1, & \mathbf{E}_\pi[\theta] + \frac{1}{\rho\sigma_i^2} g(\pi,h,\ell)W''_{avg}(\pi) \geq \mu_a; \\ 0, & \text{otherwise.} \end{cases}$$

When W_{avg} is convex (which we verify later), $\sigma_1 > \sigma_2$ and the previous expression of ξ^*_{ib} imply that

$$\xi^*_{1b} = 1 \quad \Longrightarrow \quad \xi^*_{2b} = 1.$$

Therefore we conjecture that the optimal strategy is a threshold type. There exist thresholds $\pi_{\text{fb},1},\pi_{\text{fb},2} \in (0,1)$ with $\pi_{\text{fb},1} > \pi_{\text{fb},2}$, such that

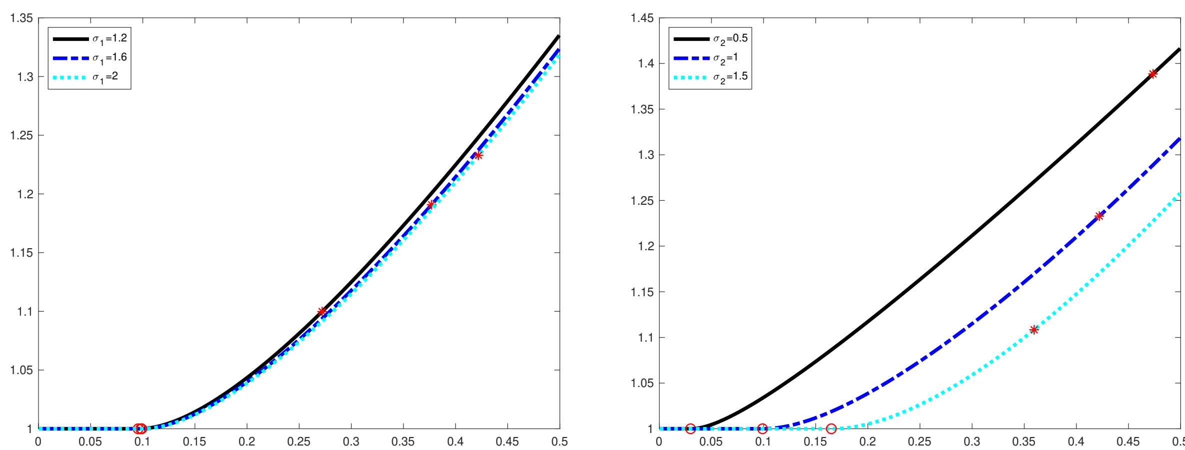
$$\xi^*_{ib}(t) = 1_{\{\pi_t > \pi_{\text{fb},i}\}}, \quad i \in \{1,2\}. \tag{8}$$

Proposition 12 in the online appendix verifies the optimality of ξ^*_{ib} for the first-best problem with asymmetric buyers.

The structure of this candidate optimal policy is simple. There are two thresholds $\pi_{\text{fb},1}$ and $\pi_{\text{fb},2}$ with $0 < \pi_{\text{fb},2} < \pi_{\text{fb},1} < 1$. Both buyers purchase the new product b when the public belief is higher than $\pi_{\text{fb},1}$, no buyer purchases b when the public belief is lower than $\pi_{\text{fb},2}$, and only the buyer with the better learning technology purchases the new product when $\pi_{\text{fb},2} < \pi < \pi_{\text{fb},1}$. The intuition for why the threshold for the better learner is lower is simple. First, note that information is valuable for overall welfare, because it allows better consumption choices for the consumers. (This mathematically translates into the convexity of the value function.) Second, the higher signal precision of the better learner implies that he can trade-off exploitation in favor of information generation at more favorable terms, and therefore it is optimal to start to do so for more pessimistic public beliefs. These cutoffs are free boundaries, which are determined by value matching and smooth pasting conditions presented in the online appendix.

The following proposition summarizes our results for the first best problem with asymmetric buyers. Denote as π_{myopic} the belief such that $\mathbf{E}_{\pi_{\text{myopic}}}[\theta] = \mu_a$.

Proposition 4. *The first-best policy is in cutoff strategies with $\pi_{\text{myopic}} > \pi_{\text{fb},1} > \pi_{\text{fb},2}$.*



(a) Different σ_1 (b) Different σ_2

Figure 2: Average value and cutoffs in the first best problem with asymmetric buyers. In both panels, the horizontal coordinate of the red circle is $\pi_{\text{fb},2}$, and horizontal coordinate of the red asterisk is $\pi_{\text{fb},1}$. In the left panel, the average value decreases and $\pi_{\text{fb},1}$ increases with σ_1. In the right panel, the average value decreases, $\pi_{\text{fb},2}$ increases, and $\pi_{\text{fb},1}$ decreases with σ_2. Other paremeters: $h = 2$, $\ell = 0$, $\mu_a = 1$, $\rho = 0.5$, $\sigma_2 = 1$ (left panel), and $\sigma_1 = 2$ (right panel).

Figure 2 illustrates how the average values and cutoffs depend on σ_1 and σ_2.

In the next sections, we compare these first-best cutoffs with the ones obtained under the two different competition structures presented above: a monopolist selling both products and competition between two sellers.

4 Analysis: Decentralized Outcome

We aim to find the equilibrium strategies when buyers are symmetric and asymmetric in their learning technology σ_i. Notably, we aim to discover how the market structure and competition between sellers affect learning, trading volume, and efficiency.

In what follows, we will show that, if the buyers have asymmetric learning technologies, a monopolistic market structure is efficient, while competition induces a welfare loss. Notably, monopoly is efficient both when the monopolist can observe the learning technologies of the buyers and when the monopolist cannot. Formally, to deal with both cases, we use a notation that allows the seller to discriminate between buyers, depending on their learning technology. Therefore the choice variable for the monopolist is the vector $(p_{a,i}, p_{b,i})_{i=1,\ldots,n}$. Similarly, under competition, the choice variable for seller a is $(p_{a,i})_{i=1,\ldots,n}$ and for seller b is $(p_{b,i})_{i=1,\ldots,n}$.

We note that the negative result we will obtain for competition is only reinforced by the assumption that the sellers can apply price discrimination. Indeed, it is well known (e.g., the textbook treatment of Wilson (1993)) that, even in static markets, the combination of market power for the seller, asymmetric consumers, and impossibility to discriminate between consumers create inefficiencies. However, inefficiencies are usually avoided when the seller can discriminate. Instead, we will show that, in dynamic markets with learning, discrimination is insufficient to eliminate the inefficiencies induced by competition.

To analyze this model, we restrict our attention to Markov perfect equilibria. Given the timing of offers, the pricing strategy of seller k is a progressively measurable function from the belief space to the real numbers $p_{k,i} \in \Pi_{ki}$, $i = 1, \ldots, n$, and the strategy of buyer i is a pair of progressively measurable functions that maps the beliefs and the posted prices into purchasing choices $\xi_{ik} \in \Xi_{ik}$, $k = 1, 2$.

We state the relevant equilibrium notion for the case of competition. An analogous definition that takes into account that the choice variable of the monopolist has two dimensions is used for the study of the monopoly case.

Definition 1. *A collection of Markov strategies* (ξ^*, p^*) *is a Markov Perfect Equilibrium if for all* $k \in \{a,b\}, i \in \{1,\ldots,n\}, (p_{k,1}, p_{k,2}) \in \Pi_{k1} \times \Pi_{k2}, (\xi_{ia}, \xi_{ib}) \in \Xi_{ia} \times \Xi_{ib}, \pi \in (0,1)$,

$$U_k^S(p^*, \xi^*, \pi) \geq U_k^S(p_k, p_{-k}^*, \xi^*, \pi) \qquad \textit{and} \qquad U_i^B(p^*, \xi^*, \pi) \geq U_i^B(p^*, \xi_{-i}^*, \xi_i, \pi).$$

4.1 Monopoly

We start by proving that the revenue-maximizing policy of a monopolist is *efficient*; that is, the induced consumption pattern maximizes the total surplus, independent of the learning technologies. To prove this result, we first derive the buyers and the monopolist's HJB equations. Recall that we assume that the sellers and, in this particular case, the monopolist have all the bargaining power with offers in take-it-or-leave-it forms. Therefore the HJB equation of a buyer i captures the comparison between the two products at the posted prices:[9]

$$v_i(\pi) = \max\Bigg\{\underbrace{\mu_a - p_{a,i}(\pi)}_{\text{flow gain of } a} + \underbrace{g(\pi,h,\ell)\sum_{j\neq i}\frac{\xi_{jb}(\pi, p_a(\pi), p_b(\pi))}{2\rho\sigma_j^2}v_i''(\pi)}_{\text{learning gain from others buying } b},$$
$$\underbrace{\mathbf{E}_\pi[\theta] - p_{b,i}(\pi)}_{\text{flow gain of } b} + \underbrace{g(\pi,h,\ell)\left(\sum_{j\neq i}\frac{\xi_{jb}(\pi, p_a(\pi), p_b(\pi))}{2\rho\sigma_j^2} + \frac{1}{2\rho\sigma_i^2}\right)v_i''(\pi)}_{\text{learning gain from } i \text{ and others buying } b}, 0\Bigg\}. \tag{9}$$

Each term of the above HJB equation has two parts. If buyer i buys the product a of known quality, then $\mu_a - p_{a,i}(\pi)$ is the instant (expected) flow payoff, and $g(\pi,h,\ell)\sum_{j\neq i}\frac{\xi_{jb}(t)}{2\rho\sigma_j^2}v_i''(\pi)$ is the expected continuation payoff (which is due to learning). A similar decomposition holds when buyer i buys the risky product b of unknown quality, but then the expected

[9] Note that the HJB solution to smooth functions can also be extended to the weaker viscosity solutions as well; see the last section of the online appendix.

amount of information generated is increased by a term inversely proportional to σ_i^2 and the (expected) flow payoff becomes $\mathbf{E}_\pi[\theta] - p_b(\pi)$.

The monopolist's HJB equation can be obtained similarly in terms of the behavior of the buyer:

$$w_m(\pi) = \sup_{p_{a,i},p_{b,i}} \left\{ \sum_{i=1}^{n} \Big(\xi_{ib}\left(\pi, p_a, p_b\right) p_{b,i} + \xi_{ia}\left(\pi, p_a, p_b\right) p_{a,i} + g\left(\pi, h, \ell\right) \frac{\xi_{ib}\left(\pi, p_a, p_b\right)}{2\rho\sigma_i^2} w_m''(\pi) \Big) \right\}. \tag{10}$$

Next, we explore the case of symmetric and asymmetric learning technologies. Under the monopolistic structure we are currently analyzing, the two cases lead to similar welfare conclusions, but we separate them because of the critical difference they feature under competition.

4.1.1 Symmetric buyers

We start characterizing the equilibrium prices posted by the monopolist in a cutoff equilibrium when buyers are symmetric.

Lemma 2. *In every symmetric equilibrium with cutoff π_m^*, the prices are as follows. If $\pi < \pi_m^*$*

$$p_a(\pi) = \mu_a \qquad \textit{and} \qquad p_b(\pi) \geq \mathbf{E}_\pi[\theta]. \tag{11}$$

If $\pi \geq \pi_m^*$

$$p_a(\pi) \geq \mu_a \qquad \textit{and} \qquad p_b(\pi) = \mathbf{E}_\pi[\theta]. \tag{12}$$

To prove this result, as a preliminary observation, notice that the problem of the monopolist reduces to the choice between either selling the product of known quality or selling the one of unknown quality. Indeed, given this choice, there is no reason to charge less than the buyers' maximal willingness to pay, since the informational content generated by the use of the product is unaffected by the price. Therefore, if $\pi < \pi_m^*$, we have $p_a(\pi) = \mu_a$. Since by the definition of cutoff equilibrium the monopolist is selling the product of known quality at those beliefs, it immediately follows from the value function of the buyer that $p_b(\pi) \geq \mathbf{E}_\pi[\theta] + \frac{g(\pi,h,\ell)}{2\rho\sigma_i^2}v''(\pi)$. Similarly, if $\pi \geq \pi_m^*$, the monopolist sets a price of product b equal to the willingness to pay $p_b(\pi) = \mathbf{E}_\pi[\theta] + \frac{g(\pi,h,\ell)}{2\rho\sigma_i^2}v''(\pi)$. It follows from the value function of the buyers that $p_a(\pi) \geq \mu_a - \frac{g(\pi,h,\ell)}{2\rho\sigma_i^2}v''(\pi)$. However, if we plug these prices into the value function of the buyers, we obtain that v is identically equal to 0, and so is its second derivative v''. This, together with the previously computed prices, implies the result.

Next, how do we find the threshold π_m^*? Recall that, once π reaches π_m^* from above, π stops at π_m^*, product b fails, and only product a is offered from then on. Given the pricing strategy of the monopolist characterized in Lemma 2, we know the value function of the monopolist for beliefs below the threshold. Therefore, to find the threshold, we combine a smooth pasting and a value matching condition with the second-order ODE given by the diffusion process derived in Lemma 1. The main takeaway is that the monopolist who sells both products chooses which one to deliver to the market using the same belief threshold as in the welfare-maximizing benchmark, that is, $\pi_m^* = \pi_{\text{fb}}$. As a result, the monopoly achieves efficiency, as summarized by the following proposition.

Proposition 5. *If $\sigma_1 = \sigma_2$, the revenue maximizing equilibrium is specified by the cutoff*

$$\pi_m^* = \pi_{\text{fb}}.$$

The above result is not surprising, because a monopolist with the power to make take-it-or-leave-it offers can extract all the surplus from symmetric buyers. But how robust is this result? Interestingly, we next show that it depends neither on the symmetry nor the ability to price discriminate according to the learning technology of the buyers.

4.1.2 Asymmetric Buyers

The following result shows that, under monopoly, efficiency is still achieved, even when buyers' learning technologies are heterogeneous.

Proposition 6. *If $\sigma_1 > \sigma_2$, the following holds under a monopolistic market structure. (i) There is a revenue-maximizing and efficient equilibrium with $p_{a,1}(\pi) = p_{a,2}(\pi)$ and $p_{b,1}(\pi) = p_{b,2}(\pi)$ for all $\pi \in (0,1)$. (ii) There is no efficient equilibrium in which $p_{a,1}(\pi) = p_{a,2}(\pi) = p_{b,1}(\pi) = p_{b,2}(\pi)$ for all $\pi \in (0,1)$.*

A reasonable concern might be that the above *no distortion result* about the monopolistic market structure with asymmetric buyers may be driven by the fact that the monopolist is allowed to use first-degree price discrimination. Part (i) of Proposition 6 shows this is *not* the case, since the first-best is achieved with a pricing strategy that does not condition on the learning skill of the buyers.

This result highlights a key difference between asymmetries in the learning technologies and asymmetries in the valuation of the new product (i.e., heterogeneous parameters h and ℓ across buyers as in models a la Bergemann and Välimäki (1997)). In the latter case, it is well known that the incentive compatibility of the buyers induces inefficient revenue-maximizing allocations. In our model of asymmetric learning technologies, this does not happen.[10]

The intuition behind the result is as follows. The willingness to pay for product a is the same for both buyers and equal to μ_a. But their willingness to pay for product b at belief π potentially differs: it equals the instant expected flow of utility $\mathbf{E}_\pi[\theta]$ plus the value of learning (i.e., v_i'') multiplied by the amount of information produced by the buyer. Even if $\mathbf{E}_\pi[\theta]$ is common across all the agents, differences in the learning components may create incentive compatibility issues. However, in our proof, we show that the monopolist can always obtain the total surplus by setting the price of the products equal to their expected flow of utility. Indeed, when the monopolist uses this pricing strategy, all the agents have zero value of information (i.e., v'' is constantly zero) and therefore have the same willingness to pay, eliminating any incentive compatibility issue. However, it is an immediate consequence of Proposition 4 that, if every type of price differentiation is banned and the monopolist must use the same price for the two products, a distortion may arise.

Of course, other fairness concerns may arise, since, in a monopolistic market, the inefficiency is eliminated but the entire surplus accrues to the monopolist. Next, we show that if we introduce competition to obtain a more fair surplus division, efficiency is lost.

[10] In a different setting Corrao, Flynn and Sastry (2021) shows a similar irrelevance for incentive compatibility when the agents are heterogenous in their attention costs.

4.2 Competition

In the case of duopolistic competition between the sellers, the value function of seller $k \in \{a,b\}$ is the solution to the following HJB equation:

$$w_k(\pi) = \sup_{p_k} \left\{ \sum_{i=1}^{n} \xi_{ik}(\pi, p_k, p_{-k}(\pi)) p_{k,i} + g(\pi, h, \ell) \sum_{i=1}^{n} \frac{\xi_{ib}(\pi, p_k, p_{-k}(\pi))}{2\rho\sigma_i^2} w_k''(\pi) \right\}. \quad (13)$$

We next show that, in sharp contrast to the monopolistic market structures, symmetric and asymmetric markets will have very different welfare implications. Once again, we consider separately the case of symmetric and asymmetric buyers.

4.2.1 Symmetric buyers

When the buyers have the same learning technologies and the equilibrium is symmetric, the HJB equation of seller k simplifies to:

$$w_k(\pi) = \sup_{p_k} \left\{ \underbrace{p_k \mathrm{Vol}_k(\pi, p_k, p_{-k})}_{\text{flow gain}} + \underbrace{\mathrm{Vol}_b(\pi, p_k, p_{-k}) \frac{g(\pi, h, \ell)}{2\rho\sigma^2} w_k''(\pi)}_{\text{learning gain from product } b} \right\}, \qquad k \in \{a,b\}. \quad (14)$$

The right-hand side of the HJB equation has two terms. The first is the expected flow payoff $p_k \mathrm{Vol}_k$ (given that the volume of seller k's sale is Vol_k), and the second is his continuation payoff that depends on Vol_b (i.e., volume of seller b's sale) via $\mathrm{Vol}_b \frac{g(\pi,h,\ell)}{2\rho\sigma^2} w_k''(\pi)$.

We start with a preliminary caveat on equilibrium multiplicity.

Remark 1 (Equilibrium selection)**.** *Since our focus is on the different efficiency properties of monopoly and competition, we want to consider the competition equilibrium with **minimal departure** from monopoly. As a monopoly is a situation in which the surplus accruing to the (unique) seller is maximal, the minimal departure is obtained by focusing on the equilibrium that maximizes profits of sellers. In what follows, we consider the equilibrium that is **more favorable to the sellers**. Since we are going to highlight the difference between monopoly and competition, our findings will be more surprising the less we depart from the monopoly with our equilibrium selection.*

Next, we characterize the seller's profit-maximizing pricing strategy in a symmetric cutoff equilibrium.

