

DISTRIBUTIONS OF POSTERIOR QUANTILES MADE EASY

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ABSTRACT. We offer a simple analysis of the problem of choosing a statistical experiment to optimize the induced distribution of posterior medians, or more generally q -quantiles for any $q \in (0, 1)$. We show that all implementable distributions of the posterior q -quantile are implemented by a single experiment, the *q -quantile matching experiment*, which pools pairs of states across the q -quantile of the prior in a positively assortative manner, with weight q on the lower state in each pair. A dense subset of implementable distributions of posterior q -quantiles can be uniquely implemented by perturbing the q -quantile matching experiment. A linear functional is optimized over distributions of posterior q -quantiles by taking the optimal selection from each set of q -quantiles induced by the q -quantile matching experiment.

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1. INTRODUCTION

Several problems of recent economic interest amount to characterizing the set of distributions of posterior quantiles that can be induced by some statistical experiment, or to finding a distribution in this set that maximizes some objective. These problems include *apparent overconfidence* (Benoît and Dubra, 2011)—e.g., what distributions of medians of individuals’ beliefs about their own abilities are consistent with Bayesian updating?; *partisan gerrymandering* (Friedman and Holden 2008; Kolotilin and Wolitzky 2020b)—e.g., what is the highest distribution of district median voters attained by any districting plan?; and *quantile persuasion* (Kolotilin and Wolitzky, 2020a)—e.g., what experiment maximizes the expected action of a receiver who minimizes the expected absolute deviation of her action from the unknown state of the world?¹

We show that this is an easy problem. For any $q \in (0, 1)$, there is a single experiment—the *q-quantile matching experiment*—that simultaneously implements all implementable distributions of the posterior q -quantile. For example, if the state is uniformly distributed on $[0, 1]$ and the relevant quantile is the median, the q -quantile matching experiment is the *median-matching experiment* that, whenever the true state is $\theta \in [0, 1/2]$, reveals only that the state is either θ or $1/2 + \theta$.² In general, the q -quantile matching experiment pools pairs of states across the q -quantile of the prior in a positively assortative manner, with weight q on the lower state in each pair.

To see why the q -quantile matching experiment implements all implementable distributions of the posterior q -quantile, consider again the median-matching experiment with a uniform state. When the experiment reveals that the state is θ or $1/2 + \theta$ with equal probability, every value $x \in [\theta, 1/2 + \theta]$ is a posterior median. In particular, the median-matching experiment simultaneously implements (i) the distribution $\underline{H}(x) = \max\{0, 2x - 1\}$, (ii) the distribution $\overline{H}(x) = \min\{2x, 1\}$, and (iii) every distribution H satisfying $\underline{H} \leq H \leq \overline{H}$. Conversely, simple Markov-type inequalities imply that every implementable distribution is bounded by $\underline{H}$ and $\overline{H}$. Moreover, the set of implementable distributions of unique posterior quantiles is essentially the same:

¹Yang and Zentefis (2023) explore these and other applications. Kolotilin and Wolitzky (2020b) consider a more general gerrymandering model, which reduces to optimizing the distribution of posterior quantiles in a special case. Kolotilin and Wolitzky (2020a) introduce quantile persuasion as a special case of a more general persuasion model, which is further developed in Kolotilin, Corrao, and Wolitzky (2023).

²The median-matching experiment coincides with the median one-to-one matching introduced by Kremer and Maskin (1996) and further studied by Legros and Newman (2002).

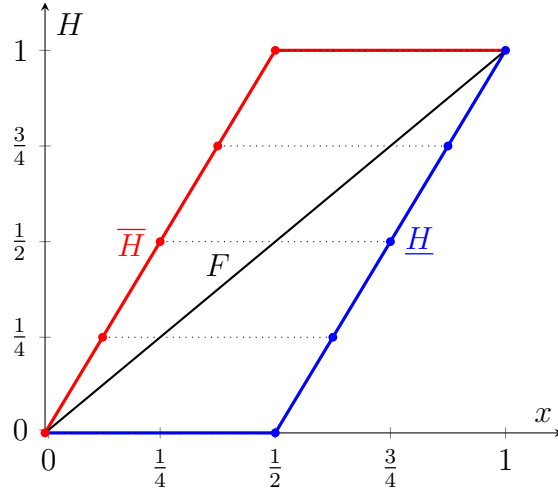


FIGURE 1. Implementable Distributions of Posterior Medians

Notes: When the prior F is uniform on $[0, 1]$, $\bar{H}$ and $\underline{H}$ are the lowest and highest implementable distributions of posterior medians. A distribution H is implementable iff $\underline{H} \leq H \leq \bar{H}$. Optimizing a linear functional over distributions of posterior medians requires taking the optimal selection from each horizontal dotted line. For example, the blue (red) dots are the optimal selections for an increasing (decreasing) objective function.

