Problem 1.1 *(Phase Transitions in the Erdös-Renyi Model)*
Consider an Erdös-Renyi random graph $G(n, p)$.

(a) Let A_l denote the event that node 1 has at least $l \in \mathbb{Z}^+$ neighbors. Do we observe a phase transition for this event? If so, find the threshold function and justify your reasoning.

(b) Let B denote the event that a cycle with k edges (for a fixed k) emerges in the graph. Do we observe a phase transition for this event? If so, find the threshold function and justify your reasoning.

Problem 1.2 *(Problem 1.2 from Jackson)*
Fix the probability of any given link forming in an Erdös-Renyi network to be p with $0 < p < 1$. Fix some arbitrary network g on k nodes. Now, consider a sequence of random networks indexed by the number of nodes n, as $n \to \infty$. Show that the probability that a copy of the k-node network g is a subnetwork of the random network on the n nodes goes to 1 as n goes to infinity.

[*Hint:* Partition the n nodes into as many separate groups of k nodes as possible (with some leftover nodes) and consider the subnetworks that form on each of these groups. Using independence of link formation, show that the probability that none of these match the desired network goes to 0 as n grows.]

Problem 1.3 *(Clustering in the Configuration Model)*

(a) Consider a graph g with n nodes generated according to the configuration model with a particular degree distribution $P(d)$. Show that the overall clustering coefficient is given by

$$Cl(g) = \frac{\langle d \rangle}{n} \left[\frac{\langle d^2 \rangle - \langle d \rangle}{\langle d \rangle^2} \right]^2,$$

where $\langle d \rangle$ is the expected degree under distribution $P(d)$, i.e., $\langle d \rangle = \sum_d dP(d)$ and similarly $\langle d^2 \rangle = \sum_d d^2 P(d)$.

(b) *(Optional for Bonus)*: Consider a power-law degree distribution $P(d)$ given by

$$P(d) = cd^{-\alpha} \qquad \text{for } \alpha < 3.$$

Show that the overall clustering coefficient satisfies

$$Cl(g) \sim n^{-\beta}, \qquad \beta = \frac{3\alpha - 7}{\alpha - 1}.$$

Discuss the monotonicity properties of the overall clustering coefficient as a function of n for different values of α.

Problem 1.4 *(Clustering in the Small World Model)*

(a) Consider the small-world model of Watts and Strogatz with rewiring probability p. Show that when $p = 0$, the overall clustering coefficient of this graph is given by

$$Cl(g) = \frac{3k - 3}{4k - 2}.$$

(b) *(Optional for Bonus)*: Show that when $p > 0$, the overall clustering coefficient is given by

$$Cl(g) = \frac{3k - 3}{4k - 2} (1 - p)^3.$$