Lemma 3. *In the symmetric cutoff equilibrium with the highest sellers' profits, the prices are as follows.*
If $\pi < \pi^*$*:*

$$p_a(\pi) = \mu_a - \mathbf{E}_\pi[\theta] \qquad \text{and} \qquad p_b(\pi) = 0. \tag{15}$$

If $\pi \geq \pi^*$*:*

$$p_a(\pi) = \frac{g(\pi,h,\ell)}{2\rho\sigma^2} w_a''(\pi) \qquad \text{and} \qquad p_b(\pi) = \mathbf{E}_\pi[\theta] - \mu_a + \frac{g(\pi,h,\ell)}{2\rho\sigma^2}\Big(v''(\pi) + w_a''(\pi)\Big). \tag{16}$$

To understand this pricing rule, let us first focus on the $\pi > \pi^*$ case, where all buyers purchase from seller b. The buyer's HJB equation (9) implies that

$$\mu_a - p_a(\pi) + (n-1)\frac{g(\pi,h,\ell)}{2\rho\sigma^2} v''(\pi) \leq \mathbf{E}_\pi[\theta] - p_b(\pi) + n\frac{g(\pi,h,\ell)}{2\rho\sigma^2} v''(\pi), \tag{17}$$

for $\pi > \pi^*$. At the equilibrium, due to price competition between sellers, (17) holds with equality. Indeed, if the right-hand side was larger, it would be profitable for seller b to slightly increase p_b, collecting higher per unit revenues and selling to the same number of buyers. As a result, we must have

$$\underbrace{p_a(\pi) - p_b(\pi)}_{\text{price difference}} = \underbrace{\mu_a - \mathbf{E}_\pi[\theta]}_{\text{utility difference}} \underbrace{-\frac{g(\pi,h,\ell)}{2\rho\sigma^2} v''(\pi)}_{\text{Information value}}. \tag{18}$$

The difference in prices is the sum between the utility difference and an information value. We will show later that v is concave. Therefore the information value is positive, which widens the price difference and compensates buyer b to use the new product b by lowering its price.

Meanwhile, given all buyers purchase from b when $\pi > \pi^*$, the seller a's HJB equation (13) implies that

$$w_a(\pi) = n\frac{g(\pi,h,\ell)}{2\rho\sigma^2}w_a''(\pi) \geq np_a(\pi), \tag{19}$$

where the inequality means decreasing the price p_a to win over all buyers is suboptimal, compared to letting all buyers purchase from seller b and collecting information gain $n\frac{g(\pi,h,\ell)}{2\rho\sigma^2}w_a''(\pi)$ instead. Similarly, the seller b's HJB equation (13) implies that

$$w_b(\pi) = np_b(\pi) + n\frac{g(\pi,h,\ell)}{2\rho\sigma^2}w_b''(\pi) \geq 0, \tag{20}$$

where the inequality means selling to all buyers is better than the alternative not selling at all.

Combining the indifference condition (18) and the inequalities in (19) and (20), we obtain admissible intervals where equilibrium prices must reside:

$$p_a(\pi) \in \Big[\mu_a - \mathbf{E}_\pi[\theta] - \frac{g(\pi,h,\ell)}{2\rho\sigma^2}\Big(v''(\pi) + w_b''(\pi)\Big),\ \frac{g(\pi,h,\ell)}{2\rho\sigma^2}w_a''(\pi)\Big], \tag{21}$$

$$p_b(\pi) \in \Big[-\frac{g(\pi,h,\ell)}{2\rho\sigma^2}w_b''(\pi),\ \mathbf{E}_\pi[\theta] - \mu_a + \frac{g(\pi,h,\ell)}{2\rho\sigma^2}\Big(v''(\pi) + w_a''(\pi)\Big)\Big]. \tag{22}$$

Choosing p_a and p_b as the high end of their admissible intervals, the profit for seller b is maximized, we obtain the pricing rule in (16).

When $\pi < \pi^*$, an argument similar to those above yields the admissible intervals for prices, similar to (21) and (22) but with their high and low ends switched. Then the pricing rule (15) follows because there is no learning when $\pi < \pi^*$, hence v, w_a, and w_b are all linear functions.

When the belief is below the cutoff π^*, seller a is serving the entire market, and the higher the perceived quality of the new product b, the lower the price seller a can ask. When the belief is higher than the threshold, the price charged by seller b is composed by three terms. First, it depends on the perceived difference in the quality of the products: $\mathbf{E}_\pi[\theta] - \mu_a$. Second, as we will show later, v is concave. Seller b rewards the buyer for purchasing the new product by lowering p_b by $\frac{g(\pi,h,\ell)}{2\rho\sigma^2}v''(\pi)$, which is buyer's exploration cost for using the new product. Third, due competition between sellers, seller b can charge $\frac{g(\pi,h,\ell)}{2\rho\sigma^2} w_a''(\pi)$ additionally due to seller a's pricing, which is the benefit that seller a receives from learning.

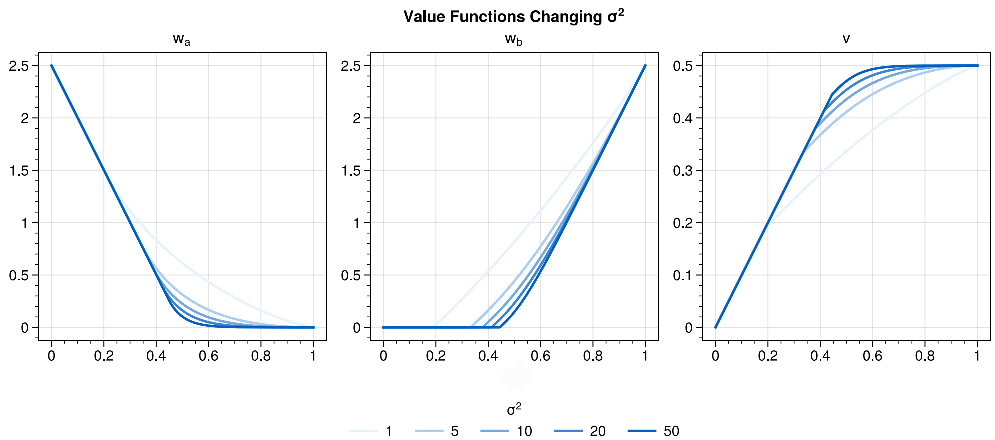
After identifying the equilibrium pricing rules, we derive the equilibrium cutoff π^* by combining the HJB equation (13) with smooth-pasting and value-matching conditions at π^* for the value function of seller a. We show that $\pi^* = \pi_{\text{fb}}$. As a consequence, efficiency is obtained under competition when buyers are symmetric. The following proposition summarizes.

Proposition 7. *The following holds. (i) The cutoff equilibrium with the highest sellers' profits is efficient (i.e., welfare-maximizing) and has a unique cutoff π^*. (ii) The consumer surplus is strictly higher than under monopoly. (iii) The value function of the two sellers is convex, and the value function of the buyers is concave.*

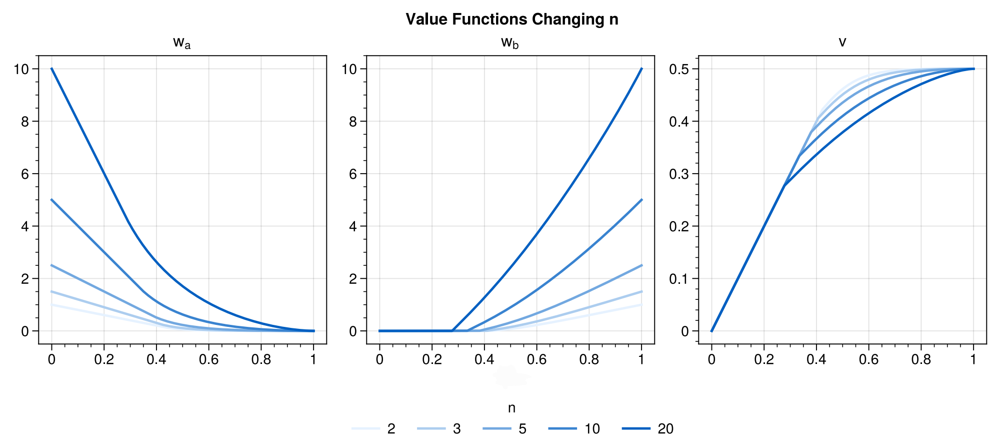
It is important to understand the economic intuition behind the result. Initially, it does not seem surprising. Standard reasoning from static markets tell us that, since the buyers are symmetric, the sellers have no reason to price discriminate. In static markets, it is well known that the absence of incentives to price discriminate (or the possibility for the seller to perfectly discriminate) is sufficient to guarantee that the market power of the seller does not induce inefficiencies. One may think that the same is happening here. Our next result below shows that this is not so: if buyers are asymmetric, the possibility to price discriminate does not amend inefficiencies. Indeed, in markets with learning externalities, efficiency is obtained only if the seller also can internalize the learning externality of the other market participants. The proof of Proposition 7 shows that, when buyers are symmetric, this is the case. However, when buyers are asymmetric in the learning technologies, the next section shows that the externality is not internalized, resulting in inefficiency.

4.2.2 Asymmetric buyers

First, we present the result that the decentralized outcome induced by competition is no longer efficient with asymmetric buyers, and we then explore the nature of the inefficiency.



(a) This figure plots the value functions w_a, w_b and v when σ^2 changes. As σ^2 increases, the cutoff π^* moves to the right.



(b) This figure plots the value functions w_a, w_b and v when n changes. As n increases, the cutoff π^* moves to the left.

Figure 3: Competition equilibrium with symmetric buyers. Other parameters $n = 5, h = 1, \ell = 0, \mu_a = .5$ (Explicit characterizations of the value functions are provided in the proof of Proposition 7.)

Proposition 8. *If $\sigma_1 > \sigma_2$, the equilibrium with the highest sellers' profit is inefficient. However, there is efficiency for the top learners:*

$$\pi_2^* = \pi_{\text{fb},2}.$$

To understand the intuition of this result, we start from the buyers' problem. The HJB equation for the buyers is almost the same as in the symmetric case, with the only difference that the learning component involved in the trade-off between the two products is now buyer-specific:

$$v_i(\pi) = \max\left\{\mu_a - p_{a,i}(\pi) + g(\pi,h,\ell)\sum_{j\neq i}\frac{\xi_{jb}(\pi,p(\pi))}{2\rho\sigma_j^2}v_i''(\pi),\right.$$
$$\left.\mathbf{E}_\pi[\mu_b] - p_{b,i}(\pi) + g(\pi,h,\ell)\sum_{j=1}^{2}\frac{\xi_{jb}(\pi,p(\pi))}{2\rho\sigma_j^2}v_i''(\pi)\right\}.$$

Price competition between sellers imposes indifference between the arguments of the above maximization. Indeed, in the right-hand side of the above HJB, if the second argument were larger, a profitable deviation for seller b would be to slightly increase $p_{b,i}$, collecting higher per-unit revenues and selling to the same number of buyers. An analogous profitable deviation would obtain for seller a if the first argument were strictly larger than the second. Therefore:

$$p_{b,1}(\pi) - p_{a,1}(\pi) = \mathbf{E}_\pi\left[\theta\right] - \mu_a + \frac{1}{2\rho\sigma_1^2}g(\pi,h,\ell)v_1''(\pi), \tag{23}$$

for buyer 1 and

$$p_{b,2}(\pi) - p_{a,2}(\pi) = \mathbf{E}_\pi\left[\theta\right] - \mu_a + \frac{1}{2\rho\sigma_2^2}g(\pi,h,\ell)v_2''(\pi), \tag{24}$$

for buyer 2. As we will show later, both v_1 and v_2 are concave. Thus equations (23) and (24) together give us the rebate that seller b awards the buyers to compensate them for their exploration.

Plugging the indifference conditions of the buyers into the HJB equations of sellers, we show that seller a's and b's problems can be characterized by the following HJB equations.

$$\begin{aligned} w_a(\pi) = &\max\left\{\mu_a - \mathbf{E}_\pi[\theta] - \frac{1}{2\rho\sigma_2^2}g(\pi,h,\ell)\left(v_2''(\pi) + w_b''(\pi)\right), \frac{1}{2\rho\sigma_2^2}g(\pi,h,\ell)w_a''(\pi)\right\} \\ &+ \max\left\{\mu_a - \mathbf{E}_\pi[\theta] - \frac{1}{2\rho\sigma_1^2}g(\pi,h,\ell)\left(v_1''(\pi) + w_b''(\pi)\right), \frac{1}{2\rho\sigma_1^2}g(\pi,h,\ell)w_a''(\pi)\right\}. \end{aligned} \tag{25}$$

$$w_b(\pi) = \max\left\{\mathbf{E}_\pi[\theta] - \mu_a + \frac{1}{2\rho\sigma_2^2}g(\pi,h,\ell)\big(v_2''(\pi) + w_a''(\pi)\big) + \frac{1}{2\rho\sigma_2^2}g(\pi,h,\ell)w_b''(\pi), 0\right\}$$
$$+ \max\left\{\mathbf{E}_\pi[\theta] - \mu_a + \frac{1}{2\rho\sigma_1^2}g(\pi,h,\ell)\big(v_1''(\pi) + w_a''(\pi)\big) + \frac{1}{2\rho\sigma_1^2}g(\pi,h,\ell)w_b''(\pi), 0\right\}. \quad (26)$$

In each maximization of (25) and(26), when the first term is larger than the second, then a product is sold to a corresponding agent. For example, when $\mu_a - \mathbf{E}_\pi[\theta] - \frac{1}{2\rho\sigma_2^2}g(v_2'' + w_b'') > \frac{1}{2\rho\sigma_2^2}gw_a''$, it is better for the seller to sell the product a to the buyer 2 with the price $\mu_a - \mathbf{E}_\pi[\theta] - \frac{1}{2\rho\sigma_2^2}g(v_2'' + w_b'')$ than let b sell the product b to the buyer 2, in which case, the seller a's value is $\frac{1}{2\rho\sigma_2^2}gw_a''$.

We conjecture that the cutoff π_2^* is pinned down by

$$\mathbf{E}_{\pi^*}[\theta] - \mu_a + \frac{1}{2\rho\sigma_2^2}g(\pi_2^*,h,\ell)\big(v_2''(\pi_2^*) + w_a''(\pi_2^*)\big) + \frac{1}{2\rho\sigma_2^2}g(\pi_2^*,h,\ell)w_b''(\pi_2^*) = 0, \quad (27)$$

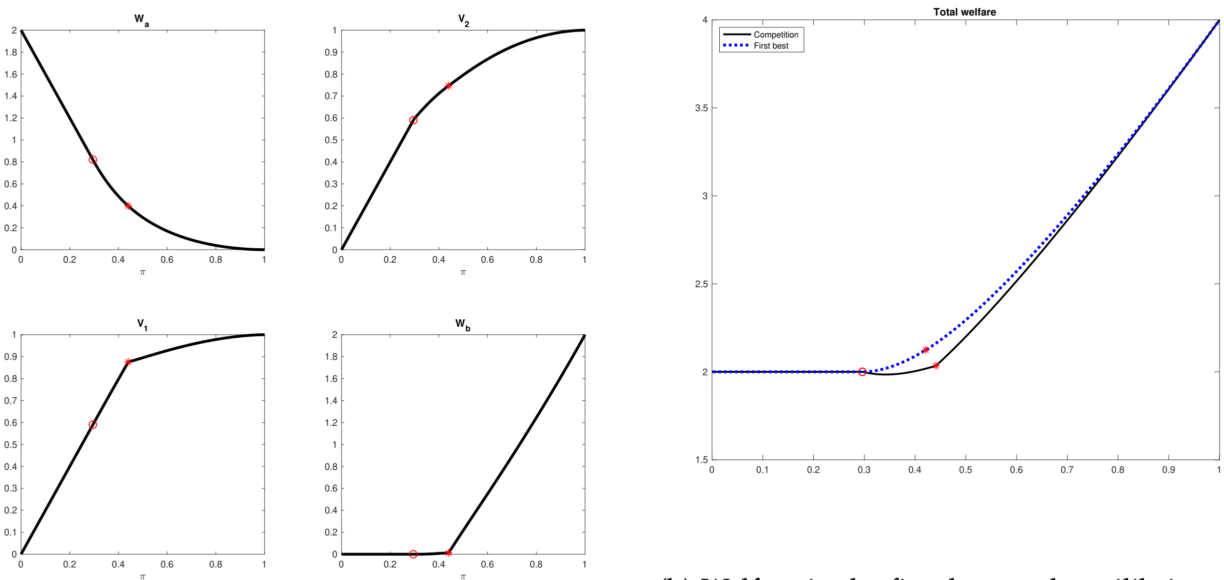
where both seller a and b are indifferent to sell to buyer 2. Meanwhile π_1^* is pinned down by

$$\mathbf{E}_{\pi^*}[\theta] - \mu_a + \frac{1}{2\rho\sigma_1^2}g(\pi_1^*,h,\ell)\big(v_1''(\pi_1^*) + w_a''(\pi_1^*)\big) + \frac{1}{2\rho\sigma_1^2}g(\pi_1^*,h,\ell)w_b''(\pi_1^*) = 0, \quad (28)$$

where both seller a and b are indifferent to sell to buyer 1.

When $\pi < \pi_2^*$, both the left-hand side (LHS) of (27) < 0 and the LHS of (28) < 0. Seller a is willing to sell to both buyers, and seller b stays out of the market. When $\pi \in (\pi_2^*, \pi_1^*)$, the LHS of (27) > 0 and the LHS of (28) < 0. Seller a is only willing to sell to buyer 1, and seller b is only willing to sell to buyer 2. When $\pi > \pi_1^*$, both the LHS of (27) > 0 and the LHS of (28) > 0. Seller b is willing to sell to both buyers.

Figure 4 presents a numeric example. Panel (a) presents value function w_a, w_b, v_1, and v_2. The plots show that w_a and w_b are convex and v_1 and v_2 are concave. w_b is strictly positive in the region (π_2^*, π_1^*), even though its value is close to zero in this example. Panel (b) compares the welfare between the first-best case and the equilibrium with competition and asymmetric buyers. We see that the first-best welfare is higher than that in equilibrium. Moreover, $\pi_{\text{fb},2} = \pi_2^*$, but $\pi_{\text{fb},1} < \pi_1^*$. In this example, the conjecture in (27) and (28) are verified in Figure 5. The same qualitative results holds with different parameter values.



(a) Value functions w_a, w_b, v_1, and v_2. Horizontal coordinate of the red circle is π_2^*, horizontal of the red asterisk is π_1^*.

(b) Welfare in the first-best and equilibrium with competition and asymmetric buyers. Horizontal coordinate of the red circle is $\pi_{\text{fb},2} = \pi_2^*$. Horizontal coordinate of the bottom red asterisk is π_1^*. Horizontal coordinate of the top red asterisk is $\pi_{\text{fb},1}$.

Figure 4: Equilibrium with competition and asymmetric buyers. Parameters are $h = 2$, $\ell = 0$, $\mu_a = 1$, $\sigma_1 = 4$, $\sigma_2 = 3$, and $\rho = 0.5$.

The inefficiency at π_1^* (with respect to $\pi_{\text{fb},1}$) follows from not considering the learning externality that consumption by the worst learner induces over the best learner. Indeed, when seller b serves a buyer with the bad learning technology, an informational impact is produced for every market participant. The profit maximizing price-setting by seller b internalizes the learning impact for him and the buyer, and competition incorporates the learning externality of seller a into the price. However, the learning externality for the best learner is not internalized.