any desired selection from each set of q -quantiles induced by the q -quantile matching experiment can be uniquely selected by assigning it an extra ε probability. Finally, optimizing a linear functional over distributions of posterior quantiles simply requires taking the optimal selection from each set of q -quantiles induced by the q -quantile matching experiment. See Figure 1 for an illustration of our results.³

The current paper is closely related to [Benoît and Dubra \(2011\)](#) and [Yang and Zentefis \(2023\)](#). Both papers establish results that are very similar to our Theorem 1. Our main contribution is to provide a much simpler alternate proof by means of the q -quantile matching experiment. [Yang and Zentefis \(2023\)](#) additionally characterize the extreme points of the set of implementable distributions of posterior quantiles, which they use (together with a rather intricate argument summarized at the end of Section 2) to derive essentially the same result as our Theorem 1. In contrast, we show that directly characterizing the set of implementable and optimal distributions of

³Similar figures in the literature include Figure 1 of [Owen and Grofman \(1988\)](#), Figure 2 of [Kamenica and Gentzkow \(2011\)](#), and Figure 3 of [Yang and Zentefis \(2023\)](#).

posterior quantiles is much easier than characterizing the extreme points of this set.⁴ Nonetheless, Yang and Zentefis's extreme point characterization remains useful, for example, for identifying distributions of posterior quantiles that can arise as unique solutions to a linear optimization problem.

2. IMPLEMENTABLE DISTRIBUTIONS OF POSTERIOR QUANTILES

This section uses the q -quantile matching experiment to characterize the set of implementable distributions of posterior q -quantiles.

Let $\Theta = [\underline{\theta}, \bar{\theta}]$, with $\underline{\theta} < \bar{\theta}$, be a compact interval in $\mathbb{R}$; and let $\Delta(\Theta)$ be the set of probability distribution functions on Θ , endowed with the weak* topology. Fix a prior distribution $F \in \Delta(\Theta)$ and a quantile of interest $q \in (0, 1)$. Following Kamenica and Gentzkow (2011), define an *experiment* as a distribution $\tau \in \Delta(\Delta(\Theta))$ of posterior distributions $G \in \Delta(\Theta)$ such that $\int G d\tau(G) = F$. For each posterior G , define the set of q -quantiles of G as

$$X(G) = \{x \in \Theta : G(x^-) \leq q \leq G(x)\},$$

where $G(x^-)$ denotes the left limit $\lim_{\theta \uparrow x} G(\theta)$, with the convention $G(\underline{\theta}^-) = 0$. In addition, for each G , define its generalized inverse G^{-1} as

$$G^{-1}(p) = \inf\{\theta \in \Theta : G(\theta) \geq p\}, \quad \text{for all } p \in [0, 1].$$

That is, $G^{-1}(p)$ is the smallest p -quantile of G .

A distribution H of q -quantiles is *implementable* by an experiment τ if there exists a selection χ from X such that the distribution of $\chi(G)$ induced by τ is H . A distribution H of q -quantiles is *uniquely implementable* by an experiment τ if H is implementable by τ and $X(G)$ is a singleton for τ -almost all G . Let $\mathcal{H}$ and $\mathcal{H}^*$ be the sets of implementable and uniquely implementable distributions of q -quantiles.

The following theorem characterizes $\mathcal{H}$ and $\mathcal{H}^*$.