This can also be understood from equation (28). This equation represents the indif-

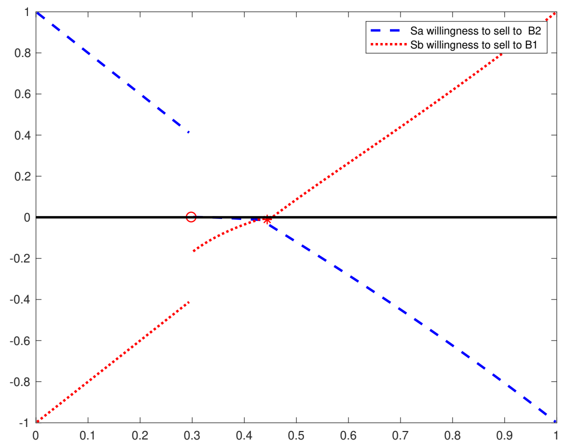


Figure 5: Seller indifference conditions. The blue dashed line plots the negative of the left-hand side of (27). When it is positive, seller a is willing to sell to buyer 2. The red dotted line represents the left-hand side of (28). When it is positive, seller b is willing to sell to buyer 1. These two lines cross zero at π_2^* and π_1^* respectively, confirming numerically (27) and (28).

ference condition for sellers a and b to sell to buyer 1. It does not incorporate information from buyer 2, who is already using the new product. This pricing between sellers a, b, and buyer 1 creates an externality to buyer 2. Indeed, under the first-best maximization, the welfare maximizing condition imposes

$$\mathbf{E}_{\pi_{\text{fb},1}}[\theta] - \mu_a + \frac{1}{2\rho\sigma_1^2} g(\pi_{\text{fb},1}, h, \ell)\left(w_a'' + w_b'' + v_1'' + v_2''\right)(\pi_{\text{fb},1}) = 0.$$

However, our numeric example in Figure 4 shows that $\lim_{\pi\uparrow\pi_1^*} v_2''(\pi) < 0$. Combined with (28) and the previous equation, we conclude that $\pi_1^* \neq \pi_{\text{fb},1}$.

Meanwhile, for the lower threshold, the welfare maximizing condition in the first-best problem implies

$$\mathbf{E}_{\pi_{\text{fb},2}}[\theta] - \mu_a + \frac{1}{2\rho\sigma_2^2} g(\pi_{\text{fb},2}, h, \ell)\left(w_a'' + w_b'' + v_1'' + v_2''\right)(\pi_{\text{fb},2}) = 0.$$

We prove in Proposition 8 that $v_1''(\pi_2^*) = 0$. Combined with (27) and the previous equation, we conclude that $\pi_2^* = \pi_{\text{fb},2}$.

The externality issue does not arise in the symmetric case, because, at the unique threshold (the same for the smallest threshold π_2^* here), no buyer has a strictly positive value of information i.e., $v''(\pi_2^*) = 0$. Indeed, around the unique threshold in the homogeneous case (and the lowest threshold of the heterogeneous one), the market agents are almost certain to receive a string of bad signals that eliminates experimentation with good b, making useless the information previously generated.

The efficiency for the top learners' part of the result highlights the way in which distortions arise. Competition does not affect the number and quality of the innovations that are given a chance, that is, the threshold for being tested by the fraction of the market that produces better (more precise) information about the product (i.e., to start the beta phase). However, what is affected is the confidence in the quality of the product that is required to start to serve the entire market.

In the final part of this section, we note that the extent of inefficiency is not monotone in the difference between the learning technologies of buyers. Proposition 7 already guarantees that efficiency holds in the case of symmetric buyers, and the next result shows that the distortion disappears also in the limit, where one of the two buyers does not produce any valuable information (through his experience) about the product's quality.

Proposition 9. *Fix σ_2. The equilibrium with a buyer that does not generate any information about the quality of the product is efficient:*

$$\lim_{\sigma_1 \to \infty} (\pi_1 - \pi_{\text{fb},1}) = 0$$

The intuition behind the previous result is that, if serving the general public does not produce additional information about the product, there is no learning externality to consider when deciding whether the product is ready for the entire market.

Finally, note that the nonmonotonicity in the difference in learning technologies implied by the previous proposition is a robust feature of the model, and it continues to arise even with multiple levels of learning technologies. However, what is lost in the more general case is the stark conclusion that, if the worst learners become completely uninformative, then efficiency is fully restored. Indeed, as long as there are two types of learners with variance of the signal strictly between zero and plus infinity, competition induces some inefficiency.

5 Discussion

In this section, we first show that inefficiency is restored if the seller can offer multilateral contracts. Then we explore the determinants and comparative statics of the transition period, beta phase, that it is peculiar to our model with asymmetries in the learning technologies.

5.1 Multilateral Contracts

Even though competition introduces inefficiency to the market with asymmetric buyers, there is a way, however, to eliminate distortion and maintain competition. This is achieved by increasing the commitment power of the sellers. More precisely, suppose that now seller $k \in \{a, b\}$ can commit to offering a multilateral contract of the following form: one-unit of product k will be delivered to buyer $i \in \{1, 2\}$ if buyer i makes a transfer $t^i_{k,i}$ to seller k and buyer $j \neq i$ makes a transfer $t^j_{k,i}$ to seller k.

That is, we allow the seller to ask a buyer to pay or to be compensated for the fact that the product is delivered to another consumer. We notice that the buyer who does not receive the product may benefit from the transfer because of the learning externality that is generated by the use of product b. More precisely, we set

$$t^2_{a,1}(\pi) = -v_2''(\pi)\frac{g(\pi,h,\ell)}{2\rho\sigma_1^2}, \qquad t^1_{a,2}(\pi) = -v_1''(\pi)\frac{g(\pi,h,\ell)}{2\rho\sigma_2^2},$$
$$t^2_{b,1}(\pi) = v_2''(\pi)\frac{g(\pi,h,\ell)}{2\rho\sigma_1^2}, \qquad t^1_{b,2}(\pi) = v_1''(\pi)\frac{g(\pi,h,\ell)}{2\rho\sigma_2^2}.$$

Because v_1 and v_2 are concave, when seller b sells to buyer 1, he also compensates buyer 2 the amount $-v_2''(\pi)\frac{g(\pi,h,\ell)}{2\rho\sigma_1^2}$, which is the externality cost to buyer 2. When the seller a sells to buyer 1 instead, buyer 2 transfers to seller a the amount $-v_2''(\pi)\frac{g(\pi,h,\ell)}{2\rho\sigma_1^2}$ which is the externality cost he would shoulder if seller b sells to buyer 1 instead.

Although there are service markets in which a similar structure may be implemented in the form of a subscription to a platform that shares buyers' experiences, we think that, in most cases, assuming such commitment power is unwarranted. Still, if we allow for this possibility, the competition outcome becomes efficient.

Proposition 10. *When sellers are allowed to use multilateral contracts, the equilibrium is efficient.*

5.2 Beta phase

We consider two given thresholds, $\pi_1 > \pi_2$. Therefore insights in this section apply both to the case of competition and the first-best.

In the beta phase, only the customers with better learning technologies buy the new product. When the public belief becomes sufficiently optimistic (i.e., $\pi > \pi_1$), all buyers purchase the new product; when the public belief is sufficiently pessimistic (i.e., π reaches π_2, π stays there forever), no buyers purchase the new product in the future and it fails. We explicitly characterize the expected length of the beta phase and the probability of new product failure or full adoption in the following proposition. Let $\tau = \inf\{t : \pi_t \notin (\pi_2, \pi_1)\}$ denote the time needed to exit the beta phase.

Proposition 11. *Let $\pi_2 < \pi_0 < \pi_1$. Define $\sigma(y) := \frac{y(1-y)(h-\ell)}{\sigma_2}$. Then,*

$$\mathbf{E}_{\pi_0}[\tau] = \frac{\pi_1 - \pi_0}{\pi_1 - \pi_2} \int_{\pi_2}^{\pi_0} (y - \pi_2) \frac{2dy}{\sigma^2(y)} + \frac{\pi_0 - \pi_2}{\pi_1 - \pi_2} \int_{\pi_0}^{\pi_1} (\pi_1 - y) \frac{2dy}{\sigma^2(y)}.$$

Particularly, the expected length of the beta phase is strictly increasing (decreasing) in the initial opinion for sufficiently low (high) initial opinions π_0.
Moreover, the following hold.

- *The probability of discarding the new product as a failure is $Pr_{\pi_0}\{\text{discarding}\} = Pr_{\pi_0}\{\pi(\tau) = \pi_2\} = \frac{\pi_1 - \pi_0}{\pi_1 - \pi_2}$.*
- *The probability that the new product serves the whole market is $Pr_{\pi_0}\{full\,adoption\} = Pr_{\pi_0}\{\pi(\tau) = \pi_1\} = \frac{\pi_0 - \pi_2}{\pi_1 - \pi_2}$.*

Particularly, $\partial_{\pi_0} Pr_{\pi_0}\{\text{discarding}\} < 0$, and $\partial_{\pi_0} Pr_{\pi_0}\{full\,adoption\} > 0$.

The above result explicitly characterizes the expected length of the beta phase and the probabilities that the new product either serves the whole market or is discarded as a failure in terms of the endogenous thresholds π_1 and π_2. Moreover, it produces intuitive comparative statics based on the initial market belief about the new product. In particular, the expected length of the beta phase increases with the initial market belief when this belief is initially sufficiently small (i.e., $\lim_{\pi_0 \searrow \pi_2} \partial_{\pi_0} \mathbf{E}_{\pi_0}[\text{beta phase}] > 0$), and the expected length of the beta phase decreases with the initial market belief when this belief is initially sufficiently large (i.e., $\lim_{\pi_0 \searrow \pi_2} \partial_{\pi_0} \mathbf{E}_{\pi_0}[\text{beta phase}] > 0$). Moreover, the probability of discarding the new product as a failure decreases in the initial market belief and the probability that the new product starts to serve the whole market increases in the initial market belief.

6 Conclusion

We study the interaction between the market structure (monopoly versus oligopoly) and asymmetry in learning technologies in a dynamic product market. In this market, a new product of unknown quality competes against an established one. Public information on the unknown quality evolves dynamically due to Bayesian learning. We establish that the optimal policy is characterized by a sequence of belief thresholds, with a beta phase in which only best learners explore the new product.

We consider the efficiency implications of different market structures. Under a monopolistic market structure in which the same seller sells the new and old products, the equilibrium is always efficient. In sharp contrast, however, if two sellers compete in marketing their own products, efficiency is achieved only if the buyers are symmetric in their learning technologies.

We identify the inefficiency as a learning externality that consumption of the product by one buyer generates for the other buyers. The equilibrium inefficiency has two features: (i) efficiency for the top learners, (i.e., the threshold for starting to serve the best learners, that is, to enter into a Beta phase, remains the efficient one); (ii) nonmonotonicity (i.e., distortions are not monotone in the extent of the asymmetry). We show that, in markets in which the sellers can use multilateral contracts, the distortion disappears. We also analyze the learning progression in the market belief and the expected time of the new product failure or full adoption to the entire market.

7 Appendix

Before proving Proposition 4, we first present a verification result, proved in the Online Appendix. For notational convenience, we set $\pi_{\text{fb},0} = 1$, $\pi_{\text{fb},3} = 0$, $\sigma_0^2 = \infty$, and $\sigma_3^2 = 0$.

Proposition 12. *Suppose that there exist $\pi_{\text{fb},1}, \pi_{\text{fb},2} \in (0,1)$, $\pi_{\text{fb},1} > \pi_{\text{fb},1}$ and a bounded convex function W_{avg} which satisfies the following conditions:*

(i) W_{avg} is twice continuously differentiable in $(0, \pi_{\text{fb},2})$, $(\pi_{\text{fb},2}, \pi_{\text{fb},1})$, $(\pi_{\text{fb},1}, 1)$, and for $i \in \{1,2\}$ satisfies

$$W_{avg}(\pi) = \mu_a + \frac{1}{2}\sum_{j=i+1}^{2}\left(\mathbf{E}_\pi[\theta] - \mu_a\right) + \frac{1}{2\rho} g(\pi, h, \ell) W''_{avg}(\pi) \sum_{j=i+1}^{2} \frac{1}{\sigma_j^2}, \quad \pi \in (\pi_{\text{fb},i+1}, \pi_{\text{fb},i}); \tag{29}$$

(ii) W_{avg} satisfy value matching and smooth pasting conditions

$$\lim_{\pi \uparrow \pi_{\text{fb},i}} W_{avg}(\pi) = \lim_{\pi \downarrow \pi_{\text{fb},i}} W_{avg}(\pi), \tag{30}$$

$$\lim_{\pi \uparrow \pi_{\text{fb},i}} W'_{avg}(\pi) = \lim_{\pi \downarrow \pi_{\text{fb},i}} W'_{avg}(\pi), \quad \text{for } i \in \{1,2\}. \tag{31}$$

(iii) $\frac{1}{\rho\sigma_{i+1}^2} g(\pi, h, \ell) W''_{avg}(\pi) > \mu_a - \mathbf{E}_\pi[\theta] > \frac{1}{\rho\sigma_i^2} g(\pi, h, \ell) W''_{avg}(\pi)$, for any $\pi \in (\pi_{\text{fb},i+1}, \pi_{\text{fb},i})$ and $i \in \{1,2\}$.

Then, W_{avg} is the first best value function and the first best optimal strategy is given in (8).

We now construct a function W_{avg} satisfying conditions in Proposition 12 to prove Proposition 4.

Proof of Proposition 4. First, we consider (29) in the three regions created by the posited cutoffs and solve the individually using the Wronskian approach of second order ODEs (Zaitsev and Polyanin (2002)).

- Case 1: If $\pi \in (\pi_{\text{fb},2}, \pi_{\text{fb},1})$ then

$$W_{avg,1}(\pi) = \frac{1}{2}(\mu_a + \mathbf{E}[\mu_b]) + \zeta_1 \pi^{\frac{1}{2}(\lambda+1)}(1-\pi)^{-\frac{1}{2}(\lambda-1)} + \zeta_2 \pi^{-\frac{1}{2}(\lambda-1)}(1-\pi)^{\frac{1}{2}(\lambda+1)}$$

with $\lambda = \sqrt{1 + \frac{8\rho\sigma_2^2}{(h-\ell)^2}}$.

- Case 2: If $\pi \geq \pi_{\text{fb},1}$, the general solution has the form

$$W_{avg,0}(\pi) = \mathbf{E}_\pi[\theta] + \zeta^* \pi^{\frac{1}{2}(\lambda+1)}(1-\pi)^{-\frac{1}{2}(\lambda-1)} + \zeta_0 \pi^{-\frac{1}{2}(\bar{\lambda}-1)}(1-\pi)^{\frac{1}{2}(\bar{\lambda}+1)}.$$

However, note that W_{avg} is bounded by h on $\pi \geq \pi_{\text{fb},1}$, thus $\zeta^* = 0$ as $\lim_{\pi\to 1}(1-\pi)^{\frac{1}{2}(1-\lambda)}$ explodes in this region: $W_{avg,0}(\pi) = \mathbf{E}_\pi[\theta] + \zeta_0 \pi^{-\frac{1}{2}(\bar{\lambda}-1)}(1-\pi)^{\frac{1}{2}(\bar{\lambda}+1)}$ where $\bar{\lambda} = \sqrt{1 + \frac{8\rho\sigma_1^2\sigma_2^2}{(h-\ell)^2(\sigma_1^2+\sigma_2^2)}}$.

- Case 3: If $\pi < \pi_{\text{fb},2}$, we trivially have $W_{avg}(\pi) = \mu_a$.

There are five unknowns $\zeta_0, \zeta_1, \zeta_2, \pi_{\text{fb},1}, \pi_{\text{fb},2}$ that we need to identify W_{avg} on $(0,1)$. Therefore, we are going to use five conditions. We already have four conditions in (30) and (31). The fifth equation will be a super-contact condition. Next claim, proved in the Online Appendix, establishes its necessity.

Claim 1. *The value matching and Proposition 12 (iii) imply that*

$$W''_{avg,1}(\pi_{\text{fb},1}) = W''_{avg,0}(\pi_{\text{fb},1}). \tag{32}$$

With these five conditions, we pin down the five unknowns, hence W_{avg}, which is identified as $W_{avg,0}, W_{avg,1}$, or μ_a on corresponding intervals, satisfies Proposition 12 (i) and (ii). The convexity of W_{avg} is verified numerically.

To show that W_{avg} constructed satisfies Proposition 12 (iii), we first note that (32) together with (72) and (73) later implies that $\mathbf{E}_{\pi_{\text{fb},1}}[\theta] - \mu_a + \frac{1}{\rho\sigma_1^2} g(\pi_{\text{fb},1}, h, \ell) W''_{avg}(\pi_{\text{fb},1}) = 0$. Moreover, from the value matching at $\pi_{\text{fb},2}$, we have $\mathbf{E}_{\pi_{\text{fb},2}}[\theta] - \mu_a + \frac{1}{\rho\sigma_2^2} g(\pi_{\text{fb},2}, h, \ell) W''_{avg}(\pi_{\text{fb},2}+) = 0$, where $W''_{avg}(\pi_{\text{fb},2}+) = \lim_{\pi \downarrow \pi_{\text{fb},2}} W''_{avg}(\pi)$. The convexity of W_{avg} and the two equations above imply that $\pi_{\text{fb},1}, \pi_{\text{fb},2} < \pi_{\text{myoptic}}$. Therefore Proposition 12 (iii) is satisfied in $(0, \pi_{\text{fb},2})$, because $W''_{avg} = 0$ there.

We claim that $\mathbf{E}_\pi[\theta] - \mu_a + \frac{1}{\rho\sigma_2^2} g(\pi,h,\ell) W''_{avg}(\pi) > 0$ when $\pi \in (\pi_{\text{fb},2}, \pi_{\text{fb},1})$. If not, there exists $\tilde{\pi} \in (\pi_{\text{fb},2}, \pi_{\text{fb},1})$ such that $\mathbf{E}_{\tilde{\pi}}[\theta] - \mu_a + \frac{1}{\rho\sigma_2^2} g(\tilde{\pi},h,\ell) W''_{avg}(\tilde{\pi}) \le 0$. By the equation satisfied by W_{avg} in $(\pi_{\text{fb},2}, \pi_{\text{fb},1})$ (see (29)), we have $W_{avg}(\tilde{\pi}) \le \mu_a$. This contradicts with the fact that W_{avg} is an increasing function in $(\pi_{\text{fb},2}, \pi_{\text{fb},1})$, because $W_{avg}(\pi_{\text{fb},2}) = \mu_a$, $W'_{avg}(\pi_{\text{fb},2}) = 0$, and W_{avg} is convex.

We claim that $\mathbf{E}_\pi[\theta] - \mu_a + \frac{1}{\rho\sigma_1^2} g(\pi,h,\ell) W''_{avg}(\pi) < 0$ when $\pi \in (\pi_{\text{fb},2}, \pi_{\text{fb},1})$. If not, there exists $\tilde{\pi} \in (\pi_{\text{fb},2}, \pi_{\text{fb},1})$ such that $\mathbf{E}_{\tilde{\pi}}[\theta] - \mu_a + \frac{1}{\rho\sigma_1^2} g(\tilde{\pi},h,\ell) W''_{avg}(\tilde{\pi}) = 0$. Then W_{avg} would satisfy its equation in $(\pi_{\text{fb},2}, 1)$. This contradicts with the assumption that $\tilde{\pi} < \pi_{\text{fb},1}$. The inequality $\mathbf{E}_\pi[\theta] - \mu_a + \frac{1}{\rho\sigma_1^2} g(\pi,h,\ell) W''_{avg}(\pi) > 0$ when $\pi \in (\pi_{\text{fb},1}, 1)$ can be proved similarly.

This concludes the proof of Proposition 4. □

Proof of Proposition 7. For $\pi > \pi^*$, it follows from (14) that

$$w_a(\pi) = n \frac{g(\pi,h,\ell)}{2\rho\sigma^2} w''_a(\pi). \tag{33}$$

Using Wronskian approach of second order ODEs (Zaitsev and Polyanin (2002)) we have $w_a(\pi) = \zeta_1 \left[\pi^{\frac{1}{2}(1-\lambda_n)} (1-\pi)^{\frac{1}{2}(1+\lambda_n)} \right] + \zeta_2 \left[\pi^{\frac{1}{2}(1+\lambda_n)} (1-\pi)^{\frac{1}{2}(1-\lambda_n)} \right]$, where $\lambda_n = \sqrt{1 + 8\frac{\sigma^2 \rho}{n(h-\ell)^2}}$. Because w_a is bounded, $\zeta_2 = 0$, otherwise the second term on the right-hand side above explodes as $\pi \to 1$. Hence,

$$w_a(\pi) = \zeta_1 \left[\pi^{\frac{1}{2}(1-\lambda_n)} (1-\pi)^{\frac{1}{2}(1+\lambda_n)} \right]. \tag{34}$$

Claim 2. *Given p_a as in Lemma 3, the equilibrium in which sellers' value is maximized is pinned down by the value matching $\lim_{\pi \uparrow \pi^*} w_a(\pi) = \lim_{\pi \downarrow \pi^*} w_a(\pi)$, and the smooth pasting condition $\lim_{\pi \uparrow \pi^*} w'_a(\pi) = \lim_{\pi \downarrow \pi^*} w'_a(\pi)$ for some threshold level π^*.*

For the rest proof, we use the value matching and smooth pasting condition to pin down π^* and functions v, w_a, and w_b. First, the value matching and smooth pasting for seller a at π^* give

$$w_a(\pi^*) = n\Big(\mu_a - \mathbf{E}_{\pi^*}[\theta]\Big), \tag{35}$$

$$w_a'(\pi^*) = n\,\frac{\partial}{\partial \pi}\Big(\mu_a - \mathbf{E}_{\pi}[\theta]\Big)|_{\pi=\pi^*} = n(\ell - h). \tag{36}$$

Given (34), and the previous two equations, we have

$$\frac{1}{2}\left(\frac{1-\lambda_n}{\pi^*} - \frac{1+\lambda_n}{1-\pi^*}\right) = \frac{w_a'(\pi^*)}{w_a(\pi^*)} = \frac{\ell - h}{\mu_a - \mathbf{E}_{\pi^*}[\theta]}. \tag{37}$$

Substituting $\mathbf{E}_{\pi^*}[\theta] = \pi^* h + (1-\pi^*)\ell$ and then solving (37) with respect to π^* implies $\pi^* = \frac{\mu_a - \ell + (\ell - \mu_a)\lambda_n}{2\mu_a - (\ell + h) + (\ell - h)\lambda_n}$. Comparing to (6), we conclude $\pi^* = \pi_{\text{fb}}$, finishing the proof of (ii).

Moreover, we obtain from (35) that $\zeta_1 = n\frac{2(h-\mu_a)}{\lambda_n - 1}\left(\frac{\pi^*}{1-\pi^*}\right)^{\frac{1}{2}(1+\lambda_n)}$. As a result, $w_a''(\pi) = nW_{avg}''(\pi)$ for $\pi > \pi^*$(see Proposition 3), proving the convexity of w_a.

To simplify the notation, we denote $g(\pi) = g(\pi, h, \ell)$ in the rest of the proof. It follows from (33) and (34) that $w_a'(\pi) = \left(\frac{1-\lambda_n}{2\pi} - \frac{1+\lambda_n}{2(1-\pi)}\right)w_a(\pi)$ and $w_a''(\pi) = \frac{2\sigma^2\rho}{ng(\pi)}w_a(\pi)$. Combining (9) and (16), we obtain that v satisfies

$$\begin{aligned} v(\pi) = \mu_a + \frac{g(\pi)}{2\rho\sigma^2}\big((n-1)v''(\pi) - w_a''(\pi)\big) &= \mu_a + (n-1)\frac{g(\pi)}{2\rho\sigma^2}v''(\pi) - \frac{w_a(\pi)}{n} \\ &= \mu_a + (n-1)\frac{g(\pi)}{2\rho\sigma^2}v''(\pi) - \frac{\zeta_1}{n}\pi^{\frac{1-\lambda_n}{2}}(1-\pi)^{\frac{1+\lambda_n}{2}} \end{aligned} \tag{38}$$

with initial condition $v(\pi^*) = \mathbf{E}_{\pi^*}[\theta] = h\pi^* + l(1-\pi^*)$ due to the value matching at π^*.

Next, we identify a unique bounded function v satisfying the previous differential equation and the initial condition. To this end, observe that the function $f(\pi) = \pi^{\frac{1-\lambda_n}{2}}(1-\pi)^{\frac{1+\lambda_n}{2}}$ obeys the differential equation $f''(\pi) = \frac{\lambda_n^2 - 1}{4}f(\pi) = \frac{2\rho\sigma^2}{n(h-\ell)^2}f(\pi)$. Therefore, define the function $\tilde{v}(\pi) = v(\pi) - \mu_a - A_1 f(\pi)$ for some constant A_1. Plug $\tilde{v}$ into (38) and using the previous equation, we obtain $A_1 = \frac{n-1}{g}\frac{g}{n}A_1 - \frac{\zeta_1}{n}$ so that $A_1 = -\zeta_1 = -\frac{2n(h-\mu_a)}{\lambda_n - 1}\left(\frac{\pi^*}{1-\pi^*}\right)^{\frac{1+\lambda_n}{2}}$, and moreover, $\tilde{v}$ satisfies the equation $\tilde{v}(\pi) = (n-1)\frac{g(\pi)}{2\rho\sigma^2}\tilde{v}''(\pi)$.

Denote $\lambda_{n-1} = \sqrt{1 + 8\frac{\sigma^2 \rho}{(n-1)(h-\ell)^2}}$. Then the solution space for the previous differential equation is parametrized by two constants A_2 and C_2, $\tilde{v}(\pi) = A_2 \pi^{\frac{1-\lambda_{n-1}}{2}}(1-\pi)^{\frac{1+\lambda_{n-1}}{2}} + C_2 \pi^{\frac{1+\lambda_{n-1}}{2}}(1-\pi)^{\frac{1-\lambda_{n-1}}{2}}$. The boundedness of $\tilde{v}$ and λ_{n-1} implies $C_2 = 0$.

Putting everything together, and using $w_a(\pi) = \zeta_1 f(\pi)$, v can be expressed as

$$v(\pi) = \mu_a - w_a(\pi) + A_2 \pi^{\frac{1-\lambda_{n-1}}{2}}(1-\pi)^{\frac{1+\lambda_{n-1}}{2}} \tag{39}$$

where A_2 is chosen to satisfy the initial condition $v(\pi^*) = \pi^* h + (1-\pi^*)l$. Recall that $w_a(\pi^*) = n(\mu_a - (\pi^* h + (1-\pi^*)l) = n(\mu_a - v(\pi^*))$. Therefore, $v(\pi^*) = \mu_a - n\mu_a + nv(\pi^*) + A_2(\pi^*)^{\frac{1-\lambda_{n-1}}{2}}(1-\pi^*)^{\frac{1+\lambda_{n-1}}{2}}$ implies $A_2 = \frac{(n-1)(\mu_a - v(\pi^*))}{(\pi^*)^{\frac{1-\lambda_{n-1}}{2}}(1-\pi^*)^{\frac{1+\lambda_{n-1}}{2}}}$. From the closed form solution of v, we immediately get (iii), and we see that v is concave and that $\lim_{\pi \to 1} v(\pi) = \mu_a$.

Finally, we solve for w_b from (79). To this end, we obtain from (39) that $\frac{g(\pi)}{2\rho\sigma^2}(v''(\pi) + w_a''(\pi)) = \frac{A_2}{n-1}\pi^{\frac{1-\lambda_{n-1}}{2}}(1-\pi)^{\frac{1+\lambda_{n-1}}{2}}$. Plugging the previous expression into (79) and defining $\tilde{w}_b(\pi) = w_b(\pi) - n(\pi h + (1-\pi)l - \mu_a) + A_2 \frac{n}{n-1}\pi^{\frac{1-\lambda_{n-1}}{2}}(1-\pi)^{\frac{1+\lambda_{n-1}}{2}}$, we obtain that $\tilde{w}_b$ satisfies the equation $\tilde{w}_b(\pi) = n\frac{g(\pi)}{2\rho\sigma^2}\tilde{w}_b{}''(\pi)$. Since $\tilde{w}_b$ is also bounded, $\tilde{w}_b = nB_2 \pi^{\frac{1-\lambda_n}{2}}(1-\pi)^{\frac{1+\lambda_n}{2}}$ for some constant B_2. Therefore, $w_b(\pi) = n(\pi h + (1-\pi)l - \mu_a) - A_2 \frac{n}{n-1}\pi^{\frac{1-\lambda_{n-1}}{2}}(1-\pi)^{\frac{1+\lambda_{n-1}}{2}} + nB_2 \pi^{\frac{1-\lambda_n}{2}}(1-\pi)^{\frac{1+\lambda_n}{2}}$. To determine B_2, we can use the initial condition $w_b(\pi^*) = 0$. Thus, $B_2 = \frac{n(\mu_a - v(\pi^*))}{(\pi^*)^{\frac{1-\lambda_n}{2}}(1-\pi^*)^{\frac{1+\lambda_n}{2}}}$. From this closed form solution we see that w_b is convex, concluding the proof. □

Proof of Proposition 8. To simplify notation, we omit the argument for functions g, v_1, v_2, w_a, and w_b.

First, observe that by Proposition 4, the claim holds trivially unless the equilibrium is a cutoff one with $\pi_1^* > \pi_2^*$, under the usual interpretation that buyer i buys from seller b if and only if $\pi \geq \pi_i^*$. We are going to show that $\pi_1^* \neq \pi_{\text{fb},1}$, while $\pi_2^* = \pi_{\text{fb},2}$.

The proof has three main parts. We first simplify and relate the value functions of the market participants using the optimal pricing and purchasing strategies, obtaining a system of four nonlinear equations for the three regions, the one where none buys from b, the one where only buyer 2 buys from b, and the one where both buyers buy from b.

This step also proves that the value of information for buyer 1 at the low cutoff is zero, i.e., $v_1''(\pi_2^*) = 0$.

Next, we combine the derived equations with value matching, smooth pasting, and supercontact conditions to numerically solve for the value function. The solution shows that at the high cutoff, there is nonnull value of information for buyer 2, $v_2''(\pi_1^*) \neq 0$.

Finally, we conclude by showing that the good learner, who does not transit from product a to b at the high cutoff, is not incorporated in the price at the high cutoff. Therefore the high cutoff is not the same as the efficient one.

Let us first write down HJB equations for v_1, v_2, w_a, w_b on $(1, \pi_2^*)$, (π_2^*, π_1^*), and $(\pi_1^*, 1)$.

Following (9), the HJB equation for buyer i is

$$v_i = \max\left\{\mu_a - p_{a,i} + \frac{1}{2\rho} g \frac{\xi_{-ib}}{\sigma_{-i}^2} v_i'', \mathbf{E}_\pi[\theta] - p_{b,i} + \frac{1}{2\rho} g\left(\frac{\xi_{-ib}}{\sigma_{-i}^2} + \frac{1}{\sigma_i^2}\right) v_i''\right\}. \tag{40}$$

Following (13), the HJB equation for the seller a is

$$w_a = \sup_{p_{a,1}, p_{a,2}} \left\{p_{a,1} 1_{\{\xi_{1a}=1\}} + p_{a,2} 1_{\{\xi_{2a}=1\}} + \frac{1}{2\rho} g\left[\frac{1}{\sigma_1^2} 1_{\{\xi_{1b}=1\}} + \frac{1}{\sigma_2^2} 1_{\{\xi_{2b}=1\}}\right] w_a''\right\}. \tag{41}$$

The HJB equation for the seller b is

$$w_b = \sup_{p_{b,1}, p_{b,2}} \left\{p_{b,1} 1_{\{\xi_{1b}=1\}} + p_{b,2} 1_{\{\xi_{2b}=1\}} + \frac{1}{2\rho} g\left[\frac{1}{\sigma_1^2} 1_{\{\xi_{1b}=1\}} + \frac{1}{\sigma_2^2} 1_{\{\xi_{2b}=1\}}\right] w_b''\right\} \tag{42}$$

The indifference condition for the buyer i implies

$$p_{b,i} - p_{a,i} = \mathbf{E}_\pi[\theta] - \mu_a + \frac{1}{2\rho} g \frac{1}{\sigma_i^2} v_i'', \quad i = 1, 2. \tag{43}$$

When b sells to the buyer i, $p_{a,i} \leq \frac{1}{2\rho\sigma_i^2} g w_a''$ so that it is optimal for a not to sell to buyer i. Following our seller profit maximization pricing rule and the indifference condition (43), we set

$$p_{a,i} = \frac{1}{2\rho\sigma_i^2} g w_a'', \quad p_{b,i} = \mathbf{E}_\pi[\theta] - \mu_a + \frac{1}{2\rho\sigma_i^2} g\left[v_i'' + w_a''\right]. \tag{44}$$

When a sells to the buyer i, $p_{b,i} \le -\frac{1}{2\rho\sigma_i^2} g w_b''$ so that it is optimal for b not to sell to buyer i. The same argument as the previous case yields

$$p_{a,i} = \mu_a - \mathbf{E}_\pi[\theta] - \frac{1}{2\rho\sigma_i^2} g\left[v_i'' + w_b''\right], \quad p_{b,i} = -\frac{1}{2\rho\sigma_i^2} g w_b''. \tag{45}$$

Let us specialize (40), (41), and (42) in three regions.

1. $\underline{\pi < \pi_2^*}$: Both buyers purchase a.

v_2 satisfies

$$v_2 = \max\left\{\mu_a - p_{a,2}, \mathbf{E}_\pi[\theta] - p_{b,2} + \frac{1}{2\rho\sigma_2^2} g v_2''\right\} = \mathbf{E}_\pi[\theta] + \frac{1}{2\rho\sigma_2^2} g(v_2'' + w_b''), \tag{46}$$

where the second equality follows from (45).

v_1 satisfies

$$v_1 = \max\left\{\mu_a - p_{a,1}, \mathbf{E}_\pi[\theta] - p_{b,1} + \frac{1}{2\rho\sigma_1^2} g v_1''\right\} = \mathbf{E}_\pi[\theta] + \frac{1}{2\rho\sigma_1^2} g(v_1'' + w_b''), \tag{47}$$

where the second equality follows from (45).

w_a satisfies

$$w_a = p_{a,1} + p_{a,2} = 2\Big(\mu_a - \mathbf{E}_\pi[\theta]\Big) - \frac{1}{2\rho} g\left[\frac{1}{\sigma_1^2} v_1'' + \frac{1}{\sigma_2^2} v_2'' + \left(\frac{1}{\sigma_1^2} + \frac{1}{\sigma_2^2}\right) w_b''\right], \tag{48}$$

where the second equality follows from (45).

w_b satisfies

$$w_b = 0. \tag{49}$$

We conjecture that $v_1'', v_2'', w_a'', w_b''$ are all zero in this region. Then

$$v_1 = \mathbf{E}_\pi[\theta], \quad v_2 = \mathbf{E}_\pi[\theta], \quad w_a = 2\Big(\mu_a - \mathbf{E}_\pi[\theta]\Big), \quad w_b = 0. \tag{50}$$

2. $\underline{\pi \in (\pi_2^*, \pi_1^*)}$: Buyer 1 purchases a, buyer 2 purchases b.

v_2 satisfies

$$v_2 = \max\left\{\mu_a - p_{a,2}, \mathbf{E}_\pi[\theta] - p_{b,2} + \frac{1}{2\rho\sigma_2^2} g v_2''\right\} = \mu_a - \frac{1}{2\rho\sigma_2^2} g w_a'', \tag{51}$$

where the equality uses (44).

v_1 satisfies

$$\begin{aligned} v_1 =& \max\left\{\mu_a - p_{a,1} + \frac{1}{2\rho\sigma_2^2} v_1'', \mathbf{E}_\pi[\theta] - p_{b,1} + \frac{1}{2\rho} g\left(\frac{1}{\sigma_1^2} + \frac{1}{\sigma_2^2}\right) v_1''\right\} \\ =& \mathbf{E}_\pi[\theta] + \frac{1}{2\rho} g\left(\frac{1}{\sigma_1^2} + \frac{1}{\sigma_2^2}\right) v_1'' + \frac{1}{2\rho\sigma_1^2} g w_b'', \end{aligned} \tag{52}$$

where the second equality follows from (45).

w_a satisfies

$$w_a = p_{a,1} + \frac{1}{2\rho} g \frac{1}{\sigma_2^2} w_a'' + \mu_a - \mathbf{E}_\pi[\theta] - \frac{1}{2\rho\sigma_1^2} g(v_1'' + w_b'') + \frac{1}{2\rho\sigma_2^2} g w_a'', \tag{53}$$

where the second equality follows from (45).

w_b satisfies

$$w_b = p_{b,2} + \frac{1}{2\rho} g \frac{1}{\sigma_2^2} w_b'' = \mathbf{E}_\pi[\theta] - \mu_a + \frac{1}{2\rho\sigma_2^2} g\left(v_2'' + w_a'' + w_b''\right), \tag{54}$$

where the second equality follows from (44).

3. $\underline{\pi > \pi_1^*}$: Both buyers purchase b.

v_2 satisfies

$$\begin{aligned} v_2 =& \max\left\{\mu_a - p_{a,2} + \frac{1}{2\rho} g \frac{1}{\sigma_1^2} v_2'', \mathbf{E}_\pi[\theta] - p_{b,2} + \frac{1}{2\rho} g\left(\frac{1}{\sigma_1^2} + \frac{1}{\sigma_2^2}\right) v_2''\right\} \\ =& \mu_a - \frac{1}{2\rho\sigma_2^2} g w_a'' + \frac{1}{2\rho\sigma_1^2} g v_2'', \end{aligned} \tag{55}$$

where the second equality uses (44).

v_1 satisfies

$$
\begin{aligned}
v_1 =& \max\Big\{\mu_a - p_{a,1} + \frac{1}{2\rho\sigma_2^2}v_1'', \mathbf{E}_\pi[\theta] - p_{b,1} + \frac{1}{2\rho}g\Big(\frac{1}{\sigma_1^2} + \frac{1}{\sigma_2^2}\Big)v_1''\Big\} \\
=& \mu_a - \frac{1}{2\rho\sigma_1^2}gw_a'' + \frac{1}{2\rho\sigma_2^2}gv_1'',
\end{aligned}
\tag{56}
$$

where the second equality follows from (44).

w_a satisfies

$$
w_a = \frac{1}{2\rho}g\Big(\frac{1}{\sigma_1^2} + \frac{1}{\sigma_2^2}\Big)w_a''. \tag{57}
$$

w_b satisfies

$$
\begin{aligned}
w_b =& p_{b,1} + p_{b,2} + \frac{1}{2\rho}g\Big(\frac{1}{\sigma_1^2} + \frac{1}{\sigma_2^2}\Big)w_b'' \\
=& 2\Big(\mathbf{E}_\pi[\theta] - \mu_a\Big) + \frac{1}{2\rho}g\Big[\frac{1}{\sigma_1^2}v_1'' + \frac{1}{\sigma_2^2}v_2'' + \Big(\frac{1}{\sigma_1^2} + \frac{1}{\sigma_2^2}\Big)(w_a'' + w_b'')\Big],
\end{aligned}
\tag{58}
$$

where the second equality follows from (44).