Theorem 1. *Let $\underline{H}(x) = \max\{0, (F(x) - q)/(1 - q)\}$ and $\overline{H}(x) = \min\{F(x)/q, 1\}$ for all $x \in \Theta$. Then $\mathcal{H} = \{H \in \Delta(\Theta) : \underline{H} \leq H \leq \overline{H}\}$. Moreover, if F has a positive*

⁴Following an influential paper of Kleiner, Moldovanu, and Strack (2021), there is much current interest in using extreme points in economic optimization problems. While characterizing a convex set by its extreme points can be useful for optimizing a linear functional over the set, sometimes the optimization problem itself is easier than the problem of characterizing the extreme points. The current paper shows that this is the case for the problem of optimizing over distributions of posterior quantiles.

density on Θ then the closure of $\mathcal{H}^*$ is $\mathcal{H}$; so, for any continuous objective function $V : \Theta \rightarrow \mathbb{R}$, we have

$$\sup_{H \in \mathcal{H}^*} \int V(x) dH(x) = \max_{H \in \mathcal{H}} \int V(x) dH(x). \quad (1)$$

Figure 1 illustrates the set $\mathcal{H}$. To see the intuition for Theorem 1, consider the case where F is uniform on $[0, 1]$ and $q = 1/2$.⁵ First, by simple Markov-type inequalities, any implementable H must satisfy $\underline{H} \leq H \leq \overline{H}$. For example, if the posterior median is less than x with probability p , then θ must be less than x with probability at least $p/2$. When $F(x) = x$, this implies that $p \leq 2x$, so the probability that the posterior median is less than x is at most $\min\{2x, 1\} = \overline{H}(x)$.⁶

Conversely, to see that any H satisfying $\underline{H} \leq H \leq \overline{H}$ is implementable, define the *median-matching experiment* τ^* that induces only posteriors G_θ that assign equal probability to some $\theta \in [0, 1/2]$ and to $1/2 + \theta$. The set of medians of such a posterior is $X(G_\theta) = [\theta, 1/2 + \theta]$. At the same time, $H \leq \overline{H}$ implies that $H^{-1}(2\theta) \geq \theta$, and $H \geq \underline{H}$ implies that $H^{-1}(2\theta) \leq 1/2 + \theta$, so we have $H^{-1}(2\theta) \in [\theta, 1/2 + \theta]$. Thus, $\chi(G_\theta) = H^{-1}(2\theta)$ is a selection from $X(G_\theta)$. Finally, the distribution of $\chi(G_\theta)$ induced by τ^* is H , because the states that induce medians below x under τ^* with selection $\chi(G_\theta)$ are precisely those in $[0, H(x)/2]$ and $[1/2, 1/2 + H(x)/2]$, and the measure of these states is $H(x)$.

Finally, for unique implementation, for any small ε and any implementable and absolutely continuous H , we explicitly construct a modification of the median-matching experiment τ_ε^* that uniquely implements the distribution $(1 - \varepsilon)H + \varepsilon F$ of medians, by making every posterior G_θ a convex combination of the median matching distribution that assigns equal probability to θ and $1/2 + \theta$ and the degenerate distribution at the unique median $\chi(G_\theta) = H^{-1}(2\theta) \in [\theta, 1/2 + \theta]$. This step occupies the bulk of the proof of Theorem 1. A simple approximation argument then completes the proof.

The literature contains several close antecedents of Theorem 1. In a discrete setting in the context of partisan gerrymandering, [Friedman and Holden \(2008\)](#) essentially show that $\underline{H}$ is the highest implementable distribution of posterior medians. In a discrete setting in the context of apparent coverconfidence, [Benoît and](#)

⁵Taking F to be uniform is essentially without loss, by the Skorohod representation theorem. Taking $q = 1/2$ is also without loss, up to a simple replacement of variables.

⁶This argument is closely related to [Kamenica and Gentzkow's \(2011\)](#) “prosecutor-judge” example. As in their example, the key observation is that, if the prior probability of an event (e.g., the event that $\theta \leq x$) is x , then the maximum probability that the posterior probability of this event is at least $1/2$ is $\min\{2x, 1\}$.

Dubra (2011) show that the set of implementable distributions of posterior medians is $\mathcal{H} = \{H \in \Delta(\Theta) : \underline{H} \leq H \leq \overline{H}\}$. In a general setting in the contexts of quantile persuasion and partisan gerrymandering, respectively, Kolotilin and Wolitzky (2020a) and Kolotilin and Wolitzky (2020b) show that $\underline{H}$ is the highest implementable distribution of posterior medians. Finally, in a general setting, Yang and Zentefis (2023) show that the set of implementable distributions of posterior medians is $\mathcal{H} = \{H \in \Delta(\Theta) : \underline{H} \leq H \leq \overline{H}\}$. Relative to Yang and Zentefis (2023), our contribution is to provide a much simpler proof based on the q -quantile matching experiment.