After writing down HJB equations on different regions, we claim that these equations are equivalent to the following four HJB equation:

$$
\begin{aligned}
w_a =& \max\Big\{\mu_a - \mathbf{E}_\pi[\theta] - \frac{1}{2\rho\sigma_2^2}g(v_2'' + w_b''), \frac{1}{2\rho\sigma_2^2}gw_a''\Big\} \\
&+ \max\Big\{\mu_a - \mathbf{E}_\pi[\theta] - \frac{1}{2\rho\sigma_1^2}g(v_1'' + w_b''), \frac{1}{2\rho\sigma_1^2}gw_a''\}.
\end{aligned}
\tag{59}
$$

$$
\begin{aligned}
w_b =& \max\Big\{\mathbf{E}_\pi[\theta] - \mu_a + \frac{1}{2\rho\sigma_2^2}g(v_2'' + w_a'') + \frac{1}{2\rho\sigma_2^2}gw_b'', 0\Big\} \\
&+ \max\Big\{\mathbf{E}_\pi[\theta] - \mu_a + \frac{1}{2\rho\sigma_1^2}g(v_1'' + w_a'') + \frac{1}{2\rho\sigma_1^2}gw_b'', 0\Big\}.
\end{aligned}
\tag{60}
$$

$$v_1 = \min\Big\{\mu_a - \frac{1}{2\rho\sigma_1^2} g w_a'', \mathbf{E}_\pi[\theta] + \frac{1}{2\rho\sigma_1^2} g(v_1'' + w_b'')\Big\} + \frac{1}{2\rho\sigma_2^2} g v_1'' 1_{\{\text{first term in the } v_2 \text{ equation is smaller}\}} \tag{61}$$

$$v_2 = \min\Big\{\mu_a - \frac{1}{2\rho\sigma_2^2} g w_a'', \mathbf{E}_\pi[\theta] + \frac{1}{2\rho\sigma_2^2} g(v_2'' + w_b'')\Big\} + \frac{1}{2\rho\sigma_1^2} g v_2'' 1_{\{\text{first term in the } v_1 \text{ equation is smaller}\}}. \tag{62}$$

To see how equations (59) - (62) are equivalent to HJB equations of v_1,v_2,w_a, and w_b on different regions, we first consider the case where the LHS of (27),(28) < 0. In this case, first terms in both maximization of (59) are larger than the second term in the same maximization problem. As a result, (59) agrees with (48). Meanwhile, the first term in both maximization of (60) are smaller than the second term, so that (60) agrees with (49). Both first term in the minimization of (61) and (62) are smaller than their corresponding second term. Therefore equations (61) and (62) agree with (47) and (46), respectively. Therefore, equations (59) - (62) agree with the HJB equations of v_1,v_2,w_a,w_b on $(0,\pi_2^*)$. Using the similar argument, we can show these equations also agree on (π_2^*,π_1^*) and $(\pi_1^*,1)$.

In what follows, we derive two consequences of (27) and (28).

1. When $\pi \in (\pi_2^*,\pi_1^*)$, we obtain from adding up (51) to (54) that

$$\begin{aligned} v_1 + v_2 + w_a + w_b =& \mathbf{E}_\pi[\theta] + \mu_a + \frac{1}{2\rho\sigma_2^2} g[v_1'' + v_2'' + w_a'' + w_b''] \\ =& 2\mu_a + \frac{1}{2\rho\sigma_2^2} g v_1'' + \mathbf{E}_\pi[\theta] - \mu_a + \frac{1}{2\rho\sigma_2^2} g[v_2'' + w_a'' + w_b'']. \end{aligned}$$

Evaluate the previous equation at π_2^*. If $v_1 + v_2 + w_a + w_b$ is continuous at π_2^*, the left hand side is $2\mu_a$. The right-hand side, equation (27) implies that

$$v_1''(\pi_2^*) = 0. \tag{63}$$

2. When $\pi \in (\pi_2^*, \pi_1^*)$, it follows from (53) that

$$\begin{aligned} w_a =& \mu_a - \mathbf{E}_\pi[\theta] - \frac{1}{2\rho\sigma_1^2} g(v_1'' + w_b'') + \frac{1}{2\rho\sigma_2^2} g w_a'' \\ =& \mu_a - \mathbf{E}_\pi[\theta] - \frac{1}{2\rho\sigma_1^2} g(v_1'' + w_a'' + w_b'') + \frac{1}{2\rho}\Big(\frac{1}{\sigma_1^2} + \frac{1}{\sigma_2^2}\Big) g w_a''. \end{aligned}$$

Evaluate the previous equation at π_1^*, use (28) and the equation of w_a when $\pi > \pi_1^*$ (see (53)). We obtain that when w_a is continuous at π_1^*, w_a'' is also continuous at π_1^*.

Solving the system of equations (59) - (62) is numerically challenging. However, we can solve the HJB equations for v_1, v_2, w_a, and w_b on three regions and combine the solutions with appropriate value matching, smooth pasting, and super-contact conditions. The two observations above motivate us to impose the following value matching, smooth pasting, and super contact conditions: (1) Value matching for all the value functions at π_1 and π_2. (2) Smooth pasting of w_a at π_1 and π_2. (3) Super-contact of w_a at π_1, i.e., $\lim_{\pi \uparrow \pi_1} w_a''(\pi) = \lim_{\pi \downarrow \pi_1} w_a''(\pi)$. (4) Smooth pasting of v_2 at π_1. (5) Smooth pasting of v_1 at π_2. (6) Smooth pasting of w_b at π_2. Our numeric solutions are presented in Figures 4 and 5. The proofs for $\pi_2^* = \pi_{\text{fb},2}$ and $\pi_1^* \neq \pi_{\text{fb},1}$ are already presented in the main text. □

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Online Appendix: Omitted proofs

A more formal version of the following proof can be established using the stochastic filtering theory in Kallianpur (2013) Ch. 8. Here we present a less technical version to establish intuition.

Proof of Lemma 1. Define $d\mathfrak{C}_{bi} := \frac{\theta}{\sigma_i}dt + dZ_{it}$, for $i = 1, \cdots, n$, where $\theta \in \{h, \ell\}$. So, $\left(d\mathfrak{C}_{bi}\right)^2 = \frac{\theta^2}{\sigma_i^2}(dt)^2 + dZ_{it}^2 + 2dZ_{it}dt = dt$. Recall that $dZ_{it} \perp dZ_{jt}$, for all $i \neq j$. We denote as $m_i(t)$ the consumer with the i-th worst learning technology that is consuming product b at time t. Then, by Bayes' rule, we have

$$\pi_{t+dt} = \frac{\pi_t \Pr\left(\frac{h}{\sigma_{m_1(t)}}, \frac{h}{\sigma_{m_2(t)}}, \cdots, \frac{h}{\sigma_{m_{\text{Vol}_b}(t)}}\right)}{\pi_t \Pr\left(\frac{h}{\sigma_{m_1(t)}}, \frac{h}{\sigma_{m_2(t)}}, \cdots, \frac{h}{\sigma_{m_{\text{Vol}_b}(t)}}\right) + (1-\pi_t)\Pr\left(\frac{\ell}{\sigma_{m_1(t)}}, \frac{\ell}{\sigma_{m_2(t)}}, \cdots, \frac{\ell}{\sigma_{m_{\text{Vol}_b}(t)}}\right)} \tag{64}$$

where for $\theta \in \{h, \ell\}$:

$$\begin{aligned}\Pr\left(\frac{\theta}{\sigma_{m_1(t)}}, \frac{\theta}{\sigma_{m_2(t)}}, \cdots, \frac{\theta}{\sigma_{m_{\text{Vol}_b}(t)}}\right) &= \prod_{i=1}^{\text{Vol}_b(t)} \frac{1}{\sqrt{2\pi dt}} \exp\left(\frac{-1}{2dt}\left(d\mathfrak{C}_{bm_i(t)} - \frac{\theta}{\sigma_{m_i(t)}}dt\right)^2\right) \\ &= \left(\frac{1}{\sqrt{2\pi dt}}\right)^{\text{Vol}_b(t)} \exp\left(\frac{-1}{2dt}\sum_{i=1}^{\text{Vol}_b(t)}\left(d\mathfrak{C}_{bm_i(t)} - \frac{\theta}{\sigma_{m_i(t)}}dt\right)^2\right).\end{aligned} \tag{65}$$

Using (64) we also have

$$\begin{aligned} d\pi_t &= \pi_{t+dt} - \pi_t \\ &= \pi_t(1-\pi_t)\frac{\Pr\left(\frac{h}{\sigma_{m_1(t)}}, \frac{h}{\sigma_{m_2(t)}}, \cdots, \frac{h}{\sigma_{m_{\text{Vol}_b}(t)}}\right) - \Pr\left(\frac{\ell}{\sigma_{m_1(t)}}, \frac{\ell}{\sigma_{m_2(t)}}, \cdots, \frac{\ell}{\sigma_{m_{\text{Vol}_b}(t)}}\right)}{\pi_t \Pr\left(\frac{h}{\sigma_{m_1(t)}}, \frac{h}{\sigma_{m_2(t)}}, \cdots, \frac{h}{\sigma_{m_{\text{Vol}_b}(t)}}\right) + (1-\pi_t)\Pr\left(\frac{\ell}{\sigma_{m_1(t)}}, \frac{\ell}{\sigma_{m_2(t)}}, \cdots, \frac{\ell}{\sigma_{m_{\text{Vol}_b}(t)}}\right)}.\end{aligned} \tag{66}$$

To simplify (66) we note that for $i = 1, \cdots, \mathrm{Vol}_b(t)$:

$$\frac{-1}{2dt}\left(d\mathfrak{C}_{bm_i(t)} - \frac{\theta}{\sigma_{m_i(t)}}dt\right)^2 = \frac{\theta}{\sigma_{m_i(t)}}d\mathfrak{C}_{bm_i(t)} - \frac{1}{2}\left(\frac{\theta}{\sigma_{m_i(t)}}\right)^2 dt - \frac{1}{2},$$

where the equality follows because $\left(d\mathfrak{C}_{bm_i(t)}\right)^2 = dt$. Using the above equality, plugging (65) into (66) implies that

$$d\pi_t = \pi_t(1-\pi_t)$$
$$\frac{\exp\left(\sum_{i=1}^{\mathrm{Vol}_b(t)}\left(\frac{h}{\sigma_{m_i(t)}}d\mathfrak{C}_{bm_i(t)} - \frac{1}{2}\left(\frac{h}{\sigma_{m_i(t)}}\right)^2 dt\right)\right) - \exp\left(\sum_{i=1}^{\mathrm{Vol}_b(t)}\left(\frac{\ell}{\sigma_{m_i(t)}}d\mathfrak{C}_{bm_i(t)} - \frac{1}{2}\left(\frac{\ell}{\sigma_{m_i(t)}}\right)^2 dt\right)\right)}{\pi_t \exp\left(\sum_{i=1}^{\mathrm{Vol}_b(t)}\left(\frac{h}{\sigma_{m_i(t)}}d\mathfrak{C}_{bm_i(t)} - \frac{1}{2}\left(\frac{h}{\sigma_{m_i(t)}}\right)^2 dt\right)\right) + (1-\pi_t)\exp\left(\sum_{i=1}^{\mathrm{Vol}_b(t)}\left(\frac{\ell}{\sigma_{m_i(t)}}d\mathfrak{C}_{bm_i(t)} - \frac{1}{2}\left(\frac{\ell}{\sigma_{m_i(t)}}\right)^2 dt\right)\right)} \tag{67}$$

Using Taylor expansion (removing the higher order terms), we further have for $\theta \in \{h, \ell\}$:

$$\begin{aligned}\exp\left(\sum_{i=1}^{\mathrm{Vol}_b(t)}\left(\frac{\theta}{\sigma_{m_i(t)}}d\mathfrak{C}_{bm_i(t)} - \frac{1}{2}\left(\frac{\theta}{\sigma_{m_i(t)}}\right)^2 dt\right)\right) &= 1 + \sum_{i=1}^{\mathrm{Vol}_b(t)}\left(\frac{\theta}{\sigma_{m_i(t)}}d\mathfrak{C}_{bm_i(t)} - \frac{1}{2}\left(\frac{\theta}{\sigma_{m_i(t)}}\right)^2 dt\right) \\ &\quad + \frac{1}{2}\left(\sum_{i=1}^{\mathrm{Vol}_b(t)}\frac{\theta}{\sigma_{m_i(t)}}d\mathfrak{C}_{bm_i(t)} - \frac{1}{2}\left(\frac{\theta}{\sigma_{m_i(t)}}\right)^2 dt\right)^2 \\ &= 1 + \sum_{i=1}^{\mathrm{Vol}_b(t)}\frac{\theta}{\sigma_{m_i(t)}}d\mathfrak{C}_{bm_i(t)} \end{aligned} \tag{68}$$

where the last equality follows because $dZ_{it}dZ_{jt} = 0$ for $i \neq j$ and $(dt)^k = 0$, for $k > 1$.

Next, plugging (68) into (67) implies

$$
\begin{aligned}
d\pi_t &= \pi_t(1-\pi_t)\frac{\sum_{i=1}^{\mathrm{Vol}_b(t)} \frac{(h-\ell)}{\sigma_{m_i(t)}} d\mathfrak{C}_{bm_i(t)}}{1+\sum_{i=1}^{\mathrm{Vol}_b(t)}\left(\frac{\pi_t h+(1-\pi_t)\ell}{\sigma_{m_i(t)}}\right) d\mathfrak{C}_{bm_i(t)}} \\
&= \pi_t(1-\pi_t)\left(\sum_{i=1}^{\mathrm{Vol}_b(t)} \frac{(h-\ell)}{\sigma_{m_i(t)}} d\mathfrak{C}_{bm_i(t)}\right)\left(1-\sum_{i=1}^{\mathrm{Vol}_b(t)}\left(\frac{\pi_t h+(1-\pi_t)\ell}{\sigma_{m_i(t)}}\right) d\mathfrak{C}_{bm_i(t)}\right) \\
&= \pi_t(1-\pi_t)(h-\ell)\left(\sum_{i=1}^{\mathrm{Vol}_b(t)}\left(\frac{1}{\sigma_{m_i(t)}} d\mathfrak{C}_{bm_i(t)} - \frac{\pi_t h+(1-\pi_t)\ell}{\sigma^2_{m_i(t)}} dt\right)\right) \qquad (69)
\end{aligned}
$$

where the second equality follows by the Taylor expansion of the denominator, and the last equality follows because $d\mathfrak{C}_{bm_i(t)} d\mathfrak{C}_{bm_j(t)} = 0$ when $i \neq j$ and $(d\mathfrak{C}_{bm_i(t)})^2 = dt$. Finally, note that

$$
\mathbf{E}_{\pi_t}\left[\sum_{i=1}^{\mathrm{Vol}_b(t)}\left(\frac{1}{\sigma_{m_i(t)}} d\mathfrak{C}_{bm_i(t)} - \frac{\pi_t h+(1-\pi_t)\ell}{\sigma^2_{m_i(t)}} dt\right)\right] = \sum_{i=1}^{\mathrm{Vol}_b(t)} \mathbf{E}_{\pi_t}\left[\frac{1}{\sigma_{m_i(t)}} d\mathfrak{C}_{bm_i(t)} - \frac{\pi_t h+(1-\pi_t)\ell}{\sigma^2_{m_i(t)}} dt\right] =
$$

$$
\mathbf{Var}_{\pi_t}\left[\sum_{i=1}^{\mathrm{Vol}_b(t)}\left(\frac{1}{\sigma_{m_i(t)}} d\mathfrak{C}_{bm_i(t)} - \frac{\pi_t h+(1-\pi_t)\ell}{\sigma^2_{m_i(t)}} dt\right)\right] = \sum_{i=1}^{\mathrm{Vol}_b(t)} \frac{1}{\sigma^2_{m_i(t)}} dt
$$

Therefore,

$$
\sum_{i=1}^{\mathrm{Vol}_b(t)}\left(\frac{1}{\sigma_{m_i(t)}} d\mathfrak{C}_{bm_i(t)} - \frac{\pi_t h+(1-\pi_t)\ell}{\sigma^2_{m_i(t)}} dt\right) \sim \sqrt{\sum_{i=1}^{\mathrm{Vol}_b(t)} \frac{1}{\sigma^2_{m_i(t)}}} dZ_t
$$

where Z_t is the standard BM. Therefore,

$$
d\pi_t = \pi_t(1-\pi_t)(h-\ell)\sqrt{\sum_{i=1}^{\mathrm{Vol}_b(t)} \frac{1}{\sigma^2_{m_i(t)}}} dZ_t = \pi_t(1-\pi_t)(h-\ell)\sqrt{\sum_{i=1}^{n} \frac{\xi_{ib}(t)}{\sigma_i^2}} dZ_t,
$$

finishing the proof.

□

The following result characterizes $MO\left(\pi_0, \beta, T, m_1, ..., m_j\right)$ in terms of the learning abilities of $\{m_1, ..., m_j\}$, that is, $\sigma_{m_1}, \cdots, \sigma_{m_j}$, the target belief β, and the horizon T.

Proposition 13. *We have:*

$$MO\left(\pi_0, \beta, T, m_1, ..., m_j\right) = (1-\beta)\pi_0\Phi(\lambda_1) - \beta(1-\pi_0)\Phi(\lambda_0)$$

where

$$\lambda_1 = \frac{1}{(h-\ell)\sqrt{\sum_{i=1}^{j}\frac{T}{\sigma_{m_i}^2}}}\left[\ln\left(\frac{\frac{\pi_0}{1-\pi_0}}{\frac{\beta}{1-\beta}}\right) + \frac{(h-\ell)^2}{2}\left(\sum_{i=1}^{j}\frac{T}{\sigma_{m_i}^2}\right)\right],$$
$$\lambda_0 = \frac{1}{(h-\ell)\sqrt{\sum_{i=1}^{j}\frac{T}{\sigma_{m_i}^2}}}\left[\ln\left(\frac{\frac{\pi_0}{1-\pi_0}}{\frac{\beta}{1-\beta}}\right) - \frac{(h-\ell)^2}{2}\left(\sum_{i=1}^{j}\frac{T}{\sigma_{m_i}^2}\right)\right],$$

and $\Phi(\cdot)$ denotes the CDF of a standard normal random variable.

Proof. To prove the proposition we first need to prove the following lemma.

Lemma 4. *Let buyers $\{m_1, ..., m_j\}$ use the risky product b in the time interval $[0, T]$. Define $U := 1-\pi$ and $L := \pi$. Given the result of Lemma 1, let π solves $d\pi = \pi(1-\pi)(h-\ell)\sqrt{\sum_{i=1}^{j}\frac{1}{\sigma_{m_i}^2}}dZ$ where Z is a BM under the probability measure $\mathbb{P}$, for a given π_0.*[11] *This is equivalent to the followings:*

Part 1. There is a standard BM, Z^U, so that the process $\gamma_y := \frac{L}{U}$ solves $\frac{d\gamma_y}{\gamma_y} = (h-\ell)\sqrt{\sum_{i=1}^{j}\frac{1}{\sigma_{m_i}^2}}dZ^U$, and $\gamma_{y_0} > 0$.

Part 2. There is a standard BM, Z^L, so that the process $\gamma_z := \frac{U}{L}$ solves $\frac{d\gamma_z}{\gamma_z} = (h-\ell)\sqrt{\sum_{i=1}^{j}\frac{1}{\sigma_{m_i}^2}}dZ^L$ and $\gamma_{z0} > 0$.

Proof of Lemma 4*.* Note that whenever is clear the dependence of the process to time is removed, for ease of notation.

Given Lemma 1, applying Ito's lemma gives

$$d\gamma_y = \gamma_y(h-\ell)\sqrt{\sum_{i=1}^{j}\frac{1}{\sigma_{m_i}^2}}\left[dZ + (h-\ell)\sqrt{\sum_{i=1}^{j}\frac{1}{\sigma_{m_i}^2}}\pi_t dt\right].$$

[11] Note that the solution π to the SDE is unique both in strong and weak sense, see, e.g., section 5.2 in Øksendal (2003).

Define Z^U by $dZ^U = dZ + (h-\ell)\sqrt{\sum_{i=1}^{j}\frac{1}{\sigma_{m_i}^2}}\pi_t dt$ with $Z_0^U = 0$. We next show that Z^U is indeed a standard BM. Note that Z is a BM under the probability measure $\mathbb{P}$. Define $\mathbb{P}_U$ as

$$\frac{d\mathbb{P}_U}{d\mathbb{P}} = \frac{U_T}{U_0} = \frac{1-\pi_T}{1-\pi_0}.$$

By Lemma 1 we have $\frac{d[1-\pi_t]}{1-\pi_t} = -(h-\ell)\sqrt{\sum_{i=1}^{j}\frac{1}{\sigma_{m_i}^2}}\pi_t dZ$. Therefore, by the results in section 5.2 in Øksendal (2003),

$$1-\pi_t = (1-\pi_0)e^{-(h-\ell)\sqrt{\sum_{i=1}^{j}\frac{1}{\sigma_{m_i}^2}}\int_0^t \pi_s dZ_s - \frac{1}{2}\left((h-\ell)\sqrt{\sum_{i=1}^{j}\frac{1}{\sigma_{m_i}^2}}\right)^2 \int_0^t \pi_s^2 ds}.$$

Moreover,

$$\frac{d\mathbb{P}_U}{d\mathbb{P}} = e^{-(h-\ell)\sqrt{\sum_{i=1}^{j}\frac{1}{\sigma_{m_i}^2}}\int_0^T \pi_s dZ_s - \frac{1}{2}\left((h-\ell)\sqrt{\sum_{i=1}^{j}\frac{1}{\sigma_{m_i}^2}}\right)^2 \int_0^T \pi_s^2 ds}.$$

So, given the above relation, Girsanov theorem shows that Z^U is indeed a $\mathbb{P}_U$-BM.

Next, we argue that Part 1 proves Part 2. Let Z^U be a $\mathbb{P}_U$ BM. Since $\gamma_z = \gamma_y^{-1}$, by Ito's lemma, we get

$$d\gamma_z = (h-\ell)\sqrt{\sum_{i=1}^{j}\frac{1}{\sigma_{m_i}^2}}\gamma_z\left[-dZ^U + (h-\ell)\sqrt{\sum_{i=1}^{j}\frac{1}{\sigma_{m_i}^2}}dt\right].$$

Moreover,

$$\frac{d\mathbb{P}_L}{d\mathbb{P}_U} = \frac{\frac{L_T}{L_0}}{\frac{U_T}{U_0}} = \frac{\gamma_{y_T}}{\gamma_{y_0}} = e^{(h-\ell)\sqrt{\sum_{i=1}^{j}\frac{1}{\sigma_{m_i}^2}}Z_T^U - \frac{1}{2}\left((h-\ell)\sqrt{\sum_{i=1}^{j}\frac{1}{\sigma_{m_i}^2}}\right)^2 T},$$

therefore, by Girsanov theorem, the process Z^L defined by $dZ^L = -dZ^U + (h-\ell)\sqrt{\sum_{i=1}^{j}\frac{1}{\sigma_{m_i}^2}}dt$ and $Z_0^L = 0$ is a $\mathbb{P}_L$-BM.

Finally, we show the converse holds as well. That is, the conditions of Part 2 proves that $d\pi = \pi(1-\pi)(h-\ell)\sqrt{\sum_{i=1}^{j}\frac{1}{\sigma_{m_i}^2}}dZ$ where Z is a BM under $\mathbb{P}$. Ito's lemma gives

$$d\pi_t = (h-\ell)\sqrt{\sum_{i=1}^{j}\frac{1}{\sigma_{m_i}^2}}\pi_t(1-\pi_t)\left[-dZ_t^L + (h-\ell)\sqrt{\sum_{i=1}^{j}\frac{1}{\sigma_{m_i}^2}}(1-\pi_t)dt\right].$$

Let Z be the process defined by $dZ = -dZ^L + (h-\ell)\sqrt{\sum_{i=1}^{j}\frac{1}{\sigma_{m_i}^2}}(1-\pi_t)dt$ with $Z_0 = 0$. Since,

$$\frac{d\pi_t}{\pi_t} = -(h-\ell)\sqrt{\sum_{i=1}^{j}\frac{1}{\sigma_{m_i}^2}}(1-\pi_t)dZ^L + \left((h-\ell)\sqrt{\sum_{i=1}^{j}\frac{1}{\sigma_{m_i}^2}}\right)^2(1-\pi_t)^2dt,$$

we have

$$\pi_t = \pi_0 e^{-(h-\ell)\sqrt{\sum_{i=1}^{j}\frac{1}{\sigma_{m_i}^2}}\int_0^t(1-\pi_s)dZ_s^L + \frac{1}{2}\left((h-\ell)\sqrt{\sum_{i=1}^{j}\frac{1}{\sigma_{m_i}^2}}\right)^2\int_0^t(1-\pi_s)^2ds}$$

and

$$\frac{d\mathbb{P}}{d\mathbb{P}_L} = \frac{\pi_0}{\pi_T} = e^{(h-\ell)\sqrt{\sum_{i=1}^{j}\frac{1}{\sigma_{m_i}^2}}\int_0^t(1-\pi_s)dZ_s^L - \frac{1}{2}\left((h-\ell)\sqrt{\sum_{i=1}^{j}\frac{1}{\sigma_{m_i}^2}}\right)^2\int_0^t(1-\pi_s)^2ds}.$$

Therefore, by Girsanov theorem, Z is a BM under $\mathbb{P}$. □

To finish the proof we also need another lemma, described below.

Lemma 5. *Define the event $\mathcal{A} := \{\omega \in \Omega : \pi_T(\omega) > \beta\}$. Then,*

$$\mathbf{E}_{\pi_t}[\max\{\pi_T-\beta, 0\}] = (1-\beta)L_t \cdot P_L\{\mathcal{A}|\mathcal{F}_t\} - \beta U_t \cdot P_U\{\mathcal{A}|\mathcal{F}_t\}$$

Proof of Lemma 5. Note that

$$\pi_T - \beta = (1-\beta)L_T - \beta U_T$$

Therefore, we have

$$\begin{aligned}\mathbf{E}_{\pi_t}[\max\{\pi_T-\beta, 0\}] &= (1-\beta)\mathbf{E}\left[L_T\mathbf{1}_{\mathcal{A}}|\mathcal{F}_t\right] - \beta\mathbf{E}\left[U_T\mathbf{1}_{\mathcal{A}}|\mathcal{F}_t\right] \\ &= (1-\beta)L_t \cdot P_L\{\mathcal{A}|\mathcal{F}_t\} - \beta U_t \cdot P_U\{\mathcal{A}|\mathcal{F}_t\}\end{aligned}$$

finishing the proof. □

Next, equipped with the above lemmas, we finish the proof of the proposition. Let Z^L and Z^U be BMs as in Lemma 4, thus (using the exponential martingale formula) we have

$$\gamma_{y_t} = \gamma_{y_0} e^{(h-\ell)\sqrt{\sum_{i=1}^{j} \frac{1}{\sigma_{m_i}^2}} Z_t^U - \frac{1}{2}\left((h-\ell)\sqrt{\sum_{i=1}^{j} \frac{1}{\sigma_{m_i}^2}}\right)^2 t}$$

and

$$\gamma_{z_t} = \gamma_{z_0} e^{(h-\ell)\sqrt{\sum_{i=1}^{j} \frac{1}{\sigma_{m_i}^2}} Z_t^L - \frac{1}{2}\left((h-\ell)\sqrt{\sum_{i=1}^{j} \frac{1}{\sigma_{m_i}^2}}\right)^2 t}.$$

Therefore, since Z^U is a BM, we have

$$\begin{aligned}
\mathbb{P}_U\{\mathcal{A}|\mathcal{F}_t\} &= P_U\left\{\gamma_{y_T} > \frac{\beta}{1-\beta} | \gamma_{y_t}\right\} = P_U\left\{\ln\frac{\gamma_{y_T}}{\gamma_{y_t}} > \ln\frac{\frac{\beta}{1-\beta}}{\gamma_{y_t}}\right\} \\
&= P_U\left\{(h-\ell)\sqrt{\sum_{i=1}^{j}\frac{1}{\sigma_{m_i}^2}}(Z_T^U - Z_t^U) > \ln\frac{\frac{\beta}{1-\beta}}{\gamma_{y_t}} + \frac{1}{2}\left((h-\ell)\sqrt{\sum_{i=1}^{j}\frac{1}{\sigma_{m_i}^2}}\right)^2 (T-t)\right\} \\
&= \Phi\left(\frac{1}{(h-\ell)\sqrt{\sum_{i=1}^{j}\frac{1}{\sigma_{m_i}^2}}\sqrt{T-t}}\left[-\ln\frac{\frac{\beta}{1-\beta}}{\gamma_{y_t}} - \frac{1}{2}\left((h-\ell)\sqrt{\sum_{i=1}^{j}\frac{1}{\sigma_{m_i}^2}}\right)^2 (T-t)\right]\right)
\end{aligned} \tag{70}$$

where the last equality follows by Lemma 4. Similarly,

$$\begin{aligned}
\mathbb{P}_L\{\mathcal{A}|\mathcal{F}_t\} &= P_L\left\{\gamma_{z_T} > \frac{1-\beta}{\beta} | \gamma_{y_t}\right\} \\
&= P_L\left\{(h-\ell)\sqrt{\sum_{i=1}^{j}\frac{1}{\sigma_{m_i}^2}}(Z_T^L - Z_t^L) > \ln\frac{\frac{1-\beta}{\beta}}{\gamma_{z_t}} + \frac{1}{2}\left((h-\ell)\sqrt{\sum_{i=1}^{j}\frac{1}{\sigma_{m_i}^2}}\right)^2 (T-t)\right\} \\
&= \Phi\left(\frac{1}{(h-\ell)\sqrt{\sum_{i=1}^{j}\frac{1}{\sigma_{m_i}^2}}\sqrt{T-t}}\left[-\ln\frac{\frac{\beta}{1-\beta}}{\gamma_{y_t}} + \frac{1}{2}\left((h-\ell)\sqrt{\sum_{i=1}^{j}\frac{1}{\sigma_{m_i}^2}}\right)^2 (T-t)\right]\right)
\end{aligned} \tag{71}$$

where the last equality follows by Lemma 4. Equations (70) and (71) along with Lemma

5 finish the proof of the first part of the Proposition (note that the proposition is stated for when $t = 0$). □

Proof of Propositions 2 and 3. They are per capita versions of Theorem 1 in Bergemann and Välimäki (2000). □

Proof of Proposition 12. First, the condition (iii), $\sigma_1 > \sigma_2$, and the convexity of W_{avg} imply that

$$0 > \mathbf{E}_\pi[\theta] - \mu_a + \frac{1}{\rho\sigma_2^2} g(\pi,h,\ell)W''_{avg}(\pi) > \mathbf{E}_\pi[\theta] - \mu_a + \frac{1}{\rho\sigma_1^2} g(\pi,h,\ell)W''_{avg}(\pi),\ \pi \in (0,\pi_{\text{fb},2}),$$

$$\mathbf{E}_\pi[\theta] - \mu_a + \frac{1}{\rho\sigma_2^2} g(\pi,h,\ell)W''_{avg}(\pi) > 0 > \mathbf{E}_\pi[\theta] - \mu_a + \frac{1}{\rho\sigma_1^2} g(\pi,h,\ell)W''_{avg}(\pi),\ \pi \in (\pi_{\text{fb},2},\pi_{\text{fb},1}),$$

$$\mathbf{E}_\pi[\theta] - \mu_a + \frac{1}{\rho\sigma_2^2} g(\pi,h,\ell)W''_{avg}(\pi) > \mathbf{E}_\pi[\theta] - \mu_a + \frac{1}{\rho\sigma_1^2} g(\pi,h,\ell)W''_{avg}(\pi) > 0,\ \pi \in (\pi_{\text{fb},2},1).$$

Therefore, combining the previous inequalities with the condition (i), we obtain that W_{avg} satisfies (7) and the optimizers $\xi^*_{ib}(\pi) = 1_{\{\pi > \pi_{\text{fb},i}\}}$, $i = 1,2$.

Next, we verify W_{avg} is the first best value function and ξ^*_{ib} in (8) is the first best optimal strategy. To this end, because $W_{avg} \in C^2(\pi_{\text{fb},i+1},\pi_{\text{fb},i})$, $i \in \{0,1,2\}$, and W_{avg} satisfies (31), Karatzas and Shreve (1998) Chapter 3 Problem 6.24 implies that Itô's formula can be applied to this type of piece-wise C^2 function. Consider any strategy $\{\xi_{ib}\}_{i=1,2}$. Itô's formula implies that

$$d\Big\{e^{-\rho t}W_{avg}(\pi_t) + \int_0^t \rho e^{-\rho s}\Big(\mu_a + \frac{1}{2}\sum_{i=1}^{2}\xi_{ib}(s)\big(\mathbf{E}_{\pi(s)}[\theta] - \mu_a\big)\Big)ds\Big\}$$

$$= \rho e^{-\rho t}\Big\{ - W_{avg}(\pi_t) + \frac{1}{2\rho} g(\pi,h,\ell)W''_{avg}(\pi_t)\sum_{i=1}^{2}\frac{\xi_{ib}(t)}{\sigma_i^2} + \Big(\mu_a + \frac{1}{2}\sum_{i=1}^{2}\xi_{ib}(t)\big(\mathbf{E}_{\pi(t)}[\theta] - \mu_a\big)\Big)\Big\}dt$$
$$+ \text{martingale},$$

whose drift is nonpositive due to (7). Therefore, the process

$$e^{-\rho t}W_{avg}(\pi_t) + \int_0^t \rho e^{-\rho s}\Big(\mu_a + \frac{1}{2}\sum_{i=1}^{2}\xi_{ib}(s)\big(\mathbf{E}_{\pi(s)}[\theta] - \mu_a\big)\Big)ds$$

is a local supermartingale. Because both W_{avg} and $\mu_a + \frac{1}{2}\sum_{i=1}^{2}\xi_{ib}(t)\big(\mathbf{E}_{\pi(t)[\theta]} - \mu_a\big)$ are bounded, the previous process is a supermartingale as well. As a result, for any T,

$$W_{avg}(\pi_t) \geq \mathbf{E}_t\Big[e^{-\rho(T-t)}W_{avg}(\pi_T) + \int_t^T \rho e^{-\rho(s-t)}\Big(\mu_a + \frac{1}{2}\sum_{i=1}^{2}\xi_{ib}(s)\big(\mathbf{E}_{\pi(s)}[\theta] - \mu_a\big)\Big)\, ds\Big].$$

Sending $T \to \infty$, because W_{avg} is bounded, it satisfies the transversality condition

$$\lim_{T\to\infty} \mathbf{E}_t[e^{-\rho(T-t)}W_{avg}(\pi_T)] = 0.$$

The previous two expressions combined yield

$$W_{avg}(\pi_t) \geq \mathbf{E}_t\Big[\int_t^\infty \rho e^{-\rho(s-t)}\Big(\mu_a + \frac{1}{2}\sum_{i=1}^{2}\xi_{ib}(s)\big(\mathbf{E}_{\pi(s)}[\theta] - \mu_a\big)\Big)\, ds\Big], \quad \text{for any strategy } \xi_{ib}.$$

When the strategy is chosen as in (8), the inequality above is an equality, confirming the optimality of ξ^*_{ib}. □

Proof of Claim 1. Sending π approaching $\pi_{\text{fb},1}$ from the left and right, $W_{avg,0}$ and $W_{avg,1}$ satisfy

$$W_{avg,1}(\pi_{\text{fb},1}) = \mu_a + \frac{1}{2}\big(\mathbf{E}_{\pi_{\text{fb},1}}[\theta] - \mu_a\big) + \frac{1}{2\rho\sigma_2^2} g(\pi_{\text{fb},1}, h, \ell) W''_{avg,1}(\pi_{\text{fb},1}) \tag{72}$$

$$\begin{aligned} W_{avg,0}(\pi_{\text{fb},1}) = &\mu_a + \frac{1}{2}\big(\mathbf{E}_{\pi_{\text{fb},1}}[\theta] - \mu_a\big) + \frac{1}{2\rho\sigma_2^2} g(\pi_{\text{fb},1}, h, \ell) W''_{avg,0}(\pi_{\text{fb},1}) \\ &+ \frac{1}{2}\big(\mathbf{E}_{\pi_{\text{fb},1}}[\theta] - \mu_a\big) + \frac{1}{2\rho\sigma_1^2} g(\pi_{\text{fb},1}, h, \ell) W''_{avg,0}(\pi_{\text{fb},1}) \end{aligned} \tag{73}$$

Suppose that $W''_{avg,1}(\pi_{\text{fb},1}) < W''_{avg,0}(\pi_{\text{fb},1})$. Then the right-hand side of (72) is less than the first line on the right-hand side of (73). However, the left-hand sides of (72) and (73) have the same value due to the value matching, the second line on the right-hand side of (73) must be strictly less than 0. This contradicts with Proposition 12 (iii) that

$$\mathbf{E}_\pi[\theta] - \mu_a + \frac{1}{2\rho\sigma_1^2} g(\pi, h, \ell) W''_{avg,0}(\pi) > 0 \text{ for } \pi \text{ in a right neighborhood of } \pi_{\text{fb},1}.$$

Suppose that $W''_{avg,1}(\pi_{\text{fb},1}) > W''_{avg,0}(\pi_{\text{fb},1})$. Then the condition

$$\mathbf{E}_{\pi_{\text{fb},1}}[\theta] - \mu_a + \frac{1}{2\rho\sigma_1^2} g(\pi_{\text{fb},1}, h, \ell) W''_{avg,0}(\pi_{\text{fb},1}) \geq 0$$

in Proposition 12 (iii) implies

$$\mathbf{E}_{\pi_{\mathrm{fb},1}}[\theta] - \mu_a + \frac{1}{2\rho\sigma_1^2} g(\pi_{\mathrm{fb},1}, h, \ell) W''_{avg,1}(\pi_{\mathrm{fb},1}) > 0$$

Due to the continuity of $W''_{avg,1}$ on $(\pi_{\mathrm{fb},2}, \pi_{\mathrm{fb},1})$, the previous inequality implies

$$\mathbf{E}_{\pi}[\theta] - \mu_a + \frac{1}{2\rho\sigma_1^2} g(\pi, h, \ell) W''_{avg,1}(\pi) > 0 \text{ in a left neighborhood of } \pi_{\mathrm{fb},1}.$$

However, this contradicts with the condition Proposition 12 (iii) on $(\pi_{\mathrm{fb},2}, \pi_{\mathrm{fb},1})$.