In particular, Theorem 1 alternatively follows from Theorems 2, 3, and Proposition 2 in Yang and Zentefis (2023), but their proof is very different from ours. They first characterize the extreme points of the set $\mathcal{H} = \{H \in \Delta(\Theta) : \underline{H} \leq H \leq \overline{H}\}$. Then they construct a distinct experiment that implements each extreme point—this construction is complicated, because the extreme points H of $\mathcal{H}$ turn out to have a relatively complex structure involving a countable set of intervals where H is constant. They then invoke the Choquet theorem to conclude that each $H \in \mathcal{H}$ (being a convex combination of extreme points of $\mathcal{H}$) is implementable by an experiment (corresponding to the convex combination of the experiments that implement the extreme points). Their proof for the unique implementation case follows a similar logic: they first find a set $\mathcal{H}_\varepsilon \subset \mathcal{H}$ such that $\mathcal{H}_\varepsilon$ converges to $\mathcal{H}$ as $\varepsilon \rightarrow 0$; then construct a distinct experiment that uniquely implements each extreme point of $\mathcal{H}_\varepsilon$; then invoke Choquet to show that each $H \in \mathcal{H}_\varepsilon$ is uniquely implementable; and then take $\varepsilon \rightarrow 0$ to conclude that $\mathcal{H}^*$ is dense in $\mathcal{H}$.

3. OPTIMAL DISTRIBUTIONS OF POSTERIOR QUANTILES

This section uses the q -quantile matching experiment to characterize the distributions of posterior q -quantiles that optimize a continuous linear functional.

Theorem 2. *Fix a continuous function $V : \Theta \rightarrow \mathbb{R}$. Then H (uniquely) maximizes $\int V(x)dH(x)$ on $\mathcal{H}$ iff $H^{-1}(p)$ (uniquely) maximizes V on $[\overline{H}^{-1}(p), \underline{H}^{-1}(p)]$ for (almost) all $p \in [0, 1]$. Consequently, the value of the maximization problem is*

$$\max_{H \in \mathcal{H}} \int_{\Theta} V(x)dH(x) = \int_0^1 \max\{V(x) : x \in [\overline{H}^{-1}(p), \underline{H}^{-1}(p)]\} dp. \quad (2)$$

Up to some technical details, Theorem 2 follows immediately from Theorem 1. Consider again the case where F is uniform and $q = 1/2$. Since the median matching

experiment τ^* implements all implementable distributions of medians, optimization just requires selecting an optimal median $\chi(G_\theta) \in \arg \max_{x \in [\theta, 1/2 + \theta]} V(x)$ for each posterior G_θ induced by τ^* , as illustrated in Figure 1. The value of the maximization problem is thus $2 \int_0^{1/2} \max_{x \in [\theta, 1/2 + \theta]} V(x) d\theta$, and a distribution H of medians is optimal iff $H^{-1}(2\theta) \in \arg \max_{x \in [\theta, 1/2 + \theta]} V(x)$ for all $\theta \in [0, 1/2]$.

Corollary 5 of [Yang and Zentefis \(2023\)](#) solves the maximization problem (2) in the special cases where V is quasi-concave and quasi-convex. The solution follows easily from Theorem 2. To see how, suppose that F is uniform, $q = 1/2$, and V is quasi-concave with a maximum at $x^* \in [0, 1]$. Then, for each interval $[\theta, 1/2 + \theta]$, it is optimal to select x^* if $x^* \in [\theta, 1/2 + \theta]$, θ if $x^* < \theta$, and $1/2 + \theta$ if $x^* > 1/2 + \theta$. This induces the distribution of posterior medians

$$H(x) = \begin{cases} \underline{H}(x), & x < x^*, \\ \overline{H}(x), & x \geq x^*. \end{cases}$$

Next, suppose that V is quasi-convex with $V(x^\dagger) = V(1/2 + x^\dagger)$ for some $x^\dagger \in [0, 1/2]$. Then, for each interval $[\theta, 1/2 + \theta]$, it is optimal to select θ if $x^\dagger > \theta$ and $1/2 + \theta$ if $x^\dagger < \theta$. This induces the distribution of posterior medians

$$H(x) = \begin{cases} \overline{H}(x), & x < x^\dagger, \\ 2x^\dagger, & x \in [x^\dagger, 1/2 + x^\dagger], \\ \underline{H}(x), & x \geq 1/2 + x^\dagger. \end{cases}$$