Combining the previous two cases, we verify the claim. □

Proof of Proposition 5. First, notice that the continuation value of each market participant is always nonnegative since they all have a strategy that guarantees a deterministic zero payoff. At the same time, observe that, given the pricing strategies of Lemma 2 for some cutoff π_m, it is optimal for the buyers to use the strategies

$$\xi_{i,a}(\pi, p_a, p_b) = 1 \quad \text{if and only if } \mu_a - p_{a,i} = \max\{\mu_a - p_{a,i}, \mathbf{E}_\pi[\theta] - p_{b,i}, 0\} \tag{74}$$

$$\xi_{i,b}(\pi, p_a, p_b) = 1 \quad \text{if and only if } \mathbf{E}_\pi[\theta] - p_{b,i} = \max\{\mu_a - p_{a,i}, \mathbf{E}_\pi[\theta] - p_{b,i}, 0\}. \tag{75}$$

Moreover, the induced expected discounted utility for the buyers is equal to 0. Therefore, by setting the cutoff equal to the welfare-maximizing one, the monopolist obtains the first-best welfare. Since we have noted that the continuation utilities of all market participants have to be nonnegative, using that cutoff is optimal for the monopolist. □

Proof of Proposition 6. (i) The proof follows the same lines of Lemma 2 and Proposition 5. In particular, consider the pricing strategy

$$p_a = \mu_a \qquad \textit{and} \qquad p_b = \mathbf{E}_\pi[\theta]. \tag{76}$$

It is immediate to see that under this pricing strategy the buyers have a value function that is identically 0, and they are always indifferent between the two products. Therefore, by letting

$$\xi_{ib}(t) = 1_{\{\pi_t \geq \pi_{i,\mathrm{fb}}\}} \tag{77}$$

we obtain an equilibrium that is welfare-maximizing. (ii) The result where the monopolist is forced to use the same price for both products is trivial, because by Proposition 4 the first-best features two different thresholds. □

Proof of Claim 2. Consider an equilibrium in which the seller a can control ξ_{ia} directly via the choice of prices that satisfy the conditions of Lemma 3. Then seller a's value satisfies the HJB equation

$$w_a(\pi) = \sup_{\xi_{ia}\in\{0,1\}} \left\{ np_a(\pi)1_{\{\xi_{ia}=1\}} + \frac{1}{2\rho} g \frac{n}{\sigma^2} w_a''(\pi)1_{\{\xi_{ib}=1\}} \right\}.$$

This equation is equivalent to

$$0 = \max\left\{ np_a(\pi) - w_a(\pi), \frac{1}{\sigma^2} g w_a''(\pi) - \frac{2\rho w_a(\pi)}{n} \right\},$$

which is exactly the variational inequality for the optimal stopping problem

$$w_a(\pi) = \sup_{\tau} \mathbf{E}_\pi \left[e^{-\rho\tau} np_a(\pi_\tau) \right], \tag{78}$$

where τ is a stopping time chosen by the seller a. We can interpret the value of this optimal stopping problem as the best value for the seller a when he can control the failure time τ of the product b. We anticipate that w_a is convex. Then the optimal stopping time is a threshold type where the threshold is pinned down by the value matching and the smooth pasting conditions. Combining (14) and (16), w_b satisfies the equation

$$w_b(\pi) = n\big(\mathbf{E}_\pi[\theta] - \mu_a\big) + \frac{n}{2\rho\sigma^2} g(\pi, h, \ell)\big(v''(\pi) + w_a''(\pi) + w_b''(\pi)\big). \tag{79}$$

Therefore, it is immediate to see that by maximizing w_a'' among the possible equilibria, we are also choosing the equilibrium with the highest profits for seller b. By equation (34), we have that the value of ζ_1 that make w_a'' larger make w_a' smaller. Therefore, the best possible scenario seller b is the one in which w_a' is as small as possible (and therefore w_a'' is as large as possible). However, for the maximization problem of w_a in (14), the notion of viscosity solution prohibits a concave kink. (see, e.g., Proposition 1 in Online Appendix D of Achdou et al. (2017)). Therefore w_a' is minimal when w_a satisfies the smooth pasting condition we imposed at π^*. □

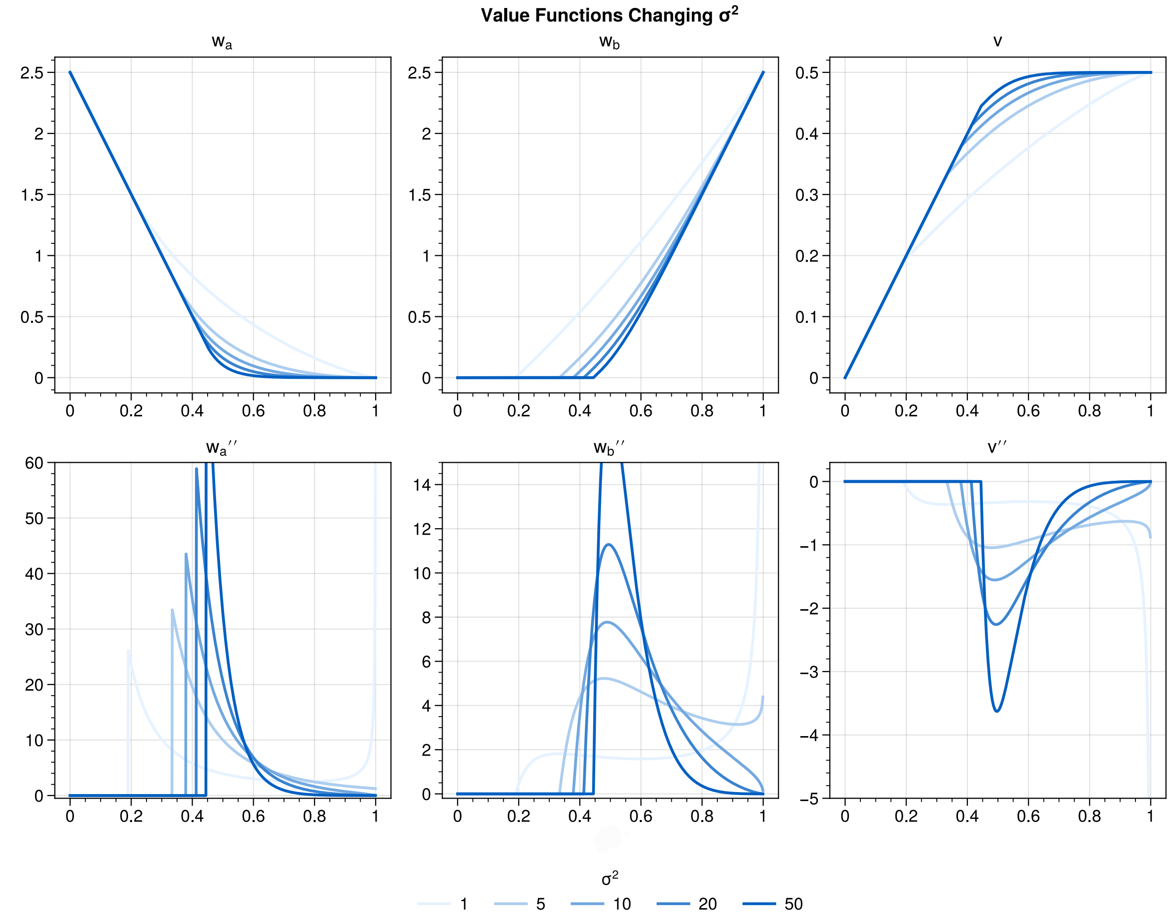


Figure 7: This figure plots the value functions w_a,w_b and v and their second derivatives w_a'',w_b'' and v'' when σ^2 changes, fixing other parameters to $n = 5, h = 1, \ell = 0, \mu_a = .5$ (using the explicit characterizations of the value functions in the proof of Proposition 7). As shown in the figure, with increasing σ^2 the cutoff π^*, expectedly, moves to the right (i.e., it increases). Moreover, it shows that v'' is concave (i.e., $v'' \leq 0$), w_a'' and w_b'' are convex (i.e., $w_a'' \geq 0$ and $w_b'' \geq 0$).

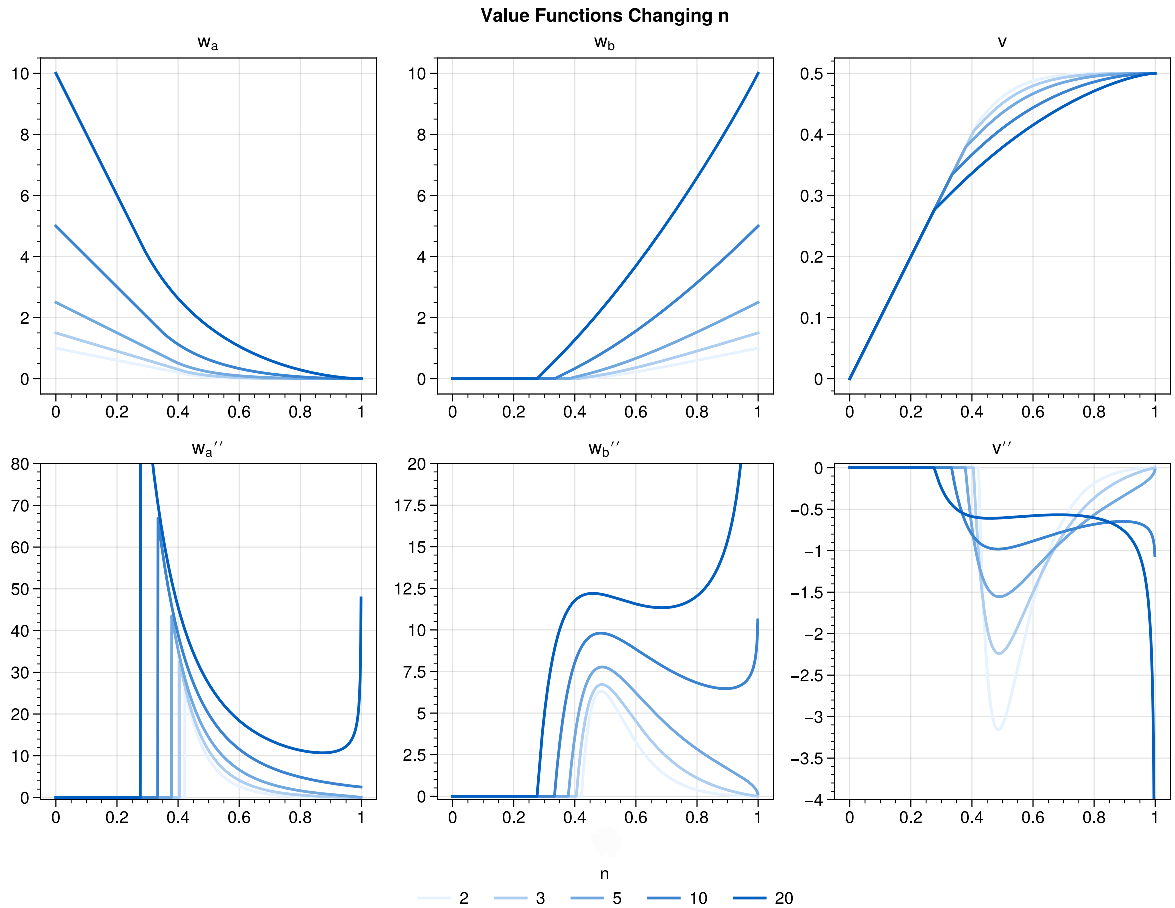


Figure 8: This figure plots the value functions w_a,w_b and v and their second derivatives w_a'',w_b'' and v'' when n changes, fixing other parameters to $\sigma^2 = 10, h = 1, \ell = 0, \mu_a = .5$ (using the explicit characterizations of the value functions in the proof of Proposition 7). As shown in the figure, with increasing n the cutoff π^*, expectedly, moves to the left (i.e., it decreases). Moreover, it shows that v'' is concave (i.e., $v'' \leq 0$), w_a'' and w_b'' are convex (i.e., $w_a'' \geq 0$ and $w_b'' \geq 0$).

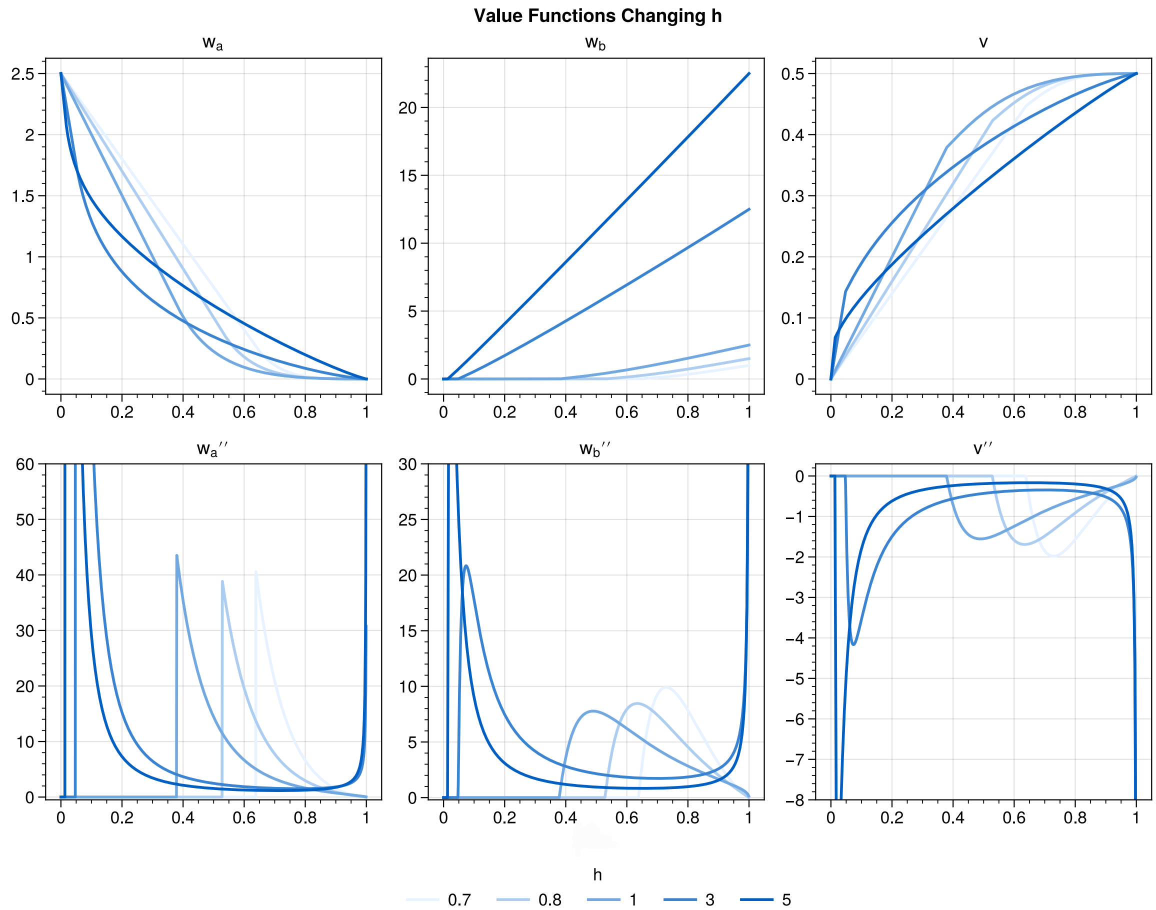


Figure 9: This figure plots the value functions w_a, w_b and v and their second derivatives w_a'', w_b'' and v'' when h changes, fixing other parameters to $\sigma^2 = 10, n = 5, \ell = 0, \mu_a = .5$ (using the explicit characterizations of the value functions in the proof of Proposition 7). As shown in the figure, with increasing h the cutoff π^*, expectedly, moves to the left (i.e., it decreases). Moreover, it shows that v'' is concave (i.e., $v'' \leq 0$), w_a'' and w_b'' are convex (i.e., $w_a'' \geq 0$ and $w_b'' \geq 0$).

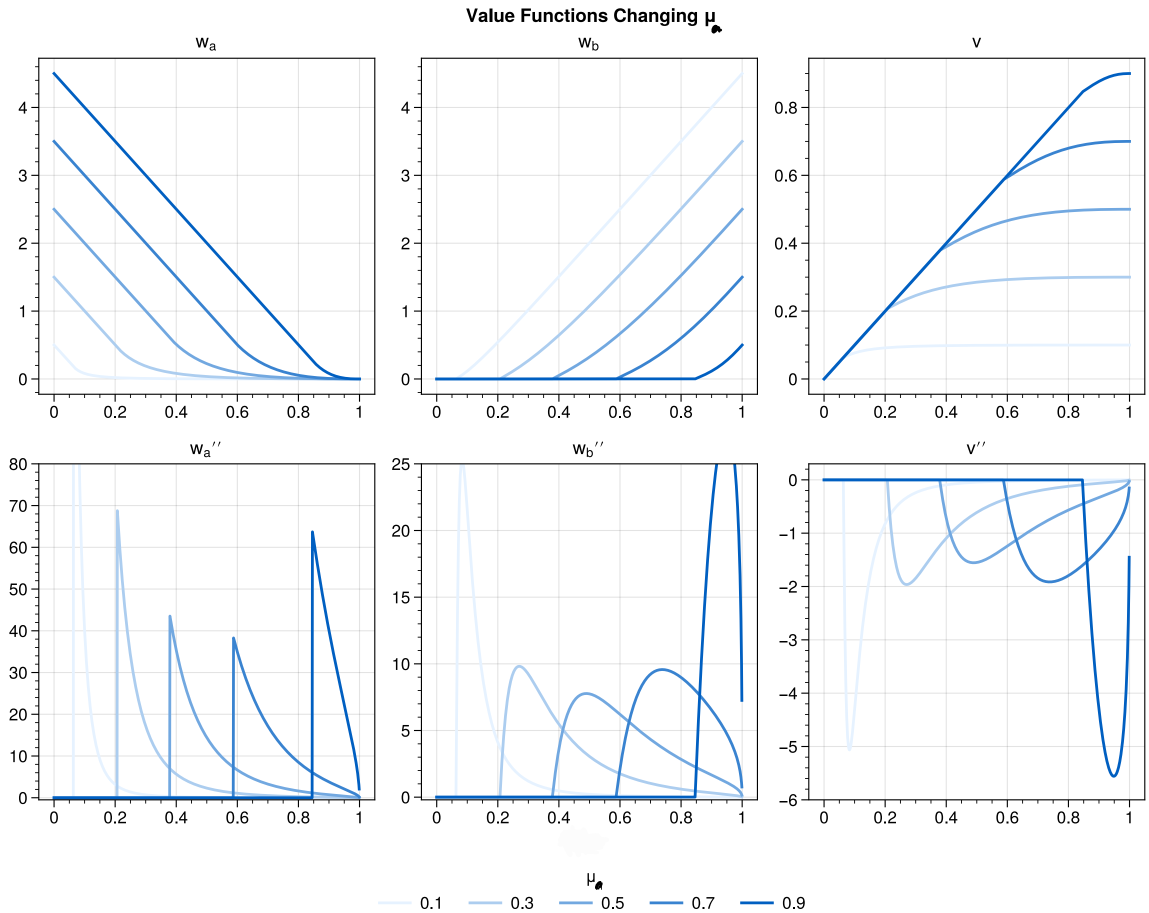


Figure 10: This figure plots the value functions w_a, w_b and v and their second derivatives w_a'', w_b'' and v'' when μ_a changes, fixing other parameters to $\sigma^2 = 10, n = 5, \ell = 0, h = 1$ (using the explicit characterizations of the value functions in the proof of Proposition 7). As shown in the figure, with increasing μ_a the cutoff π^*, expectedly, moves to the right (i.e., it increases). Moreover, it shows that v'' is concave (i.e., $v'' \leq 0$), w_a'' and w_b'' are convex (i.e., $w_a'' \geq 0$ and $w_b'' \geq 0$).

Proof of Proposition 9. The result follows immediately by rewriting equation (28) as

$$\left(\mu_a - \mathbf{E}_{\pi_1^*}[\theta]\right) 2\rho\sigma_1^2 = g\left(\pi_1^*, h, \ell\right)\left(v_1''\left(\pi_1^*\right) + w_b''\left(\pi_1^*\right) + w_a''\left(\pi_1^*\right)\right). \tag{80}$$

Notice that as σ_1 goes to infinity the LHS goes to ∞ unless $\pi_1^* \to \pi_{myopic}$. Observe that the value functions of the agents are uniformly bounded by $2h$ across all the values of σ_1. Therefore, in a right neighborhood of π_1 the HJB we have that

1. w_a'' is bounded by equation (57)

2. v_1'' is bounded by equation (56) and point 1.

3. v_2'' is bounded by equation (55) and point 1.

4. w_b'' is bounded by equation (58) and points 1,2 and 3,

proving that the RHS is not going to ∞ and $\pi_1^* \to \pi_{myopic}$. Since the same holds for the first best, we obtain the result. □

Proposition 10. *When sellers are allowed to use multilateral contracts, the equilibrium is efficient.*

Proof of Proposition 10. Let $p_{a,1}$, $p_{a,2}$,$p_{a,1}$, $p_{b,2}$ denote the equilibrium prices under competition without multilateral contracts (cf. Proposition 8). It is just bookeeping to check that the following profile of Markov strategies is a welfare maximizing equilibrium.

- Seller a continues to use the pricing strategies in the equilibrium without multilateral contracts:

$$\begin{aligned} t_{a,1}^1(\pi) &= p_{a,1}(\pi) \\ t_{a,2}^2(\pi) &= p_{a,2}(\pi) \\ t_{a,1}^2(\pi) &= -v_2''(\pi)\frac{g(\pi,h,\ell)}{2\rho\sigma_1^2} \\ t_{a,2}^1(\pi) &= -v_1''(\pi)\frac{g(\pi,h,\ell)}{2\rho\sigma_2^2}. \end{aligned}$$

- Seller b asks for the transfers:

$$\begin{aligned} t_{b,1}^1(\pi) &= p_{b,1}(\pi) \\ t_{b,2}^2(\pi) &= p_{b,2}(\pi) \\ t_{b,1}^2(\pi) &= v_2''(\pi)\frac{g(\pi,h,\ell)}{2\rho\sigma_1^2} \\ t_{b,2}^1(\pi) &= v_1''(\pi)\frac{g(\pi,h,\ell)}{2\rho\sigma_2^2}. \end{aligned}$$

- Buyers accept the multilateral contract $\left(t_{b,i}^1, t_{b,i}^2\right)$, $i \in \{1,2\}$ if $\pi \geq \pi_{\text{fb},i}$ and the multilateral contract $\left(t_{a,i}^1, t_{a,i}^2\right)$ otherwise.

Indeed, recall that under the first best satiation, welfare maximizing condition imposes

$$\mathbf{E}_{\pi_{\text{fb},1}}[\theta] - \mu_a + \frac{1}{2\rho\sigma_1^2} g(\pi_{\text{fb},1})\big(w_a'' + w_b'' + v_1'' + v_2''\big)(\pi_{\text{fb},1}) = 0.$$

Instead the equilibrium cutoff is determined by where both the seller a and the seller b are indifferent to sell to buyer 1. Seller a's indifference condition:

$$\begin{aligned} t_{a,1}^1(\pi_1^*) + t_{a,1}^2(\pi_1^*) =& \mu_a - \mathbf{E}_{\pi_1^*}[\theta] - \frac{1}{2\rho\sigma_1^2} g(\pi^*)\big(v_1''(\pi_1^*) + w_b''(\pi_1^*)\big) - \frac{1}{2\rho\sigma_1^2} g(\pi_1^*) v_2''(\pi_1^*) \\ =& -\frac{1}{2\rho\sigma_1^2} g(\pi_1^*) w_a''(\pi_1^*). \end{aligned}$$

Seller b's indifference condition:

$$\begin{aligned} t_{b,1}^1(\pi_1^*) + t_{b,1}^2(\pi_1^*) =& \mathbf{E}_{\pi_1^*}[\theta] - \mu_a + \frac{1}{2\rho\sigma_1^2} g(\pi_1^*)\big(v_1''(\pi_1^*) + w_a''(\pi_1^*)\big) + \frac{1}{2\rho\sigma_1^2} g(\pi_1^*) v_2''(\pi_1^*) \\ =& -\frac{1}{2\rho\sigma_1^2} g(\pi_1^*) w_b''(\pi_1^*). \end{aligned}$$

Similar equalities hold at π_2^* as well.

□

Proposition 11. *Let $\pi_2 < \pi_0 < \pi_1$. Define $\sigma(y) := \frac{y(1-y)(h-\ell)}{\sigma_2}$. Then,*

$$\mathbf{E}_{\pi_0}[\tau] = \frac{\pi_1 - \pi_0}{\pi_1 - \pi_2} \int_{\pi_2}^{\pi_0} (y - \pi_2) \frac{2dy}{\sigma^2(y)} + \frac{\pi_0 - \pi_2}{\pi_1 - \pi_2} \int_{\pi_0}^{\pi_1} (\pi_1 - y) \frac{2dy}{\sigma^2(y)}.$$

Particularly, the expected length of the beta phase is strictly increasing (decreasing) in the initial opinion for sufficiently low (high) initial opinions π_0.
Moreover, the following hold.

- *The probability of discarding the new product as a failure is $Pr_{\pi_0}\{\text{discarding}\} = Pr_{\pi_0}\{\pi(\tau) = \pi_2\} = \frac{\pi_1 - \pi_0}{\pi_1 - \pi_2}$.*
- *The probability that the new product serves the whole market is $Pr_{\pi_0}\{\textit{full adoption}\} = Pr_{\pi_0}\{\pi(\tau) = \pi_1\} = \frac{\pi_0 - \pi_2}{\pi_1 - \pi_2}$.*

Particularly, $\partial_{\pi_0} Pr_{\pi_0}\{\text{discarding}\} < 0$, and $\partial_{\pi_0} Pr_{\pi_0}\{\textit{full adoption}\} > 0$.

Proof of Proposition 11. We first note that the corresponding comparative statistics are immediate from the explicit characterizations of $\mathbf{E}_{\pi_0}[\tau]$, $Pr_{\pi_0}\{\text{full adoption}\}$ and $Pr_{\pi_0}\{\text{discarding}\}$

and the fact that the endogenous π_2 and π_1 do not depend on π_0 (see Proposition 4). Next, we explicitly derive $\mathbf{E}_{\pi_0}[\tau]$, $Pr_{\pi_0}\{\text{full adoption}\}$ and $Pr_{\pi_0}\{\text{discarding}\}$. To prove this statement we prepare two preliminary results.

Lemma [Extended Feynman-Kac Formula]. *Let $\Phi(x), f(x), F(x)$, $x \in [\pi_2, \pi_1]$, be continuous functions (f is non-negative). Let $u(x)$, $x \in [\pi_2, \pi_1]$ be a solution to*

$$\frac{\sigma^2(x)}{2}u''(x) - (\lambda + f(x))u(x) = -\lambda\Phi(x) - F(x), \quad x \in [\pi_2, \pi_1]$$

and $u(\pi_2) = \Phi(\pi_2)$ and $u(\pi_1) = \Phi(\pi_1)$ then

$$u(x) = \mathbf{E}_x\left[\Phi\Big(\pi_{\tau\wedge\mathcal{H}(\pi_2,\pi_1)}\Big)e^{-\int_0^{\tau\wedge\mathcal{H}(\pi_2,\pi_1)}}f(\pi_s)ds + \int_0^{\tau\wedge\mathcal{H}(\pi_2,\pi_1)} F(\pi_s)e^{-\int_0^s f(\pi_r)dr}ds\right]$$

where τ is random variable with the density $\lambda e^{-\lambda t}\mathbf{1}_{t\in[0,\infty)}$.

The proof of this result follows by a simple extension of the celebrated Feynman-Kac formula, omitted. Next, for ease of notation let us define $\mathcal{H}(\pi_2, \pi_1) = \inf\{t : \pi_t \notin (\pi_2, \pi_1)\}$.

Lemma 6. $\mathbf{E}_{\pi_0}[\mathcal{H}(\pi_2, \pi_1)] < \infty$.

Proof. The proof follows from the extended Feynman-Kac Formula. To show it, consider a family of functions $\{u_\lambda(x) : x \in [\pi_2, \pi_1]\}_{\lambda\geq 0}$ that are solution to the following $\lambda-$parametric problem:

$$\frac{\sigma^2(x)}{2}u''(x) - \lambda u(x) = -1, \quad x \in [\pi_2, \pi_1] \tag{81}$$

and $u(\pi_2) = u(\pi_1) = 0$. From the extended Feynman-Kac Formula it follows that $u_\lambda(x) = \mathbf{E}_x[\tau \wedge \mathcal{H}(\pi_2, \pi_1)]$ for $\lambda > 0$. Next, we argue that $\sup_{\lambda>0} u_\lambda(x) \leq u_0(x)$, where $u_0(x)$ solves (81) when $\lambda = 0$.

Next, since $\lim_{\lambda\to 0}\tau = \infty$ thus $\lim_{\lambda\to 0}\tau \wedge \mathcal{H}(\pi_2, \pi_1) = \mathcal{H}(\pi_2, \pi_1)$. Therefore $\mathbf{E}_{\pi_0}[\mathcal{H}(\pi_2, \pi_1)] < \infty$, finishing the proof. □

Next, we present two useful corollaries.

Corollary 1. *Let $f(x)$ and $F(x)$, $x \in [\pi_2, \pi_1]$, be continuous functions and $f(x)$ be non-negative. Let the function Φ be defined only at two points π_2 and π_1. Then the function*

$$q(x) = \mathbf{E}_x\left[\Phi(\pi_{\mathcal{H}(\pi_2,\pi_1)})e^{-\int_0^{\mathcal{H}(\pi_2,\pi_1)} f(\pi_s)ds} + \int_0^{\mathcal{H}(\pi_2,\pi_1)} F(\pi_s)e^{-\int_0^s f(\pi_r)dr}ds\right] \tag{82}$$

is the solution of the following problem

$$\frac{\sigma^2(x)}{2}q''(x) - f(x)q(x) + F(x) = 0, \quad x \in [\pi_2, \pi_1], \tag{83}$$

and $q(\pi_2) = \Phi(\pi_2)$ *and* $q(\pi_1) = \Phi(\pi_1)$.

The proof of this corollary follows directly from extended Feynman-Kac Formula by assuming $\lambda = 0$, replacing $u(x)$ with $q(x)$.

Corollary 2. *The solution of the problem*

$$\frac{\sigma^2(x)}{2}q''(x) + F(x) = 0, \quad x \in [\pi_2, \pi_1],$$

$q(\pi_2) = \Phi(\pi_2)$ *and* $q(\pi_1) = \Phi(\pi_1)$ *has the following form*

$$q(x) = \frac{\pi_1 - x}{\pi_{\text{fb},1} - \pi_2}\left(\Phi(\pi_2) + \int_{\pi_2}^{x}(y - \pi_2)\frac{2F(y)}{\sigma^2(y)}dy\right) + \frac{x - \pi_2}{\pi_1 - \pi_2}\left(\Phi(\pi_1) + \int_{x}^{\pi_1}(\pi_1 - y)\frac{2F(y)}{\sigma^2(y)}dy\right).$$

The proof of this corollary directly follows from extended Feynman-Kac Formula.

Using the above two corollaries, we have $\Pr_{\pi_0}\{\pi_\tau = \pi_2\} = \frac{\pi_1-\pi_0}{\pi_1-\pi_2}$ and $\Pr_{\pi_0}\{\pi_\tau = \pi_1\} = \frac{\pi_0-\pi_2}{\pi_1-\pi_2}$. These results follow from the above corollaries by assuming $F = f = 0, \Phi(\pi_2) = 1$ and $\Phi(\pi_1) = 0$.

In addition

$$\mathbf{E}_{\pi_0}[\tau] = \mathbf{E}_{\pi_0}[\mathcal{H}(\pi_2, \pi_1)] = = \frac{\pi_1 - \pi_0}{\pi_1 - \pi_2}\int_{\pi_2}^{\pi_0}(y - \pi_2)\frac{2dy}{\sigma^2(y)} + \frac{\pi_0 - \pi_2}{\pi_1 - \pi_2}\int_{\pi_0}^{\pi_1}(\pi_1 - y)\frac{2dy}{\sigma^2(y)}$$

which follows by the above corollaries by assuming $F = 1, f = 0, \Phi(\pi_2) = \Phi(\pi_1) = 0$ (implying $q(\pi_0) = \mathbf{E}_{\pi_0}[\mathcal{H}(\pi_2, \pi_1)]$ is the solution to (83)).

By these results, the proof of the proposition is now complete.

□

Viscosity solution

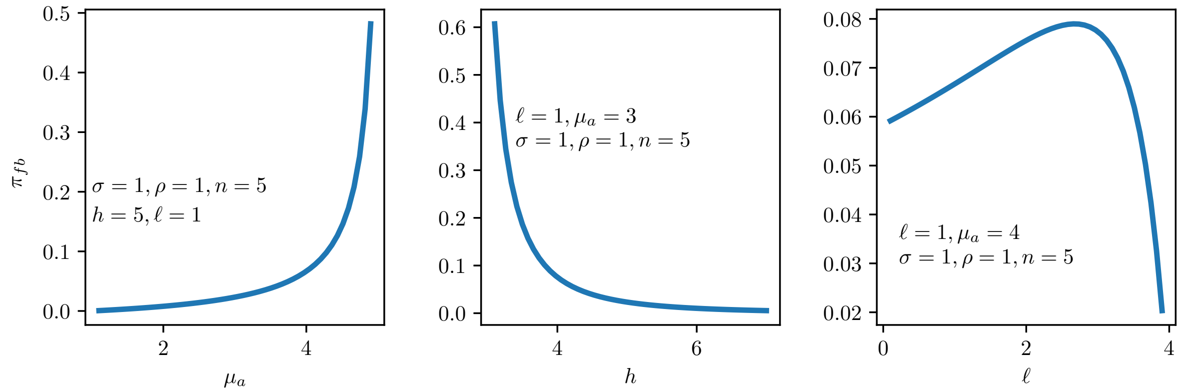
We show in this section that the buyer i's value function can be characterized as a viscosity solution to its associated dynamic programming equation. The same statement holds for sellers' value functions as well.

Define buyer i's value function for a given pricing strategy of the sellers as

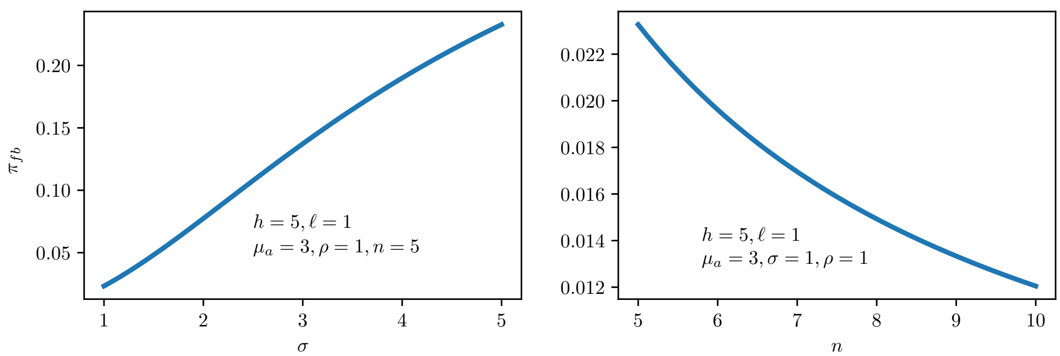
$$v_i(\pi) = \sup_{\xi_{ik}, k\in\{a,b\}} \mathbf{E}\left[\int_0^\infty \rho e^{-\rho t}\, \xi_{ik}(t)\Big(dC_{ki}(t) - p_{k,i}(t)dt\Big)\right]. \tag{84}$$

Theorem 1. *If p is an equilibrium pricing strategy for the seller, then v_i is a viscosity solution to the HJB equation in Section 4.*

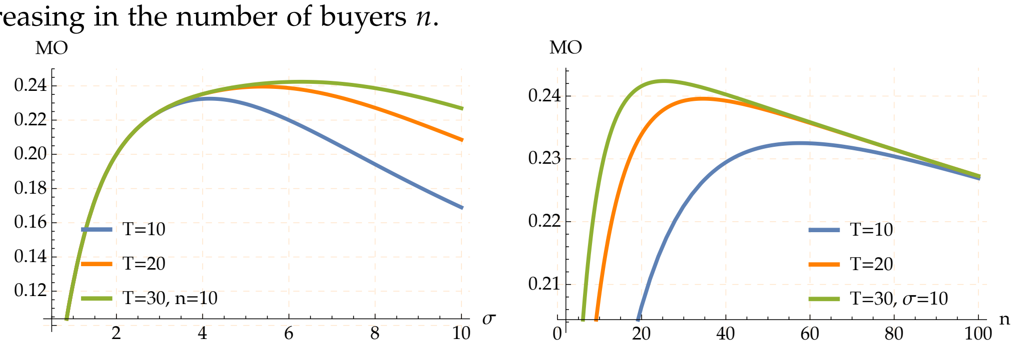
Proof. The result is standard in the theory of viscosity solutions. If v_i is locally bounded, the results follow from Propositions 4.3.1 and 4.3.2 in Pham (2009). To verify that v_i is locally bounded, we will show v_i is globally bounded on $[0,1]$. To this end, choosing a sub-optimal strategy $\xi_{ik} \equiv 0$ guarantees that v_i is nonnegative. Moreover, the continuation value of each market participant is always weakly smaller than W, which is globally bounded from above by nh. Therefore, v_i is globally bounded from above by nh as well. □



(a) The first-best belief threshold π_{fb} is increasing in the consumption utility of the old product and decreasing in the high consumption utility h and nonmonotone in the low consumption utility ℓ of the new product.



(b) The first-best belief threshold π_{fb} is increasing in the volatility σ^2 of the unknown product and decreasing in the number of buyers n.



(c) These panels show how the market optimism $MO(\pi_{\text{fb}}, \pi_{\text{fb}}, T)$ evolves with changing the volatility σ and the number of of buyers for different horizons $T = 10, 20$ and 30. As is shown, market optimism is nonmonotone in changing σ and n.

Figure 6: First-best equilibrium with symmetric buyers