From the perspective of optimization, it is natural to ask whether each extreme point of $\mathcal{H}$ is exposed, meaning that it is the unique maximizer in $\mathcal{H}$ of $\int V(x) dH(x)$ for some continuous function V . It turns out that some extreme points are not exposed. To see this, suppose that F is uniform and $q = 1/2$. By Theorem 1 in [Yang and Zentefis \(2023\)](#), the distribution H^* that assigns equal probability to $1/4$ and $1/2$ is an extreme point of $\mathcal{H}$. By our Theorem 2, if H^* maximizes $\int V(x) dH(x)$ on $\mathcal{H}$ for some continuous function V , then $V(1/4) \geq V(x)$ for all $x \in [\theta, 1/2 + \theta]$ and all $\theta \in [0, 1/4]$, and, similarly, $V(1/2) \geq V(x)$ for all $x \in [\theta, 1/2 + \theta]$ and all $\theta \in [0, 1/2]$. Thus, $V(1/4) = V(1/2) \geq V(x)$ for all $x \in [0, 1]$. But then the distribution $H^{**} \in \mathcal{H}$ that assigns probability 1 to $1/2$ also maximizes $\int V(x) dH(x)$, which shows that H^* is not an exposed point of $\mathcal{H}$.

4. PROOFS

Proof of Theorem 1. Consider any experiment $\tau \in \Delta(\Delta\Theta)$ and any measurable selection $\chi(G)$ from $X(G)$. Let H be the distribution of $\chi(G)$ induced by τ . Then, for each $x \in \Theta$, we have

$$\begin{aligned} F(x) &= \int G(x) d\tau(G) = \int \mathbf{1}\{G(x) \geq q\} G(x) d\tau(G) + \int \mathbf{1}\{G(x) < q\} G(x) d\tau(G) \\ &\geq \int \mathbf{1}\{G(x) \geq q\} q d\tau(G) \geq \int \mathbf{1}\{\chi(G) \leq x\} q d\tau(G) = qH(x), \end{aligned}$$

showing that $H \leq \overline{H}$. A symmetric argument shows that $H \geq \underline{H}$.

For the converse, let ω be uniformly distributed on $[0, 1]$. By Theorem 14.1 in [Billingsley \(2017\)](#), the distribution function of $F^{-1}(\omega)$ is F . Define the *q-quantile matching experiment* as an experiment τ^* such that for τ^* -almost all G , there exists $\omega \in [0, q]$ such that $G = G_\omega$, defined as the distribution that assigns probability q to $F^{-1}(\omega)$ and assigns probability $1 - q$ to $F^{-1}(q + \frac{1-q}{q}\omega)$.⁷ Formally, τ^* is defined by

$$\tau^*(M) = \int_0^q \mathbf{1}\{q\delta_{F^{-1}(\omega)} + (1-q)\delta_{F^{-1}(q + \frac{1-q}{q}\omega)} \in M\} \frac{d\omega}{q}, \quad \text{for all } M \subset \Delta(\Theta),$$

where hereafter δ_x , with $x \in \Theta$, denotes the degenerate distribution at x , so that $\delta_x(\theta) = \mathbf{1}\{\theta \geq x\}$ for all $\theta \in \Theta$. Note that τ^* is a well-defined experiment because $\int G d\tau^*(G) = F$. Indeed, for all $\theta \in \Theta$, we have

$$\begin{aligned} \int G(\theta) d\tau^*(G) &= \int_0^q \left(q\delta_{F^{-1}(\omega)} + (1-q)\delta_{F^{-1}(q + \frac{1-q}{q}\omega)} \right) (\theta) \frac{d\omega}{q} \\ &= \begin{cases} \int_0^{F(\theta)} q \frac{d\omega}{q}, & F(\theta) < q, \\ \int_0^q q \frac{d\omega}{q} + \int_0^{\frac{q}{1-q}(F(\theta)-q)} (1-q) \frac{d\omega}{q}, & F(\theta) \geq q, \end{cases} = F(\theta), \end{aligned}$$

where the second equality holds because $F^{-1}(\omega) \leq \theta$ iff $\omega \leq F(\theta)$, and $F^{-1}(q + \frac{1-q}{q}\omega) \leq \theta$ iff $\omega \leq \frac{q}{1-q}(F(\theta) - q)$. Note also that, for each $\omega \in [0, q]$, the set of q -quantiles of G_ω is $X(G_\omega) = [F^{-1}(\omega), F^{-1}(q + \frac{1-q}{q}\omega)]$.

Now fix a distribution $H \in \Delta(\Theta)$ satisfying $\underline{H} \leq H \leq \overline{H}$. Note that, for each $\omega \in [0, q]$, since $H \leq \overline{H}$, we have $H^{-1}(\frac{\omega}{q}) \geq \overline{H}^{-1}(\frac{\omega}{q}) = F^{-1}(\omega)$; and, since $H \geq \underline{H}$, we have $H^{-1}(\frac{\omega}{q}) \leq \underline{H}^{-1}(\frac{\omega}{q}) = F^{-1}(q + \frac{1-q}{q}\omega)$. Thus, $H^{-1}(\frac{\omega}{q}) \in X(G_\omega)$. We can

⁷For example, when F is atomless, we can let $\omega = F(\theta)$, so that the q -quantile matching experiment induces posteriors that assign probability q to θ and assign probability $1 - q$ to $F^{-1}(q + \frac{1-q}{q}F(\theta))$, for $\theta \in [0, F^{-1}(q)]$.

therefore define a selection $\chi(G)$ from $X(G)$ by letting $\chi(G) = H^{-1}(\frac{\omega}{q})$ in the τ^* -almost sure event that $G = G_\omega$ for some $\omega \in [0, q]$, and (for concreteness) letting $\chi(G) = \min X(G)$ otherwise. Finally, the distribution of $\chi(G)$ induced by τ^* is H , because, for all $x \in \Theta$, we have

$$\int \mathbf{1}\{\chi(G) \leq x\} d\tau^*(G) = \int \mathbf{1}\{H^{-1}(\frac{\omega}{q}) \leq x\} \frac{d\omega}{q} = \int_0^{qH(x)} \frac{d\omega}{q} = H(x).$$

For unique implementation, assume that F has a positive density on $\Theta = [\underline{\theta}, \bar{\theta}]$. Fix any $H \in \mathcal{H}$. Consider a sequence of partitions of Θ given by $\theta_{i,n} = \underline{\theta} + (\bar{\theta} - \underline{\theta})\frac{i}{2^n}$, with $i = 0, 1, \dots, 2^n$. Define a sequence $H_n \in \Delta(\Theta)$ by

$$H_n(x) = H(\theta_{i-1,n}) \frac{F(\theta_{i,n}) - F(x)}{F(\theta_{i,n}) - F(\theta_{i-1,n})} + H(\theta_{i,n}) \frac{F(x) - F(\theta_{i-1,n})}{F(\theta_{i,n}) - F(\theta_{i-1,n})},$$

for all $i = 1, \dots, 2^n$ and all $x \in [\theta_{i-1,n}, \theta_{i,n}]$. Note that H_n is well-defined, because F is strictly increasing on Θ . Since $H \in \mathcal{H}$, we have $H_n \in \mathcal{H}$. Moreover, H_n has a simple density function h_n with respect to F , given by

$$h_n(x) = \frac{H(\theta_{i,n}) - H(\theta_{i-1,n})}{F(\theta_{i,n}) - F(\theta_{i-1,n})},$$

for all $i = 1, \dots, 2^n$ and all $x \in (\theta_{i-1,n}, \theta_{i,n})$.

Next, for each $e \in (0, 1)$ and each n , there exists an experiment $\tau_{e,n}^* \in \Delta(\Delta(\Theta))$ satisfying the following two properties. First, for $\tau_{e,n}^*$ -almost all G , there exists $x \in \Theta$ such that $G = G^x$ where

$$G^x = \frac{(1-e)h_n(x)(q\delta_{F^{-1}(qH_n(x))} + (1-q)\delta_{F^{-1}(q+(1-q)H_n(x))}) + e\delta_x}{(1-e)h_n(x) + e}.$$

This implies that $X(G^x)$ is the singleton $\{x\}$, because $e > 0$ and $F^{-1}(qH_n(x)) \leq x \leq F^{-1}(q + (1-q)H_n(x))$ (which holds because $\underline{H}(x) \leq H_n(x) \leq \bar{H}(x)$). Second, the distribution of unique quantiles $\chi(G^x) = x$ induced by $\tau_{e,n}^*$ is $(1-e)H_n + eF$.

Formally, $\tau_{e,n}^*$ is defined by

$$\tau_{e,n}^*(M) = \int_0^1 \mathbf{1}\{G^x \in M\} ((1-e)h_n(x) + e) dF(x), \quad \text{for all } M \subset \Delta(\Theta).$$

Note that $\tau_{e,n}^*$ is a well-defined experiment because $\int G d\tau_{e,n}^*(G) = F$. Indeed, for all $\theta \in \Theta$, we have

$$\int G(\theta) d\tau_{e,n}^*(G) = \int_0^1 ((1-e)h_n(x)(q\delta_{F^{-1}(qH_n(x))} + (1-q)\delta_{F^{-1}(q+(1-q)H_n(x))}) + e\delta_x)(\theta) dF(x)$$

$$\begin{aligned}
&= (1-e) \begin{cases} \int_0^{H_n^{-1}(\frac{F(\theta)}{q})} q dH_n(x), & F(\theta) < q, \\ \int_0^{H_n^{-1}(1)} q dH_n(x) + \int_0^{H_n^{-1}(\frac{F(\theta)-q}{1-q})} (1-q) dH_n(x), & F(\theta) \geq q, \end{cases} \\
&\quad + e \int_0^\theta dF(x) = F(\theta),
\end{aligned}$$

where the second equality holds because $F^{-1}(qH_n(x)) \leq \theta$ iff $x \leq H_n^{-1}(\frac{F(\theta)}{q})$, and $F^{-1}(q + (1-q)H_n(x)) \leq \theta$ iff $x \leq H_n^{-1}(\frac{F(\theta)-q}{1-q})$; and the third equality holds because $H_n(H_n^{-1}(p)) = p$ for all $p \in [0, 1]$, by continuity of H_n (which holds because H_n has a density with respect to F and F has a density with respect to the Lebesgue measure). Finally, the distribution of unique quantiles $\chi(G)$ induced by $\tau_{e,n}^*$ is $(1-e)H_n + eF$, because, for all $y \in \Theta$, we have

$$\int \mathbf{1}\{\chi(G) \leq y\} d\tau_{e,n}^*(G) = \int_0^y ((1-e)h_n(x) + e)dF(x) = (1-e)H_n(y) + eF(y).$$

Now fix a continuous function V on the compact set Θ . By continuity, for all $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that (i) $|V(x) - V(y)| \leq \varepsilon$ for all $x, y \in [\theta_{i-1,n}, \theta_{i,n}]$, all $i = 1, \dots, 2^n$, and all $n \geq N$, and (ii) $e|V(x) - V(y)| \leq \varepsilon$ for all $x, y \in \Theta$ and all $e \in (0, \frac{1}{N}]$. Then, for all $n \geq N$ and $e \in (0, \frac{1}{N}]$, we have

$$\begin{aligned}
&\left| \int V(x) dH(x) - \int V(x) d((1-e)H_n + eF)(x) \right| \\
&\leq (1-e) \left| \int V(x) d(H - H_n)(x) \right| + e \left| \int V(x) d(H - F)(x) \right| \leq \varepsilon + \varepsilon.
\end{aligned}$$

Since this holds for any continuous V , it follows that $(1 - \frac{1}{n})H_n + \frac{1}{n}F$ converges weakly to H . In turn, since we have seen that the experiment $\tau_{1/n,n}^*$ uniquely implements $(1 - \frac{1}{n})H_n + \frac{1}{n}F$, we conclude that $\mathcal{H}^*$ is dense in $\mathcal{H}$. Finally, $\mathcal{H}$ is compact, as $\Delta(\Theta)$ is compact by Theorem 15.11 in [Aliprantis and Border \(2006\)](#), and $\mathcal{H}$ is the intersection over $x \in \Theta$ of the closed subsets $\mathcal{H}_x := \{H \in \Delta(\Theta) : \underline{H}(x) \leq H(x) \leq \overline{H}(x)\}$ of $\Delta(\Theta)$. Thus, the closure of $\mathcal{H}^*$ is $\mathcal{H}$, and hence (1) holds for any continuous V . $\square$

Proof of Theorem 2. By Theorem 14.1 in [Billingsley \(2017\)](#), for each $H \in \Delta(\Theta)$, we have

$$\int_{\Theta} V(x) dH(x) = \int_0^1 V(H^{-1}(p)) dp.$$

By Theorem 1, for each $H \in \mathcal{H}$ and $p \in [0, 1]$, we have $\overline{H}^{-1}(p) \leq H^{-1}(p) \leq \underline{H}^{-1}(p)$. Now consider the relaxed problem of finding a measurable function $J : [0, 1] \rightarrow \Theta$ to

$$\begin{aligned} & \text{maximize } \int V(J(p))dp \\ & \text{subject to } \overline{H}^{-1}(p) \leq J(p) \leq \underline{H}^{-1}(p), \quad \text{for all } p \in [0, 1]. \end{aligned}$$

One solution of the relaxed problem is

$$J^*(p) = \min \arg \max \{V(x) : x \in [\overline{H}^{-1}(p), \underline{H}^{-1}(p)]\}, \quad \text{for all } p \in [0, 1].$$

Indeed, since J^* is defined as the minimum selection from the $\arg \max$, it follows that (i) J^* is non-decreasing (and hence measurable), because $\overline{H}^{-1}$ and $\underline{H}^{-1}$ are non-decreasing, (ii) J^* is left-continuous, because $\overline{H}^{-1}$ and $\underline{H}^{-1}$ are left-continuous and V is continuous, (iii) $J^*(1) \leq \bar{\theta}$, because $\underline{H}^{-1}(1) \leq \bar{\theta}$, and (iv) $J^*(0) = \underline{\theta}$, because $\overline{H}^{-1}(0) = \underline{H}^{-1}(0) = \underline{\theta}$. This implies that $J^* = H^{*-1}$, where $H^* \in \Delta(\Theta)$ is given by

$$H^*(x) = \sup\{p \in [0, 1] : J^*(p) \leq x\}, \quad \text{for all } x \in \Theta.$$

Moreover, since $\overline{H}^{-1} \leq J^* \leq \underline{H}^{-1}$, it follows that $H^* \in \mathcal{H}$, so H^* solves the original problem, and its value coincides with the value of the relaxed problem, yielding (2). Consequently, $H \in \mathcal{H}$ maximizes $\int V(x)dH(x)$ on $\mathcal{H}$ iff $H^{-1}(p)$ maximizes V on $[\overline{H}^{-1}(p), \underline{H}^{-1}(p)]$ for almost all $p \in [0, 1]$. Moreover, by continuity of V and left-continuity of $\overline{H}^{-1}$ and $\underline{H}^{-1}$, $H^{-1}(p)$ maximizes V on $[\overline{H}^{-1}(p), \underline{H}^{-1}(p)]$ for almost all $p \in [0, 1]$ iff it does so for all $p \in [0, 1]$.

Next, if V has a unique maximum on $[\overline{H}^{-1}(p), \underline{H}^{-1}(p)]$ for almost all $p \in [0, 1]$, then J^* is the unique solution of the relaxed problem that satisfies properties (i)–(iv), and hence H^* is the unique solution of the original problem. Conversely, if there exists a non-negligible set $P \subset [0, 1]$ such that V has multiple maxima on $[\overline{H}^{-1}(p), \underline{H}^{-1}(p)]$ for each $p \in P$, then there are multiple solutions of the relaxed problem that satisfy properties (i)–(iv). For example, $J^\dagger$ defined as the maximum selection from the $\arg \max$ also solves the relaxed problem, and so does J^{**} defined by $J^{**}(p) = J^\dagger(p^-)$ for all $p \in (0, 1]$ and $J^{**}(0) = \underline{\theta}$. But, by construction, J^{**} satisfies properties (i)–(iv) and is not equal to J^* . Then $J^{**} = H^{**,-1}$ where $H^{**} \in \Delta(\Theta)$ is given by

$$H^{**}(x) = \sup\{p \in [0, 1] : J^{**}(p) \leq x\}, \quad \text{for all } x \in \Theta.$$

Thus, $H^{**} \neq H^*$ also solves the original problem. □

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